NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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INTRODUCTION

In the sequel, $K$ is a non-archimedean valued field, complete, with residue class field $k$. Our aim is to present reasonable definitions for a function $f : X \to K$ to be "monotone". ($X$ is a subset of $K$). Since $K$ admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \to \mathbb{R}$, equivalent to "$f$ is monotone", and such that these statements have translations in $K$ that make sense. This way we obtain several definitions of "$f : X \to K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of $K" ", "the sign of a nonzero element of $K" .

In Section 2 we define several notions of monotony. E.g.,

$f \in M_D(X)$ if $x$ between $y$ and $z$ implies $f(x)$ between $f(y)$ and $f(z)$ and

$f \in M_S(X)$ if $f(x)$ between $f(y)$ and $f(z)$ implies $x$ between $y$ and $z$.

Also monotone functions of type $\sigma$ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M_D(X)$ (or $f \in M_S(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
mation on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of $K$ and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: $f' > 0 \iff f$ increasing.

The notion of pseudo-ordering ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

Notations. Let $p$ be a prime. By $\mathbb{F}_p$ we mean the field of $p$ elements. By $\mathbb{Q}_p$ the non-archimedean valued field of the $p$-adic numbers. For a field $L$ we denote its characteristic by $\chi(L)$. Let $E$ be a vector space over $K$ and $S \subseteq E$. By $[S]$ we denote the smallest $K$-linear subspace of $E$ that contains $S$. 
1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let $x, y \in K$. Then the smallest ball in $K$ containing $x$ and $y$ is denoted by $[x, y]$. A subset $C$ of $K$ is called convex if $x, y \in C$ implies $[x, y] \subset C$.

Sometimes we use a more geometric terminology. Instead of $z \in [x, y]$ we will say that $z$ is between $x$ and $y$ and instead of $z \notin [x, y]$ we use the expression: $x$ and $y$ are at the same side of $z$. Notice that $[x, y] = [y, x]$ for all $x, y \in K$ and that $z \in [x, y] \iff |z-x| \leq |x-y| \iff |z-y| \leq |x-y| \iff z = \lambda x + (1-\lambda)y$ for some $\lambda \in K$, $|\lambda| \leq 1$. If $x \neq y$ then the $\lambda$ in this last expression is unique (viz. $\lambda = \frac{z-y}{x-y}$).

Examples of convex sets are: the empty set, singletons, balls, $K$. It is an easy exercise to show that these are the only convex subsets of $K$. So formally we may write each convex subset of $K$ as

$$\{x \in K : |x-a| < r\} \quad (a \in K, 0 \leq r \leq \infty)$$

or as

$$\{x \in K : |x-a| \leq r\} \quad (a \in K, 0 \leq r \leq \infty)$$

Notice that the only unbounded convex subset of $K$ is $K$ itself.

Sometimes we need the notion of convexity with respect to a subset $X$ of $K$. A subset $C \subset X$ is called convex in $X$ if $x, y \in C$ implies $[x, y] \cap X \subset C$ or, equivalently, if $C$ is the intersection of $X$ with a convex subset of $K$.

Let $x, y, z \in K$. By the strong triangle inequality we have that the "triangle" $x, y, z$ is isosceles, say $|x-y| = |y-z|$. Then $|x-z| \leq |x-y|$, so $z$ is between $x$ and $y$ and $x$ is between $y$ and $z$. If also $|x-y| = |x-z|$
then $y$ is between $x$ and $z$. Otherwise, $x$ and $z$ are at the same side of $y$.

The relation $\sim$ defined on $K^* := K \setminus \{0\}$ by

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \text{ (}x,y \in K^*)$$

is an equivalence relation. We have $x \sim y$ iff $0 \notin [x,y]$ i.e. iff $|x-y| < |x|$ (=$|y|$) i.e. iff $|xy^{-1}| < 1$. Define

$$K^+ := \{x \in K : |1-x| < 1\}$$

Then $K^+$ is a multiplicative subgroup of $K^*$, $K^+ = \{x \in K^* : x \sim 1\}$ and is called the set of the positive elements of $K$. The relation $\sim$ is also induced by the canonical group homomorphism

$$\pi : K^* \to K^*/K^+.$$ Thus, $x \sim y$ if and only if $\pi(x) = \pi(y)$ (=$x,y \in K^*$). Therefore it is natural to view the group $\Sigma := K^*/K^+$ as being the group of signs of elements of $K^*$, and we call $\pi(x)$ the sign of the element $x \in K^*$. If $x \in K^*$ then $\pi(x) = \{y : |y-x| < |x|\} = xK^+$. For $x \in K^*$, $a \in \Sigma$ we sometimes write $xa$ to indicate the element $\pi(x)a$ of $\Sigma$. In particular, for $a \in \Sigma$ the opposite sign of $a$, $-a$, is defined as $(-1)a$. Then $-a = \{-x : x \in a\}$. (Notice that in case $\chi(K) = 2$ we have $a = -a$.)

Let $a \in \Sigma$. Then for $x,y \in a$ we have $|x| = |y|$ so we can define the absolute value of $a$, $|a|$ as follows

$$|a| := |x| \quad (x \in \pi^{-1}(a)).$$

In the sequel we also need addition between elements of $\Sigma$. Let us first observe that for any $a,b \in \Sigma$ the sum

$$a+b := \{x+y : x \in a, y \in b\}$$

is always a ball with radius $\max(|a|,|b|)$. (I.e., of the form...
\{x : |x-b| < \max(|a|,|\beta|)\}. Now a+\beta contains 0 if and only if $$a = -\beta.$$ Otherwise a+\beta is again an element of \(E\). (Proof: Let \(a \in a, \ b \in \beta\). Then \(|a+b| = \max(|a|,|b|)\). If also \(x \in a, \ y \in \beta\) then \(|x+y-(a+b)| \leq \max(|x-a|,|y-b|) < \max(|a|,|b|) = |a+b|\). Thus a+\beta contains the ball with center a+b and radius \(\max(|a|^{-1},|\beta|^{-1})\), so a+\beta is equal to this ball.)

Let us define

\[a \circ \beta := a+\beta = \{x+y : x \in a, \ y \in \beta\} \quad (a, \beta \in E, \ a \neq -\beta).\]

We have

**THEOREM 1.2** Let \(E, || : E \to \mathbb{R}, \circ : E \times E \setminus \{(a,-a) : a \in E\} \to E\) be as above. Let \(a, \beta, \gamma \in E\). Then

(i) \(|a\beta| = |a| \cdot |\beta| \cdot |a^{-1}| = |a|^{-1}\).

(ii) If \(a \circ \beta\) is defined then so is \(\beta \circ a\) and \(a \circ \beta = \beta \circ a\).

(iii) If \((a \circ \beta) \circ \gamma\) and \(a \circ (\beta \circ \gamma)\) are defined then

\((a \circ \beta) \circ \gamma = a \circ (\beta \circ \gamma)\).

(iv) If \(a \circ \beta\) or \(\gamma \circ \gamma\) is defined then so is the other and \(\gamma(a \circ \beta) = \gamma a \circ \gamma\).

(v) If \(a \circ \beta\) is defined then \(|a \circ \beta| = \max(|a|,|\beta|)\). Conversely if \(|s| = \max(|a|,|\beta|)\) for some \(s \in a+\beta\) then \(a \circ \beta\) is defined.

(vi) \(|a| < |\beta|\) if and only if \(a \circ \beta = \beta\).

(vii) Let \(n \in \{1,2,\ldots,\chi(k)-1\}\) if \(\chi(k) \neq 0\), let \(n \in \mathbb{N}\) otherwise. Then we define \(\oplus_n a\) inductively as follows.

\[\oplus_1 a : a, \quad \oplus_k a := \oplus_{k-1} a \circ a \quad (k \leq n).\]

Then

\[\oplus_n a = na.\]

**Proof.** (i), (ii) are clear. (iii) is almost trivial: if \(x \in a, \ y \in \beta, \ z \in \gamma\) then \(x+y+z \in a+\beta+\gamma\) and the latter set can be regarded as
(a ⊕ β) ⊕ γ or as a ⊕ (β ⊕ γ). (It is worth noticing that (a ⊕ β) ⊕ γ may be defined whereas a ⊕ (β ⊕ γ) is not. Choose β = −γ and |a| > |β|. Then (a ⊕ β) ⊕ γ = a ⊕ γ = α, β ⊕ γ is not defined.)

(iv) is clear. If a ⊕ β is defined then for x ∈ α, y ∈ β we have |x + y| ≥ max(|x|, |y|) whence |x + y| = max(|x|, |y|). So |α ⊕ β| = max(|α|, |β|). Conversely, if a ⊕ β is not defined, then (we saw earlier that) α ⊕ β is a ball with center zero and radius max(|α|, |β|).

Thus we have (v). We prove (vi). If |a| < |β| then α ⊕ β = β so α ⊕ β = β. Conversely, if a ⊕ β = β then choose a ∈ α, b ∈ β. Then a + b ∈ β hence a + b ∼ b i.e., ab−1 + 1 ∈ K+ implying |ab−1| < 1 or |a| < |b|. Hence |α| < |β|. (Note: from (vi) it follows that α ⊕ β = α′ ⊕ β does not imply α = α′.) To prove (vii) let a : α and observe that for any k ≤ n,

if ∅ α is defined, (k−1)a ⊕ α. Hence |(k−1)a+a| = |ka| = |a| = |α|,

so ∅ α+a does not contain 0, hence ∅ α ⊕ α is defined.

Now na is by definition π(n)α. So na ∼ na and na ⊕ α. Since both na

and ∅ α are signs they must coincide.

We now define relations that resemble "ordering".

**DEFINITION 1.3** Let α ∈ Σ and x, y ∈ K. Then we say that x is greater than y in the sense of α, notation x >α y, if x−y ∈ α.

We have the following rules

**THEOREM 1.4** (i) If x, y ∈ K, x ≠ y then there is exactly one α ∈ Σ for which x >α y.

(ii) x >α x for no α.

(iii) If x >α y then for all s ∈ K: x+s >α y+s (x, y ∈ K, α ∈ Σ)

(iv) If x >α y and s >β 0 then xs >αβ ys (x, y, s ∈ K, α, β ∈ Σ)
(In particular \( x > y \) implies \(-x < -y\)).

(v) If \( x > y \), \( y > z \) and if \( \alpha \otimes \beta \) is defined then \( x > \alpha \otimes \beta \).

Proof. Easy.

The group \( \Sigma_1 := \{ \alpha \in \Sigma : |\alpha| = 1 \} \) is a subgroup of \( \Sigma \), isomorphic to the multiplicative group \( \mathbb{K}^* \). If \( \mathbb{K} \) has discrete valuation and if \( s \in \mathbb{K} \) and \( |s| \) is the largest value that is smaller than \( 1 \), then for each \( \alpha \in \Sigma \) there is \( x \in \mathbb{Z} \) such that \( \alpha = s^n \alpha_1 \) where \( \alpha_1 \in \Sigma_1 \). It follows easily that the map \( (n, \alpha) \mapsto s^n \alpha \) \((n \in \mathbb{Z}, \alpha \in \Sigma_1)\) is an isomorphism of \( \mathbb{Z} \times \Sigma_1 \) onto \( \Sigma \). Thus, in case \( \mathbb{K} \) has discrete valuation, \( \Sigma \) is isomorphic to \( \mathbb{Z} \times \Sigma_1 \), or, for that matter, to \( \mathbb{K}^* \times \mathbb{K}^* \).

If \( \mathbb{K} \) is a local field we can even define a group embedding \( \rho : \Sigma \to \mathbb{K}^* \) such that \( \pi \rho \) is the identity. (Thus, we can pick an element in every \( \alpha \) \((\alpha \in \Sigma)\) such that the resulting set is a subgroup of \( \mathbb{K}^* \)). Let \( s \in \mathbb{K} \), \( |s| < 1 \) such that \( |s| \) generates the value group and let \( q \) be the cardinality of \( \mathbb{K} \). Let \( x \in \mathbb{K}^* \). Then there is a unique \( n \in \mathbb{Z} \) such that \( x = s^n x_1 \) where \( |x_1| = 1 \).

Define

\[
\nu(x) = s^n \lim_{n \to \infty} x_1^n
\]

It is easy to verify that \( \nu \) is a homomorphism of \( \mathbb{K}^* \) into \( \mathbb{K}^* \), that \( \pi(\nu(x)) = \pi(x) \) for all \( x \in \mathbb{K}^* \) and that \( \nu(x) = 1 \) if and only if \( x \in \mathbb{K}^+ \).

Therefore the map \( \rho \) making the diagram

\[
\begin{array}{ccc}
\mathbb{K}^* & \longrightarrow & \mathbb{K}^* \\
\pi & \downarrow & \nu \downarrow \\
\Sigma & \longrightarrow & \mathbb{K}^*
\end{array}
\]

commute solves the problem.

**Example 1.5** The signs of \( \Theta_p \). Let \( \Theta \) be a primitive \((p-1)\)th root of
unity. Then \( \{0^i p^n : i \in \{0,1,\ldots,p-2\}, n \in \mathbb{Z} \} \) is a subgroup of \( \mathbb{Q}_p^* \) isomorphic to \( \mathbb{Z} \). If

\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0,1,0,\ldots,p-2\}, a_n \neq 0)
\]

is an element of \( \mathbb{Q}_p^* \), its sign, interpreted as an element of \( \mathbb{Q}_p \) is

\[
\pi(x) = a_n p^n.
\]
2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function \( f : [0,1] \to \mathbb{R} \) the following statements are equivalent.

(a) \( f \) is monotone (i.e., either \( x > y \implies f(x) \geq f(y) \) for all \( x,y \)
or \( x > y \implies f(x) \leq f(y) \) for all \( x,y \)).

(b) If \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \)
\( (x,y,z \in [0,1]) \)

(c) If \( C \subseteq \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex.

Thus we define

DEFINITION 2.1 Let \( X \subseteq K \). We say that \( f \in M_b(X) \) if for all \( x,y,z \in X \), \( x \) between \( y \) and \( z \) implies \( f(x) \) is between \( f(y) \) and \( f(z) \). In other words, \( f \in M_b(X) \) if and only if for all \( x,y,z \)
\[ |x-y| \leq |y-z| \implies |f(x)-f(y)| \leq |f(y)-f(z)|. \]

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

THEOREM 2.2 Let \( X \subseteq K \) and let \( f : X \to K \). Then the following statements are equivalent

(a) \( f \in M_b(X) \).

(b) For each convex \( C \subseteq K \), \( f^{-1}(C) \) is convex in \( X \).

(c) For all \( x,y,z \in X \) : \( |x-y| = |x-z| \implies |f(x)-f(y)| = |f(x)-f(z)|. \)

(d) For all \( x,y,z \in X \) : \( |f(x)-f(y)| > |f(x)-f(z)| \rightarrow |x-y| > |x-z|. \)

(e) For all \( x,y,z \in X \) : \( |f(x)-f(y)| \neq |f(x)-f(z)| \rightarrow |x-y| \neq |x-z|. \)
Proof. (a) $\Rightarrow$ (b). Let $x,y \in f^{-1}(C)$ and let $z \in [x,y] \cap X$. Then $|z-x| \leq |x-y|$, so $|f(z)-f(x)| \leq |f(x)-f(y)|$ i.e., $f(z) \in [f(x),f(y)] \subset C$. Hence $z \in f^{-1}(C)$.

(b) $\Rightarrow$ (a). Let $x,y,z \in X$ and $|x-y| \leq |x-z|$. The set $[f(x),f(z)]$ is convex, hence $f^{-1}([f(x),f(z)])$ is convex in $X$ and contains $x$ and $z$, so it must contain $y$. Thus $f(y) \in [f(x),f(z)]$.

Clearly, (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d). We prove (a) $\Rightarrow$ (c). Now (a) $\Rightarrow$ (c) is clear by symmetry. Suppose (c) and let $|x-y| \leq |x-z|$. It suffices to consider the case $|x-y| < |x-z|$. Then $|y-z| = |x-z|$, so by (c) we have $|f(y)-f(z)| = |f(x)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|,|f(z)-f(y)|) = |f(x)-f(z)|$. 

THEOREM 2.3 Let $X \subset K$. Then

(i) For each $a,b \in K$ the map $x \mapsto ax+b$ is in $M_b(X)$.
(ii) If $f \in M_b(X)$, $\lambda \in K$ then $\lambda f \in M_b(X)$.
(iii) $M_b(X)$ is closed under pointwise limits.
(iv) If $f \in M_b(X)$ and $g : f(X) \to K$ is in $M_b(f(X))$, then $g \circ f \in M_b(X)$.
(v) If $f \in M_b(X)$ and $f(a) = f(b)$ for some $a,b \in X$, then $f$ is constant on $[a,b] \cap X$.

Proof. Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of $M_b$-functions. For more, see the sequel.

(1) The constant functions.

(2) Isometries (e.g., exp.).

(3) Choose in every $a \in \Sigma$ an element $x_a$. Define $\phi : K \to K$ as follows

$$
\phi(x) = \begin{cases} 
0 & \text{if } x = 0 \\
x_a & \text{if } x \in a \quad (a \in \Sigma)
\end{cases}
$$
(Essentially, $\phi|K^*$ is the sign function $\pi$ of section 1).

We prove that $\phi \in M^*_b(K)$. Since $\phi$ is continuous it suffices to check that $\phi|K^*$ is in $M^*_b(K^*)$. Now for all $x, y \in K^*$ we have $\phi(x)-\phi(y) = 0$ if $|xy^{-1}| < 1$ and $|\phi(x)-\phi(y)| = |x-y| \iff |x-y| = \max(|x|, |y|)$. Now take $x, y, z \in K^*$ such that $|x-y| \leq |x-z|$. If $\phi(x) = \phi(z)$ then $|1-x^{-1}y| \leq |1-x^{-1}z| < 1$ so $\phi(x) = \phi(y)$.

If $\phi(x) \neq \phi(z)$ then $|\phi(x)-\phi(y)| \leq |x-y| \leq |x-z| = |\phi(x)-\phi(z)|$.

(4) Let $r > 0$ and choose in every ball $B$ of radius $r$ a center $x_B$.

The function defined via

$$\psi(x) = x_B \quad (x \in B)$$

is in $M^*_b(K)$. The proof is easy.

(5) (A nowhere continuous $M^*_b$-function). Let $K$ be a field such that $\#K = \#k$ (e.g., a discretely valued field where $\#k$ has the power of the continuum). Let $\sigma : K \rightarrow k$ be a bijection and let $\tau : k \rightarrow K$ such that $|\tau x - \tau y| = 1$ whenever $x \neq y$. Then $f := \tau \circ \sigma$ satisfies: $|f(x)-f(y)| = 1$ ($x, y \in K, x \neq y$).

Clearly $f$ is everywhere discontinuous, $f \in M^*_b(K)$.

(6) Let $X \subset K$. We can strengthen the definition of an $M^*_b$-function into

if $|x-y| \leq |z-t|$ then $|f(x)-f(y)| \leq |f(z)-f(t)|$ \quad ($x, y, z, t \in X$)

(some "uniform" $M^*_b$-condition) and we obtain a space, called $M^*_{ub}(X)$.

Clearly, the examples mentioned in (1), (2), (4), (5) are in $M^*_{ub}(K)$, whereas the example in (3) is not. (Choose $x, y \in K$ with $|x| > 1$, $|x-y| = 1$. Then $|1-0| \leq |x-y|$, but $1 = |\phi(1)-\phi(0)| > |\phi(x)-\phi(y)| = 0$.)

Notice that $\phi$ is locally constant on $K^*$, but not on $K$.

(7) The discontinuous function $f$ of (5) has the property that $f(K)$ consists only of isolated points. This is not accidental. If $f \in M^*_b(K)$
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing \( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points, so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that the image of an everywhere discontinuous \( M_b \)-function consists only of isolated points.

(8) For each \( n \), let \( \sigma_n : K + K \) be the example of 2.4, (4) above where \( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b(X) \) we have

\[
\sigma_n \circ f \in M_b(X) \quad \text{and} \quad \lim_{n \to \infty} \sigma_n \circ f = f \quad \text{uniformly.}
\]

Hence, if \( f \) is continuous then it can uniformly be approximated by locally constant \( M_b \)-functions.

A monotone function \( f : [0,1] \to \mathbb{R} \) maps convex sets into sets that are relatively convex in \( f([0,1]) \). In our situation we do not have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If \( f : [0,1] \to \mathbb{R} \) maps convex sets into (relatively) convex sets then \( f \) need not be monotone: any Darboux continuous function has the above property. In fact a function \( f : [0,1] \to \mathbb{R} \) is Darboux continuous if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We define

**Definition 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \to K \). Then \( f \) is called weakly Darboux continuous if for every relatively convex set \( C \subseteq X \) the set \( f(C) \) is convex in \( f(X) \).

\( f \) is called Darboux continuous if for every relatively convex set \( C \subseteq X \) the set \( f(C) \) is convex (in \( K \)).

We have the following obvious remarks.

1) \( f : X \to K \) is Darboux continuous if and only if \( f \) is weakly Darboux continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) such that for every open ball \( B \subseteq \mathbb{Z}_p \), 
\[
f(B) = \mathbb{Z}_p.
\]
Let \( A \subseteq \mathbb{Z}_p \) be defined as follows. \( x = 2^n \cdot a_n \) (\( a_n \in \{0,1,...,p-1\} \)) is in \( A \) if \( a_{2n} = a_{2n+2} = ... = 0 \) for some \( n \). Define \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) via 
\[
f(x) = \begin{cases} 
a_{2N+1} + a_{2N+3}p + a_{2N+5}p^2 + ... & \text{if } x \in A \text{ and } N = \min(n : a_{2n} = a_{2n+2} = ... = 0) \\
0 & \text{if } x \not\in A
\end{cases}
\]
Then \( f \) maps every non-empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.

(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on \( \mathbb{Z}_p \) is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous.

To see this, observe that all open compact subsets of \( \mathbb{Z}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p \) sending \( \{x : |x| < 1\} \) into \( \{x : |x| = 1\} \) and \( \{x : |x| = 1\} \) into \( \{x : |x| < 1\} \).

If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( M_p \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4,(7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \to \mathbb{R} \) the following conditions are equivalent.

(a) \( f \) is strictly monotone (i.e., injective and monotone).

(b) \( f \) is injective. For a convex set \( C \subseteq [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all x, y, z ∈ [0, 1]: if f(x) is between f(y) and f(z) then x is between y and z.

(δ) For all x, y, z ∈ [0, 1]: f(x) is between f(z) if and only if x is between y and z.

Translating (α) – (δ) into the non-archimedean situation we arrive at the following conditions. Let X ⊂ K and f : X → K

(α') f ∈ M_b(X) and f is injective.

(β') f is weakly Darboux continuous and injective.

(γ') for all x, y, z ∈ X, |x - y| < |x - z| implies |f(x) - f(y)| < |f(x) - f(z)|.

(δ') f ∈ M_b(X) and f satisfies (γ').

It will turn out that the conditions (α') – (γ') although not equivalent are closely related. We start with (γ'):

DEFINITION 2.6 Let X ⊂ K, f : X → K. We say that f ∈ M_s(X) if for all x, y, z ∈ X, f(x) ∈ [f(y), f(z)] implies x ∈ [y, z].

THEOREM 2.8 Let X ⊂ K, f : X → K. Then the following statements are equivalent:

(a) f ∈ M_s(X).

(b) f is injective and weakly Darboux continuous.

(c) f is injective and f⁻¹ ∈ M_b(f(X)).

(d) For all x, y, z ∈ X, |f(x) - f(y)| = |f(x) - f(z)| → |x - y| = |x - z|.

(e) For all x, y, z ∈ X, |x - y| < |x - z| → |f(x) - f(y)| < |f(x) - f(z)|.

(f) For all x, y, z ∈ X, |x - y| ≠ |x - z| → |f(x) - f(y)| ≠ |f(x) - f(z)|.
Proof. The implications (α) → (ε) → (ζ) → (δ) are clear from the definitions.

(δ) → (γ): injectivity follows from |f(x) − f(x)| = |f(x) − f(y)| + |x − x| = |x − y|. Use 2.2.(γ).

(γ) → (β): Let g : f(X) → X be the inverse of f. Let C ⊂ X be convex in X. Then since g ∈ M_δ, g^-1(C) is convex in f(X). But g^-1(C) = f(C).

Finally, we prove (β) → (α). Let f(x) ∈ [f(y), f(z)] by (β) the set f([y, z] ∩ X) is convex in f(X) and it contains f(y), f(z), hence f(x) ∈ [f(y), f(z)] ∩ X ⊂ f([y, z] ∩ X). Since f is injective, x ∈ [y, z] ∩ X and we are done.

We also have (compare 2.3)

THEOREM 2.9 Let X ⊂ K. Then

(i) For a, b ∈ K, a ≠ 0 the map x ↦ ax + b is in M_s(X).

(ii) If f ∈ M_s(X), λ ∈ K, λ ≠ 0 then λf ∈ M_s(X).

(iii) If f_1, f_2, ..., ∈ M_s(X), lim f_n = f pointwise, f injective then f ∈ M_s(X).

(iv) If f ∈ M_s(X), g ∈ M_s(f(X)) then g ◦ f ∈ M_s(X).

Proof. Obvious verifications.

Returning to our conditions (α') - (δ') we see that (δ') is equivalent to (γ'), that (α') means f^-1 ∈ M_s(f(X)) and that (δ') means f ∈ M_b(X) ∩ M_s(X).

Our f of example 2.4 (5) is in M_b, injective but not in M_s. Its inverse yields an example of an M_s-function that is not in M_b. Thus, in general, we have neither one of the implications (α') → (γ'), (γ') → (α'), (β') → (δ'), (α') → (δ'). But our counterexample is
rather weird (f is nowhere continuous and the domain of $f^{-1}$ is discrete). We can do better.

**EXAMPLE 2.10** Let $K$ have discrete valuation and let $k$ be infinite.

Then there exists a homeomorphism of the unit ball of $K$ that is in $M_d$ but not in $M_s$. (The inverse map is in $M_g$ but not in $M_d$.)

**Proof.** Set $X = \{ x \in K : |x| \leq 1 \}$ and let $R$ be a full set of representatives of the equivalence relation $x \sim y$ iff $|x-y| < 1$ in $X$. Then $R$ is infinite. Let $\pi \in K$ be such that $|\pi|$ is the largest value that is smaller than 1. The map

$$f(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \quad (a_i \in R \text{ for each } i)$$

is a bijection of $R^\mathbb{N}$ onto $X$. We may suppose that $0 \in R$.

Since $R$ is infinite we can define injections

$$\tau_1 : R \setminus \{0\} \to R$$
$$\tau_2 : R \to R$$

such that $\text{im } \tau_1 \cap \text{im } \tau_2 = \emptyset$, $\text{im } \tau_1 \cup \text{im } \tau_2 = R$.

For $x = \sum_{n=0}^{\infty} a_n \pi^n \in X$ (a $n \in R$ for each n) set

$$f(x) := \begin{cases} 
\tau_1(a_0) + a_1 \pi + \ldots = x - a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\
\tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0 
\end{cases}$$

A simple inspection of the definition shows that $f$ is a bijection of $X$ onto $X$. If $a, b \in R$, $a \neq b$ then $|a\pi - b\pi| < |b\pi - a|$, whereas $|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1$ and $|f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1$, so $f \notin M_g(X)$. Finally, let $x, y, z \in X$ and $|x-y| \leq |x-z|$. We prove that $|f(x) - f(y)| \leq |f(x) - f(z)|$. If $|f(x) - f(z)| = 1$ there is nothing to prove, so suppose $|f(x) - f(z)| < 1$. Set $x = \sum_{n=0}^{\infty} a_n \pi^n$, $y = \sum_{n=0}^{\infty} b_n \pi^n$, $z = \sum_{n=0}^{\infty} c_n \pi^n$. 


If \( a_0 = 0 \) then also \( \tau_2 (a_1) = \tau_2 (c_1) \) so \( a_1 = c_1 \), hence \( |x-z| \leq |\pi|^2 \). Since \( |x-y| \leq |x-z| \) we have also \( b_0 = 0, b_1 = a_1 \).

So, \( f(x) - f(y) = \frac{x-y}{\pi}, f(x) - f(z) = \frac{y-z}{\pi} \) whence \( |f(x) - f(y)| \leq |f(x) - f(z)| \).

If \( a_0 \neq 0 \) then \( \tau_1 (a_0) = \tau_1 (c_0) \) so \( a_0 = c_0 \). Then also \( c_0 = a_0 = b_0 \).

Then \( f(x) - f(y) = x-y, f(x) - f(z) = x-z \) whence \( |f(x) - f(y)| \leq |f(x) - f(z)| \).

Let \( X \subseteq K \). If \( f \in M_S (X) \) then \( f^{-1} \in M_B (f(X)) \). Conversely, if \( f \in M_B (X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) then \( g \in M_S (f(X)) \). This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let \( X \subseteq K \) and \( f : X \to K \). \( f \) is called weakly monotone \((f \in M_w (X)) \) if for all \( x, y, z \in X \)

\[
|x-y| < |x-z| + |f(x) - f(y)| \leq |f(x) - f(z)|
\]

\( f \) is called strongly monotone \((f \in M_{bs} (X)) \) if

\[
f \in M_S (X) \cap M_B (X).
\]

Clearly, \( f \in M_{bs} (X) \) if and only if \( f^{-1} \in M_{bs} (f(X)) \). Also, if \( f \in M_w (X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) we have \( g \in M_w (f(X)) \).

Obviously we have \( M_B (X) \cup M_S (X) \subseteq M_w (X) \) and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of \( M_w \)-functions, not for the sake of \( M_w \) itself but in order to get results that are valid for \( M_B, M_S \) simultaneously. The functions in \( M_{bs} \) behave reasonable and they may be viewed as the non-archimedean equivalents of strict monotone functions in the real case.
THEOREM 2.12 Let $X \subseteq K$ and $f : X \to K$. Then the following conditions are equivalent.

(a) $f \in M_{bs}(X)$.

(b) $f$ is injective and $C \mapsto f(C)$ is a 1-1 correspondence between the relatively convex subsets of $X$ and those of $f(X)$.

(c) For all $x, y, z \in X$, $|x - y| < |x - z| \iff |f(x) - f(y)| < |f(x) - f(z)|$.

(d) For all $x, y, z \in X$, $|x - y| = |x - z| \iff |f(x) - f(y)| = |f(x) - f(z)|$.

(e) For all $x, y, z \in X$, $|x - y| \leq |x - z| \iff |f(x) - f(y)| \leq |f(x) - f(z)|$.

(f) $f \in M_s(X)$, $f^{-1} \in M_s(f(X))$.

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An $M_w$-function that is not in $M_s \cup M_b$). Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be any function, constant on the cosets of $\{x \in \mathbb{Z}_p : |x| < 1\}$. Then $f \in M_w(\mathbb{Z}_p)$. Clearly $f \notin M_s(\mathbb{Z}_p)$, $f \in M_b(\mathbb{Z}_p)$ if and only if the points of $f(\mathbb{Z}_p)$ are equidistant.

(2) (Continuity of monotone functions). Let $X \subseteq K$.

(a) Let $f \in M_w(X)$. If $f(X)$ has no isolated points, then $f$ is continuous.

Proof. Let $a \in X$ and $\varepsilon > 0$. Then there is $z \in X$ such that $z \neq a$, $|f(z) - f(a)| < \varepsilon$. Let $\delta := |z - a|$. Then for all $x \in X$ with $|x - a| < \delta$ we have, by the weak monotony of $f$, $|f(x) - f(a)| \leq |f(z) - f(a)| < \varepsilon$.

It follows that if $X$ and $Y$ do not have isolated points and if $f$ is an $M_w$-bijection of $X$ onto $Y$, then $f$ is a homeomorphism of $X$ onto $Y$. 

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 Conversely, it is easy to construct homeomorphisms of \( \mathbb{Z}_p \) that are not in \( M_w(\mathbb{Z}_p) \).

(b) If \( K \) is a local field then every \( f \in M_w(X) \) is continuous. (See 5.1 (i)).

(c) If \( K \) has discrete valuation then every \( f \in M_s(X) \) is continuous.

(Example 2.4 (5) shows that such a statement is not true for \( f \in M_b(X) \).)

(Proof. If \( f \) were not continuous at some \( a \in X \) then there would be an \( \varepsilon > 0 \) such that for some sequence converging to \( a \) we had \( |f(x_n) - f(a)| \geq \varepsilon \). We may suppose that \( |x_1 - a| > |x_2 - a| > \ldots \). Since the valuation is discrete we have \( \lim_{n \to \infty} |f(x_n) - f(a)| = 0 \), a contradiction.)

(d) In 5.14 we shall give an example of a function in \( M_{bs}(X) \) that is not continuous. (Of course, \( K \) will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" \( M_w, \ldots \)-conditions.

Thus, by definition, \( f \in M_{uw}(X) \) if for all \( x,y,z,t \in X \)

\[
|x-y| < |z-t| \Rightarrow |f(x) - f(y)| \leq |f(z) - f(t)|
\]

\( f \in M_{us}(X) \) if for all \( x,y,z,t \in X \)

\[
|x-y| < |z-t| \Rightarrow |f(x) - f(y)| < |f(z) - f(t)|
\]

\( f \in M_{ubs}(X) \) if for all \( x,y,z,t \in X \)

\[
|x-y| < |z-t| \leftrightarrow |f(x) - f(y)| < |f(z) - f(t)|.
\]

Notice that \( f \in M_{ubs}(X) \) means that \( |f(x) - f(y)| \) is a strictly increasing function of \( |x-y| \). Examples of such functions are isometries, but also the function \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) defined via

\[
\sum a_n p^n \mapsto \sum a_n p^{2n} \quad (\sum a_n p^n \in \mathbb{Z}_p)
\]

\( (|f(x) - f(y)| = |x-y|^2 \) for all \( x,y \in \mathbb{Z}_p \).)

Monotone functions : \( \mathbb{R} \to \mathbb{R} \) are divided into two classes: the
increasing functions and the decreasing functions. For the non-archimedean case we may ask for a similar classification. First we try to express the situation in the real case in such a way that it can be translated. Let \( a \in \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be strictly monotone. If \( x \) runs through some side of \( a \) then \( f(x) \) runs through some fixed side of \( f(a) \). So there is a map \( \sigma : \{-1,1\} \to \{-1,1\} \) such that \( \sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a)) \) \((x \neq a)\). Apparently, the only \( \sigma \)'s that can occur are the identity and \( \sigma(x) = -x \) \((x \in \{-1,1\})\). Moreover it turns out that the map \( \sigma \) is independent of the choice of \( a \).

The two maps \( \sigma \) that can occur can be interpreted as multiplication maps (with 1 and \(-1\) respectively) or as the bijections \( \{1,1\} \to \{-1,1\} \) and there seems to be no philosophical reason to make any decision of preference.

As an example, let us consider a function \( f \in M_\Sigma(K) \). Let \( a \in K \), let \( \alpha \in \Sigma \). If \( x \in a+\alpha \) and \( y \in a+\alpha \) ("\( x, y \) are at the same side of \( a \)"") then \( x-a, y-a \in \alpha \), so \(|x-y| < |y-a|\). Since \( f \in M_\Sigma(K) \) we have

\[
|f(x)-f(y)| < |f(y)-f(a)|,
\]

whence \( |f(x)-f(a)-(f(y)-f(a))| < |f(y)-f(a)| \), so \( f(x)-f(a) \) and \( f(y)-f(a) \) have the same sign. We may conclude that there is a map \( \alpha_\alpha : \Sigma \to \Sigma \) such that for all \( x \in K \)

\[
x \in a+\alpha + f(x) \in f(a)+\alpha_\alpha(a) \quad (\alpha \in \Sigma).
\]

Unfortunately, it turns out that in general \( \alpha_\alpha \) may be different from \( \beta_\beta \) if \( \alpha \neq \beta \), even for isometrical maps. For example, let \( p \neq 2 \) and let \( \tau \) be a permutation of \( \{0,1,2,\ldots,p-1\} \) and define \( f : \mathbb{F}_p \to \mathbb{F}_p \) by

\[
\Sigma a_n \mathbb{F}_p \to \Sigma \tau(a_n) \mathbb{F}_p \quad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n).
\]

Suppose we had a \( \sigma : \Sigma \to \Sigma \) such that for all \( x, y \in \mathbb{F}_p \), \( x-y \in \alpha \) implies \( f(x)-f(y) \in \sigma(\alpha) \). Let \( \alpha = \theta^i \mathbb{F}_p \) (see 1.5). Then \( x-y \in \alpha \) means
where \( a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n b_n = \theta^i \mod p \).

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n)) p^n + \ldots \), so \( \sigma(a) = \theta^j p^n \) where \( \tau(a_n) - \tau(b_n) = \theta^j \mod p \) (\( j \) depending on \( i \) and \( n \)).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = \ldots = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \subset K \), \( f : X \to K \) and let \( \sigma : \Sigma \to \Sigma \). We say that \( f \) is monotone of type \( \sigma \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[
\alpha \ni x - y \Rightarrow f(x) - f(y) \in \sigma(\alpha).
\]

(In other words if \( x \alpha y \) implies \( f(x) > f(y) \) \( \sigma(\alpha) \).

see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \to \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( \alpha \), \( x \alpha y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \subset K \), \( f : X \to K \), \( \beta \in \Sigma \). We say that \( f \) is monotone of type \( \beta \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[
\alpha \ni x - y \Rightarrow f(x) - f(y) \in \alpha \beta.
\]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \)
where $o : \Sigma \times \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$ is monotone of type $\beta$ iff the sign of $\frac{f(x) - f(y)}{x - y}$ is constant $\beta$ for all $x, y \in X, x \neq y$. This leads to

**DEFINITION 2.16** Let $X \subset K$, $f : X \to K$. $f$ is called increasing if $f$ is monotone of type 1. In other words, $f$ is increasing if for all $x, y \in X, x \neq y$ the difference quotient $\frac{f(x) - f(y)}{x - y}$ is positive, i.e., if

$$\left| \frac{f(x) - f(y)}{x - y} - 1 \right| < 1.$$  

In the next section we shall study the monotone functions of type $\alpha$ and we will give a partial answer to the question for which maps $\sigma : \Sigma \to \Sigma$ there exists an $f : K \to K$ that is monotone of type $\alpha$. 
3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $\sigma$.

DEFINITION 3.1. Let $X \subseteq K$, $f: X \to K$. Let $\phi(x,y) := \frac{f(x) - f(y)}{x - y}$ ($x, y \in X$, $x \neq y$). $f$ is called

**positive** if $f(X) \subseteq K^+$

**strictly positive** if $\sup_{x \in X} |f(x) - 1| < 1$

**increasing** if $\phi(x,y) \in K^+$ for all $x, y \in X$, $x \neq y$

**strictly increasing** if $\sup \{|1 - \phi(x,y)| : x, y \in X, x \neq y\} < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. Let $X \subseteq K$.

(i) If $f: X \to K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a, b \in K$ the function $x \mapsto ax + b$ is increasing if and only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly increasing).

(iii) If $f: X \to K$ is (strictly) increasing and $f$ is (strictly) positive then $\frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \to K$ form a convex set.

(v) If $f: X \to K$ and $g: f(X) \to K$ are (strictly) increasing then so is $g \circ f$.

(vi) If $f: X \to K$ is (strictly) increasing then so is $f^{-1}: f(X) \to K$.

(vii) If $f_1, f_2, \ldots : X \to K$ are increasing and $f := \lim_{n \to \infty} f_n$ pointwise

then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function
\[ \exp x = 1 + x + \frac{x^2}{2!} + \ldots \]
defined on \( \{ x \in \mathbb{R} : |x| < p^{1-p} \} \) if \( \chi(k) = p \), \( x(0) = 0 \) and on \( \{ x \in \mathbb{R} : |x| < 1 \} \) if \( \chi(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f : X + K \) be a \( C^\ell \)-function (i.e., \( f \) can continuously be extended to a function on \( X \times X \), assume that \( X \subset K \) has no isolated points. See [2]) and suppose \( f'(a) \in K^+ \) for some \( a \in X \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta, |y-a| < \delta, x \neq y \) implies
\[ \frac{|f(x)-f(y)|}{x-y} - f'(a) | \leq \frac{1}{2}. \]

For such \( x,y \) we have
\[ \frac{|f(x)-f(y)|}{x-y} - 1 \leq \frac{|f(x)-f(y)|}{x-y} - f'(a) | \leq \max(\frac{1}{2}, |f'(a)|) < 1. \]

(3) The space \( C(\mathbb{R}) \) of all continuous functions \( \mathbb{R} \rightarrow \mathbb{R} \), is a Banach space with respect to the sup norm \( || \cdot ||_{\infty} \). Let \( e_0 := \xi \p_{\mathbb{R}} \) and for \( n \geq 1 \) let \( e_n := \xi \p_{B_n} \) where \( B_n := \{ x \in \mathbb{R} : |x-n| < \frac{1}{n} \} \). It is proved in [1] that \( e_0, e_1, \ldots \) is an orthonormal base of \( C(\mathbb{R}) \) i.e., for each \( f \in C(\mathbb{R}) \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that
\[ f = \sum_{n=0}^{\infty} \lambda_n n. \]
The coefficients $\lambda_n$ can be reconstructed from $f$ via

$$
\lambda_0 = f(0) \\
\lambda_n = f(n) - f(n_-) \quad (n \in \mathbb{N})
$$

where $n_-$ is defined as $a_0 + a_1 p + \ldots + a_{s-1} p^{s-1}$ if $n = a_0 + a_1 p + \ldots + a_s p^s$ ($a_s \neq 0$) in base $p$.

Our aim is here to describe a necessary and sufficient condition for the $\lambda_n$ in order that $f = \sum \lambda_n e_n$ is increasing. We show

$$
f = \sum \lambda_n e_n \text{ is increasing if and only if for all } n \in \mathbb{N}, \quad |\lambda_n - (n-n_-)| < |n-n_-|.
$$

Proof. First observe that $f$ is increasing if and only if for all $x \in \mathbb{Z}_p$

$$
f(x) = x + g(x)
$$

where $|g(x,y)| < 1$ for all $x,y \in \mathbb{Z}_p$, $x \neq y$.

As

$$
x = \sum_{n \geq 1} (n-n_-) e_n(x) \quad (x \in \mathbb{Z}_p)
$$

it suffices to show that for $g = \sum \lambda_n e_n \in C(\mathbb{Z}_p)$ we have $|\phi(g)| < 1$ if and only if $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Suppose first $|\phi(g)| < 1$. Then for all $n \in \mathbb{N}$, $|f(n) - f(n_-)| < 1$, so $|\lambda_n| = |f(n) - f(n_-)| < |n-n_-|$. Conversely, let $g = \sum \lambda_n e_n$ and let $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Let $x,y \in \mathbb{Z}_p$ and let $|x-y| = p^{-k}$ for some $k \in \{0,1,2,\ldots\}$. Since $e_n(a) = e_n(b)$ if and only if $|a-b| < \frac{1}{n}$ we have

$$
e_n(x) = e_n(y) \quad \text{for} \quad n < p^k.
$$
Therefore
\[ |g(x)-g(y)| = | \sum_{n=1}^{\infty} \lambda_n (e_n(x)-e_n(y)) | = \sum_{n=1}^{\infty} | \lambda_n (e_n(x)-e_n(y)) | \]
\[ \leq \max_{n \geq k} | \lambda_n | \max_{n \geq k} |n-n_-| = \| x-y \| \]
so $|g| < 1$.

(4) Let $K$ have dense valuation and let $k$ be (countably) infinite. Let $X$ be the unit ball of $K$ and let $B_i$ (i$\in$N) be the balls in $X$ with radius $1^{-}$. Choose $c_1, c_2, \ldots \in K$ such that $|c_1| < |c_2| < \ldots$, $\lim |c_n| = 1$. For $n \in \mathbb{N}$ define a function $f_n : X \to K$ via

\[ f_n(x) = \begin{cases} x + c_i & \text{if } x \in B_i \quad (1 \leq i \leq n) \\ x & \text{elsewhere} \end{cases} \]

Then each $f_n$ is strictly increasing ($|f_n(x,y)-1| \leq \max_{1 \leq i,j \leq n} |c_i-c_j| \leq |c_n| < 1$). The sequence $f_1, f_2, \ldots$ converges pointwise to an increasing function $f$. But $f$ is not strictly increasing:

\[ \sup_{x \neq y} |f(x,y)-1| = \sup_{i,j} |c_i-c_j| = 1. \]

(Compare 3.2, (vii) and (viii)).

Increasing functions are closely related to functions $g$ for which $|g(x)-g(y)| < |x-y|$ ($x \neq y$) (if $f$ is increasing, set $g(x) := f(x)-x$).

**DEFINITION 3.4.** Let $(X, \rho)$ be an ultrametric space. A map $g : X \to X$ is called a pseudocontraction if $\rho(f(x), f(y)) < \rho(x, y)$ ($x, y \in X, x \neq y$).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X \to X$ has a fix point. We have

**Lemma 3.5.** Let $(X, \rho)$ be an ultrametric space. Then the following conditions are equivalent.

1. $X$ is spherically complete.
2. Each pseudocontraction $X \to X$ has a fix point.
3. Each pseudocontraction $X \to X$ has a unique fix point.

**Proof.** If $\sigma: X \to X$ is a pseudocontraction and if $x, y$ are fix points and $x \neq y$, then $\rho(x, y) = \rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction. Thus, we have (2) $\Rightarrow$ (3). We prove (1) $\Rightarrow$ (2). Let $B \subseteq X$ be a ball (i.e., either $B = \{x \in X : \rho(x, a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X : \rho(x, a) < r\}$ for some $a \in X$, $r > 0$). We call $B$ invariant if $\sigma(B) \subseteq B$. Now we observe two facts

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x), \sigma(a)) < \rho(x, a) \leq \max(\rho(x, \sigma(a)), \rho(\sigma(a), a)) = \rho(a, \sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \notin V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x, y)$ does not depend on $x, y$, since for $z \in B_1$, $u \in B_2$, $\rho(x, z) < \rho(x, y)$ and $\rho(y, u) < \rho(x, y)$, so $\rho(z, u) = \rho(x, y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a small-
least invariant ball $S$. If $a \in S$, $\sigma(a) \neq a$ then $\{x \in S : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$ is invariant and does not contain $a$, a contradiction. Hence, $\sigma$ has a fix point (actually, $S$ is a singleton).

We prove $(\beta) \rightarrow (a)$. If $X$ were not spherically complete, there exist balls $B_1 \ni B_2 \ni \ldots$ such that $\cap B_n = \emptyset$. Choose $x_n \in B \setminus B_{n+1}$ ($n \in \mathbb{N}$), set $B_0 := X$ and define

$$
\sigma(x) := \begin{cases} 
 x & \text{if } x \in B \setminus B_{n+1} 
\end{cases} (n \in \{0, 1, 2, \ldots\}).
$$

Then $\sigma$ has obviously no fix point. Let $x \in B \setminus B_{n+1}$ and $y \in B \setminus B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in $B_{m+1}$, whereas $x \in B \setminus B_{n+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$. Then $\sigma$ is a pseudocontraction without a fix point. Contradiction.

**COROLLARY 3.6.** The following conditions are equivalent.

1. $K$ is spherically complete.
2. If $C \subset K$ is convex, $f : C \rightarrow C$ is increasing then $f$ is surjective.
3. If $C \subset K$ is convex, $f : C \rightarrow K$ is increasing then $f(C)$ is convex.
4. An increasing $f : K \rightarrow K$ is surjective.

**Proof.** $(a) \rightarrow (\beta)$. Choose $a \in C$ and consider the map $\sigma : x \rightarrow x - f(x) + a$ ($x \in C$). Then $\sigma : C \rightarrow C$. $C$ is spherically complete, $\sigma$ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: $f$ is surjective.

$(\beta) \rightarrow (\gamma)$. For a suitable $s \in K$, $f-s$ sends $C$ into $C$. $(\gamma) \rightarrow (\delta)$ is clear.

$(\delta) \rightarrow (a)$. Let $\sigma : K \rightarrow K$ be a pseudocontraction. Then $x \rightarrow x - \sigma(x)$
is increasing hence is surjective. So then is $x \in K$ for which $x-\sigma(x) = 0$, i.e., $\sigma$ has a fix point. By 3.5, $K$ is spherically complete.

In case $f$ is strictly increasing we do not have to require that $K$ is spherically complete:

**Theorem 3.7.** Let $C \subseteq K$ be convex and let $f: C \to K$ be strictly increasing. Then $f(C)$ is convex. If $f(C) \subseteq C$, then $f(C) = C$.

**Proof.** Reread the proof of (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (y) above. $\sigma$ now is a contraction. $C$ is complete. Apply the Banach contraction theorem.

Let $X$ be a subset of $\mathbb{R}$ and let $f: X \to \mathbb{R}$ be a bounded increasing function. Then $f$ can be extended to an increasing function $\mathbb{R} \to \mathbb{R}$ by setting $f(x) := \inf f$ if $x < y$ for all $y \in X$ and $f(x) := \sup \{f(y): y \leq x, y \in X\}$ for all other $x \in \mathbb{R}$. In our situation we can prove

**Theorem 3.8.** The following conditions are equivalent.

(a) $K$ is spherically complete.

(b) For every $X \subseteq K$ an increasing function $f: X \to K$ can be extended to an increasing $\overline{f}: K \to K$.

(y) Let $X \subseteq K$, and let $f: X \to K$ be a strictly increasing function. Then $f$ can be extended to a strictly increasing function $\overline{f}: K \to K$ such that

$$\sup_{x,y \in X} \left| \frac{\overline{f}(x) - \overline{f}(y)}{x-y} - 1 \right| = \sup_{x,y \in X} \left| \frac{f(x) - f(y)}{x-y} - 1 \right|$$

**Proof.** (a) $\Rightarrow$ (b). Let a $\notin X$. By Zorn's Lemma it suffices to define $\overline{f}$ such that $\overline{f}$ is increasing on $X \cup \{a\}$. We are done if we can find $a \in K$ such that for $x \in X$
\[
\left| \frac{a-f(x)}{a-x} \right| -1 < 1
\]

i.e., \( a \in B_x := B_{f(x) - (a-x)} (|a-x|) (x \in X) \).

Now \( B_x \cap B_y \neq \emptyset (x,y \in X) \) since the distance of their centers is

\[
|f(x) - (a_x) - f(y) - (a_y)| = |f(x) - f(y) - (x-y)| = |\phi f(x,y) - 1||x-y| < \\
< \max(|x-a|,|a-y|).
\]

So if, say, \(|x-a| \leq |y-a| \) we see that \(|f(x) - (a-x) - f(y) - (a-y)| < |y-a| \) whence \( f(x) - (a-x) \in B_y \). By the spherical completeness of \( K \) we have \( \bigcap_{x \in X} B_x \neq \emptyset \). Choose \( a \in \bigcap_{x \in X} B_x \).

\((\beta) \Rightarrow (\alpha)\). Suppose \( K \) is not spherically complete. By 3.6, \((\delta) \Rightarrow (\alpha)\) there is a non surjective increasing function \( f: K \to K \). Then its inverse \( g: f(K) \to K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K \to K \).

\((\beta) \leftrightarrow (\gamma)\) follows from the fact that \( (\text{with } \Phi(x) = x \text{ for all } x) \)

\[
f \mapsto (1-c)\Phi + cf \quad (c \in K, |c| < 1)
\]

is a 1-1 correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \( |1-\Phi(g)| < |c| \).

We will now investigate the relation between increasingness of \( f \) and positivity of \( f_! \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \to K \). Let \( \lambda \in K, 0 < |\lambda| < 1 \). Then \( x \mapsto x - \lambda \sigma(x) \) is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \) of \( K \) without isolated points and if \( f \) is differentiable then for each
\[ x \in X, f'(x) = \lim_{y \to x} \frac{f(x,y)}{y-x} \in K^+. \] So \( f' \) is positive. If, addition, \( f \)
\[ y \to x \]
is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f : X \to K \) be a (strictly) positive Baire class 1 function. Then does \( f \) have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subseteq K \) and let \( f : X \to K \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots : X \to K \) such that \( |g_n| \leq 1 - \frac{1}{n} \) for each \( n \) and
\[
\sum g_n (\text{pointwise}).
\]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots : X \to K \) such that
\[
f = \lim f_n \text{ pointwise. There exist locally constant functions } h_1, h_2, \ldots : X \to K \text{ such that } |f_n - h_n| \leq 2^{-n}, \text{ hence } f = \lim h_n \text{ pointwise. Define } t_1, t_2, \ldots : X \to K \text{ as follows}
\]
\[
t_n(x) := \begin{cases}
h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\
0 & \text{if } |h_n(x)| > 1 - \frac{1}{n}
\end{cases}
\]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N}. (\{x \in X : |h_n(x)| \leq 1 - \frac{1}{n}\} \) is closed and open in \( X \). \( |t_n| \leq 1 - \frac{1}{n} \) and \( \lim t_n = f \). Now let \( g_1 := t_1 \), and \( g_n := t_n - t_{n-1} \) \( \forall n \geq 2 \). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n}, f = \)
\[
\sum_{n=1}^{\infty} (t_n + t_n) = \sum g_n.
\]
LEMMA 3.10. Let \( X \subset K \) have no isolated points and let \( f : X \to K \) be a Baire class 1 function, \( |f(x)| < 1 \) for all \( x \in X \). Then \( f \) has an antiderivative \( F \) for which
\[
\frac{|F(x) - F(y)|}{x-y} < 1 \quad (x,y \in X, x \neq y).
\]

Proof. By Lemma 3.9, \( f = \sum_{n=1}^{\infty} f_n \), where each \( f_n \) is locally constant, \( |f_n| \leq 1 - \frac{1}{n} \). By [2] 3.9 each \( f_n \) has an antiderivative \( F_n \) for which
\[
|F_n(x) - F_n(y)| \leq \max (|f_n(x)|, \frac{1}{2n}) |x-y| \quad (x,y \in X).
\]

By [2] 3.7, \( F_n = \sum F_n \) is an antiderivative of \( \sum f_n = f \). Now for \( x,y \in X, x \neq y \):
\[
|F(x) - F(y)| \leq \sup_n |F_n(x) - F_n(y)| \leq \sup_n \max (|f_n(x)|, \frac{1}{2n}) |x-y|
\]
\[
\leq |x-y| \max (|f_n(x)|, \frac{1}{2}). \text{ Now for each } x \in X, |f_n(x)| < 1 \text{ for each } n
\]
and \( \lim_{n} |f_n(x)| = 0 < 1 \). Hence \( \max_n |f_n(x)| < 1 \). It follows that
\[
|F(x) - F(y)| < |x-y|.
\]

THEOREM 3.11. Let \( X \subset K \) have no isolated points and let \( f : X \to K \) be (strictly) positive. Then \( f \) has a (strictly) increasing antiderivative.

Proof. The function \( x \mapsto f(x) - 1 \) has, by 3.10, an antiderivative \( H \) such that \( |\phi(H)| < 1 \). Let \( F(x) = x + H(x) \ (x \in X) \). Then \( F' = f \) and \( \phi(F) = 1+\phi(H) \).

Thus, if \( f \) is positive then \( F \) is increasing. If \( f \) is strictly positive then \( |f(x)-1| < r < 1 \) for all \( x \in X \) and, by a trivial extension of 3.10, we may choose \( H \) such that \( |\phi(H)| < r \). It follows that \( |\phi(F)-1| < r \), so \( F \) is strictly increasing.
We collect the results in

COROLLARY 3.12. Let $X \subset K$ have no isolated points. Then

(i) **If** $f: X \to K$ **is differentiable and (strictly) increasing**

then $f'$ **is a (strictly) positive Baire class 1 function.**

(ii) **If** $g: X \to K$ **is a (strictly) positive Baire class 1 func­

tion then** $g$ **has a (strictly) increasing antiderivative.**

(iii) **If** $f: X \to K$ **is differentiable and if** $f'$ **is (strictly) po-

sitive then** $f = g + h$ **where** $g$ **is (strictly) increasing and

where** $h' = 0$.

**Note.** We cannot strengthen 3.12 (iii) by replacing "$h' = 0" by "$h$ is

locally constant". In fact, if $X = \mathbb{Z}$ then every locally constant func­

tion has bounded difference quotients. If our statements were true, then every differentiable $f: \mathbb{Z} \to \mathbb{Q}$ for which $f'$ is positive would have bounded difference quotients.

But consider the function $f: \mathbb{Z} \to \mathbb{Q}$ defined via

$$f(x) := \begin{cases} x-p^n & \text{if } |x-p^n| < p^{-3n} \quad (n \in \{0,1,2,\ldots\}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}$. Let $x := p^n$ and $y := p^n + p^{3n} \quad (n \in \mathbb{N})$. Then

$f(x_n) = p^n - p^{2n}$, $f(y_n) = p^n + p^{3n}$, so $|f(x_n) - f(y_n)| = |p^{2n}| = p^{-2n}$,

whereas $|x_n - y_n| = |p^{3n}| = p^{-3n}$. So

$$\lim_{n \to \infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = \lim_{n \to \infty} p^n = \infty.$$

We now study the connection between increasing $C^1$-functions and continuous positive functions.

If $f$ is a (strictly) increasing $C^1$-function then clearly $f'$ is a con-

tinuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f: X \to K$ be continuous and positive. Let $P: C(X) \to C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X).$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_n = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in IN$, let $x_n := \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x) - x| < r_n$, $\sigma_n(x) \in R_n$. See [2] 5.3, 5.4.)

From [2] 5.4, it follows that $Pf$ is a $C^1$-antiderivative of $f$. It suffices to prove that $Pf$ is (strictly) increasing. Let $x, y \in X, x \neq y$, $|x-y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x-y| < r_s$. We have $x_1 = y_1, \ldots, x_s = y_s, x_{s+1} \neq y_{s+1}$. Further $|x_{n+1} - x_n| \leq |x-y|$ ($n > s$), $|y_{n+1} - y_n| \leq |x-y|$ ($n > s$), $|x_{s+1} - y_{s+1}| \leq |x-y|$. Hence (using the identity $x = \sum (x_{n+1} - x_n) + x_1, y = \sum (y_{n+1} - y_n) + y_1, x_1 = y_1$)

$$|Pf(x) - Pf(y) - (x-y)| = |(f(x_1) - 1)(x_{s+1} - y_{s+1}) + \sum_{n>s} (f(x_n) - 1)(x_{n+1} - x_n) - \sum_{n>s} (f(y_n) - 1)(y_{n+1} - y_n)|.$$

If $|f(x) - 1| < \alpha$ for all $x \in X$, we have since $\lim |f(x_n) - 1|$ exists, $\sup_{n>s} |f(x_n) - 1| < \alpha$, similarly, $\sup_{n>s} |f(y_n) - 1| < \alpha$. So we get $|Pf(x) - Pf(y) - (x-y)| < \alpha |x-y|$. Now suppose $|x-y| \geq r_1$. Then since for all $n: |x_{n+1} - x_n| < r_1, |x_1 - y_1| = |x-y|$ we get (again under the assumption $|f(x) - 1| < \alpha$ for all $x \in X$):
\[(Pf)(x)-(Pf)(y)-(x-y) = \sum_{n=1}^{\infty} f(x_n^{-1})(x_{n+1}^{-n}-x_n^{-n}) - \sum_{n=1}^{\infty} f(y_n^{-1})(y_{n+1}^{-n}-y_n^{-n}) \leq |x| |x-y|.
\]

We have proved:

**THEOREM 3.13.** Let \(X \subseteq T\) have no isolated points. Then the map \(P\) defined via

\[
(Pf)(x) = x_n + \sum_{n=1}^{\infty} f(x_n^{-1})(x_{n+1}^{-n}-x_n^{-n}) \quad (f \in C(X), x \in X)
\]

maps (strictly) positive functions into (strictly) increasing functions.

**COROLLARY 3.14.** Let \(X \subseteq T\) have no isolated points. Then if \(f \in C^1(X)\) and \(f'\) is (strictly) positive, then \(f = j*h\) where \(j\) is (strictly) increasing and \(h\) is locally constant.

**Proof.** By 3.12 we have \(f = j*h\) where \(j\) is (strictly) increasing and \(h' = 0\). Now by [2] Cor. 5.2 bis there is a locally constant function \(l: X \rightarrow T\) with \(|l(h-1)|_\infty < \frac{1}{2}\). Then \(s = j*(h-1)\) is (strictly) increasing, so we have \(f = s+1\), where \(s\) is (strictly) increasing and \(l\) is locally constant.

**Note.** We may also define convex functions. Let \(X \subseteq T\). A function \(f: X \rightarrow T\) is called convex if the second order difference quotient is positive. I.e., if for all \(x,y,z \in X\) \((x \neq y, y \neq z, x \neq z)\) we have

\[\frac{f(x)-f(y)}{y-z} \leq \frac{f(x)-f(z)}{y-z} \in K^+\]

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function \( f \) the function \( x \mapsto \Phi f(x,y) \) defined on \( X \setminus \{y\} \) is an isometry, hence can be continuously extended to the whole of \( X \). Define \( \Phi f(y,y) = \lim_{x \uparrow y} \Phi f(x,y) \) (\( y \in X \)). Thus, \( f \) is differentiable. For all \( x, y, z, t \in X \) we have

\[
\left| \Phi f(x,y) - \Phi f(z,t) \right| \leq \max \left( \left| \Phi f(x,y) - \Phi f(z,y) \right|, \left| \Phi f(z,y) - \Phi f(z,t) \right| \right) \leq \max(|x-z|, |y-t|).
\]

Hence, \( \Phi f \) is uniformly continuous on \( X \) i.e., \( f \) is strongly uniformly differentiable in the sense of [2] page 67.

For each \( y \in X \) the function \( x \mapsto \Phi f(x,y) \) is increasing on \( X \).

If \( \chi(K) \neq 2 \) then convexity of \( f \) implies increasingness of \( \Phi f' \).

(Proof.

\[
\lim_{y \uparrow x} \frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} = \frac{f'(x)\Phi f(x,y) - f'(x')\Phi f(x',y)}{x-x'} \in K^+ (x \neq x')
\]

\[
\lim_{y \uparrow x'} \frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} = \frac{\Phi f(x,x') - f'(x')}{x-x'} \in K^+ (x \neq x')
\]

so \( \frac{f'(x) - f'(x')}{x-x'} \in 2K^+ (x \neq x') \), whence \( \Phi (\Phi f')(x,x') \in K^+ \) if \( x \neq x' \).

Of course, if \( f \in C^2(X) \) (see [2] 8.1) then convexity of \( f \) implies positivity of \( D^2 f \) ([2] 8.4). So if \( \chi(K) \neq 2 \) then \( \Phi f'' = D^2 f \) ([2] 8.14) is positive. If \( \chi(K) = 2 \) then \( f'' = 0 \) for all \( C^2 \)-functions.

Note. The functions that are monotone of type \( \beta \) (\( \beta \in \mathcal{E} \)), see Def. 2.15, are easy to describe: \( f \) is monotone of type \( \beta \) if and only if \( b^{-1} f \) is increasing for any \( b \in \beta \).

We now turn to the functions \( X \times K \) that are of type \( \sigma \) where \( \sigma : \mathcal{E} \to \mathcal{E} \). (2.14). For examples of such \( f \), where \( \sigma \) is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this
section we will assume that $X$ is an open convex subset of $K$. This im-
plies that the set $\{a \in \Sigma : \text{there is } x \in X \text{ such that } x > a\}$ is inde-
pendent of $a \in X$. Thus, $X$ is homogeneous in the sense that $(a + \alpha) \cap X \neq \emptyset$
for some $a \in X$, $\alpha \in \Sigma$ then for each $b \in X$, $(b + \alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y\}$. Then for each $a \in X$

$$
\Sigma(X) = \{a \in \Sigma : x > a \text{ for some } x \in X\}.
$$

Either $\Sigma(X) = K$ or $\Sigma(X) = \{a \in \Sigma : |a| < r\}$ for some $r > 0$ or $\Sigma(X) =
\{a \in \Sigma : |a| \leq r\}$ for some $r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see
1.2) i.e., if $a, \beta \in \Sigma(X)$ and $a \oplus \beta$ is defined then $a \oplus \beta \in \Sigma(X)$.

To be sure that $f$ is monotone both of type $\sigma$ and type $\tau$ implies $\sigma = \tau$ we define

**DEFINITION 3.15.** (Let $X \subset K$ be open, convex and) let $\sigma : \Sigma(X) \to \Sigma$.

$f : X \to K$ is called monotone of type $\sigma$ if for all $x, y \in X$ and $a \in \Sigma(X)$

$$
\begin{align*}
\sigma(a) &= \sigma(a) \\
(x > y + f(x)) &> f(y).
\end{align*}
$$

**THEOREM 3.16.** Let $f : X \to K$ be monotone of type $\sigma : \Sigma(X) \to \Sigma$. Then

(i) $\sigma(-a) = -\sigma(a)$ $(a \in \Sigma(X))$.

(ii) Let $a, \beta \in \Sigma(X)$. If $\sigma(a) \oplus \sigma(\beta)$ is defined then so is

$$
\begin{align*}
\sigma(a) \oplus \sigma(\beta) &= \sigma(a \oplus \beta).
\end{align*}
$$

(iii) Let $a, \beta \in \Sigma(X)$. If $|a| < |\beta|$ then $|\sigma(a)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then

$$
\sigma(sa) = s \sigma(a) (a \in \Sigma(X))\).
$$

(v) If $\beta \in \Sigma(X)$, $|\beta| = 1$, $\beta$ contains an element of the prime
field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $a \in \Sigma(X)$. 

(vi) $f \in \mathcal{M}_{us}(X)$ (i.e., for all $x,y,z,t \in X, |x-y| < |z-t|$ implies $|f(x)-f(y)| < |f(z)-f(t)|$).

(vii) $f$ is either nowhere continuous or uniformly continuous on $X$.

Proof.

(i) Let $x,y \in X$ such that $x > y$. Then $f(x)-f(y) \in \sigma(a); f(y)-f(x) \in -\sigma(a)$. But also $y > x$, hence $f(y)-f(x) \in -\sigma(a)$. So $-\sigma(a)$ and $\sigma(-a)$ are not disjoint and they must coincide.

(ii) Suppose $\sigma(a) \oplus \sigma(\beta)$ is defined. If $\alpha \oplus \beta$ were not, then $\beta = -\alpha$ so, by (i), $\sigma(\beta) = \sigma(-\alpha) = -\sigma(a)$. Hence also $\alpha \oplus \beta$ is defined. Choose $x,y \in X$ with $x > y$. There is $z \in X$ such that $y > z$. Then $x-y \in \alpha, \beta, y-z \in \alpha \oplus \beta$. Further $f(x)-f(y) \in \sigma(a), f(y)-f(z) \in \sigma(\beta)$ so $f(x)-f(z) \in \sigma(a) \oplus \sigma(\beta)$. Also $x-z \in \alpha \oplus \beta$, so $f(x)-f(z) \in \sigma(\alpha \oplus \beta)$.

The signs $\sigma(a) \oplus \sigma(\beta)$ and $\sigma(\alpha \oplus \beta)$ are not disjoint and they must coincide.

(iii) Let $|\alpha| < |\beta|$. Choose $x,y,z$ such that $x-y \in \alpha, y-z \in \beta$. Then (see 1.2 and preamble) $f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(a)+\sigma(\beta)$, $x-z \in \alpha + \beta = \alpha \oplus \beta = \beta$, so $f(x)-f(z) \in \sigma(\beta)$. Thus $[\sigma(a)+\sigma(\beta)] \cap \sigma(\beta)$ is not empty. If $\sigma(a) \oplus \sigma(\beta)$ were not defined then $\sigma(a) = -\sigma(\beta)$ and $\sigma(a) + \sigma(\beta)$ would be a ball with center 0 and radius $|\sigma(\beta)|^{-1}$, but then $[\sigma(a)+\sigma(\beta)] \cap \sigma(\beta)$ would be empty. Hence $\sigma(a) \oplus \sigma(\beta)$ is defined and by (ii) we have $\sigma(a) \oplus \sigma(\beta) = \sigma(\beta)$. By (1.2) (vi), $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) Let $x(K) \neq 0$. Then $s = n \cdot 1$ for some $n \in \{1,2,\ldots,x(K)-1\}$, so by 1.2 (vii), $s \alpha = n \alpha = \ominus \alpha$, $s \sigma(a) = n \sigma(a) = \ominus \sigma(a)$. By a repeated application of (ii), we see $\sigma(\ominus \alpha) = \ominus \sigma(\alpha)$. Hence $\sigma(s \alpha) = s \sigma(\alpha)$. Let $x(k) = 0$. Let $s$ be of the form $n \cdot 1$ for some $n \in \text{IN}$. By a similar reasoning as above, $\sigma(s \alpha) = s \sigma(a)$. We may identify the prime field of $K$ with $\mathbb{Q}$.
Now observe that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) \((n \in \mathbb{N})\), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1,1,2,\ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N}, n = s \mod p \) \((1 \leq s \leq p-1)\) then \( n\alpha = s\alpha \) for all \( \alpha \), so \( \sigma(n\alpha) = \sigma(sa) = \sigma(a) = n\sigma(a) \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and if \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Xi, |\alpha| < |\beta| \). By (iii) \( |\sigma(a)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( z-t \in \alpha \) for some \( \alpha \) hence \( f(z)-f(t) \in \sigma(a) \). Thus, \( f(z) \neq f(t) \).

(vii) Let \( p := \inf \{|f(x)-f(y)| \}. \) If \( p > 0 \) then clearly \( f \) is nowhere continuous. If \( p = 0 \), let \( \varepsilon > 0 \). There is \( a,b \in X, a \neq b \) such that \( |f(a)-f(b)| < \varepsilon \). By (vi), for all \( x,y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \varepsilon \). Then \( f \) is uniformly continuous.

A natural question that can be raised is the following. If \( f : X \to K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f : X \to K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
(β) \( f \in M_d(X) \).

(γ) \( f \in M_{\text{abs}}(X) \).

(δ) If, for \( \alpha, \beta \in \Sigma(X) \), \( \alpha \oplus \beta \) is defined then so is \( \sigma(\alpha) \oplus \sigma(\beta) \) (and \( \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta) \)).

(ε) If \( \alpha, \beta \in \Sigma(X) \), \( |\sigma(\alpha)| < |\sigma(\beta)| \) then \( |\alpha| < |\beta| \).

Proof. We prove \( (\alpha) \rightarrow (\varepsilon) \rightarrow (\gamma) \rightarrow (\beta) \rightarrow (\delta) \rightarrow (\alpha) \).

(α) \rightarrow (ε). Let \( |\sigma(\alpha)| < |\sigma(\beta)| \) then \( \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta) \) (1.2.(vi)). By 3.16, (iii), \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta) \). Since \( \sigma \) is injective, \( \alpha \oplus \beta = \beta \) so (again 1.2.(vi)) \( |\alpha| < |\beta| \).

(ε) \rightarrow (γ). Let \( |x-y| \leq |z-t| \) \( (x,y,z,t \in X) \). We prove \( |f(x)-f(y)| \leq |f(z)-f(t)| \). If \( z = t \) there is nothing to prove. Assume \( z \neq t \) and \( |f(x)-f(y)| > |f(z)-f(t)| \). Then (\( f \) is injective), supposing \( x-y \in \alpha \), \( z-t \in \beta \) for some \( \alpha, \beta \in \Sigma(X) \), we have \( f(x)-f(y) \in \sigma(\alpha) \), \( f(z)-f(t) \in \sigma(\beta) \) and \( |\sigma(\alpha)| > |\sigma(\beta)| \). By (ε), \( |\alpha| > |\beta| \) i.e., \( |x-y| > |z-t| \). Contradiction.

(γ) \rightarrow (β). Trivial.

(β) \rightarrow (δ). Suppose \( \sigma(\alpha) \oplus \sigma(\beta) \) is not defined. Then \( |\sigma(\alpha)| = |\sigma(\beta)| \) and, by 3.16 (iii), \( |\alpha| = |\beta| \). Choose \( x,y,z \) such that \( x-y \in \alpha \), \( y-z \in \beta \). Then \( f(x)-f(z) \in \sigma(\alpha) + \sigma(\beta) \) so \( |f(x)-f(z)| < |\sigma(\alpha)| = |f(x)-f(y)| \).

Since \( f \in M_d(X) \), \( |x-z| < |x-y| \) hence, since \( x-z \in \alpha \oplus \beta \), \( x-y \in \alpha \):

\[ |\alpha \oplus \beta| < |\alpha| \]. But \( |\alpha \oplus \beta| = \max(|\alpha|,|\beta|) \), a contradiction.

(δ) \rightarrow (α). Suppose \( \sigma(\alpha) = \sigma(\beta) \) and \( \alpha \neq \beta \). Then \( \alpha \oplus (-\beta) \) is defined. By (δ), also \( \sigma(\alpha) \oplus \sigma(-\beta) \) is defined. But \( \sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha) \), so \( \sigma(\alpha) \oplus -\sigma(\alpha) \) is defined, a contradiction.

THEOREM 3.18. Let \( k \) be a prime field. Then, if \( f : X \rightarrow k \) is monotone of type \( \sigma \) then \( \sigma \) is injective.
Proof. Suppose \( \sigma(a) = \sigma(b) \) for some \( a, b \in \Sigma(X) \). Then \( |\sigma(a)| = |\sigma(b)| \) so, by 3.16 (iii), \( |a| = |b| \). There is \( t \in K, |t| = 1 \) such that \( b = ta \). Since \( K \) is a prime field we may suppose \( t \in \{1, 2, \ldots, p-1\} \) if \( k = \mathbb{F}_p \) and \( t \in \mathbb{Q}^* \) if \( k = \mathbb{Q} \). So, by 3.16 (iv), \( \sigma(b) = \sigma(ta) = t\sigma(a) = ta(\beta) \). For \( x \in \sigma(\beta) \) we have \( tx \in \sigma(\beta) \), so \( tx \cdot x^{-1} \in K^+ \) i.e., \( |t-1| < 1 \). It follows easily that \( t = 1 \). Hence, \( a = \beta \).

We now like to determine all \( \sigma : \Sigma \to \Sigma \) that "can occur" as the type of a monotone function in case \( K = \mathbb{Q}_p \). We use the fact that \( \Sigma \) can be identified with the following subgroup of \( \mathbb{Q}_p^* \)

\[
\{ \theta^i_p n : i \in \{0,1,2,\ldots,p-2\}, n \in \mathbb{Z} \}
\]

where \( \theta \) is a primitive \((p-1)\)th root of 1. (See 1.5.)

First, let \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) be monotone of some type \( \sigma : \Sigma \to \Sigma \). By 3.18, \( \sigma \) is injective. By 3.17, (e), 3.16 (iii) we have \( |a| < |\beta| \iff |\sigma(a)| < |\sigma(\beta)| \) and \( |a| = |\beta| \iff |\sigma(a)| = |\sigma(\beta)| \), so \( |\sigma(a)| \) is a strictly increasing function of \( |a| \).

Set

\[
\sigma(\theta^i_p n_p) = \theta^s(i,n)_p \lambda(i,n) \quad (\theta^i_p n \in \Sigma)
\]

Where \( s : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \{0,1,2,\ldots,p-2\} \) and \( \lambda : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \mathbb{Z} \). We see that \( |\sigma(\theta^i_p n_p)| = |\sigma(\theta^j_p n_p)| \) for all \( i, j \in \{0,1,2,\ldots,p-2\} \) hence \( \lambda(i,n) = \lambda(j,n) \) for all \( i, j \in \{0,1,2,\ldots,p-2\} \). Then

\[
\sigma(\theta^i_p n_p) = \theta^s(i,n)_p \lambda(n)
\]

where \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is a strictly increasing function (in the classical sense).

By 3.16 (v), \( \sigma(\theta^i_p n_p) = \theta^i\sigma(p^n) = \theta^i \theta(0,n)_p \lambda(n) \).
Thus, $\sigma$ is of the form

$$(*) \, \theta^i_p n \rightarrow \theta^i s(n)_p \lambda(n)$$

where $s : \mathbb{N} \rightarrow \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

Conversely, if we are given a map $\sigma$ of the form $(*)$ then it is easy to construct an $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, monotone of type $\sigma$. In fact, let $x \in \mathbb{Q}_p$, $x = \sum a_n p^n$, where $a_n \in \{0,1,\ldots,p^{B-2}\}$ for each $n$ and $a_{-n} = 0$ for large $n$. Then set

$$f(x) = \sum_{n \in \mathbb{Z}} a_n \theta^i s(n)_p \lambda(n).$$

Now let $x = \sum a_n p^n$, $y = \sum b_n p^n$ and $\pi(x-y) = \theta^i m$ for some $i \in \{0,1,\ldots,p-2\}$, $m \in \mathbb{Z}$. Then $a_n - b_n$ for $n < m$ and $a_m - b_m = \theta^i \mod p$. So the sign of $a_m - b_m$ is $\theta^i$. $f(x)-f(y) = \sum (a_n - b_n) \theta^i s(n)_p \lambda(n) = (a_m - b_m) \theta^i s(m)_p \lambda(m) + r$, where $n > m$. Then $|r| < |f(x)-f(y)|$. The sign of $f(x)-f(y)$ is the sign of $(a_m - b_m) \theta^i s(m)_p \lambda(m)$ which is $\theta^i s(m)_p \lambda(m)$. So $\pi(f(x)-f(y)) = \theta^i s(m)_p \lambda(m) = \sigma(\theta^i p)$. Thus, $f$ is monotone of type $\sigma$. We have found

**THEOREM 3.19.** The set $\{\sigma : \Sigma \rightarrow \Sigma : \text{there is } f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, \text{monotone of type } \sigma\}$ is equal to the set of all $\sigma : \Sigma \rightarrow \Sigma$ of the form

$$\theta^i p n \rightarrow \theta^i s(n)_p \lambda(n)$$

where $s : \mathbb{Z} \rightarrow \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

**Remark.** With the notations as in 3.19, let $\mu(n) := \lambda(n)-n$. Then $\mu : \mathbb{Z} \rightarrow \mathbb{Z}$ is increasing ($\mu(n+1) = \lambda(n+1)-(n+1) \geq \lambda(n)+1-(n+1) = \mu(n)$).

We then have two possibilities for a function $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, monotone of type $\sigma$. 
(a) \( \lim_{n \to \infty} \mu(n) = \infty. \) Then \( |\sigma(a)| = |a||p^\mu(n)|, \) \( (a = 0^i p^n), \) so \( \lim_{|a| \to 0} \frac{|\sigma(a)|}{a} = 0. \)

Thus \( \lim_{x-y \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0. \)

(b) \( \mu \) is bounded above. Then \( \mu(n) \) is constant, \( c, \) for \( n \geq n_0. \) (For example, if \( \sigma \) is bijective then we have even \( \mu(n) = c \) for all \( n. \)) Thus, for sufficiently small \( |a| \) \( (a = 0^i p^n \in \Sigma) \) we have

\[
|\sigma(a)| = |p^\lambda(n)| = |p^{n+c}| = |p^c| |a|.
\]

So, there is \( r \) such that \( |x-y| < r \) implies \( |f(x)-f(y)| = |p^c||x-y|. \)

In this case we then have: there is \( r > 0 \) and \( \lambda \in \mathbb{Q}_p \) such that on each ball in \( \mathbb{Q}_p \) of radius \( r, \lambda^{-1} f \) is an isometry.

We now construct an example of a function \( f \) monotone of type \( \sigma, \) where \( \sigma \) is not injective. Let \( p = 3 \) mod 4 and let \( K := \mathbb{Q}_p(\sqrt{-1}). \) The elements of \( K \) can be written as \( a+bi \) \( (a,b \in \mathbb{Q}_p) \) and \( |a+bi| = \max(|a|,|b|). \)

The value group of \( K \) is the same as the one of \( \mathbb{Q}_p, \) the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.18). Let \( X \) be the unit ball of \( K, \) let

\[
S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}.
\]

For each \( x \in X \) there is a unique \( \bar{x} \in S \) such that \( |x-\bar{x}| < 1. \) As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s,t \in S^*, \ s \neq t. \)

Define a function \( h : S \to K \) as follows

\[
h(a+bi) = \frac{1}{p^c} a \quad (a+bi \in S)
\]
and let $f : X \to K$ be defined via

$$f(x) = x + h(x) \quad (x \in X).$$

We claim that $f$ is monotone of type $\sigma$ where

$$\sigma(\pi(a+bi)) = \pi\left(\frac{1}{p} a\right) \text{ if } a+bi \in S, \ a \neq 0$$

$$\sigma(a) = a \quad \text{ elsewhere.}$$

(Clearly, $\sigma$ is a well defined map $\Xi(X) \to K$, $\sigma$ is not injective since, for example, $\sigma(\pi(1)) = \sigma(\pi(1+i))$).

**Proof.** Let $|a| < 1$ and $x-y \in \alpha$, then $|x-y| < 1$ so $x = y$, $h(x) = h(y)$. It follows that $f(x)-f(y) = x-y \in \alpha = \sigma(\alpha)$.

Now let $|a| = 1$ be of the form $\pi(bi), \ b \in \{1,2,\ldots,p-1\}$ and let $x-y \in \alpha$. Say, $\overline{x} = r+si, \ \overline{y} = t+ui \ (r,s,t,u \in \{0,1,2,\ldots,p-1\})$. Then also $\overline{x-y} \in \alpha$, so $|r+si-t+ui| < 1$ hence $r = t$. Thus, $h(x) = \frac{1}{p} r = h(y)$, and we have $f(x)-f(y) = x-y \in \alpha = \sigma(\alpha)$.

Finally, let $|a| = 1$, $a = \pi(a+bi)$, where $a \neq 0 \ (a,b \in \{0,1,2,\ldots,p-1\})$ and let $x-y \in \alpha$. Set $\overline{x} = r+si, \ \overline{y} = t+ui$. Then $\overline{x-y} \in \alpha$, so $r-t = a \mod p$. We find $h(x) = \frac{1}{p} r, h(y) = \frac{1}{p} t$, so $|h(x)-h(y)| = \frac{1}{p} |a| < \frac{1}{|p||a|}$ i.e. $h(x)-h(y) = \pi\left(\frac{1}{p} a\right)$. Since $||\pi(x-y)|| < 1$, we find $f(x)-f(y) = x-y-(h(x)-h(y)) 

\epsilon = \pi\left(\frac{1}{p} a\right) = \sigma(\pi(a+bi)) = \sigma(\alpha)$.

Concluding:

**EXAMPLE 3.20.** Let $p = 3 \mod 4$ and $K = \mathbb{Q}_p(\sqrt{-1})$. Then there exists a function $f : \{x \in K : |x| \leq 1\} \to K$, monotone of some type $\sigma$, where $\sigma$ is not injective.

In case $K$ has discrete valuation we have some extra information.
THEOREM 3.21. Let $K$ have discrete valuation and let $f : X \to K$ be monotone of type $\sigma \in \Sigma(X)$. Then

(i) $f$ is continuous.

(ii) If $\sigma$ is injective, then $f$ and $\phi(f)$ are bounded on bounded sets.

(iii) If $\sigma$ is injective and if $f(X)$ is convex then there is a $\lambda \in K$ such that $\lambda f$ is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2)(c). If $\sigma$ is injective then by 3.16 (iii) and 3.17 (e), $|\sigma(a)|$ is a strictly increasing function of $|a|$. Suppose $X$ is bounded. Then $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r \in K^*$. Let $|\sigma(a)| = s$ whenever $|a| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|a| = |\pi| r$. Then $|a| < r$ so $|\sigma(a)| < s$, hence $|\sigma(a)| \leq |\pi|^n s$. By induction, it follows that $|\sigma(a)| \leq |\pi|^n s$ whenever $|a| = |\pi|^n r$, so $|\sigma(a)| \leq |a| \cdot s |\pi|^{-1}$. So the difference quotients of $f$ are bounded by $sr^{-1}$. Then clearly $f$ is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $\Sigma(f(X))$ has the form $\{a \in \Sigma : |a| \leq s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then $\sigma$ induces an injection of $\{p \in |K^*| : p \leq r\}$ onto $\{a \in |K^*| : |a| \leq s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(a)| = c|a|$ for some $c \in \mathbb{R}$ i.e., $|f(x) - f(y)| = c|x - y|$ for all $x, y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type $\sigma$, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) $K$ is a local field ($f$ is continuous).

(2) $K$ is finite and $X = \{x \in K : |x| \leq 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be
representatives modulo \{x \in K: |x| < 1\}, let \(M = \max |f(a_i)|\). For each \(x, y \in X\) we have \(i, j\) for which \(|x-a_i| < 1, |y-a_j| < 1\). Since \(f \in M(X)\), we have \(|f(x)-f(a_i)| < M, |f(y)-f(a_j)| < M\) whence \(|f(x)-f(y)| \leq M: f\) is bounded.)

(3) \(K\) is discrete, \(\sigma\) is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let \(k\) be isomorphic to the algebraic closure of \(\mathbb{F}_p\). Let \(X\) be the unit ball of \(K\). Then there exists a function \(f: X \to K\), monotone of type \(\sigma\), for some \(\sigma: \Sigma(X) \to \Sigma\) such that

(i) \(\sigma\) is not injective.

(ii) \(f, \Phi(f)\) are unbounded.

Proof.

As an \(\mathbb{F}_p\)-vector space, \(k\) has a countable base \(e_1, e_2, \ldots\). For any \(\lambda \in \mathbb{F}_p\), \(\lambda = n_1 \mathbb{1}_K\) for some \(n \in \{0, 1, 2, \ldots, p-1\}\). (Here for a field \(L\), \(\mathbb{1}_L\) is the unit element of \(L\).) Define \(n_k := n_1\). Choose \(c_1, c_2, \ldots \in K\) such that

\[1 < |c_1| < |c_2| < \ldots \lim_{n \to \infty} |c_n| = \infty,\]

and define a map \(h: k \to K\) via

\[h(\Sigma \lambda \mathbb{1}_n e_n) = \Sigma \lambda \mathbb{1}_n c_n \quad (\Sigma \lambda_n e_n \in k)\]

Define \(f: X \to K\) by

\[f(x) = x + h(x) \quad (x \in X)\]

(Here \(x\) is the image of \(x\) under the canonical map \(X \to k\)).

Then clearly \(f\) is unbounded and so is \(\Phi(f)\).

Let us identify \{\(a \in \Sigma: |a| = 1\}\) with \(k^*\) in the obvious way. We claim that \(f\) is monotone of type \(\sigma\) where
In fact, let \( x-y \in \alpha \) and \(|\alpha| < 1 \). Then \( h(x) = h(y) \) so \( f(x) - f(y) = x-y \in \sigma(\alpha) \). Now let \( x-y \in \alpha \) where \(|\alpha| = 1 \). Then set \( \bar{x} = \sum_{n} \alpha_n e_n \), \( \bar{y} = \sum_{n} \mu_n e_n \). Let \( r = \max\{n : \lambda_n \neq \mu_n\} \). Then \( \bar{x} - \bar{y} = \sum_{n=1}^{r} (\lambda_n - \mu_n) e_n = \sigma_0((\lambda_{-r} - \mu_{-r}) e_{-r}) \).

On the other hand, \( f(x) - f(y) = x-y - (h(x) - h(y)) = x-y - \sum_{n=1}^{r} (\tilde{\lambda}_n - \tilde{\mu}_n) c_n \). Thus \( \pi(f(x) - f(y)) = \pi((\tilde{\lambda}_{-r} - \tilde{\mu}_{-r}) c_{-r}) \).

Now we have \( \tilde{\lambda}_{-r} - \tilde{\mu}_{-r} \equiv 0 \pmod{p} \), so \( \pi(\tilde{\lambda}_{-r} - \tilde{\mu}_{-r}) = \pi(\tilde{\lambda}_{-r} - \tilde{\mu}_{-r}) \). It follows that \( f(x) - f(y) \in \sigma(\alpha) \).

Obviously, \( \sigma \) is not injective.

We will consider briefly the differentiable functions, monotone of type \( \sigma \). Let \( f : X \to K \) be such a function. If \( f'(a) = 0 \) for some \( a \in X \), then \( \lim_{|a| \to 0} \frac{\sigma(a)}{|a|} = 0 \), so \( f' = 0 \) uniformly on \( X \). Now let \( f'(a) \neq 0 \). Then if \( x \) is sufficiently close to \( a \), we have \( |\frac{f(x) - f(a)}{x-a} - f'(a)| < |f'(a)| \).

Thus for \(|a|\) small enough we have \( f'(a) \in \frac{\sigma(a)}{a} \) i.e. \( \frac{\sigma(a)}{a} \) is constant.

This implies that \( \pi(f'(x)) \) does not depend on \( x \) (\( f' \) has constant sign) and that locally \( f \) is monotone of type \( \beta \) for some \( \beta \in \Sigma \).

We end this section with a discussion on the question which maps \( \sigma : \Sigma \to \Sigma \) do occur as a type of a monotone function.

**Lemma 3.23.** Let \( \sigma : \Sigma \to \Sigma \). Suppose \( \sigma \) satisfies

if \( \alpha \oplus \beta \) is defined then \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \). (\( \alpha, \beta \in \Sigma \)).

Then

(i) \( \sigma(-\alpha) = -\sigma(\alpha) \) (\( \alpha \in \Sigma \)).
(ii) If $\sigma(\alpha)$ is defined then so is $\alpha \oplus \beta$.

(iii) If $|\alpha| < |\beta|$ then $|\sigma(\alpha)| < |\sigma(\beta)|$.

... $\sigma$ is injective.

(v) If $|\alpha| = |\beta|$ then $|\sigma(\alpha)| = |\sigma(\beta)|$.

Proof. (i) is trivial if $\chi(k) = 2$, so suppose $\chi(k) \neq 2$ and let $-\sigma(\alpha) \neq \sigma(\alpha')$ for some $\alpha \in \Sigma$. Then we have the identity $(\alpha \oplus \alpha) \oplus (-\alpha) = \alpha$, so $\sigma(\alpha \oplus \alpha) \oplus \sigma(-\alpha) = \sigma(\alpha)$, whence $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha) = \sigma(\alpha)$.

Now by 1.2 (iii) $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ (this last expression is defined).

If not, then $-\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(-\alpha)$. Now $\sigma(\alpha) \oplus \gamma = -\sigma(\alpha)$ has only one solution namely $\gamma = -2\sigma(\alpha)$. So we then would have $\sigma(-\alpha) = -2\sigma(\alpha) = -(\sigma(\alpha) \oplus \sigma(\alpha))$, but this contradicts the existence of $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha))$.

From $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ we obtain by 1.2 (vi): $|\sigma(\alpha) \oplus \sigma(-\alpha)| < |\sigma(\alpha)|$. On the other hand, by 1.2 (v), $|\sigma(\alpha) \oplus \sigma(-\alpha)| = |\sigma(\alpha)| \lor |\sigma(-\alpha)|$.

Contradiction. (i) follows.

Now (ii) follows easily from (i): if $\alpha \oplus \beta$ were not defined then $\beta = -\alpha$ so, by (i), $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, a contradiction. Let $|\alpha| < |\beta|$, then $\alpha \oplus \beta = \beta$, so $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. By 1.2 (vi) we find $|\sigma(\alpha)| < |\sigma(\beta)|$. We proved (iii).

If $\sigma(\alpha) = \sigma(\beta)$ and $\alpha \neq \beta$ then $\sigma(\alpha \oplus (-\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, an absurdity. So $\sigma$ is injective (iv).

Finally, let $|\alpha| = |\beta|$ and $|\sigma(\alpha)| > |\sigma(\beta)|$. Then $\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(\beta) = (by (ii)) = \sigma(\alpha \oplus \beta)$. By injectivity of $\sigma$, $\alpha = \alpha \oplus \beta$, and by 1.2 (vi), we find $|\beta| < |\alpha|$.

Now we have
LEMMMA 3.24. Let $K$ be spherically complete, let $Y \subset K$ (not necessarily convex) and let $\tau : \Sigma(Y) (= \{ (x-y) : x, y \in Y, x \neq y \}) \rightarrow \Sigma$ such that $f$ is monotone of type $\tau$ (i.e., $x, y \in Y, x - y \in \alpha \in \Sigma(Y)$ then $f(x) - f(y) \in \tau(\alpha)$.

Suppose $\tau$ can be extended to a $\sigma : \Sigma \rightarrow \Sigma$ satisfying the condition of Lemma 3.23. Then $f$ can be extended to a monotone function $\bar{f} : K \rightarrow K$ of type $\sigma$.

Proof. By Zorn's lemma, it suffices to extend $f$ to $Y \cup \{ a \}$ ($a \neq Y$) such that $f(x) - f(a) \in \sigma(\pi(x-a))$, $f(a) - f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x))$ ($x \in Y$). Each $B_x$ is a ball with radius $|\pi(a-x)|$. By the spherical completeness of $K$, we are done if we can show that $B_x \cap B_y \neq \emptyset$ ($x \neq y, x, y \in Y$).

Set $\alpha := \pi(a-x)$ and $\beta := \pi(a-y)$. Let $b \in \sigma(\alpha); c \in \sigma(\beta)$. We prove:

$|f(x) + b - f(y) - c| < |\sigma(\alpha) \cup |\sigma(\beta)|. We have two cases:

1) $\alpha = \beta$. Then $a-x \in \alpha, a-y \in \alpha$ implies $|x-y| < |a-x| = |a|$, so

$|\pi(x-y)| < |a|$ whence $|\pi(f(x) - f(y))| = |\sigma(\pi(x-y))| < |\sigma(\alpha)|$ (by 3.23 (iii)), so $|f(x) - f(y)| < |\sigma(\alpha)|$. Further, $b \in \sigma(\alpha), c \in \sigma(\alpha)$ implies $|b - c| < |\sigma(\alpha)|$, hence $|f(x) + b - f(y) - c| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x-y = a-y - (a-x) \in \beta \oplus (\alpha)$, so $f(x) - f(y) + b - c \in \sigma(\beta \oplus (\alpha)) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus (\alpha)) + \sigma(\alpha) \oplus (\alpha)$, hence $|f(x) - f(y) + b - c| < |\sigma(\beta \oplus (\alpha))| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(\alpha)|, |\sigma(\beta)|$.

THEOREM 3.25. Let $K$ be spherically complete and let $\sigma : \Sigma \rightarrow \Sigma$. Suppose

$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma, \alpha \neq -\beta$).

Then there exists a function $f : K \rightarrow K$, monotone of type $\sigma$. 

\[ \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \] 
\[ (\alpha, \beta \in \Sigma, \alpha \neq -\beta). \]
Proof. Choose $Y := \{0\}$ and let $g : Y \to K$ be defined via $g(0) = 0$. Then $g$ satisfies the conditions of Lemma 3.24 so it can be extended to a function $f$ of type $\sigma$.

We now give a description of the maps $\sigma : \Sigma \to \Sigma$ mentioned in 3.23. For each $r \in |K^*|$ choose $\alpha_r \in \Sigma$ such that $|\alpha_r| = r$. Further, there is a natural isomorphism of multiplicative groups between $k^*$ and $\{a \in \Sigma : |a| = 1\}$, denoted by $l \mapsto \alpha_1 (l \in k^*)$. Of course, if $l + l' \neq 0$ then $\alpha_{l + l'} = \alpha_l + \alpha_{l'}$. Each element of $\Sigma$ can be written in only one way as $\alpha_{r,1}$ ($r \in |K^*|$, $1 \in k^*$). Now if $\sigma$ is as in 3.23 we get

$$\sigma(\alpha_{r,1}) = \alpha_{\lambda(r)} n(r,1)$$

where $\lambda : |K^*| \to |K^*|$ is strictly increasing and $n(r,1)$ is an injective group isomorphism of the additive group $k$. Conversely, if $\lambda : |K^*| \to |K^*|$ is strictly increasing and for each $r$, $l \mapsto n(r,1)$ is an injective group homomorphism $k \to k$ then

$$\alpha_{r,1} \mapsto \alpha_{\lambda(r)} n(r,1) \ (\alpha_{r,1} \in \Sigma)$$

satisfies the condition of 3.23. So we get

**THEOREM 3.26.** Let $K$ be spherically complete and let $|K| = [0,\infty)$. Then there exist a nowhere continuous $f : K \to K$, monotone of some type $\sigma : \Sigma \to \Sigma$.

Proof. With the notations as above, let $\sigma : \Sigma \to \Sigma$ be defined as follows

$$\sigma(\alpha_{r,1}) = \alpha_{r+1,1}.$$

By 3.25 there is an $f : K \to K$ monotone of type $\sigma$. Clearly $|f(x) - f(y)| \geq 1$ if $x \neq y$ so $f$ is nowhere continuous.
4. MONOTONE FUNCTIONS, GENERAL THEOREMS

In this section we study \( M_w(X) \), \( M_b(X) \), \( M_g(X) \), .... To avoid unnecessary complications we ASSUME THROUGHOUT THIS SECTION THAT \( X \) IS A CLOSED SUBSET OF \( K \) WITHOUT ISOLATED POINTS. We collect here the results on monotone functions that are valid for general \( K \). In the next section we will see what happens if we put some extra conditions on \( K \) (e.g., \( |K| \) discrete, ...).

First two elementary lemmas.

**LEMMA 4.1** Let \( f : X \to K \). Then the following conditions are equivalent

(a) \( f \in M_w(X) \) (see Def. 2.11).

(β) For all \( x, y, z \in X \), \(|x-y| < |x-z| \) implies \(|f(x)-f(z)| = |f(y)-f(z)|\).

(γ) For all \( x, y, z \in X \), \(|f(x)-f(z)| \neq |f(y)-f(z)|\) implies \(|x-y| = \max(|x-z|, |y-z|)\).

**Proof.** (α) \( \Rightarrow \) (β). |x-y| < |x-z| implies |y-z| = |x-z| > |x-y|, so

\[ |f(x)-f(y)| \leq \min(|f(x)-f(z)|, |f(y)-f(z)|). \]

It follows that \(|f(x)-f(z)| = |f(y)-f(z)|\).

(β) \( \Rightarrow \) (γ). (β) says that \(|f(x)-f(z)| \neq |f(y)-f(z)|\) implies \(|x-y| \geq |x-z|\). By symmetry, also \(|x-y| \geq |y-z|\) where \(|x-y| \geq \max(|x-z|, |y-z|)\). The opposite inequality is trivial.

(γ) \( \Rightarrow \) (α). Let \(|x-y| < |x-z|\). Then \(|x-y| \neq \max(|x-z|, |z-y|)|. Let \(|f(x)-f(z)| \leq \max(|f(x)-f(z)|, |f(y)-f(z)|)|. Then \( |f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(y)-f(z)|) = |f(x)-f(z)|\).

**LEMMA 4.2** (i) If \( f \in M_w(X) \), \( \lambda \in K \) then \( \lambda f \in M_w(X) \).
(ii) If \( f_1, f_2, \ldots \in M_w(X) \) and \( f := \lim_{n} f_n \) pointwise then \( f \in M_w(X) \).

(iii) If \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \), then \( g \in M_w(f(X)) \). In particular, if \( f \) is injective and weakly monotone then so is \( f^{-1} \).

(Notice that \( f(X) \) need not be closed and may have isolated points.)

Proof. Obvious.

Remark. Lemmas, similar to 4.1 and 4.3, but now for \( M_d(X), M_S(X), M_{bs}(X) \) have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an \( M_w \)-function need not be continuous (see 2.4(5), 3.26) we will derive properties of \( M_w \)-functions that are closely related to continuity.

**Lemma 4.3** Let \( f \in M_w(X) \). Then \( f \) is bounded on precompact subsets of \( X \).

Proof. Let \( Y \subset X \) be precompact. Assume that \( Y \) is not a singleton. Then \( Y \) is bounded and has a positive diameter \( r = \max\{|x-y| : x,y \in Y\} \).

The equivalence relation \( x \sim y \) iff \( |x-y| < r \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) \((n \geq 2)\). Choose \( a_i \in Y_i \) for each \( i \), and let 
\[
M := \max_{1 \leq i \leq n} |f(a_i)|. 
\]
We prove: \(|f| \leq M \). In fact, let \( x \in Y \). Then there is \( i \) such that \(|x-a_i| < r \). Choose \( j \neq i \). We have \(|x-a_i| < |a_i-a_j| \) whence \(|f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M \). So \(|f(x)| \leq M \).

The following lemma shows that an \( f \in M_w(X) \) at \( a \in X \) is either continuous or "very discontinuous".

**Lemma 4.4** Let \( f \in M_w(X) \) and let \( a \in X \). Then we have the following alternative.
Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots \in X$ ($x_n \neq a$ for all $n$) with $\lim x_n = a$ the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Proof. Since $(x_1, x_2, \ldots)$ is precompact the set $\{f(x_1), f(x_2), \ldots\}$ is bounded by Lemma 4.3. We are done if we can prove the following. If $x_1, x_2, \ldots, \lim x_n = a, x_n \neq a$ for all $n, \lim f(x_n)$ exists, then $f$ is continuous at $a$. Now set $a := \lim f(x_n)$. Let $y_1, y_2, \ldots \in X, \lim y_n = a$.

We prove $\lim f(y_n) = a$. (Then it follows that $a = f(a)$ since we may choose $y_n := a$ for all $n$.) Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|f(x_k) - a| < \varepsilon$. For $n$ sufficiently large we have $|y_n - x_n| < |x_k - x_m|$, so for large $m$ (depending on $n$) we have $|y_n - x_n| < |x_k - x_m|$, whence $|f(y_n) - f(x_n)| \leq |f(x_k) - f(x_m)|$. Since $\lim f(x_m) = a$ we find $\lim f(y_n) = a$.

COROLLARY 4.5 Let $f \in C_w(X)$. Then the graph of $f$
\[ \Gamma_f := \{(x, y) \in X \times K : y = f(x)\} \]
is closed in $K^2$.

Proof. Let $(x_n, f(x_n)) \in \Gamma_f$ and let $\lim x_n = x, \lim f(x_n) = a$. If $x_n = x$ for infinitely many $n$ then $a = f(x)$, so $(x, a) \in \Gamma_f$. If not then by the alternative of lemma 4.4, $f$ is continuous at $x$, so $a = f(x)$ and $(x, a) \in \Gamma_f$.

COROLLARY 4.6 Let $f \in C_w(X)$. If each bounded subset of $f(X)$ is precompact then $f$ is continuous.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let \( f \in \mathcal{M}_w(X) \) and let \( Y \subseteq f(X) \) be precompact. Then either
\[ f \text{ is constant on } f^{-1}(Y) \text{ or } f^{-1}(Y) \text{ is bounded.} \]

**Proof.** It suffices to prove: if \( Z \subseteq X \) is unbounded and \( f(Z) \) is precompact then \( f \) is constant on \( Z \). Let \( a, b \in Z \). Since \( Z \) is unbounded there are \( x_1, x_2, \ldots \in Z \) such that
\[
(*) \quad |a - b| < |x_1 - a| < |x_2 - a| < \ldots
\]
Since \( f(Z) \) is precompact we may assume (by taking a suitable subsequence) that \( a = \lim f(x_n) \) exists. From \( (*) \) we obtain
\[ |x_1 - x_2| = |x_2 - a|, \quad |x_2 - x_3| = |x_3 - a|, \ldots, \]
so
\[ |a - b| < |x_1 - a| < |x_1 - x_2| < |x_2 - x_3| < \ldots
\]
hence
\[ |f(a) - f(b)| \leq |f(x_1) - f(a)| \leq |f(x_1) - f(x_2)| \leq \ldots
\]
it follows that \( |f(a) - f(b)| = \lim_{n \to \infty} |f(x_n) - f(x_{n+1})| = 0 \) i.e., \( f(a) = f(b) \).

LEMMA 4.8 Let \( f \in \mathcal{M}_w(X) \) and let \( \alpha \in f(X) \) be a non-isolated point of \( f(X) \).

Then we have the following alternative. Either

I. There is \( a \in X \) such that for each sequence \( x_1, x_2, \ldots \) in \( X \) for which \( \lim_{n \to \infty} f(x_n) = \alpha \) we have \( \lim_{n \to \infty} x_n = a \), or

II. If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} f(x_n) = \alpha \), \( f(x_n) \neq \alpha \) for all \( n \), then \( x_1, x_2, \ldots \) is bounded and has no convergent subsequence.

**Proof.** Suppose we are not in case II. Since \( \alpha \) is not isolated in \( f(X) \) and \( f(X) \) is dense in \( f(X) \) we have a sequence \( x_1, x_2, \ldots \) in \( X \) for which \( f(x_n) \neq \alpha \) for each \( n \), and \( \lim_{n \to \infty} f(x_n) = \alpha \). Since \( f \) is not constant on \( \{x_1, x_2, \ldots\} \) it follows by Lemma 4.7 that \( \{x_1, x_2, \ldots\} \) is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by \( x_1, x_2, \ldots \) and set
a := \lim x_n. Then a \in X. Now let \( y_1, y_2, \ldots \) be a sequence in X for which \( \lim f(y_n) = a \). We prove that \( \lim y_n = a \). In fact, let \( \epsilon > 0 \).

There is \( k \in \mathbb{N} \) such that \( |x_k - a| < \epsilon \). For large \( n \) we have

\[ |f(y_n) - a| < |f(x_k) - a|, \]

so for large \( m \) (depending on \( n \)) we have

\[ |f(y_n) - f(x_m)| < |f(x_k) - f(x_m)| \]

whence \( |y_n - x_m| \leq |x_k - x_m| \), so

\[ |y_n - a| \leq |x_k - a| < \epsilon. \]

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function \( f : X \to K \) **injective at** \( a \in X \) if \( f(x) = f(a) \) for some \( x \in X \) implies \( x = a \).

Now suppose that we have \( a \in \overline{f(X)} \), not isolated, for which we are in alternative I. Then for a sequence \( x_1, x_2, \ldots \) with \( \lim f(x_n) = a \) we have \( \lim x_n = a \in X \) so \( (a, a) = \lim (x_n, f(x_n)) \), so by Cor. 4.5 we have \( a = f(a) \). Thus, \( a \in f(X) \). \( f \) is injective at \( a \); if \( f(b) = f(a) \) then since \( \lim f(b) = a \) we must have \( \lim b = a \) i.e. \( b = a \). Further, \( f \) is continuous at \( a \) (see 2.13 (2)(a)).

If each bounded subset of \( X \) is precompact we never can be in case II.

This is also true if \( f \in M_b(X) \) and \( |X| \) is discrete i.e. if \( x_1, y_1 \in X \)

\[ |x_1 - y_1| > |x_2 - y_2| > \ldots \]

then \( \lim |x_n - y_n| = 0 \). Proof: let \( a \in \overline{f(X)} \) and let \( \lim f(x_n) = a \), \( f(x_n) \neq a \) for all \( n \). Without loss of generality we may assume

\[ |a - f(x_1)| > |a - f(x_2)| > \ldots \]

hence

\[ |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \]

and, since \( f \in M_b(X) \)

\[ |x_1 - x_2| > |x_2 - x_3| > \ldots \]

Since \( |X| \) is discrete, the sequence \( x_1, x_2, \ldots \) is convergent. So we have case I. We find
THEOREM 4.9 Let either each closed and bounded subset of \( X \) be compact and \( f \in M^w(X) \), or let \( |X| \) be discrete and \( f \in M^b(X) \). Then

(i) \( f(X) \) is closed (in \( X \)).

(ii) If \( f(a) \in f(X) \) is not isolated then \( f \) is injective at \( a \), \( f \) is continuous at \( a \). In particular if \( f(X) \) has no isolated points then \( f \) is a homeomorphism \( X \cong f(X) \).

Proof. If \( a \in f(X) \setminus f(X) \) then \( a \) is not isolated. Since we are in alternative I of Lemma 4.8, \( a \in f(X) \). Contradiction. Thus, \( f(X) \) is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an \( f \) satisfying the conditions of 4.9 maps closed subsets of \( X \) (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let \( f : X \to K \).

(i) If \( f \in M^w(X) \) and if \( Y \subseteq X \) is closed and the closed and bounded subsets of \( Y \) are compact then \( f(Y) \) is spherically complete.

(ii) If \( f \in M^b(X) \) and if \( Y \subseteq X \) is spherically complete then so is \( f(Y) \).

(iii) If \( f \in M^g(X) \) and if \( A \subseteq f(X) \) is spherically complete then so is \( f^{-1}(A) \).

Proof. (i) Let \( B_1 \supseteq B_2 \supseteq \ldots \) be balls in \( f(Y) \). Choose \( y_1, y_2, \ldots \in Y \) such that \( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots \]

and, by the weak monotony of \( f \)
\[ |y_1 - y_2| \geq |y_2 - y_3| \geq \ldots \]

Suppose first that \( \lim |y_n - y_{n+1}| = 0 \). Then \( y := \lim y_n \) and there are infinitely many \( k \) for which

\[ |y_k - y_{k-1}| > |y_{k+1} - y_k| . \]

Now \( |y - y_k| \leq \max(|y_k - y_{k+1}|, |y_{k+1} - y_{k+2}|, \ldots) \leq |y_k - y_{k+1}| \). So we get for infinitely many \( k \)

\[ |y - y_k| < |y_k - y_{k-1}| \]

whence

\[ |f(y) - f(y_k)| < |f(y_k) - f(y_{k-1})| \]

so \( f(y) \in B_{k-1} \) for infinitely many \( k \), i.e., \( f(y) \in \bigcap \mathbb{B}_{k} \).

Next, suppose that \( |y_{k+1} - y_k| \geq \varepsilon > 0 \) for all \( k \). Then since \( y_1, y_2, \ldots \)

is bounded it has a convergent subsequence \( y_{n_1}, y_{n_2}, \ldots \). Let \( y := \lim y_{n_1} \).

Then we have for infinitely many \( i \)

\[ |y - y_{n_i}| < \varepsilon \leq |y_{n_i} - y_{n_i+1}| \]

whence

\[ |f(y) - f(y_{n_i})| < |f(y_{n_i}) - f(y_{n_i+1})| \]

so \( f(y) \in B_{n} \) for infinitely many \( i \), i.e., \( f(y) \in \bigcap \mathbb{B}_{k} \).

(ii) Let \( B_1 \supseteq B_2 \supseteq \ldots \) be balls in \( \mathcal{F}(Y) \) and let \( y_1, y_2, \ldots \in Y \) such that \( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots \]

and since \( f \in \mathcal{M}_b(X) \):

\[ |y_1 - y_2| > |y_2 - y_3| > \ldots \]

Since \( Y \) is spherically complete, there is \( y \in Y \) such that

\[ |y - y_n| \leq |y_n - y_{n+1}| \text{ for all } n, \text{ hence } |f(y) - f(y_n)| \leq |f(y_n) - f(y_{n+1})| \]

for all \( n \). It follows that \( f(y) \in B_n \) for all \( n \).

(iii) Let \( B_1 \supsetneq B_2 \supsetneq \ldots \) be balls in \( \mathcal{F}^{-1}(A) \). Choose \( x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_3, \ldots \).

Then \( |x_1 - x_2| > |x_2 - x_3| > \ldots \) whence \( |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \)

There is \( x \in \mathcal{F}^{-1}(A) \) such that \( |f(x) - f(x_n)| \leq |f(x_n) - f(x_{n+1})| \) for all \( n \).

Hence \( |x - x_n| \leq |x_n - x_{n+1}| \) for all \( n \), i.e., \( x \in \bigcap \mathcal{B}_n \).
DEFINITION 4.11 Let \( f : X \to K \). The oscillation function \( \omega_f : X \to [0, \infty] \) is defined by

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{ |f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}, x, y \in X \} \quad (a \in X)
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}, x \in X\}.
\]

THEOREM 4.12 Let \( f \in M_w(X) \). Then

\[
\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X).
\]

Proof. For \( x \neq a \) we have \( |f(x) - f(a)| \geq \inf_{z \neq a} |f(z) - f(a)| \) and (since \( a \) is not isolated) consequently

\[
\omega_f(a) \geq \inf_{z \neq a} |f(z) - f(a)|.
\]

Conversely, let \( z \neq a \). Then for all \( x \) such that \( |x - a| < |z - a| \) we have

\[
|f(x) - f(a)| \leq |f(z) - f(a)|
\]

so

\[
\omega_f(a) \leq |f(z) - f(a)|
\]

whence

\[
\omega_f(a) \leq \inf_{z \neq a} |f(z) - f(a)|.
\]

THEOREM 4.13 Let \( f \in M_w(X), a \in X \). If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} x_n = a \) (\( x_n \neq a \) for all \( n \)) then \( \lim_{n \to \infty} |f(x_n) - f(a)| = \omega_f(a) \).

Proof. By 4.12 we have \( \lim_{n \to \infty} |f(x_n) - f(a)| \geq \omega_f(a) \). Conversely, \( \lim_{n \to \infty} |f(x_n) - f(a)| \leq \omega_f(a) \) is clear from the definition of \( \omega_f \).
5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: K is local, k is finite, K has discrete valuation. Also we can sometimes say a little more if we assume X to be convex. For the time being, let X be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case K is a local field.

**Theorem 5.1** Let K be a local field, and let \( f \in M_w(X) \). Then

(i) \( f \) is continuous.

(ii) If \( Y \subset X \) is closed then \( f(Y) \) is closed.

(iii) If \( f(X) \) is bounded and \( f \) is not constant then \( X \) is bounded.

(iv) Let \( a \in X \). Then the following are equivalent

(a) \( f \) is not injective at \( a \)

(b) \( f \) is locally constant at \( a \)

(c) \( f(a) \) is isolated in \( f(X) \).

(v) The following conditions are equivalent

(a) \( f \) is injective

(b) \( f(X) \) has no isolated points

(c) \( f \) is a homeomorphism of \( X \) onto \( f(X) \).

**Proof.** Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for \( M^b \) and \( M^s \)-functions in case X is convex and K is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2 Let the residue class field $k$ of $K$ be finite. Let $X$ be convex and let $f \in M_b(X)$. Then

(i) If $a, b, c \in X$, $|a-b| < |a-c|$, $f(a) \neq f(c)$ then $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) If $C \subseteq X$ is convex then $f(C)$ is convex in $f(X)$ (if $f$ is weakly Darboux continuous, see 2.5).

(iii) If $f$ is injective, then $f \in M_b(X)$.

Proof. (i) Let $B := \{x \in K : |x-a| < |a-c|\}$. Then $B \subseteq X$ and $f(B) \subseteq [f(a), f(c)]$. Define an equivalence relation on $B$ by: $x \sim y$ if $|f(x)-f(y)| < |f(a)-f(c)|$.

Since $k$ is finite we get finitely many equivalence classes $B_1, B_2, \ldots, B_n$. Since $a \neq c$ we have $n \geq 2$. The diameter of $f(B)$ equals $|f(a)-f(c)|$, the distance between $f(B_i)$ and $f(B_j)$ equals $|f(a)-f(c)|$ ($i \neq j$). Since $[f(a), f(c)]$ can contain at most $q := \chi(k)$ sets having distances $|f(a)-f(c)|$ to one another we have $n \leq q$. Hence $2 \leq n \leq q$.

By 2.2 (β), each $B_i$ is convex. If $x, y \in B_i$ and $|x-y| = |a-c|$ then $B_i = B$, contradicting $n \geq 2$. Thus $B$ is a disjoint union of $n$ balls $B_1, \ldots, B_n$, where $2 \leq n \leq q$ and $|x-y| < |a-c|$ whenever $x, y \in B_i$ ($i = 1, \ldots, n$). It follows that $n = q$ and that each $B_i$ has the form $\{x \in K : |x-b_i| < |a-c|\}$ ($b_i \in B$). Hence, if $|a-b| < |a-c|$ then there is $i$ for which $a, b \in B_i$. So $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) Let $a, b \in C$ and let $a \in f(X)$ with $a \in [f(a), f(b)]$. We show that $a \in f(C)$. If $f(a) = f(b)$ this is clear. If $f(a) \neq f(b)$, set $a = f(x)$ where $x \in X$. Then $|f(x)-f(a)| \leq |f(b)-f(a)|$. If $|x-a|$ were $>|b-a|$ then $f(x) \neq f(a)$ (since $f \in M_b(X)$) and by (i) we then had $|f(b)-f(a)| < |f(x)-f(a)|$, a contradiction. Hence $|x-a| \leq |b-a|$ i.e., $x \in [a, b] \subseteq C$, so $a = f(x) \in f(C)$. 


(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on k, see 2.10.

COROLLARY 5.3 Let $K$ be a local field and let $f \in M_b(X)$ and $X$ convex. Then the following conditions are equivalent.

(a) $f \in M_s(X)$.

(b) $f$ is injective.

(c) $f \in \mathcal{M}_b(X)$.

(d) $f(X)$ has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 Let $K$ be a local field and let $X$ be the unit ball of $K$ (or any bounded convex set, for that matter). If either $f \in M_s(X)$ or $f \in M_b(X)$ then $f$ has bounded difference quotients.

Proof. $f$ is bounded, let $M := \sup\{ |f(x) - f(y)| : x, y \in X \}$. It suffices to prove that $|f(x) - f(0)| \leq M|x|$ for all $x$. Let $\pi \in K$, $|\pi| < 1$, be a generator of the value group. By induction on $n$ we prove:

if $|x| = |\pi|^n$ then $|f(x) - f(0)| \leq |\pi|^n M$.

The statement is clear for $n = 0$. Now suppose the statement is true for $0, 1, \ldots, n-1$.

Let $x \in X$, $|a| = |\pi|^n$. Then $|x-0| < |\pi^{n-1}-0|$. If $f(\pi^{n-1}) \neq f(0)$ we have either since $f \in M_s(X)$ or by 5.2(1)

$$|f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1} M$$

hence

$$|f(x) - f(0)| \leq |\pi|^n M$$

If $f(\pi^{n-1}) = f(0)$ then $|f(x) - f(0)| \leq |f(\pi^{n-1}) - f(0)| = 0$, so certainly $|f(x) - f(0)| \leq |\pi|^n M$. 
Notes.

(a) 5.4 cannot be extended to the case \( X = K \). In fact, let
\[
f : \mathbb{Q}_p \to \mathbb{Q}_p \text{ be the map } \Sigma_n p^n \mapsto \Sigma_n p^{2n}. \quad (\Sigma_n p^n \in \mathbb{Q}_p).
\]
Then
\[
f \in M_{bs}(\mathbb{Q}_p) \text{ but } |p^n f(p^{-n})| = p^n \to \infty.
\]

(b) If we lose the condition on \( K \), for example by requiring that the valuation is discrete then 3.22 and 2.4(5) show that the conclusion of 5.4 is false both for \( M_b \)-functions and \( M_s \)-functions. On the other hand, it is clear from the proof of 5.4 that a bounded \( M_s \)-function on \( X \) has bounded difference quotients.

(c) One may wonder how difference quotients of \( M_w \)-functions behave.

See the example below.

**EXAMPLE 5.5** Let \( p \neq 2 \). Then there is an \( f \in M_w(\mathbb{Z}_p \to \mathbb{Q}_p) \) that has unbounded difference quotients.

**Proof.** Let \( a_0, a_1, \ldots \) be defined via \( a_{2n} := p^n \) (\( n = 0,1,2,\ldots \)) and \( a_{2n+1} := 2p^n \) (\( n = 0,1,2,\ldots \)). Thus \( \{a_0, a_1, a_2, \ldots\} = \{1, 2, p, p^2, 2p^2, \ldots\} \).

Then \( |a_0| \geq |a_1| \geq |a_2| \geq \ldots \), \( \lim a_n = 0 \), \( |a_n - a_m| = |a_m| (n > m) \).

Set
\[
f(x) := \begin{cases} 
-a_n & \text{if } |x| = p^{-n} \quad (n = 0,1,2,\ldots) \\
0 & \text{if } x = 0
\end{cases} \quad (x \in \mathbb{Z}_p)
\]

Then the difference quotients of \( f \) are not bounded (for \( n \in \mathbb{N} : f(p^{2n}) = p^n \), so \( |p^{-2n} f(p^{2n})| = p^n \to \infty \text{ if } n \to \infty \)). We show that \( f \in M_w(\mathbb{Z}_p) \). Since \( f \) is continuous it suffices to show that if \( x, y, z \) are \( \neq 0 \), \( |x-y| < |x-z| \) then \( |f(x)-f(y)| \leq |f(x)-f(z)| \). This is clear if \( |x| = |y| \). If \( |x| < |y| \), then \( |x| < |y| < |z| \). If \( |x| > |y| \), then \( |y| < |x| < |z| \).

Let \( f(x) = a_n, f(y) = a_m, f(z) = a_t \). Then in both cases \( n \neq m, t < \min(n,m) ; |f(x)-f(y)| = |a_n - a_m| \leq |a_t| \) and \( |f(x)-f(z)| = |a_n - a_t| = |a_t| \) and we are done.
On the other hand (how surprising is life!)

THEOREM 5.6 Let $k$ be the field of two elements. Then $M_w(X) = M_b(X)$.

Proof. We prove that $|x-y| = |y-z|$ implies $|f(x)-f(y)| \leq |f(y)-f(z)|$ $(x \neq y, y \neq z, x, y, z \in X)$. There is $a \in K^*$ such that $|a(x-y)| = |a(y-z)| = 1$. So since $k = F_2$, $\overline{a(x-y)} = \overline{a(y-z)} = 1$, whence $a(x-z) = 0$ or $|a(x-z)| < 1$. Thus, $|x-z| < |x-y| = |y-z|$. Since $f \in M^w(X)$, $|f(x)-f(z)| \leq \min(|f(x)-f(y)|, |f(y)-f(z)|)$. Consequently, $|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) \leq |f(y)-f(z)|$.

Of particular interest may be monotone functions mapping convex sets onto convex sets.

THEOREM 5.7 Let $K$ be a local field, let $X$ be a bounded open convex set, and let $f : X \to X$ be surjective. Then the following are equivalent.

(a) $f \in M_b(X)$
(b) $f \in M_s(X)$
(γ) $f \in M_{bs}(X)$
(δ) $f$ is an isometry.

Proof. (a) $\Rightarrow$ (β). Since $f(X)$ has no isolated points, $f$ is a homeomorphism, by 5.1(v). Then $f \in M_s(X)$, by 5.3. (β) $\Rightarrow$ (γ). $f^{-1} \in M_b(X)$.

We just have shown (a) $\Rightarrow$ (β), so $f^{-1} \in M_s(X)$ i.e., $f \in M_b(X)$.

(γ) $\Rightarrow$ (δ). From the proof of 5.4 we have seen that $|f(x)-f(y)| \leq M|x-y|$, where $M = \sup |f(x)-f(y)| = 1$. Hence $|f(x)-f(y)| \leq |x-y|$ for all $x, y \in X$, but by the same token this also holds for $f^{-1}$. Then $f$ is an isometry. (δ) $\Rightarrow$ (a) is obvious.
COROLLARY 5.8 Let $X$ be an open convex subset of $K$ and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.

(a) $f \in M_b(X)$

(b) $f \in M_s(X)$

(c) $f_* : M_s(X)$

(d) $f$ is a scalar multiple of an isometry.

Proof. If $X$ is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If $X$ is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (a) (b) (c) is now easy. To prove (c) $\to$ (d) we may assume $f(0) = 0, f(1) = 1$. Let $X_n := \{x \in K : |x| \leq n\}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is $c_n$ such that $|f(x) - f(y)| = c_n|x-y|$ ($x,y \in X_n$). By substituting $x = 1, y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in M_{ubs}(Z)$ into conditions on the coefficients of $f$ with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(Z)$. So let the notations be as in 3.3(3), and suppose first $f \in M_{ubs}(Z)$ i.e. $|x-y| = |s-t| \implies |f(x)-f(y)| = |f(s)-f(t)|$. Let $n,m \in \mathbb{N}$. If $|n-n'| = |m-m'|$ then $|f(n)-f(n')| = |f(m)-f(m')|$, so if we write $f = \sum_n a_n e_n$ we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ ($a_k \neq 0$) then $|n-n'| = p^{-k}$ where $k = \left\lfloor \frac{\log n}{\log p} \right\rfloor$. We find

\[
\text{if } \left\lfloor \frac{\log n}{\log p} \right\rfloor < \left\lfloor \frac{\log m}{\log p} \right\rfloor \text{ then } |\lambda_n| > |\lambda_m| \\
\text{if } \left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor \text{ then } |\lambda_n| = |\lambda_m|.
\]
Moreover, if \( \left[ \frac{\log n}{\log p} \right] = \left[ \frac{\log m}{\log p} \right] = k \) and \( n - m \) is divisible by \( p^k \), i.e.,
\[
n - m = n - m \text{ then } |f(n) - f(m)| = |\lambda_n - \lambda_m|.
\]
If \( n > m \) then \( |f(n) - f(0)| = |\lambda_n| \).

We have found the first half of

**THEOREM 5.9** Let \( f = \sum \lambda_n e_n \in C(Z_p) \). In order that \( f \in M_{\text{ubs}}(Z_p) \) it is necessary and sufficient that condition (*) below holds

\[
(*) \quad |\lambda_n| \text{ is a strictly decreasing function of } \left[ \frac{\log n}{\log p} \right] \quad (n \in \mathbb{N})
\]

We have shown \( f \in M_{\text{ubs}}(Z_p) \) \( \Rightarrow \) (*) . Now suppose (*) and let \( |x - y| = p^{-k} \). We show that \( |f(x) - f(y)| = |\lambda_{n_{p^k}}| \). Now \( e_n(x) = e_n(y) \) for \( n < p^k \), so

\[
f(x) - f(y) = \sum_{n \geq p^k} \lambda_n (e_n(x) - e_n(y)).
\]

Set
\[
x = a_0 + a_1 p + \ldots + a_k p^k + a_{k+1} p^{k+1} + \ldots \quad (a_k \neq b_k)
\]
\[
y = a_0 + a_1 p + \ldots + b_k p^k + b_{k+1} p^{k+1} + \ldots
\]

Then
\[
\left| \sum_{n \geq p^k} \lambda_n e_n(x) \right| = \left| \lambda_{p^k} + \ldots + \lambda_{a_k p^k} + \lambda_{a_{k+1} p^{k+1}} + \ldots \right|
\]
\[
\left| \sum_{n \geq p^k} \lambda_n e_n(y) \right| = \left| \lambda_{p^k} + \ldots + \lambda_{b_k p^k} + \lambda_{b_{k+1} p^{k+1}} + \ldots \right|
\]

Now either \( a_k \) or \( b_k \) is \( \neq 0 \), say, \( a_k \neq 0 \). If \( b_k = 0 \) then by (*)
\[
\left| \sum_{n \geq p^k} \lambda_n e_n(y) \right| < \left| \lambda_{p^k} \right| = \left| \sum_{n \geq p^k} \lambda_n e_n(x) \right|, \text{ so } |f(x) - f(y)| = |\lambda_{n_{p^k}}|
\]
\[
eq |\lambda_{p^k}|. \text{ If } b_k \neq 0 \text{ then by } (*) \quad |\lambda_{p^k}| = |\lambda_{n_{p^k}} - \lambda_{p^k a_k p^k b_k p^k}| = |f(x) - f(y)|.
\]

**Note.** Using similar methods, we can prove: \( f = \sum \lambda_n e_n \) is in \( M_{\text{ubs}}(Z_p) \)

if and only if we have (***) for all \( n, m \in \mathbb{N} \):
If we assume only that $K$ has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let $X$ be the unit ball of a discretely valued field. Let $f : X \to X$ be surjective, $f \in M_b(X)$. Then $f$ is an isometry.

**Proof.** It is clear from previous theory that $f$ is a homeomorphism of the unit ball. It suffices to show that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in X$. (Apply this result also for $f^{-1}$. Then $f$ is an isometry.)

Let $\pi \in K$, $|\pi| < 1$, be a generator of $|K^*|$. We prove by induction

$$|f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)|.$$  

For $n = 0$ this is clear. (If $|x - 0| \leq |1 - 0|$, so $|f(x) - f(0)| \leq |f(1) - f(0)|$.

Suppose the statement is true for $n = 1$. Let $|x| = |\pi|^n$. Then

$$|x - 0| < |\pi|^{n-1} - 0|,$$

so $|f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1} |f(1) - f(0)|$, so $|f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)|$ and we are done. (In fact, we have shown that a bounded $M$-function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let $K$ have discrete valuation and let $f \in M_b(X)$. Then

the following conditions are equivalent.

(a) $f(X)$ has no isolated points.

(b) $f$ is injective and continuous.

(γ) $f$ is a homeomorphism $X \sim f(X)$. 

\[ \left( \frac{\log n}{\log p} \right) > \left( \frac{\log m}{\log p} \right) = k \quad \Rightarrow \quad |\lambda_n| < |\lambda_m| 
\]

$n - m$ divisible by $p^k$

\[ \left( \frac{\log n}{\log p} = \frac{\log m}{\log p} \right) \quad \Rightarrow \quad |\lambda_n| = |\lambda_m| = |\lambda_n - \lambda_m|.
\]
Proof. (a) $\rightarrow (\gamma)$ is 4.9(ii). $(\gamma) \rightarrow (\delta)$ is clear. $(\delta) \rightarrow (\gamma)$: if $f(a)$ were an isolated point of $f(X)$, then $\{x : f(x) = f(a)\}$ is open in $X$. Since $f$ is injective $\{a\}$ is open. But $X$ has no isolated points. Contradiction.

To show that 5.11 may not be true if $K$ has a dense valuation we construct

**Example 5.12** Let $|K| = [0,\infty)$. Then we construct an $M_{bs}$-homeomorphism sending

$$\{x \in K : \frac{1}{2} < |x| \leq 1\} \text{ onto } \{x \in K : 0 < |x| \leq 1\}.$$  

Proof. Let $\phi : [\frac{1}{2},1] \rightarrow [0,1]$ be the map $x \mapsto 2(x-\frac{1}{2})$ $(x \in (\frac{1}{2},1])$. For each $v \in (\frac{1}{2},1]$, choose $\beta_v \in K$ such that $|\beta_v| = \phi(|v|)$. Define $f : \{x \in K : \frac{1}{2} < |x| \leq 1\} \rightarrow \{x \in K : 0 < |x| \leq 1\}$ as follows

$$f(x) = \beta_{|x|} x \quad \left(\frac{1}{2} < |x| \leq 1\right)$$

Clearly, $|f(x)| = |\beta_{|x|}||x| = \phi(|x|) \in (0,1]$. The inverse of $f$ is given by $y \mapsto \beta_{\phi^{-1}(|y|)}^{-1}y$, so $f$ is a bijection. Since $f^{-1}$ can be defined in the same way as $f$ (only with $\phi^{-1}$ instead of $\phi$) it suffices to show that $f \in M_{bs}$. Let $|x-y| < |x-z|$. Suppose $|x| > |z|$. Then $|x-z| = |x|$ and $|y| = \max(|x-y|,|x|) = |x|$. Then $\beta_{|x|} = \beta_{|y|}$, so $|f(x)-f(y)| = \beta_{|x|} |x-y|$ and $|f(x)-f(z)| = |f(x)| = \beta_{|x|} |x-z|$, so we are done in this case. Suppose $|x| < |z|$. Then $|x-z| = |z|$ and $|y| = \max(|x-y|,|x|) < |z|$. Then $|f(x)-f(y)| \leq \max(|f(x)|,|f(y)|) < |f(z)| = |f(z)-f(x)|$.

Suppose $|x| = |z|$. Then $|y| = \max(|x-y|,|x|) \leq |x|$; if $|y|$ were $< |x|$ then $|x-y| = |x| = |z| < |x-z|$, a contradiction, so $|y| = |x| = |z|$, and $|f(x)-f(y)| = \beta_{|x|} |x-y|$, $|f(x)-f(z)| = \beta_{|x|} |x-z|$ whence $|f(x)-f(y)| < |f(x)-f(z)|$.  

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EXAMPLE 5.13 Extend $f$ to a surjection $g$ of $\{x \in K : |x| \leq 1\}$ onto itself by defining $g(x) = 0$ if $|x| \leq \frac{1}{2}$. We claim that $g \in M_b$. Let $|x-y| \leq |x-z|$. To check whether $|g(x)-g(y)| \leq |g(x)-g(z)|$ we only have to consider the cases $|x| \leq \frac{1}{2}$ and $|y| \leq \frac{1}{2}$ and $|x| > \frac{1}{2}$ and $|y| > \frac{1}{2}$. In the first case, $|x-y| = |y| \leq |x-z|$, so $|z| = \max(|z-x|,|x|) = |z-x| \geq |y|$. Then $|g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)|$. In the second case $|g(x)-g(y)| = |f(x)|$. If $|x| < |z|$ then $|f(x)| < |f(z)| = |f(z)-f(x)| = |g(z)-g(x)|$. If $|x| > |z|$ then $|f(x)| = |g(x)-g(z)|$.

If $|x| = |z|$ then $|f(x)-f(z)| = \beta_{|x|}|x-z| \geq \beta_{|x|}|x-y| = \beta_{|x|}|x| = |f(x)|$.

Thus we have found a continuous surjection $g : \{x \in K : |x| \leq 1\} \to \{x \in K : |x| \leq 1\}$, $g \in M_b$, such that $g = 0$ on $\{x : |x| < \frac{1}{4}\}$. (Compare 5.11).

EXAMPLE 5.14 Let $h : \{x \in K : |x| \leq 1\} \to K$ be defined as

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \ (f \text{ as in 5.12)} \\ 0 & \text{if } x = 0. \end{cases}$$

Then $h$ is a non-continuous $M_{bs}$-function.

Proof. That $h$ is not continuous at 0 is clear. Further, $h$, restricted to $\{x : 0 < |x| \leq 1\}$ is in $M_{bs}$ (see 5.12). Further, since $g \circ h$ is the identity ($g$ as in 5.12), we see that $h + M_s$. It suffices to show that $|x-y| = |x-z|$ implies $|h(x)-h(y)| = |h(x)-h(z)|$ in case $0 \in \{x, y, z\}$.

We may suppose $x \neq y$, $y \neq z$, $x \neq z$. Let $x = 0$. Then $|y| = |z|$, so $|f^{-1}(y)| = |f^{-1}(z)|$, i.e., $|h(x)-h(y)| = |h(x)-h(z)|$. Now let $y = 0$.

Then $|x| = |x-z|$. Choose $0 < |t| \leq 1$ such that $|t| < |x|$. Then $|x-t| = |x-z|$ so $|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x) - f^{-1}(z)|$, i.e., $|h(x)| = |h(x)-h(z)|$, and we are done.
6. FUNCTIONS OF BOUNDED VARIATION

In this section $X$ is the unit ball of $K$, and $BA(X) := \{f : X \to K : \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right| < \infty \}$. Let us define

$$\|f\|_\Delta := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in X, x \neq y \right\} (f \in BA(X)).$$

It will turn out that, in a natural way, $BA(X)$ can be regarded as the space of functions of bounded variation, and that $\|\|_\Delta$ plays the role of the total variation.

**Theorem 6.1** Let $f : X \to K$. Then the following are equivalent

(a) $f \in BA(X)$.

(b) $f$ is a linear combination of two increasing functions.

If $|K|$ is discrete (a), (b) are equivalent to

(c) $f$ is the difference of two bounded monotone functions of some type $\sigma$.

(d) $f \in [M_B(X)]$.

If $K$ is a local field then (a)-(d) are equivalent to

(e) $f \in [M_S(X)]$.

(f) $f \in [M_D(X)]$.

**Proof.** We only prove (a) $\Rightarrow$ (b). The rest follows from (5.10), (5.4).

So let $f \in BA(X)$ and choose $\lambda \in K$ such that $|f(x) - f(y)| < |\lambda| |x - y|$ ($x, y \in X, x \neq y$). Then $\lambda^{-1}f$ is a pseudocontraction, $f(x) = \lambda x + \lambda(\lambda^{-1}f(x) - x)$ ($x \in X$), where $x \to x$ and $x \to \lambda^{-1}f(x) - x$ are increasing.

In the real case, we can define for a function $[0,1] \to \mathbb{R}$, of bounded variation
\[ V(f) := \inf \{ \text{Var } g + \text{Var } h : f = g + h, g, h \text{ monotone} \}. \]

It is an easy exercise to show that \( f \rightarrow V(f) \) is a seminorm on the space of all functions of bounded variation and that \( V \) is equivalent to the total variation \( \text{Var} \), defined via

\[ \text{Var } f = \sup \{ \sum |f(x_k) - f(x_{k-1})| : x_0 < x_1 < \ldots < x_n \text{ is a partition of } [0,1] \}. \]

So in the non-archimedean situation we define for \( f : X \rightarrow K \)

\[ J(f) = \sup \{ |f(x) - f(y)| : x, y \in X \}. \]

(If \( f \) is considered to be "monotone" then \( J(f) \) can be interpreted as the "total variation" of \( f \).) We are led to the following definitions for \( f \in BA(X) \):

\[ \text{Var } f := \inf \{ \max (J(g), J(h)) : f = g + h, g, h \text{ are scalar multiples of increasing functions} \}. \]

(If \( |K| \) is discrete) \( \text{Var}^f := \inf \{ \max J(g), J(h) : f = g + h, g, h \text{ are in } M_s(X) \} \).

(If \( K \) is local) \( \text{Var}_1^f := \inf \{ \max J(g), J(h) : f = g + h, g, h \in M_p(X) \} \)

\[ \text{Var}_2^f := \inf \{ \max J(g), J(h) : f = g + h, g, h \in M_s(X) \}. \]

Let us first compare \( \text{Var } f \) and \( \|f\|_A \). If \( f = g + h \) and \( g, h \) are scalar multiples of increasing functions we have for \( x, y \in X, x \neq y \)

\[ \frac{|f(x) - f(y)|}{x-y} \leq \max \left( \frac{|g(x) - g(y)|}{x-y}, \frac{|h(x) - h(y)|}{x-y} \right) \leq \max (J(g), J(h)) \]

so \( \|f\|_A \leq \text{Var } f \). Conversely, if \( |\lambda| > \sup \frac{|f(x) - f(y)|}{x-y} \) then

\[ f(x) = \lambda x + \lambda (\lambda^{-1} f(x) - x) \quad (x \in X) \]

whence

\[ \text{Var } f \leq |\lambda| \]
So, if \(|K|\) is dense we have \(\Var f = \|f\|_\Delta (f \cdot B\Delta(X))\). Otherwise we have at least
\[
\|f\|_\Delta \leq \Var f \leq c \|f\|_\Delta \quad (f \in B\Delta(X))
\]
(where \(c\) is the smallest value > 1).

If \(|K|\) is discrete we clearly have \(\Var_1 f \leq \Var f\). Conversely, let \(f = g + h\), where \(g, h \in \M^\pm(X)\). It follows from the proof of 5.10 that
\[
|g(x) - g(y)| \leq M|x-y| \quad (x, y \in X)
\]
\[
|h(x) - h(y)| \leq N|x-y|
\]
where \(M = \sup |g(x) - g(y)| = \J(g)\) and \(N = \J(h)\).

So
\[
\left| \frac{f(x) - f(y)}{x-y} \right| \leq \max(\J(g), \J(h)), \text{ whence}
\]
\[
\|f\|_\Delta \leq \Var_1 f.
\]

Similar proofs work for \(\Var_2 f, \Var_3 f\). We have

**THEOREM 6.2** The seminorms \(\Var, \Var_1, \Var_2, \Var_3\), on \(B\Delta(X)\) (whenever defined) are all equivalent to \(\|\|_\Delta\).
REFERENCES

