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NON-ARCHIMEDEAN MONOTONE FUNCTIONS

by

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INTRODUCTION

In the sequel, $K$ is a non-archimedean valued field, complete, with residue class field $k$. Our aim is to present reasonable definitions for a function $f : X \to K$ to be "monotone". ($X$ is a subset of $K$). Since $K$ admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \to \mathbb{R}$, equivalent to "$f$ is monotone", and such that these statements have translations in $K$ that make sense. This way we obtain several definitions of "$f : X \to K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of $K"", "the sign of a nonzero element of $K"".

In Section 2 we define several notions of monotony. E.g.,

\[ f \in M^+_D(X) \text{ if } x \text{ between } y \text{ and } z \text{ implies } f(x) \text{ between } f(y) \text{ and } f(z) \text{ and } f \in M^-_S(X) \text{ if } f(x) \text{ between } f(y) \text{ and } f(z) \text{ implies } x \text{ between } y \text{ and } z. \]

Also monotone functions of type $\sigma$ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M^+_D(X)$ (or $f \in M^-_S(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
mation on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of K and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: \( f' > 0 \iff f \text{ increasing.} \)

The notion of pseudo-ordening ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

**Notations.** Let \( p \) be a prime. By \( \mathbb{F}_p \) we mean the field of \( p \) elements. By \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers. For a field \( L \) we denote its characteristic by \( \chi(L) \). Let \( E \) be a vector space over \( K \) and \( S \subset E \). By \( [S] \) we denote the smallest \( K \)-linear subspace of \( E \) that contains \( S. \)
1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let \( x, y \in K \). Then the smallest ball in \( K \) containing \( x \) and \( y \) is denoted by \([x, y]\). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \([x, y] \subseteq C\).

Sometimes we use a more geometric terminology. Instead of \( z \in [x, y] \) we will say that \( z \) is between \( x \) and \( y \) and instead of \( z \notin [x, y] \) we use the expression: \( x \) and \( y \) are at the same side of \( z \).

Notice that \([x, y] = [y, x] \) for all \( x, y \in K \) and that \( z \in [x, y] \iff |z-x| \leq |x-y| \iff |z-y| \leq |x-y| \iff z = \lambda x + (1-\lambda)y \) for some \( \lambda \in K \), \( |\lambda| \leq 1 \). If \( x \neq y \) then the \( \lambda \) in this last expression is unique (viz. \( \lambda = \frac{z-y}{x-y} \)).

Examples of convex sets are: the empty set, singletons, balls, \( K \). It is an easy exercise to show that these are the only convex subsets of \( K \). So formally we may write each convex subset of \( K \) as

\[
\{ x \in K : |x-a| < r \} \quad (a \in K, 0 \leq r \leq \infty)
\]
or as

\[
\{ x \in K : |x-a| \leq r \} \quad (a \in K, 0 \leq r \leq \infty)
\]

Notice that the only unbounded convex subset of \( K \) is \( K \) itself.

Sometimes we need the notion of convexity with respect to a subset \( X \) of \( K \). A subset \( C \subseteq X \) is called convex in \( X \) if \( x, y \in C \) implies \([x, y] \cap X = C\) or, equivalently, if \( C \) is the intersection of \( X \) with a convex subset of \( K \).

Let \( x, y, z \in K \). By the strong triangle inequality we have that the "triangle" \( x, y, z \) is isosceles, say \( |x-y| = |y-z| \). Then \( |x-z| \leq |x-y| \), so \( z \) is between \( x \) and \( y \) and \( x \) is between \( y \) and \( z \). If also \( |x-y| = |x-z| \)
then \( y \) is between \( x \) and \( z \). Otherwise, \( x \) and \( z \) are at the same side of \( y \).

The relation \( \sim \) defined on \( K^* := K \setminus \{0\} \) by

\[
x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*)
\]
is an equivalence relation. We have \( x \sim y \Leftrightarrow 0 \not\in [x, y] \) i.e. iff

\[
|x-y| < |x| = |y| \quad \text{i.e. iff } |xy^{-1}| < 1.
\]
Define

\[
K^+ := \{x \in K : |1-x| < 1\}
\]

Then \( K^+ \) is a multiplicative subgroup of \( K^* \), \( K^+ = \{x \in K^* : x \sim 1\} \) and is called the set of the positive elements of \( K \). The relation \( \sim \) is also induced by the canonical group homomorphism

\[
\pi : K^* \to K^*/K^+.
\]

Thus, \( x \sim y \) if and only if \( \pi(x) = \pi(y) \) \( (x, y \in K^*) \). Therefore it is natural to view the group \( \Sigma := K^*/K^+ \) as being the group of signs of elements of \( K^* \), and we call \( \pi(x) \) the sign of the element \( x \in K^* \). If \( x \in K^* \) then \( \pi(x) = \{y : |y-x| < |x|\} = xx^+ \). For \( x \in K^* \), \( a \in \Sigma \) we sometimes write \( xa \) to indicate the element \( \pi(x).a \) of \( \Sigma \). In particular, for \( a \in \Sigma \) the opposite sign of \( a \), \(-a \), is defined as \((-1)a\). Then

\[
-a = \{-x : x \in a\}. \text{ (Notice that in case } \chi(K) = 2 \text{ we have } a = -a.\)
\]

Let \( a \in \Sigma \). Then for \( x, y \in a \) we have \( |x| = |y| \) so we can define the absolute value of \( a \), \( |a| \) as follows

\[
|a| := |x| \quad (x \in \pi^{-1}(a)).
\]

In the sequel we also need addition between elements of \( \Sigma \). Let us first observe that for any \( a, b \in \Sigma \) the sum

\[
a + b := \{x+y : x \in a, y \in b\}
\]
is always a ball with radius \( \max(|a|^-1, |b|^-1) \). (i.e., of the form
\{x : |x-b| < \max(|a|,|\beta|)\}. Now \alpha+\beta contains 0 if and only if 
\alpha = -\beta. Otherwise \alpha+\beta is again an element of \Sigma. (Proof: Let \alpha, \beta \in \Sigma. Then 
|\alpha+\beta| = \max(|\alpha|,|\beta|). If also \alpha, \beta \in \mathbb{R} then 
|x+y-(\alpha+\beta)| \leq \max(|x-a|,|y-b|) < \max(|\alpha|,|\beta|) = |\alpha+\beta|. Thus \alpha+\beta contains the ball 
with center \alpha+\beta and radius \max(|\alpha|,|\beta|), so \alpha+\beta is equal to this 
ball.)

Let us define 
\alpha \Theta \beta := \alpha+\beta = \{x+y : x \in \alpha, y \in \beta\} \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).

We have

**Theorem 1.2** Let \Sigma, \mathcal{E} \to \mathbb{R}, \theta : \Sigma \times \Sigma \setminus \{(\alpha, -\alpha) : \alpha \in \Sigma\} \to \Sigma be as 
above. Let \alpha, \beta, \gamma \in \Sigma. Then

(i) \ |a\beta| = |\alpha| \cdot |\beta|, |\alpha^{-1}| = |\alpha|^{-1}.

(ii) If \alpha \Theta \beta is defined then so is \beta \Theta \alpha and \alpha \Theta \beta = \beta \Theta \alpha.

(iii) If (\alpha \Theta \beta) \Theta \gamma and \alpha \Theta (\beta \Theta \gamma) are defined then

(\alpha \Theta \beta) \Theta \gamma = \alpha \Theta (\beta \Theta \gamma).

(iv) If \alpha \Theta \beta or \gamma \Theta \gamma \beta is defined then so is the other 
and \gamma(\alpha \Theta \beta) = \gamma \alpha \Theta \gamma \beta.

(v) If \alpha \Theta \beta is defined then |\alpha \Theta \beta| = \max(|\alpha|,|\beta|). Conver-

sely if |s| = \max(|\alpha|,|\beta|) for some s \in \alpha+\beta then \alpha \Theta \beta 
is defined.

(vi) |\alpha| < |\beta| if and only if \alpha \Theta \beta = \beta.

(vii) Let \mathcal{E} \in \{1,2,\ldots,\chi(k)-1\} if \chi(k) \neq 0, let \mathcal{E} \in \mathbb{N} other-

wise. Then we define \oplus_{\mathcal{E}} \alpha inductively as follows.

\oplus_{1} \alpha : \alpha, \oplus_{k} \alpha := \oplus_{k-1} \alpha \oplus \alpha (k \leq n). Then

\oplus_{n} \alpha = n\alpha.

**Proof.** (i), (ii) are clear. (iii) is almost trivial: if \alpha \in \Sigma, \beta \in \Sigma, 
z \in \gamma \Theta \gamma \text{ then } x+y+z \in \alpha+\beta+\gamma \text{ and the latter set can be regarded as}
(a ⊗ β) ⊗ γ or as a ⊗ (β ⊗ γ). (It is worth noticing that (a ⊗ β) ⊗ γ may be defined whereas a ⊗ (β ⊗ γ) is not. Choose β = −γ and

|a| > |b|. Then (a ⊗ β) ⊗ γ = a ⊗ γ = a, β ⊗ γ is not defined.)

(iv) is clear. If a ⊗ β is defined then for x ⊔ α, y ⊔ β we have

|x + y| ≥ max(|x|, |y|) whence |x + y| = max(|x|, |y|). So |a ⊗ β| =

max(|a|, |β|). Conversely, if a ⊗ β is not defined, then (we saw

earlier that) α + β is a ball with center zero and radius max(|α|, |β|).

Thus we have (v). We prove (vi). If |a| < |β| then α + β = β so a ⊗ β = β.

Conversely, if a ⊗ β = β then choose α, β ∈ ß. Then α + β ∈ ß hence

a + b ≈ b i.e., ab−1 + 1 ∈ K + implying |ab−1| < 1 or |a| < |b|. Hence

|α| < |β|. (Note: from (vi) it follows that a ⊗ β = α' ⊗ β does not

imply α = α'.) To prove (vii) let α ⊔ α and observe that for any k ≤ n,

if ⊔ a is defined, (k−1)a ⊔ ⊔ a. Hence |(k−1)a+a| = |ka| = |a| = |a|,

k−1 k−1

so ⊔ a+a does not contain 0, hence ⊔ a ⊔ a is defined.

k−1 k−1

Now na is by definition π(n)α. So na ⊔ na and na ⊔ a. Since both na

and ⊔ a are signs they must coincide.

n

We now define relations that resemble "ordering".

DEFINITION 1.3 Let α ∈ A and x, y ∈ K. Then we say that x is greater

than y in the sense of α, notation x ⊔ α y, if x−y ∈ α.

We have the following rules

THEOREM 1.4 (i) If x, y ∈ K, x ≠ y then there is exactly one α ∈ A for

which x ⊔ α y.

(ii) x ⊔ x for no α.

(iii) If x ⊔ y then for all s ∈ K: x + s ⊔ y + s (x, y ∈ K, α ∈ A)

(iv) If x ⊔ y and s ⊔ 0 then x + s ⊔ y + s (x, y, s ∈ K, α, β ∈ A)
(In particular \( x > y \) implies \(-x \leq -y\)).

(v) If \( x > y, y > z \) and if \( \alpha \oplus \beta \) is defined then \( x > \alpha \oplus \beta \).

Proof. Easy.

The group \( \Sigma_1 := \{ \alpha \in \Sigma : |\alpha| = 1 \} \) is a subgroup of \( \Sigma \), isomorphic to multiplicative group \( k^* \). If \( K \) has discrete valuation and if \( s \in K \) and \( |s| \) is the largest value that is smaller than \( 1 \), then for each \( \alpha \in \Sigma \) there is \( x \in \mathbb{Z} \) such that \( \alpha = s^n \alpha_1 \) where \( \alpha_1 \in \Sigma_1 \). It follows easily that the map \( (n, \alpha) \mapsto s^n \alpha \ (n \in \mathbb{Z}, \alpha \in \Sigma_1) \) is an isomorphism of \( \mathbb{Z} \times \Sigma_1 \) onto \( \Sigma \). Thus, in case \( K \) has discrete valuation, \( \Sigma \) is isomorphic to \( \mathbb{Z} \times \Sigma_1 \), or, for that matter, to \( |K^*| \times k^* \).

If \( K \) is a local field we can even define a group embedding \( \rho : \Sigma \rightarrow K^* \) such that \( \pi \rho \) is the identity. (Thus, we can pick an element in every \( \alpha (\alpha \in \Sigma) \) such that the resulting set is a subgroup of \( K^* \).) Let \( s \in K, |s| < 1 \) such that \( |s| \) generates the value group and let \( q \) be the cardinality of \( k \). Let \( x \in K^* \). Then there is a unique \( n \in \mathbb{Z} \) such that \( x = s^n x_1 \) where \( |x_1| = 1 \).

Define \( \nu(x) = s^n \lim_{n \to \infty} x_1^n \). It is easy to verify that \( \nu \) is a homomorphism of \( K^* \) into \( K^* \), that \( \pi(\nu(x)) = \pi(x) \) for all \( x \in K^* \) and that \( \nu(x) = 1 \) if and only if \( x \in K^+ \).

Therefore the map \( \rho \) making the diagram

\[
\begin{array}{ccc}
K^* & \xrightarrow{\nu} & K^* \\
\pi & \downarrow & \downarrow \rho \\
\Sigma & \downarrow & \\
\end{array}
\]

commute solves the problem.

EXAMPLE 1.5 The signs of \( \phi_p \). Let \( \theta \) be a primitive \((p-1)^{th}\) root of
unity. Then \(\mathcal{O}_p^n : i \in \{0, 1, \ldots, p-2\}, n \in \mathbb{Z}\) is a subgroup of \(\mathbb{Q}_p^*\) isomorphic to \(\mathbb{Z}\). If
\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0, 1, \ldots, p^{p-2}\}, a_n \neq 0)
\]
is an element of \(\mathbb{Q}_p^*\), its sign, interpreted as an element of \(\mathbb{Q}_p\) is
\[
\pi(x) = a_n p^n.
\]
2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function \( f : [0,1] \to \mathbb{R} \) the following statements are equivalent.

(a) \( f \) is monotone (i.e., either \( x > y \) implies \( f(x) \geq f(y) \) for all \( x,y \)

or \( x > y \) implies \( f(x) \leq f(y) \) for all \( x,y \)).

(b) If \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \)

\((x,y,z \in [0,1])\).

(c) If \( C \subseteq \mathbb{R} \) is convex then \( f(C) \) is convex.

Thus we define

DEFINITION 2.1 Let \( X \subseteq K \). We say that \( f \in M_b(X) \) if for all

\( x,y,z \in X, x \text{ between } y \text{ and } z \text{ implies } f(x) \text{ between } f(y) \text{ and } f(z) \).

In other words, \( f \in M_b(X) \) if and only if for all \( x,y,z \)

\[ |x-y| \leq |y-z| \Rightarrow |f(x)-f(y)| \leq |f(y)-f(z)|. \]

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

THEOREM 2.2 Let \( X \subseteq K \) and let \( f : X \to K \). Then the following statements are equivalent

(a) \( f \in M_b(X) \).

(b) For each convex \( C \subseteq K \), \( f^{-1}(C) \) is convex in \( X \).

(c) For all \( x,y,z \in X \): \( |x-y| = |x-z| \Rightarrow |f(x)-f(y)| = |f(x)-f(z)| \).

(d) For all \( x,y,z \in X \): \( |f(x)-f(y)| > |f(x)-f(z)| \Rightarrow |x-y| > |x-z| \).

(e) For all \( x,y,z \in X \): \( |f(x)-f(y)| \neq |f(x)-f(z)| \Rightarrow |x-y| \neq |x-z| \).
Proof. (a) $\implies$ (b). Let $x,y \in f^{-1}(C)$ and let $z \in [x,y] \cap X$. Then $|z-x| \leq |x-y|$, so $|f(z)-f(x)| \leq |f(x)-f(y)|$ i.e., $f(z) \in [f(x),f(y)] \subset C$. Hence $z \in f^{-1}(C)$.

(b) $\implies$ (a). Let $x,y,z \in X$ and $|x-y| \leq |x-z|$. The set $[f(x),f(z)]$ is convex, hence $f^{-1}([f(x),f(z)])$ is convex in $X$ and contains $x$ and $z$, so it must contain $y$. Thus $f(y) \in [f(x),f(z)]$.

Clearly, (a) $\iff$ (b) and (f) $\iff$ (e). We prove (a) $\iff$ (f). Now (a) $\implies$ (f) is clear by symmetry. Suppose (f) and let $|x-y| \leq |x-z|$. It suffices to consider the case $|x-y| < |x-z|$. Then $|y-z| = |x-z|$, so by (f) we have $|f(y)-f(z)| = |f(x)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|,|f(z)-f(y)|) = |f(x)-f(z)|$.

**Theorem 2.3** Let $X \subset X$. Then

1. For each $a,b \in K$ the map $x \mapsto ax+b$ is in $M_b(X)$.
2. If $f \in M_b(X)$, $\lambda \in K$ then $\lambda f \in M_b(X)$.
3. $M_b(X)$ is closed under pointwise limits.
4. If $f \in M_b(X)$ and $g : f(X) \to K$ is in $M_b(f(X))$, then $g \circ f \in M_b(X)$.
5. If $f \in M_b(X)$ and $f(a) = f(b)$ for some $a,b \in X$, then $f$ is constant on $[a,b] \cap X$.

**Proof.** Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of $M_b$-functions. For more, see the sequel.

1. The constant functions.
2. Isometries (e.g., exp.).
3. Choose in every $a \in \Lambda$ an element $x_a$. Define $\phi : K \to K$ as follows
   \[
   \phi(x) = \begin{cases} 
   0 & \text{if } x = 0 \\
   x_a & \text{if } x \neq 0
   \end{cases} \quad (a \in \Lambda)
   \]
Essentially, \( \phi | K^* \) is the sign function \( \pi \) of section 1.

We prove that \( \phi \in M_b(K) \). Since \( \phi \) is continuous it suffices to check that \( \phi | K^* \) is in \( M_b(K^*) \). Now for all \( x, y \in K^* \) we have \( \phi(x) - \phi(y) = 0 \) if \( |xy^{-1} - 1| < 1 \) and \( |\phi(x) - \phi(y)| = |x-y| \) if \( |x-y| = \max(|x|,|y|) \). Now take \( x, y, z \in K^* \) such that \( |x-y| < |x-z| \). If \( \phi(x) = \phi(z) \) then \( |1-x^{-1}y| < |1-x^{-1}z| < 1 \) so \( \phi(x) = \phi(y) \).

If \( \phi(x) \neq \phi(z) \) then \( |\phi(x) - \phi(y)| \leq |x-y| \leq |x-z| = |\phi(x) - \phi(z)| \).

(4) Let \( r > 0 \) and choose in every ball \( B \) of radius \( r \) a center \( x_B \).

The function defined via

\[
\phi(x) = x_B \quad (x \in B)
\]

is in \( M_b(K) \). The proof is easy.

(5) (A nowhere continuous \( M_b \)-function). Let \( K \) be a field such that \( \#K = \#k \) (e.g., a discretely valued field where \( \#k \) has the power of the continuum). Let \( \sigma: K \to k \) be a bijection and let \( \tau: k \to K \) such that \( |\tau x - \tau y| = 1 \) whenever \( x \neq y \). Then \( f = \tau \circ \sigma \) satisfies: \( |f(x) - f(y)| = 1 \) \((x, y \in K, x \neq y)\).

Clearly \( f \) is everywhere discontinuous, \( f \in M_b(K) \).

(6) Let \( X \subset K \). We can strengthen the definition of an \( M_b \)-function into

\[
\text{if } |x-y| \leq |z-t| \text{ then } |f(x) - f(y)| \leq |f(z) - f(t)|
\]

(some "uniform" \( M_b \)-condition) and we obtain a space, called \( M_{ub}(X) \).

Clearly, the examples mentioned in (1), (2), (4), (5) are in \( M_{ub}(K) \), whereas the example in (3) is not. (Choose \( x, y \in K \) with \( |x| > 1 \), \( |x-y| = 1 \). Then \( |1-0| \leq |x-y| \), but \( 1 = |\phi(1) - \phi(0)| > |\phi(x) - \phi(y)| = 0 \).)

Notice that \( \phi \) is locally constant on \( K^* \), but not on \( K \).

(7) The discontinuous function \( f \) of (5) has the property that \( f(K) \) consists only of isolated points. This is not accidental. If \( f \in M_b(K) \)
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing \( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points, so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that the image of an everywhere discontinuous \( M_b \)-function consists only of isolated points.

(8) For each \( n \), let \( \sigma_n : K \rightarrow K \) be the example of 2.4, (4) above where \( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b(X) \) we have \( \sigma_n \circ f \in M_b(X) \) and \( \lim_{n \to \infty} \sigma_n \circ f = f \) uniformly. Hence, if \( f \) is continuous then it can uniformly be approximated by locally constant \( M_b \)-functions.

A monotone function \( f : [0,1] \rightarrow \mathbb{R} \) maps convex sets into sets that are relatively convex in \( f([0,1]) \). In our situation we do not have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If \( f : [0,1] \rightarrow \mathbb{R} \) maps convex sets into (relatively) convex sets then \( f \) need not be monotone: any Darboux continuous function has the above property. In fact a function \( f : [0,1] \rightarrow \mathbb{R} \) is Darboux continuous if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We define

**DEFINITION 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \rightarrow K \). Then \( f \) is called **weakly Darboux continuous** if for every relatively convex set \( C \subseteq X \) the set \( f(C) \) is convex in \( f(X) \). If \( f \) is called **Darboux continuous** if for every relatively convex set \( C \subseteq X \) the set \( f(C) \) is convex (in \( K \)).

We have the following obvious remarks.

1) \( f : X \rightarrow K \) is Darboux continuous if and only if \( f \) is weakly Darboux continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{R}_p \to \mathbb{Q}_p \) such that for every open ball \( B \subset \mathbb{R}_p \)
\[ f(B) = \mathbb{Z}_p. \]
Let \( A \subset \mathbb{R}_p \) be defined as follows: \( x = \sum a_n p^n (a_n \in \{0, 1, \ldots, p-1\}) \)
is in \( A \) if \( a_{2n} = a_{2n+2} = \ldots = 0 \) for some \( n \). Define \( f : \mathbb{R}_p \to \mathbb{Q}_p \) via
\[
 f(x) = \begin{cases} 
 a_{2N+1} + a_{2N+3}p + a_{2N+5}p^2 + \ldots & \text{if } x \in A \text{ and } N = \min \{n : a_{2n} = a_{2n+2} = \ldots = 0\} \\
 0 & \text{if } x \notin A
\end{cases}
\]
Then \( f \) maps every non empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.
(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on \( \mathbb{R}_p \) is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous.

To see this, observe that all open compact subsets of \( \mathbb{R}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{R}_p \) onto \( \mathbb{R}_p \) sending \( \{x : |x| < 1\} \) into \( \{x : |x| = 1\} \) and \( \{x : |x| = 1\} \) into \( \{x : |x| < 1\} \). If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( \mathbb{M}_p \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4,(7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \to \mathbb{R} \) the following conditions are equivalent.

(a) \( f \) is strictly monotone (i.e., injective and monotone).

(b) \( f \) is injective. For a convex set \( C \subset [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all \( x, y, z \in [0, 1] \): if \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

(δ) For all \( x, y, z \in [0, 1] \): \( f(x) \) is between \( f(z) \) if and only if \( x \) is between \( y \) and \( z \).

Translating (α) - (δ) into the non-archimedean situation we arrive at the following conditions. Let \( X \subseteq K \) and \( f : X \to K \)

(α') \( f \in M^*_D(X) \) and \( f \) is injective.

(β') \( f \) is weakly Darboux continuous and injective.

(γ') for all \( x, y, z \in X \), \(|x-y| < |x-z| \) implies \(|f(x)-f(y)| < |f(x)-f(z)|\).

(δ') \( f \in M_D(X) \) and \( f \) satisfies (γ').

It will turn out that the conditions (α') - (γ') although not equivalent are closely related. We start with (γ'):

DEFINITION 2.6 Let \( X \subseteq K \), \( f : X \to K \). We say that \( f \) is \( M_s(X) \) if for all \( x, y, z \in X \), \( f(x) \in [f(y), f(z)] \) implies \( x \in [y, z] \).

THEOREM 2.8 Let \( X \subseteq K \), \( f : X \to K \). Then the following statements are equivalent:

(α) \( f \in M^*_s(X) \).

(β) \( f \) is injective and weakly Darboux continuous.

(γ) \( f \) is injective and \( f^{-1} \in M_D(f(X)) \).

(δ) for all \( x, y, z \in X \), \(|f(x)-f(y)| = |f(x)-f(z)| \) \( \Rightarrow |x-y| = |x-z| \).

(ε) for all \( x, y, z \in X \), \(|x-y| < |x-z| \) \( \Rightarrow |f(x)-f(y)| < |f(x)-f(z)| \).

(ζ) for all \( x, y, z \in X \), \(|x-y| \neq |x-z| \) \( \Rightarrow |f(x)-f(y)| \neq |f(x)-f(z)| \).
Proof. The implications \((a) \Rightarrow (e) \Rightarrow (c) \Rightarrow (\delta)\) are clear from the definitions.

\((\delta) \Rightarrow (\gamma)\): injectivity follows from \(|f(x)-f(y)| = |f(x)-f(y)| + |x-x| = |x-y|\). Use 2.2.(\gamma).

\((\gamma) \Rightarrow (\beta)\): Let \(g : f(X) \to X\) be the inverse of \(f\). Let \(C \subseteq X\) be convex in \(X\). Then since \(g \in M_b\), \(g^{-1}(C)\) is convex in \(f(X)\). But \(g^{-1}(C) = f(C)\).

Finally, we prove \((\beta) \Rightarrow (a)\). Let \(f(x) \in [f(y),f(z)]\). By (\beta) the set \(f([y,z] \cap X)\) is convex in \(f(X)\) and it contains \(f(y),f(z)\), hence \(f(x) \in [f(y),f(z)] \cap X \subseteq f([y,z] \cap X)\). Since \(f\) is injective, \(x \in [y,z] \cap X\) and we are done.

We also have (compare 2.3)

**THEOREM 2.9** Let \(X \subseteq K\). Then

(i) For \(a,b \in K\), \(a \neq 0\) the map \(x \mapsto ax+b\) is in \(M_s(X)\).

(ii) If \(f \in M_s(X), \lambda \in K, \lambda \neq 0\) then \(\lambda f \in M_s(X)\).

(iii) If \(f_1,f_2,\ldots \in M_s(X), \lim f_n = f\) pointwise, \(f\) injective then \(f \in M_s(X)\).

(iv) If \(f \in M_s(X), g \in M_s(f(X))\) then \(g \circ f \in M_s(X)\).

Proof. Obvious verifications.

Returning to our conditions \((a') \Rightarrow (\delta')\) we see that \((\delta')\) is equivalent to \((\gamma')\), that \((a')\) means \(f^{-1} \in M_s(f(X))\) and that \((\delta')\) means \(f \in M_b(X) \cap M_s(X)\).

Our \(f\) of example 2.4 (5) is in \(M_b\), injective but not in \(M_s\). Its inverse yields an example of an \(M_s\)-function that is not in \(M_b\). Thus, in general, we have neither one of the implications \((a') \Rightarrow (\gamma'), (\gamma') \Rightarrow (a')\), \((\beta') \Rightarrow (\delta')\), \((\alpha') \Rightarrow (\delta')\). But our counterexample is
rather weird \( f \) is nowhere continuous and the domain of \( f^{-1} \) is discrete). We can do better.

**EXAMPLE 2.10** Let \( K \) have discrete valuation and let \( k \) be infinite.

Then there exists a homeomorphism of the unit ball of \( K \) that is in \( M_d \) but not in \( M_s \). (The inverse map is in \( M_s \) but not in \( M_d \)).

**Proof.** Set \( X = \{ a \in K : |a| \leq 1 \} \) and let \( R \) be a full set of representatives of the equivalence relation \( x \sim y \iff |x-y| < 1 \) in \( X \). Then \( R \) is infinite. Let \( \pi \in K \) be such that \( |\pi| \) is the largest value that is smaller than 1. The map

\[
(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \quad (a_i \in R \text{ for each } i)
\]

is a bijection of \( R^N \) onto \( X \). We may suppose that \( 0 \in R \).

Since \( R \) is infinite we can define injections

\[
\tau_1 : R \setminus \{0\} \to R \\
\tau_2 : R \to R
\]

such that \( \text{im } \tau_1 \cap \text{im } \tau_2 = \emptyset \), \( \text{im } \tau_1 \cup \text{im } \tau_2 = R \).

For \( x = \sum a_n \pi^n \in X \) (\( a_n \in R \) for each \( n \)) set

\[
f(x) := \begin{cases} 
\tau_1(a_0) + a_1 \pi + \ldots = x - a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\
\tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0
\end{cases}
\]

A simple inspection of the definition shows that \( f \) is a bijection of \( X \) onto \( X \). If \( a,b \in R \), \( a \neq b \) then \( |a\pi - b\pi| < |b\pi - a| \), whereas

\[
|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1 \quad \text{and} \quad |f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1,
\]

so \( f \notin M_s(X) \). Finally, let \( x, y, z \in X \) and \( |x-y| \leq |x-z| \). We prove that \( |f(x) - f(y)| \leq |f(x) - f(z)| \). If \( |f(x) - f(z)| = 1 \) there is nothing to prove, so suppose \( |f(x) - f(z)| < 1 \). Set \( x = \Sigma a_n \pi^n \), \( y = \Sigma b_n \pi^n \), \( z = \Sigma c_n \pi^n \).
If \( a_0 = 0 \) then also \( c_0 = 0 \) and \( \tau_2(a_1) = \tau_2(c_1) \) so \( a_1 = c_1 \), hence \( |x-z| \leq |\pi| \). Since \( |x-y| \leq |x-z| \) we have also \( b_0 = 0, b_1 = a_1 \).
So, \( f(x)-f(y) = \frac{x-y}{\pi}, f(x)-f(z) = \frac{y-z}{\pi} \) whence \( |f(x)-f(y)| \leq |f(x)-f(z)| \).

If \( a_0 \neq 0 \) then \( \tau_1(a_0) = \tau_1(c_0) \) so \( a_0 = c_0 \). Then also \( c_0 = a_0 = b_0 \).
Then \( f(x)-f(y) = x-y, f(x)-f(z) = x-z \) whence \( |f(x)-f(y)| \leq |f(x)-f(z)| \).

Let \( X \subseteq K \). If \( f \in M_s(X) \) then \( f^{-1} \in M_b(f(X)) \). Conversely, if \( f \in M_b(X) \) and \( g : f(X) \rightarrow X \) is such that \( f \circ g \) is the identity on \( f(X) \) then \( g \in M_s(f(X)) \). This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let \( X \subseteq K \) and \( f : X \rightarrow K \). \( f \) is called weakly monotone \((f \in M_w(X))\) if for all \( x, y, z \in X \)
\[
|x-y| < |x-z| + |f(x)-f(y)| \leq |f(x)-f(z)|
\]
\( f \) is called strongly monotone \((f \in M_{bs}(X))\) if
\( f \in M_s(X) \cap M_{bs}(X) \).

Clearly, \( f \in M_{bs}(X) \) if and only if \( f^{-1} \in M_{bs}(f(X)) \). Also, if \( f \in M_w(X) \)
and \( g : f(X) \rightarrow X \) is such that \( f \circ g \) is the identity on \( f(X) \) we have
\( g \in M_w(f(X)) \).

Obviously we have \( M_b(X) \cup M_s(X) \subseteq M_w(X) \) and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of \( M_w \)-functions, not for the sake of \( M_w \) itself but in order to get results that are valid for \( M_b, M_s \) simultaneously.
The functions in \( M_{bs} \) behave reasonable and they may be viewed as the non-archimedean equivalents of strict monotone functions in the real case.
THEOREM 2.12 Let \( X \subseteq K \) and \( f : X \to K \). Then the following conditions are equivalent.

(a) \( f \in M_{bs}(X) \).

(b) \( f \) is injective and \( C \mapsto f(C) \) is a 1-1 correspondence between the relatively convex subsets of \( X \) and those of \( f(X) \).

(c) For all \( x,y,z \in X \): \( |x-y| < |x-z| \iff |f(x)-f(y)| < |f(x)-f(z)| \).

(d) For all \( x,y,z \in X \): \( |x-y| = |x-z| \iff |f(x)-f(y)| = |f(x)-f(z)| \).

(e) For all \( x,y,z \in X \): \( |x-y| \leq |x-z| \iff |f(x)-f(y)| \leq |f(x)-f(z)| \).

(f) \( f \in M_s(X) \), \( f^{-1} \in M_s(f(X)) \).

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An \( M_w \)-function that is not in \( M_s \cup M_b \)). Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be any function, constant on the cosets of \( \{ x \in \mathbb{Z}_p : |x| < 1 \} \). Then \( f \in M_w(\mathbb{Z}_p) \).

Clearly \( f \notin M_s(\mathbb{Z}_p) \). \( f \in M_b(\mathbb{Z}_p) \) if and only if the points of \( f(\mathbb{Z}_p) \) are equidistant.

(2) (Continuity of monotone functions). Let \( X \subseteq K \).

(a) Let \( f \in M_w(X) \). If \( f(X) \) has no isolated points, then \( f \) is continuous.

(Proof. Let \( a \in X \) and \( \varepsilon > 0 \). Then there is \( z \in X \) such that \( z \neq a \), \( |f(z)-f(a)| < \varepsilon \). Let \( \delta := |z-a| \). Then for all \( x \in X \) with \( |x-a| < \delta \) we have, by the weak monotony of \( f \), \( |f(x)-f(a)| \leq |f(z)-f(a)| < \varepsilon \).

It follows that if \( X \) and \( Y \) do not have isolated points and if \( f \) is an \( M_w \)-bijection of \( X \) onto \( Y \), then \( f \) is a homeomorphism of \( X \) onto \( Y \).
Conversely, it is easy to construct homeomorphisms of $\mathbb{R}_p$ that are not in $M_w(\mathbb{R}_p)$.

(b) If $K$ is a local field then every $f \in M_w(X)$ is continuous. (See 5.1 (i)).

(c) If $K$ has discrete valuation then every $f \in M_s(X)$ is continuous. (Example 2.4 (5) shows that such a statement is not true for $f \in M_b(X)$.)

(Proof. If $f$ were not continuous at some $a \in X$ then there would be an $\epsilon > 0$ such that for some sequence converging to $a$ we had $|f(x_n) - f(a)| \geq \epsilon$. We may suppose that $|x_n - a| > |x_2 - a| > \ldots$. Since the valuation is discrete we have $\lim_{n \to \infty} |f(x_n) - f(a)| = 0$, a contradiction.)

(d) In 5.14 we shall give an example of a function in $M_{bs}(X)$ that is not continuous. (Of course, $K$ will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" $M_w, \ldots$-conditions.

Thus, by definition, $f \in M_uw(X)$ if for all $x, y, z, t \in X$

$$|x - y| < |z - t| \rightarrow |f(x) - f(y)| \leq |f(z) - f(t)|$$

$f \in M_us(X)$ if for all $x, y, z, t \in X$

$$|x - y| < |z - t| \rightarrow |f(x) - f(y)| < |f(z) - f(t)|$$

$f \in M_{ubs}(X)$ if for all $x, y, z, t \in X$

$$|x - y| < |z - t| \leftrightarrow |f(x) - f(y)| < |f(z) - f(t)|.$$ Notice that $f \in M_{ubs}(X)$ means that $|f(x) - f(y)|$ is a strictly increasing function of $|x - y|$. Examples of such functions are isometries, but also the function $f : \mathbb{R}_p \rightarrow \mathbb{R}_p$ defined via

$$\Sigma a_n p^n \rightarrow \Sigma a_n p^{2n} \quad (\Sigma a_n p^n \in \mathbb{Z}_p)$$

$$|f(x) - f(y)| = |x - y|^2 \quad \text{for all } x, y \in \mathbb{Z}_p.$$ Monotone functions : $\mathbb{R} \rightarrow \mathbb{R}$ are divided into two classes: the
increasing functions and the decreasing functions. For the non-archi-
medean case we may ask for a similar classification. First we try to
express the situation in the real case in such a way that it can be
translated. Let \( a \in \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be strictly monotone. If
\( x \) runs through some side of \( a \) then \( f(x) \) runs through some fixed side
of \( f(a) \). So there is a map \( \sigma : \{-1,1\} \to \{-1,1\} \) such that \( \sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a)) (x \neq a) \). Apparently, the only \( \sigma \)'s that can occur are
the identity and \( \sigma(x) = -x \ (x \in \{1,-1\}) \). Moreover it turns out that the
map \( \sigma \) is independent of the choice of \( a \).

The two maps \( \sigma \) that can occur can be interpreted as multiplication
maps (with 1 and -1 respectively) or as the bijections \( \{1,1\} \to \{-1,1\} \)
and there seems to be no philosophical reason to make any decision of
preference.

As an example, let us consider a function \( f \in M_\Sigma(K) \). Let \( a \in K, \)
let \( a \in \Sigma \). If \( x \in a+\alpha \) and \( y \in a+\alpha \) ("\( x, y \) are at the same side of \( a \)"
then \( x-a, y-a \in a \), so \( |x-y| < |y-a| \). Since \( f \in M_\Sigma(K) \) we have
\( |f(x)-f(y)| < |f(y)-f(a)| \), whence \( |f(x)-f(a)-(f(y)-f(a))| < |f(y)-f(a)| \),
so \( f(x)-f(a) \) and \( f(y)-f(a) \) have the same sign. We may conclude that
there is a map \( \sigma_a : \Sigma \to \Sigma \) such that for all \( x \in K \)
\[ x \in a+\alpha + f(x) \in f(a)+\sigma_a(\alpha) \quad (\alpha \in \Sigma). \]

Unfortunately, it turns out that in general \( \sigma_a \) may be different
from \( \sigma_b \) if \( a \neq b \), even for isometrical maps. For example, let \( p \neq 2 \)
and let \( \tau \) be a permutation of \( \{0,1,2,\ldots,p-1\} \) and define \( f : \mathbb{Z}_p \to \mathbb{Z}_p \)
by
\[ \Sigma a \cdot p^n \to \Sigma \tau(a_n) \cdot p^n \quad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n). \]
Suppose we had a \( \sigma : \Sigma \to \Sigma \) such that for all \( x,y \in \mathbb{Z}_p, x-y \in a \) implies
\( f(x)-f(y) \in \sigma(a) \). Let \( a = \theta^{p-n} \) (see 1.5). Then \( x-y \in a \) means
\[ x = a_0 + a_1 p + \ldots + a_n p^n \ldots \]
\[ y = b_0 + b_1 p + \ldots + b_n p^n \ldots \]
where \( a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n - b_n = \theta^i \) modulo \( p \).

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n)) p^n \ldots \), so \( \sigma(a) = \theta^j p^n \) where \( \tau(a_n) - \tau(b_n) = \theta^j \) mod \( p \). (\( j \) depending on \( i \) and \( n \)).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = \ldots = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \subset K \), \( f : X \to K \) and let \( \sigma : \Sigma \to \Sigma \). We say that

\( f \) is monotone of type \( \sigma \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \implies f(x) - f(y) \in \sigma(\alpha). \]

(In other words if \( x >_\alpha y \) implies \( f(x) > \sigma(\alpha) f(y) \),

see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \to \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( \alpha, x >_\alpha y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \subset K \), \( f : X \to K \), \( \beta \in \Sigma \). We say that \( f \) is monotone of type \( \beta \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \implies f(x) - f(y) \in \alpha \beta. \]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \).
where $\sigma : \Sigma \to \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$ is monotone of type $\beta$ iff the sign of $\frac{f(x)-f(y)}{x-y}$ is constant $\beta$ for all $x, y \in X, x \neq y$. This leads to

**DEFINITION 2.16** Let $X \subset K$, $f : X \to K$. $f$ is called increasing if $f$ is monotone of type 1. In other words, $f$ is increasing if for all $x, y \in X, x \neq y$ the difference quotient $\frac{f(x)-f(y)}{x-y}$ is positive, i.e., if

$$\left| \frac{f(x)-f(y)}{x-y} - 1 \right| < 1.$$

In the next section we shall study the monotone functions of type $\sigma$ and we will give a partial answer to the question for which maps $\sigma : \Sigma \to \Sigma$ there exists an $f : K \to K$ that is monotone of type $\sigma$. 
3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $\sigma$.

DEFINITION 3.1. Let $X \subseteq K$, $f : X \to K$. Let $\Phi f(x,y) := \frac{f(x)-f(y)}{x-y}$ ($x, y \in X$, $x \neq y$). $f$ is called

positive if $f(X) \subseteq K^+$

strictly positive if $\sup_{x \in X} |f(x)| < 1$

increasing if $\Phi f(x,y) \in K^+$ for all $x, y \in X$, $x \neq y$

strictly increasing if $\sup \{ |1-\Phi f(x,y)| : x, y \in X, x \neq y \} < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. Let $X \subseteq K$.

(i) If $f : X \to K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a, b \in K$ the function $x \mapsto ax + b$ is increasing if and only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly increasing).

(iii) If $f : X \to K$ is (strictly) increasing and $f$ is (strictly) positive then $-\frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \to K$ form a convex set.

(v) If $f : X \to K$ and $g : f(X) \to K$ are (strictly) increasing then so is $g \circ f$.

(vi) If $f : X \to K$ is (strictly) increasing then so is $f^{-1} : f(X) \to K$.

(vii) If $f_1, f_2, \ldots : X \to K$ are increasing and $f := \lim f_n$ pointwise then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function
\[ \exp x = 1 + x + \frac{x^2}{2!} + \ldots \]
defined on \( \{ x \in K : |x| < p^{-1} \} \) if \( \chi(k) = p, \chi(K) = 0 \) and on \( \{ x \in K : |x| < 1 \} \) if \( \chi(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f : X \to K \) be a \( C^\infty \)-function (i.e., \( \Phi f \) can continuously be extended to a function on \( X \times X \), assume that \( X \subseteq K \) has no isolated points. See \([2]\)) and suppose \( f'(a) \not\in K^+ \) for some \( a \in X \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta, |y-a| < \delta, x \neq y \) implies
\[ \left| \frac{f(x)-f(y)}{x-y} - f'(a) \right| \leq \frac{M}{4}. \]
For such \( x, y \) we have
\[ \left| \frac{f(x)-f(y)}{x-y} - 1 \right| \leq \left| \frac{f(x)-f(y)}{x-y} - f'(a) \right| + \left| f'(a) - 1 \right| \leq \max(\frac{M}{4}, |f'(a) - 1|) < 1. \]

(3) The space \( C(\mathbb{Z}_p) \) of all continuous functions \( \mathbb{Z}_p \to \mathbb{Q}_p \), is a Banach space with respect to the sup norm \( | |_\infty \). Let \( e_0 := \mathbb{Z}_p \) and for \( n \geq 1 \) let \( e_n := \mathbb{Z}_p/B_n \) where \( B_n := \{ x \in \mathbb{Z}_p : |x-n| < \frac{1}{n} \} \). It is proved in [1] that \( e_0, e_1, \ldots \) is an orthonormal base of \( C(\mathbb{Z}_p) \) i.e., for each \( f \in C(\mathbb{Z}_p) \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that
\[ f = \sum_{n=0}^{\infty} \lambda_n e_n \]
The coefficients $\lambda_n$ can be reconstructed from $f$ via

$$
\lambda_0 = f(0)
$$

$$
\lambda_n = f(n) - f(n_-) \quad (n \in \mathbb{N})
$$

where $n_-$ is defined as $a_0 + a_1p + \ldots + a_{s-1}p^{s-1}$ if $n \neq a_0 + a_1p + \ldots + a_sp^s$ ($a_s \neq 0$) in base $p$.

Our aim is here to describe a necessary and sufficient condition for the $\lambda_n$ in order that $f = \sum \lambda_n e_n$ is increasing. We show

$$
f = \sum \lambda_n e_n \text{ is increasing if and only if for all } n \in \mathbb{N}
$$

$$
|\lambda_n - (n-n_-)| < |n-n_-|.
$$

Proof. First observe that $f$ is increasing if and only if for all $x \in \mathbb{Z}_p$

$$
f(x) = x + g(x)
$$

where $|\Phi(g(x,y))| < 1$ for all $x, y \in \mathbb{Z}_p$, $x \neq y$.

As

$$
x = \sum_{n \geq 1} (n-n_-)e_n(x) \quad (x \in \mathbb{Z}_p)
$$

it suffices to show that for $g = \sum \lambda_n e_n \in C(\mathbb{Z}_p)$ we have $|\Phi(g)| < 1$ if and only if $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Suppose first $|\Phi(g)| < 1$. Then for all $n \in \mathbb{N}$, $|\frac{f(n)-f(n_-)}{n-n_-}| < 1$, so

$$
|\lambda_n| = |\frac{f(n)-f(n_-)}{n-n_-}| < |n-n_-|.
$$

Conversely, let $g = \sum \lambda_n e_n$ and let $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Let $x, y \in \mathbb{Z}_p$ and let $|x-y| = p^{-k}$ for some $k \in \{0,1,2,\ldots\}$. Since

$$
e_n(a) = e_n(b) \text{ if and only if } |a-b| < \frac{1}{n}
$$

we have

$$
e_n(x) = e_n(y) \quad \text{for } n < p^k.
$$
Therefore
\[ |g(x)-g(y)| = \left| \sum_{n=1}^{\infty} \lambda_n (e_n(x)-e_n(y)) \right| \leq \max_k |\lambda_k| < \max_k |n-n_k| = p^{-k} = |x-y| \]
so \( |g| < 1 \).

(4) Let \( K \) have dense valuation and let \( k \) be (countably) infinite. Let \( X \) be the unit ball of \( K \) and let \( B_i \) (\( i \in \mathbb{N} \)) be the balls in \( X \) with radius \( 1 \). Choose \( c_1, c_2, \ldots \in K \) such that \( |c_1| < |c_2| < \ldots \), \( \lim_{n \to \infty} |c_n| = 1 \). For \( n \in \mathbb{N} \) define a function \( f_n: X \to K \) via
\[ f_n(x) = \begin{cases} x + c_i & \text{if } x \in B_i \ (1 \leq i \leq n) \\ x & \text{elsewhere} \end{cases} \]
Then each \( f_n \) is strictly increasing (\( |f_n(x,y)-1| \leq \max_{1 \leq i, j \leq n} |c_i-c_j| \)). The sequence \( f_1, f_2, \ldots \) converges pointwise to an increasing function \( f \). But \( f \) is not strictly increasing:
\[ \sup_{x \neq y} |f(x,y)-1| = \sup_{i,j} |c_i-c_j| = 1. \]
(Compare 3.2, (vii) and (viii).)

Increasing functions are closely related to functions \( g \) for which
\[ |g(x)-g(y)| < |x-y| \ (x \neq y) \ (\text{if } f \text{ is increasing, set } g(x):= f(x)-x). \]

**DEFINITION 3.4.** Let \( (X, \rho) \) be an ultrametric space. A map \( g: X \to X \)
is called a pseudocontraction if \( \rho(f(x), f(y)) < \rho(x, y) \)
\( (x, y \in X, x \neq y) \).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X \rightarrow X$ has a fix point. We have

**Lemma 3.5.** Let $(X, \rho)$ be an ultrametric space. Then the following conditions are equivalent.

1. $X$ is spherically complete.
2. Each pseudocontraction $X \rightarrow X$ has a fix point.
3. Each pseudocontraction $X \rightarrow X$ has a unique fix point.

**Proof.** If $\sigma: X \rightarrow X$ is a pseudocontraction and if $x, y$ are fix points and $x \neq y$, then $\rho(x, y) = \rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction. Thus, we have (β) $\Rightarrow$ (γ). We prove (α) $\Rightarrow$ (β). Let $B \subset X$ be a ball (i.e., either $B = \{x \in X: \rho(x, a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X: \rho(x, a) < r\}$ for some $a \in X$, $r > 0$). We call $B$ **invariant** if $\sigma(B) \subset B$.

Now we observe two facts

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X: \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x), \sigma(a)) < \rho(x, a) \leq \max(\rho(x, \sigma(a)), \rho(\sigma(a), a)) = \rho(a, \sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \notin V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x, y)$ does not depend on $x, y$, since for $z \in B_1$, $u \in B_2$, $\rho(x, z) < \rho(x, y)$ and $\rho(y, u) < \rho(x, y)$, so $\rho(z, u) = \rho(x, y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a smal-
lest invariant ball $S$. If $a \in S$, $\sigma(a) \neq a$ then \( \{x \in S : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\} \) is invariant and does not contain $a$, a contradiction. Hence, $\sigma$ has a fix point (actually, $S$ is a singleton).

We prove ($\beta$) $\Rightarrow$ ($\alpha$). If $X$ were not spherically complete, there exist balls $B_1 \supsetneq B_2 \supsetneq \ldots$ such that $\bigcap_{n} B_n = \emptyset$. Choose $x_n \in B_n \setminus B_{n+1}$ ($n \in \mathbb{N}$), set $B_0 := X$ and define

$$\sigma(x) := x_{n+1} \text{ if } x \in B_n \setminus B_{n+1} \quad (n \in \{0, 1, 2, \ldots\}).$$

Then $\sigma$ has obviously no fix point. Let $x \in B_n \setminus B_{n+1}$ and $y \in B_m \setminus B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in $B_{m+1}$, whereas $x \in B_n \subseteq B_{m+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$. Then $\sigma$ is a pseudocontraction without a fix point. Contradiction.

**COROLLARY 3.6.** The following conditions are equivalent.

(a) $K$ is spherically complete.

(\beta) If $C \subseteq K$ is convex, $f : C \rightarrow C$ is increasing then $f$ is surjective.

(\gamma) If $C \subseteq K$ is convex, $f : C \rightarrow K$ is increasing then $f(C)$ is convex.

(\delta) An increasing $f : K \rightarrow K$ is surjective.

**Proof.** ($\alpha$) $\Rightarrow$ ($\beta$). Choose $a \in C$ and consider the map $\sigma : x \mapsto x - f(x) + a$ ($x \in C$). Then $\sigma : C \rightarrow C$. $C$ is spherically complete, $\sigma$ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: $f$ is surjective.

($\beta$) $\Rightarrow$ ($\gamma$). For a suitable $s \in K$, $f-s$ sends $C$ into $C$. ($\gamma$) $\Rightarrow$ ($\delta$) is clear.

($\delta$) $\Rightarrow$ ($\alpha$). Let $\sigma : K \rightarrow K$ be a pseudocontraction. Then $x \mapsto x - \sigma(x)$
is increasing hence is surjective. So then is \( x \in K \) for which \( x - \sigma(x) = 0 \), i.e., \( \sigma \) has a fix point. By 3.5, \( K \) is spherically complete.

In case \( f \) is strictly increasing we do not have to require that \( K \) is spherically complete:

**Theorem 3.7.** Let \( C \subset K \) be convex and let \( f : C \to K \) be strictly increasing. Then \( f(C) \) is convex. If \( f(C) \subset C \), then \( f(C) = C \).

**Proof.** Reread the proof of (a) + (b), (b) + (c) above. \( \sigma \) now is a contraction. \( C \) is complete. Apply the Banach contraction theorem.

Let \( X \) be a subset of \( IR \) and let \( f : X \to IR \) be a bounded increasing function. Then \( f \) can be extended to an increasing function \( IR \to IR \) by setting \( f(x) = \inf \{ f(y) : y \in X \} \) for all \( x \) in \( X \) and \( f(x) = \sup \{ f(y) : y \leq x, y \in X \} \) for all other \( x \in IR \). In our situation we can prove

**Theorem 3.8.** The following conditions are equivalent.

(a) \( K \) is spherically complete.

(b) For every \( X \subset K \) an increasing function \( f : X \to K \) can be extended to an increasing \( \overline{f} : K \to K \).

(c) Let \( X \subset K \), and let \( f : X \to K \) be a strictly increasing function. Then \( f \) can be extended to a strictly increasing function \( \overline{f} : K \to K \) such that

\[
\sup_{x,y \in K} |\overline{f}(x) - \overline{f}(y) - 1| = \sup_{x,y \in X} |f(x) - f(y) - 1|.
\]

\( x \neq y \)

**Proof.** (a) + (b). Let \( a \notin X \). By Zorn's Lemma it suffices to define \( \overline{f} \) such that \( \overline{f} \) is increasing on \( X \cup \{a\} \). We are done if we can find \( a \in K \) such that for \( x \in X \)
\[
\frac{|a-f(x)|}{a-x} - 1 < 1
\]

i.e., \( a \in B_x := B_x^{f(x)-f(a-x)} (|a-x|) (x \in X) \).

Now \( B_x \cap B_y \neq \emptyset (x, y \in X) \) since the distance of their centers is
\[
|f(x)-(a-x)-f(y)-(a-y)| = |f(x)-f(y)-(x-y)| = |\Phi(x,y)-1||x-y| < \max(|x-a|, |a-y|).
\]

So if, say, \(|x-a| \leq |y-a|\) we see that \(|f(x)-(a-x)-f(y)-(a-y)| < |y-a| \) whence \( f(x)-(a-x) \in B_y \). By the spherical completeness of \( K \) we have \( \cap_{x \in X} B_x \neq \emptyset \). Choose \( a \in \cap_{x \in X} B_x \).

(\( \beta \) + (\( \alpha \)). Suppose \( K \) is not spherically complete. By 3.6, (\( \delta \)) + (\( \alpha \)) there is a non surjective increasing function \( f: K + K \). Then its inverse \( g: f(K) + K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K + K \).

(\( \beta \leftrightarrow \gamma \)) follows from the fact that (with \( \Phi(x) = x \) for all \( x \))

\[
f \leftrightarrow (1-c)x + cf \quad (c \in K, |c| < 1)
\]

is a 1-1 correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \(|1-\Phi(g)| < |c|\).

We will now investigate the relation between increasingness of \( f \) and positivity of \( f! \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \rightarrow K \). Let \( \lambda \in K, 0 < |\lambda| < 1 \). Then \( x \rightarrow x-\lambda \sigma(x) \) is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \) of \( K \) without isolated points and if \( f \) is differentiable then for each
\( x \in X, f'(x) = \lim_{y \to x} f(x,y) \in K^+. \) So \( f' \) is positive. If, addition, \( f \) is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f: X \to K \) be a (strictly) positive Baire class 1 function. Then does \( f \) have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subset K \) and let \( f: X \to K \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots : X \to K \) such that \( |g_n| \leq 1 - \frac{1}{n} \) for each \( n \) and

\[
\sum_{n=1}^{\infty} g_n (\text{pointwise}).
\]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots : X \to K \) such that \( f = \lim_{n \to \infty} f_n \) pointwise. There exist locally constant functions \( h_1, h_2, \ldots : X \to K \) such that \( |f_n - h_n| \leq 2^{-n} \), hence \( f = \lim_{n \to \infty} h_n \) pointwise. Define

\[
t_1, t_2, \ldots : X \to K \text{ as follows}
\]

\[
t_n(x) := \begin{cases} h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\ 0 & \text{if } |h_n(x)| > 1 - \frac{1}{n}. \end{cases}
\]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N} \). \( \{x \in X : |h_n(x)| \leq 1 - \frac{1}{n}\} \) is closed and open in \( X \). \( |t_n| \leq 1 - \frac{1}{n} \) and \( \lim t_n = f \). Now let \( g_1 := t_1 \), and \( g_n := t_n - t_{n-1} \) \( (n \geq 2) \). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n} \), \( f = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} g_n \).
**Lemma 3.10.** Let $X \subseteq K$ have no isolated points and let $f: X \to K$ be a Baire class 1 function, $|f(x)| < 1$ for all $x \in X$. Then $f$ has an antiderivative $F$ for which

$$\frac{|F(x)-F(y)|}{|x-y|} < 1 \quad (x, y \in X, x \neq y).$$

**Proof.** By Lemma 3.9, $f = \sum_{n=1}^{\infty} f_n$, where each $f_n$ is locally constant, $|f_n| \leq 1 - \frac{1}{n}$. By [2] 3.9 each $f_n$ has an antiderivative $F_n$ for which

$$|F_n(x)-F_n(y)| \leq \max \{|f_n(x)|, \frac{1}{2n}\}|x-y| \quad (x, y \in X).$$

By [2] 3.7, $F := \sum F_n$ is an antiderivative of $\sum f_n = f$. Now for $x, y \in X, x \neq y$:

$$|F(x)-F(y)| \leq \sup_n |F_n(x)-F_n(y)| \leq \sup_n \max \{|f_n(x)|, \frac{1}{2n}\}|x-y| \leq |x-y| \max \{|f_n(x)|, \frac{1}{2n}\}. \quad \text{Now for each } x \in X, |f_n(x)| < 1 \text{ for each } n \text{ and } \lim_{n} |f_n(x)| = 0 < 1. \quad \text{Hence } \max_n |f_n(x)| < 1. \text{ It follows that }$$

$$|F(x)-F(y)| < |x-y|.$$

**Theorem 3.11.** Let $X \subseteq K$ have no isolated points and let $f: X \to K$ be (strictly) positive. Then $f$ has a (strictly) increasing antiderivative.

**Proof.** The function $x \mapsto f(x)-1$ has, by 3.10, an antiderivative $H$ such that $|\phi(H)| < 1$. Let $F(x) := f + H(x)$ ($x \in X$). Then $F' = f$ and $\phi(F) = 1 + \phi(H)$. Thus, if $f$ is positive then $F$ is increasing. If $f$ is strictly positive then $|f(x)-1| < r < 1$ for all $x \in X$ and, by a trivial extension of 3.10, we may choose $H$ such that $|\phi(H)| < r$. It follows that $|\phi(F)-1| < r$, so $F$ is strictly increasing.
We collect the results in

**COROLLARY 3.12.** Let $X \subset K$ have no isolated points. Then

(i) If $f: X + K$ is differentiable and (strictly) increasing then $f'$ is a (strictly) positive Baire class 1 function.

(ii) If $g: X + K$ is a (strictly) positive Baire class 1 function then $g$ has a (strictly) increasing antiderivative.

(iii) If $f: X + K$ is differentiable and if $f'$ is (strictly) positive then $f = g + h$ where $g$ is (strictly) increasing and where $h' = 0$.

**Note.** We cannot strengthen 3.12 (iii) by replacing "$h' = 0"$ by "$h$ is locally constant". In fact, if $X = \mathbb{Z}_p$ then every locally constant function has bounded difference quotients. If our statements were true, then every differentiable $f: \mathbb{Z}_p + \mathbb{Q}_p$ for which $f'$ is positive would have bounded difference quotients.

But consider the function $f: \mathbb{Z}_p + \mathbb{Q}_p$ defined via

$$f(x) = \begin{cases} \frac{x-p^{2n}}{p} & \text{if } |x-p^n| < p^{-3n} (n \in \{0,1,2,\ldots\}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. Let $x := p^n$ and $y := p^n + 3^n (n \in \mathbb{N})$. Then $f(x_n) = p^n - 2^n$, $f(y_n) = p^n + 3^n$, so $|f(x_n) - f(y_n)| = |2^n| = p^{-2n}$, whereas $|x_n - y_n| = |3^n| = p^{-3n}$. So

$$\lim_{n \to \infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n \to \infty} p^n = \infty.$$ 

We now study the connection between increasing $C^1$-functions and continuous positive functions.

If $f$ is a (strictly) increasing $C^1$-function then clearly $f'$ is a continuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f: X \rightarrow K$ be continuous and positive. Let $P: C(X) \rightarrow C(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$
(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (x \in X).
$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_n = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in \mathbb{N}$, let $x_n := \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x)-x| < r_n$, $\sigma_n(x) \in R_n$. See [2] 5.3, 5.4.)

From [2] 5.4, it follows that $P$ is a $C^1$-antiderivative of $f$. It suffices to prove that $P$ is (strictly) increasing. Let $x, y \in X$, $x \neq y$, $|x-y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x-y| < r_s$. We have

$$
x_1 = y_1, \ldots, x_s = y_s, x_{s+1} \neq y_{s+1}. \quad \text{Further} \quad |x_{n+1} - x_n| \leq |x-y| \quad (n \geq s),
$$

$$
|y_{n+1} - y_n| \leq |x-y| \quad (n \geq s), \quad |x_{s+1} - y_{s+1}| \leq |x-y|. \quad \text{Hence (using the identity}
$$

$$
x = \sum_{n=1}^{\infty} (x_{n+1} - x_n) + x_1, \quad y = \sum_{n=1}^{\infty} (y_{n+1} - y_n) + y_1, \quad x_1 = y_1 \quad \Rightarrow \quad |Pf(x) - Pf(y) - (x-y)| =
$$

$$
|f(x_s) - 1|(x_{s+1} - y_{s+1}) + \sum_{n=s+1}^{\infty} f(x_n)(x_{n+1} - x_n) - \sum_{n=s}^{\infty} f(y_n)(y_{n+1} - y_n).
$$

If $|f(x)-1| < \alpha$ for all $x \in X$, we have since $\lim|f(x_n)-1|$ exists,

$$
\sup_{n \geq s} |f(x_n)-1| < \alpha, \quad \text{similarly,} \quad \sup_{n \geq s} |f(y_n)-1| < \alpha.
$$

So we get $|Pf(x) - Pf(y) - (x-y)| < \alpha|x-y|$. Now suppose $|x-y| \leq r_1$. Then since for all $n: |x_{n+1} - x_n| < r_1$, $|x_1 - y_1| = |x-y|$ we get (again under the assumption $|f(x)-1| < \alpha$ for all $x \in X$):

We have proved:

**Theorem 3.13.** Let $X \subset K$ have no isolated points. Then the map $P$ defined via

$$
(Pf)(x) = x \sum_{n=1}^{\infty} f(x_n)(x_n - x) \quad (f \in C(X), \ x \in X)
$$

maps (strictly) positive functions into (strictly) increasing functions.

**Corollary 3.14.** Let $X \subset K$ have no isolated points. Then if $f \in C^1(X)$ and $f'$ is (strictly) positive, then $f = j + h$ where $j$ is (strictly) increasing and $h$ is locally constant.

**Proof.** By 3.12 we have $f = j + h$ where $j$ is (strictly) increasing and $h' = 0$. Now by [2] Cor. 5.2 bis there is a locally constant function $l: X \to K$ with $|l(h-1)|_\infty < \frac{1}{2}$. Then $s: = j + (h-1)$ is (strictly) increasing, so we have $s = s + l$, where $s$ is (strictly) increasing and $l$ is locally constant.

**Note.** We may also define convex functions. Let $X \subset K$. A function $f: X \to K$ is called convex if the second order difference quotient is positive. I.e., if for all $x, y, z \in X$ ($x \neq y, y \neq z, x \neq z$) we have

$$
\frac{f(x) - f(y)}{x-y} - \frac{f(x) - f(z)}{x-z} \quad \in \ K^+
$$

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function $f$ the function $x \mapsto f(x,y)$ defined on $X \setminus \{y\}$ is an isometry, hence can be continuously extended to the whole of $X$. Define $\overline{f}(y,y) = \lim_{x \to y} f(x,y)$ $(y \in X)$. Thus, $f$ is differentiable. For all $x,y,z,t \in X$ we have

$$|\overline{f}(x,y) - \overline{f}(z,t)| \leq \max(\{|f(x,y) - f(z,y)|, |f(z,y) - f(z,t)|\}) \leq \max(|x-z|, |y-t|).$$

Hence, $\overline{f}$ is uniformly continuous on $X$ i.e., $f$ is strongly uniformly differentiable in the sense of [2], page 67.

For each $y \in X$ the function $x \mapsto f(x,y)$ is increasing on $X$.

If $\chi(K) \neq 2$ then convexity of $f$ implies increasingness of $\phi f'.

(Proof. \)

$$\lim_{y \to x} \frac{\phi f(x,y) - \phi f(x',y)}{x-x'} = \frac{f'(x) - \phi f(x',x)}{x-x'} \in K^+(x \neq x')$$

$$\lim_{y \to x'} \frac{\phi f(x,y) - \phi f(x',y)}{x-x'} = \frac{\phi f(x,x') - f'(x')}{x-x'} \in K^+(x \neq x')$$

so

$$\frac{f'(x) - f'(x')} {x-x'} \in 2K^+(x \neq x'), \text{ hence } \phi(\phi f')(x,x') \in K^+ \text{ if } x \neq x'.)$$

Of course, if $f \in C^2(X)$ (see [2] 8.1) then convexity of $f$ implies positivity of $D_2f$ ([2] 8.4). So if $\chi(K) \neq 2$ then $\phi f'' = D_2f$ ([2] 8.14) is positive. If $\chi(K) = 2$ then $f'' = 0$ for all $C^2$-functions.

Note. The functions that are monotone of type $\beta$ ($\beta \in \Sigma$), see Def. 2.15, are easy to describe: $f$ is monotone of type $\beta$ if and only if $b^{-1}f$ is increasing for any $b \in \beta$.

We now turn to the functions $X \rightarrow K$ that are of type $\sigma$ where $\sigma : \Sigma \rightarrow \Sigma$. (2.14). For examples of such $f$, where $\sigma$ is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this section we will assume that $X$ is an open convex subset of $K$. This implies that the set $\{a \in \Sigma : \text{there is } x \in X \text{ such that } x > a\}$ is independent of $a \in X$. Thus, $X$ is homogeneous in the sense that $(a + \alpha) \cap X \neq \emptyset$ for some $a \in X$, $\alpha \in \Sigma$ then for each $b \in X$, $(b + \alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y\}$. Then for each $a \in X$

$$\Sigma(X) = \{a \in \Sigma : x > a \text{ for some } x \in X\}.$$ 

Either $\Sigma(X) = K$ or $\Sigma(X) = \{a \in \Sigma : |a| < r\}$ for some $r > 0$ or $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see 1.2) i.e., if $a, \beta \in \Sigma(X)$ and $a \oplus \beta$ is defined then $a \oplus \beta \in \Sigma(X)$.

To be sure that $f$ is monotone both of type $\sigma$ and type $\tau$ implies $\sigma = \tau$ we define

**DEFINITION 3.15.** (Let $X \subseteq K$ be open, convex and) let $\sigma : \Sigma(X) \rightarrow \Sigma$.

$f : X \rightarrow K$ is called monotone of type $\sigma$ if for all $x, y \in X$ and $\alpha \in \Sigma(X)$

$$x > y + f(x) > f(y).$$

Let $f : X \rightarrow K$ be monotone of type $\sigma : \Sigma(X) \rightarrow \Sigma$. Then

(i) $\sigma(-\alpha) = -\sigma(\alpha)$ $(\alpha \in \Sigma(X))$.

(ii) Let $\alpha, \beta \in \Sigma(X)$. If $\sigma(\alpha) \oplus \sigma(\beta)$ is defined then so is $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$.

(iii) Let $\alpha, \beta \in \Sigma(X)$. If $|\alpha| < |\beta|$ then $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then $\sigma(sa) = s\sigma(a)$ $(\alpha \in \Sigma(X))$.

(v) If $\beta \in \Sigma(X)$, $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $\alpha \in \Sigma(X)$. 

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**THEOREM 3.16.** Let $f : X \rightarrow K$ be monotone of type $\sigma : \Sigma(X) \rightarrow \Sigma$. Then

(i) $\sigma(-\alpha) = -\sigma(\alpha)$ $(\alpha \in \Sigma(X))$.

(ii) Let $\alpha, \beta \in \Sigma(X)$. If $\sigma(\alpha) \oplus \sigma(\beta)$ is defined then so is $\alpha \oplus \beta$ and $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$.

(iii) Let $\alpha, \beta \in \Sigma(X)$. If $|\alpha| < |\beta|$ then $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then $\sigma(sa) = s\sigma(a)$ $(\alpha \in \Sigma(X))$.

(v) If $\beta \in \Sigma(X)$, $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $\alpha \in \Sigma(X)$. 

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(vi) \( f \in M_{us}(X) \) (i.e., for all \( x,y,z,t \in X, |x-y| < |z-t| \) implies \( |f(x)-f(y)| < |f(z)-f(t)| \)).

(vii) \( f \) is either nowhere continuous or uniformly continuous on \( X \).

Proof.

(i) Let \( x,y \in X \) such that \( x > y \). Then \( f(x)-f(y) \in \sigma(a) \); \( f(y)-f(x) \in -\sigma(a) \).

But also \( y > x \), hence \( f(y)-f(x) \in \sigma(-a) \). So \( -\sigma(a) \) and \( \sigma(-a) \) are not disjoint and they must coincide.

(ii) Suppose \( \sigma(a) \otimes \sigma(b) \) is defined. If \( a \otimes b \) were not, then \( b = -a \) so, by (i), \( \sigma(b) = \sigma(-a) = -\sigma(a) \). Hence also \( a \otimes b \) is defined. Choose \( x,y \in X \) with \( x > y \). There is \( z \in X \) such that \( y > z \). Then \( x-y \in \alpha, y-z \in \beta \). Further \( f(x)-f(y) \in \sigma(a) \), \( f(y)-f(z) \in \sigma(b) \) so \( f(x)-f(z) \in \sigma(a) \otimes \sigma(b) \). Also \( x-z \in \alpha \otimes \beta \), so \( f(x)-f(z) \in \sigma(\alpha \otimes \beta) \).

The signs \( \sigma(\alpha) \otimes \sigma(\beta) \) and \( \sigma(\alpha \otimes \beta) \) are not disjoint and they must coincide.

(iii) Let \( |a| < |b| \). Choose \( x,y,z \) such that \( x-y \in \alpha, y-z \in \beta \). Then (see 1.2 and preamble) \( f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(a)+\sigma(\beta) \), \( x-z \in \alpha + \beta = \alpha \otimes \beta = b \), so \( f(x)-f(z) \in \sigma(\beta) \). Thus \([\sigma(a)+\sigma(\beta)] \cap \sigma(\beta) \) is not empty. If \( \sigma(a) \otimes \sigma(\beta) \) were not defined then \( \sigma(a) = -\sigma(\beta) \) and \( \sigma(a) + \sigma(\beta) \) would be a ball with center 0 and radius \( |\sigma(\beta)| \), but then \([\sigma(a)+\sigma(\beta)] \cap \sigma(\beta) \) would be empty. Hence \( \sigma(a) \otimes \sigma(\beta) \) is defined and by (ii) we have \( \sigma(a) \otimes \sigma(\beta) = \sigma(\beta) \). By (1.2) (vi), \( |\sigma(a)| < |\sigma(\beta)| \).

(iv) Let \( \chi(k) \neq 0 \). Then \( s = n \cdot 1 \) for some \( n \in \{1,2,...,\chi(k)-1\} \), so by 1.2 (vii), \( sa = na = \theta a \), \( s\sigma(a) = n\sigma(a) = \theta \sigma(a) \). By a repeated application of (ii), we see \( \sigma(\theta a) = \theta \sigma(a) \). Hence \( \sigma(sa) = sa \).

Let \( \chi(k) = 0 \). Let \( s \) be of the form \( n \cdot 1 \) for some \( n \in \mathbb{N} \). By a similar reasoning as above, \( \sigma(sa) = sa \). We may identify the prime field of \( K \) with \( \mathbb{Q} \).
Now observe that \( \{ s \in K^*: \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) (\( n \in \mathbb{N} \)), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^*: \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1, 1, 2, \ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N} \), \( n = s \mod p \) \( (1 \leq s < p) \) then \( na = sa \) for all \( a \), so \( \sigma(na) = \sigma(sa) = \sigma(a) = n\sigma(a) \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). By (iii) \( |\sigma(\alpha)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( x-t \in \alpha \) for some \( \alpha \) hence \( f(z)-f(t) \in \sigma(\alpha) \). Thus, \( f(z) \neq f(t) \).

(vii) Let \( \rho := \inf |f(x)-f(y)| \). If \( \rho > 0 \) then clearly \( f \) is nowhere continuous. If \( \rho = 0 \), let \( \varepsilon > 0 \). There is a, b \( \in X \), a \( \neq b \) such that \( |f(a)-f(b)| < \varepsilon \). By (vi), for all \( x, y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \varepsilon \). Then \( f \) is uniformly continuous.

A natural question that can be raised is the following. If \( f: X \to K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f: X \to K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
(β) \( f \in M_d(X) \).

(γ) \( f \in M^u(X) \).

(δ) If, for \( \alpha, \beta \in \Sigma(X) \), \( \alpha + \beta \) is defined then so is \( \sigma(\alpha) + \sigma(\beta) \) (and \( \sigma(\alpha) + \sigma(\beta) = \sigma(\alpha + \beta) \)).

(ε) If \( \alpha, \beta \in \Sigma(X) \), \( |\sigma(\alpha)| < |\sigma(\beta)| \) then \( |\alpha| < |\beta| \).

**Proof.** We prove (α) \( \Rightarrow \) (ε) \( \Rightarrow \) (γ) \( \Rightarrow \) (β) \( \Rightarrow \) (δ) \( \Rightarrow \) (α).

(α) \( \Rightarrow \) (ε). Let \( |\sigma(\alpha)| < |\sigma(\beta)| \) then \( \sigma(\alpha) + \sigma(\beta) = \sigma(\beta) \) (1.2.(vi)). By 3.16, (iii), \( \sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta) = \sigma(\beta) \). Since \( \sigma \) is injective, \( \alpha + \beta = \beta \) so (again 1.2.(vi)) \( |\alpha| < |\beta| \).

(ε) \( \Rightarrow \) (γ). Let \( |x-y| \leq |z-t| \) \( (x,y,z,t \in X) \). We prove \( |f(x)-f(y)| < |f(z)-f(t)| \). If \( z = t \) there is nothing to prove. Assume \( z \neq t \) and \( |f(x)-f(y)| > |f(z)-f(t)| \). Then \( f \) is injective, supposing \( x-y \in \alpha \), \( z-t \in \beta \) for some \( \alpha, \beta \in \Sigma(X) \), we have \( f(x)-f(y) \in \sigma(\alpha), f(z)-f(t) \in \sigma(\beta) \) and \( |\sigma(\alpha)| > |\sigma(\beta)| \). By (ε), \( |\alpha| > |\beta| \) i.e., \( |x-y| > |z-t| \). Contradiction.

(γ) \( \Rightarrow \) (β). Trivial.

(β) \( \Rightarrow \) (δ). Suppose \( \sigma(\alpha) + \sigma(\beta) \) is not defined. Then \( |\sigma(\alpha)| = |\sigma(\beta)| \) and, by 3.16 (iii), \( |\alpha| = |\beta| \). Choose \( x, y, z \) such that \( x-y \in \alpha, y-z \in \beta \). Then \( f(x)-f(z) \in \sigma(\alpha) + \sigma(\beta) \) so \( |f(x)-f(z)| < |\sigma(\alpha)| = |f(x)-f(y)| \).

Since \( f \in M_d(X) \), \( |x-z| < |x-y| \) hence, since \( x-z \in \alpha + \beta, x-y \in \alpha: \)
\[ |\alpha + \beta| < |\alpha| \] But \( |\alpha + \beta| = \max(|\alpha|,|\beta|) \), a contradiction.

(δ) \( \Rightarrow \) (α). Suppose \( \sigma(\alpha) = \sigma(\beta) \) and \( \alpha \neq \beta \). Then \( \alpha + (-\beta) \) is defined. By (δ), also \( \sigma(\alpha) + \sigma(-\beta) \) is defined. But \( \sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha) \), so \( \sigma(\alpha) \oplus -\sigma(\alpha) \) is defined, a contradiction.

**Theorem 3.18.** Let \( k \) be a prime field. Then, if \( f : X \to K \) is monotone of type \( \sigma \) then \( \sigma \) is injective.
Proof. Suppose \( \sigma(a) = \sigma(\beta) \) for some \( a, \beta \in \Sigma(X) \). Then \( |\sigma(a)| = |\sigma(\beta)| \) so, by 3.16 (iii), \( |a| = |\beta| \). There is \( t \in K \), \( |t| = 1 \) such that \( \beta = ta \). Since \( k \) is a prime field we may suppose \( t \in \{1,2,\ldots,p-1\} \) if \( k \cong F_p \) and \( t \in Q^* \) if \( k \cong Q \). So, by 3.16 (iv), \( \sigma(\beta) = \sigma(ta) = t\sigma(a) = t\sigma(\beta) \). For \( x \in \sigma(\beta) \) we have \( tx \in \sigma(\beta) \), so \( tx \cdot x^{-1} \in K^* \) i.e., \( |t-1| < 1 \). It follows easily that \( t = 1 \). Hence, \( a = \beta \).

We now like to determine all \( \sigma : X \rightarrow \Sigma \) that "can occur" as the type of a monotone function in case \( K = Q_p^* \). We use the fact that \( \Sigma \) can be identified with the following subgroup of \( Q_p^* \)

\[ \{ \theta^{i,n} : i \in \{0,1,2,\ldots,p-2\}, n \in Z \} \]

where \( \theta \) is a primitive \((p-1)^{th}\) root of 1. (See 1.5.)

First, let \( f : Q_p^* \rightarrow Q_p^* \) be monotone of some type \( \sigma : X \rightarrow \Sigma \). By 3.18, \( \sigma \) is injective. By 3.17, (c), 3.16 (iii) we have \( |a| < |\beta| \iff |\sigma(a)| < |\sigma(\beta)| \) and \( |a| = |\beta| \iff |\sigma(a)| = |\sigma(\beta)| \), so \( |\sigma(a)| \) is a strictly increasing function of \(|a|\).

Set

\[ \sigma(\theta^{i,n}) = \theta^{s(i,n)} \lambda(i,n) \quad (\theta^{i,n} \in \Sigma) \]

Where \( s : \{0,1,2,\ldots,p-2\} \times Z \rightarrow \{0,1,2,\ldots,p-2\} \) and \( \lambda : \{0,1,2,\ldots,p-2\} \times Z \rightarrow Z \). We see that \( |\sigma(\theta^{i,n})| = |\sigma(\theta^{j,n})| \) for all \( i, j \in \{0,1,2,\ldots,p-2\} \) hence \( \lambda(i,n) = \lambda(j,n) \) for all \( i, j \in \{0,1,2,\ldots,p-2\} \). Then

\[ \sigma(\theta^{i,n}) = \theta^{s(i,n)} \lambda(n) \]

where \( \lambda : Z \rightarrow Z \) is a strictly increasing function (in the classical sense).

By 3.16 (v), \( \sigma(\theta^{i,n}) = \theta^{i} \sigma(p^{n}) = \theta^{i} \theta^{0,n} \lambda(n) \).
Thus, \( \sigma \) is of the form

\[(*) \quad \theta^i p^n \to \theta^i s(n) p^\lambda(n)\]

where \( s : \mathbb{N} \to \{0,1,2,\ldots,p-2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

Conversely, if we are given a map \( \sigma \) of the form \((*)\) then it is easy to construct an \( f : \mathbb{Q}_p \to \mathbb{Q}_p \), monotone of type \( \sigma \). In fact, let \( x \in \mathbb{Q}_p \), \( x = \sum a_n p^n \), where \( a_n \in \{0,1,\ldots,p-2\} \) for each \( n \) and \( a_{-n} = 0 \) for large \( n \). Then set

\[ f(x) = \sum_{n \in \mathbb{Z}} a_n \theta^s(n) p^\lambda(n). \]

Now let \( x = \sum a_n p^n \), \( y = \sum b_n p^n \) and \( \pi(x-y) = \theta^i p^m \) for some \( i \in \{0,1,\ldots,p-2\} \), \( m \in \mathbb{Z} \). Then \( a_n = b_n \) for \( n < m \) and \( a_m = b_m = \theta^i \mod p \). So the sign of \( a_m - b_m \) is \( \theta^i \). \( f(x)-f(y) = \sum (a_n - b_n) \theta^s(n) p^\lambda(n) = (a_m - b_m) \theta^s(m) p^\lambda(m) + r \), where

\[ |r| < |f(x)-f(y)|. \]

The sign of \( f(x)-f(y) \) is the sign of \( (a_m - b_m) \theta^s(m) p^\lambda(m) \) which is \( \theta^i \theta^s(m) p^\lambda(m) \). So \( \pi(f(x)-f(y)) = \theta^i \theta^s(m) p^\lambda(m) = \sigma(\theta^i p^m) \). Thus, \( f \) is monotone of type \( \sigma \). We have found

**Theorem 3.19.** The set \( \{ \sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, \text{ monotone of type } \sigma \} \) is equal to the set of all \( \sigma : \Sigma \to \Sigma \) of the form

\[ \theta^i p^n \to \theta^i s(n) p^\lambda(n) \]

where \( s : \mathbb{Z} \to \{0,1,2,\ldots,p-2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

**Remark.** With the notations as in 3.19, let \( \mu(n) := \lambda(n)-n \). Then \( \mu : \mathbb{Z} \to \mathbb{Z} \) is increasing \( (\mu(n+1) = \lambda(n+1)-(n+1) \geq \lambda(n)+1-(n+1) = \mu(n)) \).

We then have two possibilities for a function \( f : \mathbb{Q}_p \to \mathbb{Q}_p \), monotone of type \( \sigma \).
(a) \( \lim_{{n \to \infty}} \mu(n) = \infty \). Then \( \left| \sigma(a) \right| = |a| |p^{{\mu(n)}}| \), \( (a = \theta_1 p^n) \), so \( \lim_{{|a| \to 0}} \left| \sigma(a) \right| = 0 \).

Thus \( \lim_{{x-y \to 0}} \left| \frac{f(x)-f(y)}{x-y} \right| = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0 \).

(b) \( \mu \) is bounded above. Then \( \mu(n) \) is constant, \( c \), for \( n \geq n_0 \). (For example, if \( \sigma \) is bijective then we have even \( \mu(n) = c \) for all \( n \)).

Thus, for sufficiently small \( |a| \) (\( a = \theta_1 p^n \in \Sigma \)) we have

\[
\left| \sigma(a) \right| = \left| p^{\lambda(n)} \right| = \left| p^{n+c} \right| = \left| p^c \right| |a|.
\]

So, there is \( r \) such that \( |x-y| < r \) implies \( |f(x)-f(y)| = |p^c||x-y| \).

In this case we then have: there is \( r > 0 \) and \( \lambda \in \mathcal{Q}_p \) such that on each ball in \( \mathcal{Q}_p \) of radius \( r \), \( \lambda^{-1}f \) is an isometry.

We now construct an example of a function \( f \) monotone of type \( \sigma \), where \( \sigma \) is not injective. Let \( p = 3 \) mod 4 and let \( K := \mathcal{Q}_p(\sqrt{-1}) \). The elements of \( K \) can be written as \( a+bi \) (\( a, b \in \mathcal{Q}_p \)) and \( |a+bi| = \max(|a|, |b|) \).

The value group of \( K \) is the same as the one of \( \mathcal{Q}_p \), the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.18). Let \( X \) be the unit ball of \( K \), let

\[
S := \{a+bi: a, b \in \{0, 1, 2, \ldots, p-1\}\}.
\]

For each \( x \in X \) there is a unique \( \overline{x} \in S \) such that \( |x-\overline{x}| < 1 \). As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s, t \in S^*, s \neq t \).

Define a function \( h : S \to K \) as follows

\[
h(a+bi) = \frac{1}{p} a \quad (a+bi \in S)
\]
and let \( f : X \rightarrow K \) be defined via

\[
f(x) = x + h(x) \quad (x \in X).
\]

We claim that \( f \) is monotone of type \( \sigma \) where

\[
\sigma(\pi(a+bi)) = \pi\left(\frac{1}{p}a\right) \text{ if } a+bi \in S, \ a \neq 0
\]

\[
\sigma(\alpha) = \alpha \quad \text{elsewhere.}
\]

(Clearly, \( \sigma \) is a well defined map \( \Xi(X) \rightarrow K \), \( \sigma \) is not injective since, for example, \( \sigma(\pi(1)) = \sigma(\pi(1+i)) \).

**Proof.** Let \( |\alpha| < 1 \) and \( x-y \in \alpha \), then \( |x-y| < 1 \) so \( \overline{x} = \overline{y} \), \( h(x) = h(y) \).

It follows that \( f(x) - f(y) = x-y \in \alpha = \sigma(\alpha) \).

Now let \( |\alpha| = 1 \) be of the form \( \pi(bi), \ b \in \{1,2,\ldots,p-1\} \) and let \( x-y \in \alpha \). Say, \( \overline{x} = r+si \), \( \overline{y} = t+ui \) \((r,s,t,u \in \{0,1,2,\ldots,p-1\})\). Then also \( \overline{x-y} \in \alpha \), so \( |r+si-t+ui| < 1 \) hence \( r = t \). Thus, \( h(x) = \frac{1}{p}r = h(y) \), and we have \( f(x) - f(y) = x-y \in \alpha = \sigma(\alpha) \).

Finally, let \( |\alpha| = 1 \), \( \alpha = \pi(a+bi) \), where \( a \neq 0 \) \((a,b \in \{0,1,2,\ldots,p-1\})\) and let \( x-y \in \alpha \). Set \( \overline{x} = r+si \), \( \overline{y} = t+ui \). Then \( \overline{x-y} \in \alpha \), so \( r-t = \alpha \mod p \).

We find \( h(x) = \frac{1}{p}r \), \( h(y) = \frac{1}{p}t \), so \( |h(x)-h(y)| = \frac{1}{p}|a| < \frac{1}{|p|}|a| \) i.e. \( h(x)-h(y) \in (\frac{1}{p}a) \). Since \( |\pi(x-y)| < 1 \), we find \( f(x)-f(y) = x-y-(h(x)-h(y)) \in (\frac{1}{p}a) = \sigma(\pi(a+bi)) = \sigma(\alpha) \).

Concluding:

**Example 3.20.** Let \( p = 3 \mod 4 \) and \( K = \mathbb{Q}_p(i^{-1}) \). Then there exists a function \( f : \{x \in K : |x| \leq 1\} \rightarrow K \), monotone of some type \( \sigma \), where \( \sigma \) is not injective.

In case \( K \) has discrete valuation we have some extra information.
THEOREM 3.21. Let $K$ have discrete valuation and let $f : X \to K$ be monotone of type $\sigma \in \Sigma(X)$. Then

(i) $f$ is continuous.

(ii) If $\sigma$ is injective, then $f$ and $\phi(f)$ are bounded on bounded sets.

(iii) If $\sigma$ is injective and if $f(X)$ is convex then there is $\lambda \in K$ such that $\lambda f$ is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2)(c). If $\sigma$ is injective then by 3.16 (iii) and 3.17 (c), $|\sigma(a)|$ is a strictly increasing function of $|a|$. Suppose $X$ is bounded. Then $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r \in K$. Let $|\sigma(a)| = s$ whenever $|a| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|a| = |\pi|r$. Then $|a| < r$ so $|\sigma(a)| < s$, hence $|\sigma(a)| \leq |\pi|r$. By induction, it follows that $|\sigma(a)| \leq |\pi|^n$ whenever $|a| = |\pi|^n r$, so $|\sigma(a)| \leq |a| \cdot \frac{s}{r^n}$. So the difference quotients of $f$ are bounded by $sr^{-1}$. Then clearly $f$ is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $E(f(X))$ has the form $\{a \in \Sigma : |a| \leq s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then $\sigma$ induces an injection of $\{p \in |K^*| : p \leq r\}$ onto $\{a \in |K^*| : |a| \leq s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(a)| = c|a|$ for some $c \in \mathbb{R}$ i.e., $|f(x) - f(y)| = |c||x - y|$ for all $x, y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type $\sigma$, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) $K$ is a local field ($f$ is continuous).

(2) $K$ is finite and $X = \{x \in K : |x| \leq 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be
representatives modulo \{x \in K : |x| < 1\}, let \( M = \max |f(a_i) - f(a_j)| \). For each \( x, y \in X \) we have \( i, j \) for which \( |x - a_i| < 1, |y - a_j| < 1 \). Since \( f \in M_s(X) \), we have \( |f(x) - f(a_i)| < M, |f(y) - f(a_j)| < M \) whence \( |f(x) - f(y)| \leq M : f \) is bounded.\)

(3) \( K \) is discrete, \( \sigma \) is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let \( k \) be isomorphic to the algebraic closure of \( \mathbb{F}_p \). Let

\[ X \] be the unit ball of \( K \). Then there exists a function

\[ f : X \to K, \] monotone of type \( \sigma \), for some \( \sigma : \Sigma(X) \to \Sigma \)

such that

(i) \( \sigma \) is not injective.

(ii) \( f, \Phi(f) \) are unbounded.

Proof.

As an \( \mathbb{F}_p \)-vector space, \( k \) has a countable base \( e_1, e_2, \ldots \). For any \( \lambda \in \mathbb{F}_p \),

\[ \lambda = n_1 \] for some \( n \in \{0, 1, 2, \ldots, p-1\} \). (Here for a field \( L \), \( 1_L \) is the unit element of \( L \).) Define \( \lambda := n_1 \). Choose \( c_1, c_2, \ldots \in K \) such that

\[ 1 < |c_1| < |c_2| < \ldots, \lim_{n \to \infty} |c_n| = \infty, \] and define a map \( h : k \to K \) via

\[ h(\sum \lambda_n e_n) = \sum \lambda_n c_n \] (\( \Sigma \lambda_n e_n \in k \))

Define \( f : X \to K \) by

\[ f(x) = x + h(x) \] (\( x \in X \))

(Here \( x \) is the image of \( x \) under the canonical map \( X \to k \)).

Then clearly \( f \) is unbounded and so is \( \Phi(f) \).

Let us identify \( \{ \alpha \in \Sigma : |\alpha| = 1 \} \) with \( k^* \) in the obvious way. We claim that \( f \) is monotone of type \( \sigma \) where
\( \sigma(a) = \begin{cases} 
\alpha \text{ if } |\alpha| < 1 
\pi(\beta_n \gamma_n) \text{ if } \alpha = \sum \lambda_n \epsilon_n, \ n = \max(m : \lambda_m \neq 0). 
\end{cases} \)

In fact, let \( x-y \in \alpha \) and \( |\alpha| < 1 \). Then \( h(x) = h(y) \) so \( f(x)-f(y) = x-y \in \sigma(\alpha) \). Now let \( x-y \in \alpha \) where \( |\alpha| = 1 \). Then set \( \bar{x} = \sum \lambda_n \epsilon_n, \ \bar{y} = \sum \mu_n \epsilon_n \).

Let \( r = \max\{n : \lambda_n \neq \mu_n\} \). Then \( \bar{x-y} = \sum \lambda_n \epsilon_n \epsilon_n = \alpha \), so \( \sigma(\alpha) = \pi(\bar{\lambda_n \epsilon_n}) \).

On the other hand, \( f(x)-f(y) = x-y-(h(x)-h(y)) = x-y-\sum (\lambda_n \epsilon_n) \epsilon_n = x-y-\sum (\lambda_n \epsilon_n) \epsilon_n = \pi(\bar{\lambda_n \epsilon_n}) \).

Now we have \( \lambda_n \epsilon_n \sim \lambda_n \epsilon_n \mod p \), so \( \pi(\lambda_n \epsilon_n) = \pi(\lambda_n \epsilon_n) \). It follows that \( f(x)-f(y) \in \sigma(\alpha) \).

Obviously, \( \sigma \) is not injective.

We will consider briefly the differentiable functions, monotone of type \( \sigma \). Let \( f : X \to \mathbb{K} \) be such a function. If \( f'(a) = 0 \) for some \( a \in X \), then \( \lim_{|a| \to 0} \frac{|\sigma(a)|}{|\alpha|} = 0 \), so \( f' = 0 \) uniformly on \( X \). Now let \( f'(a) \neq 0 \). Then if \( x \) is sufficiently close to \( a \), we have \( \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < |f'(a)| \).

Thus for \( |\alpha| \) small enough we have \( f'(a) \in \frac{\sigma(a)}{\alpha} \) i.e. \( \frac{\sigma(a)}{\alpha} \) is constant.

This implies that \( \pi(f'(x)) \) does not depend on \( x \) (\( f' \) has constant sign) and that locally \( f \) is monotone of type \( \beta \) for some \( \beta \in \Sigma \).

We end this section with a discussion on the question which maps \( \sigma : \Sigma \to \Sigma \) do occur as a type of a monotone function.

**Lemma 3.23.** Let \( \sigma : \Sigma \to \Sigma \). Suppose \( \sigma \) satisfies

\[ \text{if } \alpha \in \beta \text{ is defined then } \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta). \ (\alpha, \beta \in \Sigma). \]

Then

\[ (1) \ \sigma(-\alpha) = -\sigma(\alpha) \quad (\alpha \in \Sigma). \]
(ii) If $\sigma(a)$ is defined then so is $a \oplus b$.

(iii) If $|a| < |b|$ then $|\sigma(a)| < |\sigma(b)|$.

If $a$ is injective.

(v) If $|a| = |b|$ then $|\sigma(a)| = |\sigma(b)|$.

Proof. (i) is trivial if $x(k) = 2$, so suppose $x(k) \neq 2$ and let $-\sigma(a) \neq \sigma(-a)$ for some $a \in \Sigma$. Then we have the identity $(a \oplus a) \oplus (-a) = a$, so

$$\sigma(a \oplus a) \oplus \sigma(-a) = \sigma(a),$$

whence $(\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a) = \sigma(a)$. Now by 1.2 (iii) $\sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a))$ (this last expression is defined).

If not, then $-\sigma(a) = \sigma(a) \oplus \sigma(-a)$. Now $\sigma(a) \oplus \gamma = -\sigma(a)$ has only one solution namely $\gamma = -2\sigma(a)$. So we then would have $\sigma(-a) = -2\sigma(a) = -(\sigma(a) \oplus \sigma(a))$, but this contradicts the existence of $(\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a))$.

From $\sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a))$ we obtain by 1.2 (vi): $|\sigma(a) \oplus \sigma(-a)| < |\sigma(a)|$. On the other hand, by 1.2 (v), $|\sigma(a) \oplus \sigma(-a)| = |\sigma(a)| + |\sigma(-a)|$.

Contradiction. (i) follows.

Now (ii) follows easily from (i): if $a \oplus b$ were not defined then $b = -a$ so, by (i), $\sigma(a) \oplus \sigma(-a) = \sigma(a) \oplus -\sigma(a)$, a contradiction. Let $|a| < |b|$, then $a \oplus b = \beta$, so $\sigma(a \oplus b) = \sigma(a) \oplus \sigma(b) = \sigma(b)$. By 1.2 (vi) we find $|\sigma(a)| < |\sigma(b)|$. We proved (iii).

If $\sigma(a) = \sigma(b)$ and $a \neq b$ then $\sigma(a \oplus (-b)) = \sigma(a) \oplus \sigma(-b) = \sigma(a) \oplus -\sigma(a)$, an absurdity. So $\sigma$ is injective (iv). Finally, let $|a| = |b|$ and $|\sigma(a)| > |\sigma(b)|$. Then $\sigma(a) = \sigma(a) \oplus \sigma(b) = (by (ii)) = \sigma(a \oplus b)$. By injectivity of $\sigma$, $a = a \oplus b$, and by 1.2 (vi), we find $|b| < |a|$.

Now we have
LEMMA 3.24. Let $K$ be spherically complete, let $Y \subset K$ (not necessarily convex) and let $\tau : \Sigma(Y) (= \{ \pi(x-y) : x,y \in Y, x \neq y \}) \to \Sigma$ such that $f$ is monotone of type $\tau$ (i.e., $x,y \in Y, x-y \in \alpha \in \Sigma(Y)$ then $f(x) - f(y) \in \tau(\alpha)$.

Suppose $\tau$ can be extended to a $\sigma : \Sigma \to \Sigma$ satisfying the condition of Lemma 3.23. Then $f$ can be extended to a monotone function $\bar{f} : K \to K$ of type $\sigma$.

Proof. By Zorn's lemma, it suffices to extend $f$ to $Y \cup \{ a \} (a \notin Y)$ such that $f(x) - f(a) \in \sigma(\pi(x-a))$, $f(a) - f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x)) (x \in Y)$. Each $B_x$ is a ball with radius $|\sigma(\pi(a-x))|$. By the spherical completeness of $K$, we are done if we can show that $B_x \cap B_y \neq \emptyset (x \neq y, x,y \in Y)$.

Set $\alpha := \pi(a-x)$ and $\beta := \pi(a-y)$. Let $b \in \sigma(\alpha)$; $c \in \sigma(\beta)$. We prove:

$$|f(x) + b - f(y) - c| < |\sigma(\alpha)| \lor |\sigma(\beta)|.$$ We have two cases:

1) $\alpha = \beta$. Then $a-x \in \alpha$, $a-y \in \alpha$ implies $|x-y| < |a-x| = |\alpha|$, so $|\pi(x-y)| < |\alpha|$ whence $|\pi(f(x) - f(y))| = |\sigma(\pi(x-y))| < |\sigma(\alpha)|$ (by 3.23 (iii)), so $|f(x) - f(y)| < |\sigma(\alpha)|$. Further, $b \in \sigma(\alpha)$, $c \in \sigma(\alpha)$ implies $|b-c| < |\sigma(\alpha)|$, hence $|f(x) + b - f(y) - c| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x-y = a-y - (a-x) \in \beta \oplus (-\alpha)$, so $f(x) - f(y) + b - c \in \sigma(\beta \oplus -\alpha) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus -\alpha) + \sigma(\alpha \oplus -\beta) = \sigma(\beta \oplus (-\alpha)) - \sigma(\beta \oplus -\alpha)$, hence $|f(x) - f(y) + b - c| < |\sigma(\beta \oplus -\alpha)| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(\alpha)|, |\sigma(\beta)|)$.

THEOREM 3.25. Let $K$ be spherically complete and let $\sigma : \Sigma \to \Sigma$. Suppose

$$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).$$

Then there exists a function $f : K \to K$, monotone of type $\sigma$. 
Proof. Choose $Y := \{0\}$ and let $g : Y \to K$ be defined via $g(0) = 0$. Then $g$ satisfies the conditions of Lemma 3.24 so it can be extended to a function $f$ of type $\sigma$.

We now give a description of the maps $\sigma : \Sigma \to \Sigma$ mentioned in 3.23. For each $r \in |K^*|$ choose $a_r \in \Sigma$ such that $|a_r| = r$. Further, there is a natural isomorphism of multiplicative groups between $k^*$ and $\{a \in \Sigma : |a| = 1\}$, denoted by $l \mapsto a_1 (l \in k^*)$. Of course, if $l + l' \neq 0$ then $a_{l+l'} = a_l \otimes a_{l'}$. Each element of $\Sigma$ can be written in only one way as $a_{l_1} (r \in |K^*|, l \in k^*)$. Now if $\sigma$ is as in 3.23 we get

$$\sigma(a_{l_1}) = a_{\lambda(r)} n(r, l)$$

where $\lambda : |K^*| \to |K^*|$ is strictly increasing and $l \mapsto n(r, l)$ is an injective group endomorphism of the additive group $k$. Conversely, if $\lambda : |K^*| \to |K^*|$ is strictly increasing and for each $r, l \mapsto n(r, l)$ is an injective group homomorphism $k + k$ then

$$a_{l_1} \mapsto a_{\lambda(r)} n(r, l) (a_{l_1} \in \Sigma)$$

satisfies the condition of 3.23. So we get

**THEOREM 3.26.** Let $K$ be spherically complete and let $|K| = [0, \infty)$. Then there exist a nowhere continuous $f : K \to K$, monotone of some type $\sigma : \Sigma \to \Sigma$.

Proof. With the notations as above, let $\sigma : \Sigma \to \Sigma$ be defined as follows

$$\sigma(a_{l_1}) = a_{l+1}$$

By 3.25 there is an $f : K \to K$ monotone of type $\sigma$. Clearly $|f(x) - f(y)| \geq 1$ if $x \neq y$ so $f$ is nowhere continuous.
4. MONOTONE FUNCTIONS, GENERAL THEOREMS

In this section we study $M_w(X), M_b(X), M_s(X), \ldots$. To avoid unnecessary complications we assume throughout this section that $X$ is a closed subset of $K$ without isolated points. We collect here the results on monotone functions that are valid for general $K$. In the next section we will see what happens if we put some extra conditions on $K$ (e.g., $|K|$ discrete, ...).

First two elementary lemmas.

**LEMMA 4.1** Let $f : X \to K$. Then the following conditions are equivalent

(a) $f \in M_w(X)$ (see Def. 2.11).

(b) For all $x, y, z \in X$, $|x-y| < |x-z|$ implies $|f(x) - f(z)| = |f(y) - f(z)|$.

(c) For all $x, y, z \in X$, $|f(x) - f(z)| \neq |f(y) - f(z)|$ implies $|x-y| = \max(|x-z|, |y-z|)$.

**Proof.** (a) $\Rightarrow$ (b). $|x-y| < |x-z|$ implies $|y-z| = |x-z| > |x-y|$, so

$$|f(x) - f(y)| \leq \min(|f(x) - f(z)|, |f(y) - f(z)|).$$

It follows that $|f(x) - f(z)| = |f(y) - f(z)|$.

(b) $\Rightarrow$ (c). (b) says that $|f(x) - f(z)| \neq |f(y) - f(z)|$ implies $|x-y| \geq |x-z|$. By symmetry, also $|x-y| \geq |y-z|$ where $|x-y| \geq \max(|x-z|, |y-z|)$. The opposite inequality is trivial.

(c) $\Rightarrow$ (a). Let $|x-y| < |x-z|$. Then $|x-y| \neq \max(|x-z|, |z-y|)$ so, by (c), $|f(x) - f(z)| = |f(y) - f(z)|$. Then $|f(x) - f(y)| \leq \max(|f(x) - f(z)|, |f(y) - f(z)|) = |f(x) - f(z)|$.

**LEMMA 4.2** (i) If $f \in M_w(X), \lambda \in K$ then $\lambda f \in M_w(X)$. 

...
(ii) If \( f_1, f_2, \ldots \in \mathcal{M}_w(X) \) and \( f := \lim_{n \to \infty} f_n \) pointwise then 
\( f \in \mathcal{M}_w(X) \).

(iii) If \( f \in \mathcal{M}_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \), then \( g \in \mathcal{M}_w(f(X)) \). In particular, if \( f \) is injective and weakly monotone then so is \( f^{-1} \).

(Notice that \( f(X) \) need not be closed and may have isolated points.)

*Proof.* Obvious.

*Remark.* Lemmas, similar to 4.1 and 4.3, but now for \( \mathcal{M}_b(X), \mathcal{M}_s(X), \mathcal{M}_w(X) \) have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an \( \mathcal{M}_w \)-function need not be continuous (see 2.4(5), 3.26) we will derive properties of \( \mathcal{M}_w \)-functions that are closely related to continuity.

**Lemma 4.3** Let \( f \in \mathcal{M}_w(X) \). Then \( f \) is bounded on precompact subsets of \( X \).

*Proof.* Let \( Y \subset X \) be precompact. Assume that \( Y \) is not a singleton. Then \( Y \) is bounded and has a positive diameter \( r = \max \{|x-y| : x, y \in Y\} \).

The equivalence relation \( x \sim y \iff |x-y| < r \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) \((n \geq 2)\). Choose \( a_i \in Y_i \) for each \( i \), and let \( M := \max_{1 \leq i \leq n} |f(a_i)| \). We prove: \( |f| \leq M \). In fact, let \( x \in Y \). Then there is \( i \) such that \( |x-a_i| < r \). Choose \( j \neq i \). We have \( |x-a_i| < |a_i-a_j| \) whence \( |f(x)-f(a_i)| \leq |f(x)-f(a_j)| \leq M \). So \( |f(x)| \leq M \).

The following lemma shows that an \( f \in \mathcal{M}_w(X) \) at \( a \in X \) is either continuous or "very discontinuous".

**Lemma 4.4** Let \( f \in \mathcal{M}_w(X) \) and let \( a \in X \). Then we have the following alternative.
Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \in X \) \((x_n \neq a \text{ for all } n)\) with \( \lim x_n = a \) the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Proof. Since \( \{x_1, x_2, \ldots \} \) is precompact the set \( \{f(x_1), f(x_2), \ldots \} \) is bounded by Lemma 4.3. We are done if we can prove the following. If \( x_1, x_2, \ldots, \lim x_n = a, x_n \neq a \text{ for all } n, \lim f(x_n) \) exists, then \( f \) is continuous at \( a \). Now set \( a := \lim f(x_n) \). Let \( y_1, y_2, \ldots \in X, \lim y_n = a \). We prove \( \lim f(y_n) = a \). (Then it follows that \( a = f(a) \) since we may choose \( y_n := a \) for all \( n \).) Let \( \varepsilon > 0 \). There is \( k \in \mathbb{N} \) for which
\[
|f(x_k) - a| < \varepsilon.
\]
For \( n \) sufficiently large we have \( |y_n - a| < |x_k - a| \), so for large \( m \) (depending on \( n \)) we have \( |y_n - x_m| < |x_k - x_m| \), whence
\[
|f(y_n) - f(x_m)| \leq |f(x_k) - f(x_m)|.
\]
Since \( \lim_{m \to \infty} f(x_m) = a \) we find
\[
|f(y_n) - a| \leq |f(x_k) - a| < \varepsilon,
\]
so \( \lim_{n \to \infty} f(y_n) = a \).

COROLLARY 4.5 Let \( f \in M_w(X) \). Then the graph of \( f \)
\[
\Gamma_f := \{(x,y) \in X \times K : y = f(x)\}
\]
is closed in \( K^2 \).

Proof. Let \( (x_n, f(x_n)) \in \Gamma_f \) and let \( \lim x_n = x, \lim f(x_n) = a \). If \( x_n = x \) for infinitely many \( n \) then \( a = f(x) \), so \( (x, a) \in \Gamma_f \). If not then by the alternative of lemma 4.4, \( f \) is continuous at \( x \), so \( a = f(x) \) and \( (x, a) \in \Gamma_f \).

COROLLARY 4.6 Let \( f \in M_w(X) \). If each bounded subset of \( f(X) \) is precompact then \( f \) is continuous.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let \( f \in \mathcal{M}(X) \) and let \( Y \subseteq f(X) \) be precompact. Then either
\[ f \text{ is constant on } f^{-1}(Y) \text{ or } f^{-1}(Y) \text{ is bounded.} \]

Proof. It suffices to prove: if \( Z \subseteq X \) is unbounded and \( f(Z) \) is precompact then \( f \) is constant on \( Z \). Let \( a, b \in Z \). Since \( Z \) is unbounded there are \( x_1, x_2, \ldots \in Z \) such that
\[ (*) \quad |a - b| < |x_1 - a| < |x_2 - a| < \ldots \]
Since \( f(Z) \) is precompact we may assume (by taking a suitable subsequence) that \( a = \lim f(x_n) \) exists. From (*) we obtain
\[ |x_1 - x_2| = |x_2 - a|, \; |x_2 - x_3| = |x_3 - a|, \ldots, \]
so\[ |a - b| < |x_1 - a| < |x_1 - x_2| < |x_2 - x_3| < \ldots \]
hence\[ |f(a) - f(b)| \leq |f(x_1) - f(a)| \leq \lim_{n \to \infty} |f(x_n) - f(x_n)| = 0 \text{ i.e., } f(a) = f(b). \]

LEMMA 4.8 Let \( f \in \mathcal{M}(X) \) and let \( a \in f(X) \) be a non-isolated point of \( f(X) \).

Then we have the following alternative. Either

I. There is \( a \in X \) such that for each sequence \( x_1, x_2, \ldots \) in \( X \) for which \( \lim_{n \to \infty} f(x_n) = a \) we have \( \lim_{n \to \infty} x_n = a \), or

II. If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} f(x_n) = a, \) \( f(x_n) \neq a \) for all \( n \),

then \( x_1, x_2, \ldots \) is bounded and has no convergent subsequence.

Proof. Suppose we are not in case II. Since \( a \) is not isolated in \( f(X) \) and \( f(X) \) is dense in \( f(X) \) we have a sequence \( x_1, x_2, \ldots \) in \( X \) for which \( f(x_n) \neq a \) for each \( n \), and \( \lim_{n \to \infty} f(x_n) = a \). Since \( f \) is not constant on \( \{x_1, x_2, \ldots\} \) it follows by Lemma 4.7 that \( \{x_1, x_2, \ldots\} \) is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by \( x_1, x_2, \ldots \) and set
a := \lim_{n \to \infty} x_n. Then a \in X. Now let y_1, y_2, ... be a sequence in X for which \lim_{n \to \infty} f(y_n) = a. We prove that \lim_{n \to \infty} y_n = a. In fact, let \varepsilon > 0.

There is \sigma \in \mathbb{N} such that |x_{k}-a| < \varepsilon. For large \sigma we have

\begin{align*}
|f(y_\sigma) - a| < |f(x_\sigma) - a|, \quad \text{so for large} \ m \ (\text{depending on} \ \sigma) \ \text{we have}
\end{align*}

\begin{align*}
|f(y_\sigma) - f(x_\sigma)| < |f(x_\sigma) - f(x_m)|, \quad \text{whence} \ |y_\sigma - x_\sigma| \leq |x_\sigma - x_m|, \ \text{so}
\end{align*}

\begin{align*}
|y_\sigma - a| \leq |x_\sigma - a| < \varepsilon.
\end{align*}

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function \( f : X \to K \) injective at \( a \in X \) if \( f(x) = f(a) \) for some \( x \in X \) implies \( x = a \).

Now suppose that we have \( a \in f(X) \), not isolated, for which we are in alternative I. Then for a sequence \( x_1, x_2, ... \) with \( \lim_{n \to \infty} f(x_n) = a \) we have \( \lim_{n \to \infty} x_n = a \in X \) so \( (a, a) = \lim_{n \to \infty} (x_n, f(x_n)) \), so by Cor. 4.5 we have \( a = f(a) \). Thus, \( a \in f(X) \). \( f \) is injective at \( a \): if \( f(b) = f(a) \) then since \( \lim_{n \to \infty} f(b) = a \) we must have \( \lim_{n \to \infty} b = a \), i.e. \( b = a \). Further, \( f \) is continuous at \( a \) (see 2.13 (2) (a)).

If each bounded subset of \( X \) is precompact we never can be in case II. This is also true if \( f \in M^b_b(X) \) and \( |X| \) is discrete i.e. if \( x_1, y_1 \in X \) \( |x_1 - y_1| > |x_2 - y_2| > ... \) then \( \lim_{n \to \infty} |x_n - y_n| = 0 \). Proof: let \( a \in f(X) \) and let \( \lim_{n \to \infty} f(x_n) = a \), \( f(x_n) \neq a \) for all \( n \). Without loss of generality we may assume

\begin{align*}
|a - f(x_1)| > |a - f(x_2)| > ... \\
\text{hence} \\
|f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > ...
\end{align*}

and, since \( f \in M^b_b(X) \)

\begin{align*}
|x_1 - x_2| > |x_2 - x_3| > ...
\end{align*}

Since \( |X| \) is discrete, the sequence \( x_1, x_2, ... \) is convergent. So we have case I. We find
THEOREM 4.9 Let either each closed and bounded subset of $X$ be compact and $f \in M^w(X)$, or let $|X|$ be discrete and $f \in M^d(X)$. Then

(i) $f(X)$ is closed (in $K$).

(ii) If $f(a) \in f(X)$ is not isolated then $f$ is injective at $a$, $f$ is continuous at $a$. In particular if $f(X)$ has no isolated points then $f$ is a homeomorphism $X \rightarrow f(X)$.

Proof. If $a \in f(X) \setminus f(X)$ then $a$ is not isolated. Since we are in alternative I of Lemma 4.8, $a \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an $f$ satisfying the conditions of 4.9 maps closed subsets of $X$ (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let $f : X \rightarrow K$.

(i) If $f \in M^w(X)$ and if $Y \subseteq X$ is closed and the closed and bounded subsets of $Y$ are compact then $f(Y)$ is spherically complete.

(ii) If $f \in M^d(X)$ and if $Y \subseteq X$ is spherically complete then so is $f(Y)$.

(iii) If $f \in M^g(X)$ and if $A \subseteq f(X)$ is spherically complete then so is $f^{-1}(A)$.

Proof. (i) Let $B_1 \supsetneq B_2 \supsetneq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1 \setminus B_2$, $f(y_2) \in B_2 \setminus B_3$, $\ldots$ Then we have

$$|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots$$

and, by the weak monotony of $f$
\[ |y_1 - y_2| \geq |y_2 - y_3| \geq \ldots \]

Suppose first that \( \lim |y_n - y_{n+1}| = 0 \). Then \( y := \lim y_n \) and there are infinitely many \( n \) for which

\[ |y_k - y_{k-1}| > |y_{k+1} - y_k|. \]

Now \( |y - y_k| \leq \max(|y_k - y_{k+1}|, |y_{k+1} - y_{k+2}|, \ldots) \leq |y_k - y_{k+1}| \). So we get

for infinitely many \( n \)

\[ |y - y_k| < |y - y_{k-1}| \]

whence

\[ |f(y) - f(y_k)| \leq |f(y_k) - f(y_{k-1})| \]

so \( f(y) \in B_{k-1} \) for infinitely many \( k \), i.e., \( f(y) \in \bigcap B_k \).

Next, suppose that \( |y_{k+1} - y_k| \geq \epsilon > 0 \) for all \( k \). Then since \( y_1, y_2, \ldots \)

is bounded it has a convergent subsequence \( y_{n_1}, y_{n_2}, \ldots \). Let \( y := \lim y_{n_1} \).

Then we have for infinitely many \( i \)

\[ |y - y_{n_i}| < \epsilon \leq |y_{n_i} - y_{n_i+1}| \]

whence

\[ |f(y) - f(y_{n_i})| \leq |f(y_{n_i}) - f(y_{n_i+1})|, \]

so \( f(y) \in B_{n_i} \) for infinitely many \( i \), i.e., \( f(y) \in \bigcap B_k \).

(ii) Let \( B_1 \supset B_2 \supset \ldots \) be balls in \( f(Y) \) and let \( y_1, y_2, \ldots \in Y \) such that

\( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots \]

and since \( f \in M_b(X) \):

\[ |y_1 - y_2| > |y_2 - y_3| \ldots \]

Since \( Y \) is spherically complete, there is \( y \in Y \) such that

\[ |y - y_n| \leq |y - y_{n+1}| \text{ for all } n, \text{ hence } |f(y) - f(y_n)| \leq |f(y_n) - f(y_{n+1})| \]

for all \( n \). It follows that \( f(y) \in B_n \) for all \( n \).

(iii) Let \( B_1 \supset B_2 \supset \ldots \) be balls in \( f^{-1}(A) \). Choose \( x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_3, \ldots \).

Then \( |x_1 - x_2| > |x_2 - x_3| > \ldots \) whence \( |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \)

There is \( x \in f^{-1}(A) \) such that \( |f(x) - f(x_n)| \leq |f(x_n) - f(x_{n+1})| \) for all \( n \).

Hence \( |x - x_n| \leq |x_n - x_{n+1}| \) for all \( n \), i.e., \( x \in \bigcap B_n \).
DEFINITION 4.11 Let $f : X \to K$. The oscillation function $\omega_f : X \to [0, \infty]$ is defined by

$$
\omega_f(a) := \lim_{n \to \infty} \sup \{ |f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}, x, y \in X \} \quad (a \in X)
$$

$$
= \lim_{n \to \infty} \sup \{ |f(x) - f(a)| : |x - a| \leq \frac{1}{n}, x \in X \}.
$$

THEOREM 4.12 Let $f \in \mathcal{M}_w(X)$. Then

$$
\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X).
$$

Proof. For $x \neq a$ we have $|f(x) - f(a)| \geq \inf_{z \neq a} |f(z) - f(a)|$ and (since $a$ is not isolated) consequently

$$
\omega_f(a) \geq \inf_{z \neq a} |f(z) - f(a)|.
$$

Conversely, let $z \neq a$. Then for all $x$ such that $|x - a| < |z - a|$ we have

$$
|f(x) - f(a)| \leq |f(z) - f(a)|
$$

so

$$
\omega_f(a) \leq |f(z) - f(a)|
$$

whence

$$
\omega_f(a) \leq \inf_{z \neq a} |f(z) - f(a)|.
$$

THEOREM 4.13 Let $f \in \mathcal{M}_w(X)$, $a \in X$. If $x_1, x_2, \ldots \in X$, $\lim_{n \to \infty} x_n = a$ ($x_n \neq a$ for all $n$) then $\lim_{n \to \infty} |f(x_n) - f(a)| = \omega_f(a)$.

Proof. By 4.12 we have $\lim_{n \to \infty} |f(x_n) - f(a)| \geq \omega_f(a)$. Conversely, $\lim_{n \to \infty} |f(x_n) - f(a)| \leq \omega_f(a)$ is clear from the definition of $\omega_f$. 
5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: K is local, k is finite, K has discrete valuation. Also we can sometimes say a little more if we assume X to be convex. For the time being, let X be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case K is a local field.

THEOREM 5.1 Let K be a local field, and let $f \in M_w(X)$. Then

(i) $f$ is continuous.

(ii) If $Y \subset X$ is closed then $f(Y)$ is closed.

(iii) If $f(X)$ is bounded and $f$ is not constant then $X$ is bounded.

(iv) Let $a \in X$. Then the following are equivalent

(a) $f$ is not injective at $a$

(b) $f$ is locally constant at $a$

(c) $f(a)$ is isolated in $f(X)$.

(v) The following conditions are equivalent

(a) $f$ is injective

(b) $f(X)$ has no isolated points

(c) $f$ is a homeomorphism of $X$ onto $f(X)$.

Proof. Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for $M_b$- and $M_s$-functions in case $X$ is convex and $K$ is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2  Let the residue class field \( k \) of \( K \) be finite. Let \( X \) be convex and let \( f \in M^c_d(X) \). Then

(i) If \( a,b,c \in X, |a-b| < |a-c|, f(a) \neq f(c) \) then \( |f(a)-f(b)| < |f(a)-f(c)| \).

(ii) If \( C \subset X \) is convex then \( f(C) \) is convex in \( f(X) \) (\( f \) is weakly Darboux continuous, see 2.5).

(iii) If \( f \) is injective, then \( f \in M^g_d(X) \).

Proof. (i) Let \( B := \{ x \in K : |x-a| \leq |a-c| \} \). Then \( B \subset X \) and \( f(B) \subset [f(a),f(c)] \). Define an equivalence relation on \( B \) by: \( x \sim y \) if \( |f(x)-f(y)| < |f(a)-f(c)| \).

Since \( k \) is finite we get finitely many equivalence classes \( B_1, B_2, \ldots, B_n \). Since \( a \neq c \) we have \( n \geq 2 \). The diameter of \( f(B) \) equals \( |f(a)-f(c)| \), the distance between \( f(B_i) \) and \( f(B_j) \) equals \( |f(a)-f(c)| \) (\( i \neq j \)). Since \([f(a),f(c)]\) can contain at most \( q := \chi(k) \) sets having distances \( |f(a)-f(c)| \) to one another we have \( n \leq q \). Hence \( 2 \leq n \leq q \).

By 2.2 (\( \beta \)), each \( B_i \) is convex. If \( x,y \in B_i \) and \( |x-y| \) were \( |a-c| \) then \( B_i = B \), contradicting \( n \geq 2 \). Thus \( B \) is a disjoint union of \( n \) balls \( B_1, \ldots, B_n \), where \( 2 \leq n \leq q \) and \( |x-y| < |a-c| \) whenever \( x,y \in B_i \) (\( i = 1, \ldots, n \)). It follows that \( n = q \) and that each \( B_i \) has the form \( \{ x \in K : |x-b_i| < |a-c| \} \) (\( b_i \in B \)). Hence, if \( |a-b| < |a-c| \) then there is \( i \) for which \( a,b \in B_i \).

So \( |f(a)-f(b)| < |f(a)-f(c)| \).

(ii) Let \( a,b \in C \) and let \( a \in f(X) \) with \( a \in [f(a),f(b)] \). We show that \( a \in f(C) \). If \( f(a) = f(b) \) this is clear. If \( f(a) \neq f(b) \), set \( a = f(x) \) where \( x \in X \). Then \( |f(x)-f(a)| \leq |f(b)-f(a)| \). If \( |x-a| \) were \( > |b-a| \) then \( f(x) \neq f(a) \) (since \( f \in M^c_d(X) \)) and by (i) we then had \( |f(b)-f(a)| < |f(x)-f(a)| \), a contradiction. Hence \( |x-a| \leq |b-a| \) i.e., \( x \in [a,b] \subset C \), so \( a = f(x) \in f(C) \).
(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on k, see 2.10.

COROLLARY 5.3 Let K be a local field and let \( f \in M_b(X) \) and X convex. Then the following conditions are equivalent.

(a) \( f \in M_s(X) \).

(b) \( f \) is injective.

(c) \( f \) is injective.

(d) \( f(X) \) has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 Let K be a local field and let X be the unit ball of K (or any bounded convex set, for that matter). If either \( f \in M_s(X) \) or \( f \in M_b(X) \) then f has bounded difference quotients.

Proof. f is bounded, let \( M := \sup \{|f(x)-f(y)| : x, y \in X\} \). It suffices to prove that \( |f(x)-f(0)| \leq M|x| \) for all x. Let \( \pi \in K \), \( |\pi| < 1 \), be a generator of the value group. By induction on n we prove:

if \( |x| = |\pi|^n \) then \( |f(x)-f(0)| \leq |\pi|^n M \).

The statement is clear for \( n = 0 \). Now suppose the statement is true for \( 0, 1, \ldots, n-1 \). Let \( x \in X \), \( |x| = |\pi|^n \). Then \( |x-0| \leq |\pi|^{n-1}-0 \). If \( f(\pi^{n-1}) \neq f(0) \) we have either since \( f \in M_s(X) \) or by 5.2(1)

\[
|f(x)-f(0)| < |f(\pi^{n-1})-f(0)| \leq |\pi|^{n-1} M
\]

hence

\[
|f(x)-f(0)| \leq |\pi|^n M
\]

If \( f(\pi^{n-1}) = f(0) \) then \( |f(x)-f(0)| \leq |f(\pi^{n-1})-f(0)| = 0 \), so certainly

\[
|f(x)-f(0)| \leq |\pi|^n M.
\]
Notes.

(a) 5.4 cannot be extended to the case $X = K$. In fact, let

$$f : \mathcal{D}_a \to \mathcal{D}_a$$

be the map $\mathcal{D}_a \to \mathcal{D}_a$. Then

$$f \in \mathcal{M}_b(\mathcal{D}_a)$$

but $|p^n f(p^{-n})| = p^n \to \infty$.

(b) If we loose the condition on $K$, for example by requiring that

the valuation is discrete then 3.22 and 2.4(5) show that the

conclusion of 5.4 is false both for $\mathcal{M}_b$-functions and $\mathcal{M}_s$-functions.

On the other hand, it is clear from the proof of 5.4 that a

bounded $\mathcal{M}_s$-function on $X$ has bounded difference quotients.

(c) One may wonder how difference quotients of $\mathcal{M}_w$-functions behave.

See the example below.

EXAMPLE 5.5 Let $p \neq 2$. Then there is an $f \in \mathcal{M}_w(\mathcal{Z}_p)$ that has unbounded difference quotients.

Proof. Let $a_0, a_1, \ldots$ be defined via $a_{2n} := p^n$ ($n = 0, 1, 2, \ldots$) and

$a_{2n+1} := 2p^n$ ($n = 0, 1, 2, \ldots$). Thus $(a_0, a_1, a_2, \ldots) = (1, 2, p, p^2, 2p^2, \ldots)$.

Then $|a_0| \geq |a_1| \geq |a_2| \geq \ldots$, $\lim a_n = 0$, $|a_n - a_m| = |a_n| (n > m)$.

Set

$$f(x) :=
\begin{cases}
  a_n & \text{if } |x| = p^{-n} \quad (n = 0, 1, 2, \ldots) \\
  0 & \text{if } x = 0
\end{cases} \quad (x \in \mathcal{Z}_p)
$$

Then the difference quotients of $f$ are not bounded (for $n \in \mathbb{N}$:

$f(p^{-n}) = p^n$, so $|p^{-2n} f(p^{-2n})| = p^n \to \infty$ if $n \to \infty$). We show that

$f \in \mathcal{M}_w(\mathcal{Z}_p)$. Since $f$ is continuous it suffices to show that if $x, y, z$

are $\neq 0$, $|x - y| < |x - z|$ then $|f(x) - f(y)| \leq |f(x) - f(z)|$. This is clear

if $|x| = |y|$. If $|x| < |y|$, then $|x| > |y| < |z|$. If $|x| > |y|$, then $|y| < |x| < |z|$. Let $f(x) = a_n$, $f(y) = a_m$, $f(z) = a_t$. Then in

both cases $n \neq m, t < \min(n, m)$; $|f(x) - f(y)| = |a_n - a_m| \leq |a_t|$ and

$|f(x) - f(z)| = |a_n - a_t| = |a_t|$ and we are done.
On the other hand (how surprising is life!)

**THEOREM 5.6** Let $k$ be the field of two elements. Then $M_w(X) = M_b(X)$.

Proof. We prove that $|x-y| = |y-z|$ implies $|f(x) - f(y)| \leq |f(y) - f(z)|$ ($x \neq y, y \neq z, x,y,z \in X$). There is $a \in K^*$ such that $|a(x-y)| = |a(y-z)| = 1$. So since $k = F_2$, $\overline{a(x-y)} = \overline{a(y-z)} = 1$, whence $a(x-z) = 0$ or $|a(x-z)| < 1$. Thus, $|x-z| < |x-y| = |y-z|$. Since $f \in M_w(X)$, $|f(x) - f(z)| \leq \min(|f(x) - f(y)|, |f(y) - f(z)|)$. Consequently, $|f(x) - f(y)| \leq \max(|f(x) - f(z)|, |f(z) - f(y)|) \leq |f(y) - f(z)|$.

Of particular interest may be monotone functions mapping convex sets onto convex sets.

**THEOREM 5.7** Let $K$ be a local field, let $X$ be a bounded open convex set, and let $f : X \to X$ be surjective. Then the following are equivalent.

(a) $f \in M_b(X)$
(b) $f \in M_s(X)$
(y) $f \in M_{bs}(X)$
(6) $f$ is an isometry.

Proof. (a) $\Rightarrow$ (b). Since $f(X)$ has no isolated points, $f$ is a homeomorphism, by 5.1(v). Then $f \in M_s(X)$, by 5.3. (b) $\Rightarrow$ (y). $f^{-1} \in M_b(X)$. We just have shown (a) $\Rightarrow$ (b), so $f^{-1} \in M_s(X)$ i.e., $f \in M_b(X)$.

(y) $\Rightarrow$ (6). From the proof of 5.4 we have seen that $|f(x) - f(y)| \leq M|x-y|$, where $M = \sup |f(x) - f(y)| = 1$. Hence $|f(x) - f(y)| \leq |x-y|$ for all $x,y \in X$, but by the same token this also holds for $f^{-1}$. Then $f$ is an isometry. (6) $\Rightarrow$ (a) is obvious.
COROLLARY 5.8 Let $X$ be an open convex subset of $K$ and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.

(a) $f \in M_b(X)$

(b) $f \in M_s(X)$

(c) $f : \mathbb{M}(X) \to \mathbb{M}(Y)$ is an isometry.

(d) $f$ is a scalar multiple of an isometry.

Proof. If $X$ is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If $X$ is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (a), (b), (c) is now easy. To prove (c) $\Rightarrow$ (d) we may assume $f(0) = 0$, $f(1) = 1$. Let $X_n := \{x \in K : |x| \leq n\}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is $c_n$ such that $|f(x)-f(y)| = c_n|x-y|$ $(x,y \in X_n)$. By substituting $x = 1$, $y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in \mathbb{M}_{ubs}(Z_p)$ into conditions on the coefficients of $f$ with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(Z_p)$. So let the notations be as in 3.3(3), and suppose first $f \in \mathbb{M}_{ubs}(Z_p)$ i.e.

$|x-y| = |s-t| \iff |f(x)-f(y)| = |f(s)-f(t)|$. Let $n,m \in \mathbb{N}$. If $|n-n_-| = |m-m_-|$ then $|f(n)-f(n_-)| = |f(m)-f(m_-)|$, so if we write $f = \sum_{n}^{\infty} a_n e_n$ we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ ($a_k \neq 0$) then $|n-n_-| = p^{-k}$ where $k = \left\lfloor \frac{\log n}{\log p} \right\rfloor$. We find

if $\left\lfloor \frac{\log n}{\log p} \right\rfloor < \left\lfloor \frac{\log m}{\log p} \right\rfloor$ then $|\lambda_n| > |\lambda_m|$

if $\left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor$ then $|\lambda_n| = |\lambda_m|$.
Moreover, if $\left\lceil \frac{\log n}{\log p} \right\rceil = k$ and $n - m$ is divisible by $p^k$ i.e., $n_+ - m_+ = m_-$ then $|f(n) - f(m)| = |\lambda_p - \lambda_m|$. If $n = m$ then $|f(n - m) - f(0)| = |\lambda_{n - m}| = |\lambda_n|$. We have found the first half of

**THEOREM 5.9** Let $f = \sum_{n} e_n \in C(\mathbb{Z}_p)$. In order that $f \in M_{\text{ubs}}(\mathbb{Z}_p)$ it is necessary and sufficient that condition (*) below holds

$$\left\lceil \frac{\log n}{\log p} \right\rceil = \left\lceil \frac{\log m}{\log p} \right\rceil, \quad n \neq m, \quad n_+ = m_+ \text{ implies }$$

$$|\lambda_p - \lambda_m| = |\lambda_n| = |\lambda_m| \quad (n, m \in \mathbb{N}).$$

We have shown $f \in M_{\text{ubs}}(\mathbb{Z}_p) \implies (*)$. Now suppose (*) and let $|x - y| = p^{-k}$. We show that $|f(x) - f(y)| = |\lambda_p^k|$. Now $e_n(x) = e_n(y)$ for $n < p^k$, so

$$f(x) - f(y) = \sum_{n \geq p^k} \lambda_n (e_n(x) - e_n(y)).$$

Set

$$x := a_0 + a_1 p + \ldots + a_{k+1} p + \ldots, \quad (a_k \neq b_k)$$

$$y := a_0 + a_1 p + \ldots + b_{k+1} p + \ldots$$

Then

$$\left| \sum_{n \geq p^k} \lambda_n e_n(x) \right| = \left| \lambda_p^k + \lambda_{p^{k+1}} \right|$$

$$\left| \sum_{n \geq p^k} \lambda_n e_n(y) \right| = \left| \lambda_p^k + \lambda_{p^{k+1}} \right|$$

Now either $a_k$ or $b_k$ is $\neq 0$, say, $a_k \neq 0$. If $b_k = 0$ then by (*)

$$\left| \sum_{n \geq p^k} \lambda_n e_n(x) \right| < \left| \lambda_p^k \right| = \left| \sum_{n \geq p^k} \lambda_n e_n(x) \right|,$$

so $|f(x) - f(y)| = |\lambda_p^k|$. If $b_k \neq 0$ then by (*) $|\lambda_p^k| = |\lambda_p^k - \lambda_{p^{k+1}}| = |f(x) - f(y)|$.

Note. Using similar methods, we can prove: $f = \sum_{n \in \mathbb{Z}} e_n$ is in $M_{\text{ubs}}(\mathbb{Z})$ if and only if we have (**) for all $n, m \in \mathbb{N}$. 
If we assume only that \( K \) has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let \( X \) be the unit ball of a discretely valued field. Let \( f : X \to X \) be surjective, \( f \in M_{bs}(X) \). Then \( f \) is an isometry.

**Proof.** It is clear from previous theory that \( f \) is a homeomorphism of the unit ball. It suffices to show that \(|f(x)-f(y)| \leq |x-y|\) for all \( x, y \in X \). (Apply this result also for \( f^{-1} \). Then \( f \) is an isometry.)

Let \( \pi \in K \), \( |\pi| < 1 \), be a generator of \(|K^*| \). We prove by induction

if \(|x| = |\pi|^n\) then \(|f(x)-f(0)| \leq |\pi|^n|f(1)-f(0)|\).

For \( n = 0 \) this is clear. (\(|x-0| \leq |1-0|\), so \(|f(x)-f(0)| \leq |f(1)-f(0)|\)). Suppose the statement is true for \( n = k \). Let \(|x| = |\pi|^{k+1} \). Then

\(|x-0| < |\pi|^{k+1}-0|\), so \(|f(x)-f(0)| < |f(\pi^{k+1})-f(0)| \leq |\pi|^{k+1}|f(1)-f(0)|\), so \(|f(x)-f(0)| \leq |\pi|^k|f(1)-f(0)|\) and we are done. (In fact, we have shown that a bounded \( M_{bs} \)-function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let \( K \) have discrete valuation and let \( f \in M_b(X) \). Then

the following conditions are equivalent.

(a) \( f(X) \) has no isolated points.

(b) \( f \) is injective and continuous.

(c) \( f \) is a homeomorphism \( X \sim f(X) \).
Proof. (a) \( \iff \) (c) is 4.9(ii). (c) \( \iff \) (b) is clear. (b) \( \iff \) (c): if \( f(a) \)
were an isolated point of \( f(X) \), then \( \{x : f(x) = f(a)\} \) is open in \( X \).
Since \( f \) is injective \( \{a\} \) is open. But \( X \) has no isolated points. Con­
tradiction.

To show that 5.11 may not be true if \( K \) has a dense valuation
we construct

**EXAMPLE 5.12** Let \( |K| = [0,\infty) \). Then we construct an \( M \)-homeomorphism
sending
\[
\{x \in K : \frac{1}{2} < |x| \leq 1\} \text{ onto } \{x \in K : 0 < |x| \leq 1\}.
\]

**Proof.** Let \( \phi : [\frac{1}{2},1] \to [0,1] \) be the map \( x \mapsto 2(x-\frac{1}{2}) \) (\( x \in (\frac{1}{2},1] \)). For
each \( v \in (\frac{1}{2},1] \), choose \( \beta_v \in K \) such that \( |\beta_v| = \frac{\phi(v)}{v} \). Define
\[
f : \{x \in K : \frac{1}{2} < |x| \leq 1\} \to \{x \in K : 0 < |x| \leq 1\}
\]
as follows
\[
f(x) = \beta_{|x|} \quad (\frac{1}{2} < |x| \leq 1)
\]
Clearly, \( |f(x)| = |\beta_{|x|}| \cdot |x| = \phi(|x|) \in (0,1] \). The inverse of \( f \)
is given by \( y \mapsto \beta_{\phi^{-1}(|y|)} \phi^{-1}(|y|) \), so \( f \) is a bijection. Since \( f^{-1} \)
can be
defined in the same way as \( f \) (only with \( \phi^{-1} \) instead of \( \phi \)) it suffices
to show that \( f \in M \). Let \( |x-y| < |x-z| \).
Suppose \( |x| > |z| \). Then \( |x-z| = |x| \) and \( |y| = \max(|x-y|,|x|) = |x| \).
Then \( \beta_{|x|} = \beta_{|y|} \), so \( |f(x)| = |f(y)| = |\beta_{|x|}||x-y| \) and \( |f(x)| = |f(z)| = |\beta_{|x|}||x-z| \), so we are done in this case. Suppose \( |x| < |z| \).
Then \( |x-z| = |z| \) and \( |y| = \max(|x-y|,|x|) < |z| \). Then \( |f(x)| = |f(y)| \leq \max(|f(x)|,|f(y)|) < |f(z)| = |f(z)-f(x)| \).
Suppose \( |x| = |z| \). Then \( |y| \leq \max(|x-y|,|x|) \leq |x|; \) if \( |y| \) were < \( |x| \)
then \( |x-y| = |x| = |z| < |x-z| \), a contradiction, so \( |y| = |x| = |z| \),
and \( |f(x)| = |f(y)| = \beta_{|x|}||x-y|, |f(x)| = |f(z)| = \beta_{|x|}||x-z| \) whence
\[|f(x)-f(y)| < |f(x)-f(z)|.\]
EXAMPLE 5.13 Extend \( f \) to a surjection \( g \) of \( \{ x \in K : |x| \leq 1 \} \) onto itself by defining \( g(x) = 0 \) if \( |x| \leq \frac{1}{2} \). We claim that \( g \in M_b \). Let \(|x-y| < |x-z|\). To check whether \( |g(x) - g(y)| \leq |g(x) - g(z)| \) we only have to consider the cases \(|x| < \frac{1}{2} \) and \(|y| > \frac{1}{2} \) and \(|y| \leq \frac{1}{2} \). In the first case, \(|x-y| = |y| \leq |x-z|\), so \(|z| = \max(|z-x|,|x|) = |z-x| \geq |y|\). Then \(|g(x) - g(y)| = |f(y)| \leq |f(z)| = |g(z) - g(x)|\). In the second case \(|g(x) - g(y)| = |f(x)|\). If \(|x| < |z|\) then \(|f(x)| < |f(z)| = |f(x) - f(z)| = |g(z) - g(x)|\). If \(|x| > |z|\) then \(|f(x)| = |g(x) - g(z)|\).

Thus we have found a continuous surjection \( g : \{ x \in K : |x| \leq 1 \} \to \{ x \in K : |x| \leq 1 \}, g \in M_b \), such that \( g = 0 \) on \( \{ x : |x| \leq \frac{1}{2} \} \). (Compare 5.11).

EXAMPLE 5.14 Let \( h : \{ x \in K : |x| \leq 1 \} \to K \) be defined as

\[
h(x) = \begin{cases} 
  f^{-1}(x) & \text{if } x \neq 0 \text{ (} f \text{ as in 5.12)} \\
  0 & \text{if } x = 0.
\end{cases}
\]

Then \( h \) is a non-continuous \( M_{bs} \)-function.

Proof. That \( h \) is not continuous at 0 is clear. Further, \( h \), restricted \( \{ x : 0 < |x| \leq 1 \} \) is in \( M_{bs} \) (see 5.12). Further, since \( g \circ h \) is the identity (\( g \) as in 5.12), we see that \( h \in M_{bs} \). It suffices to show that

\(|x-y| = |x-z| \) implies \(|h(x)-h(y)| = |h(x)-h(z)|\) in case \( 0 \in \{ x,y,z \} \).

We may suppose \( x \neq y, y \neq z, x \neq z \). Let \( x = 0 \). Then \(|y| = |z|\), so

\(|f^{-1}(y)| = |f^{-1}(z)| \) i.e., \(|h(x)-h(y)| = |h(x)-h(z)|\). Now let \( y = 0 \).

Then \(|x| = |x-z|\). Choose \( 0 < |t| \leq 1 \) such that \(|t| < |x|\). Then

\(|x-t| = |x-z| \) so \(|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x) - f^{-1}(z)| \) i.e.,

\(|h(x)| = |h(x)-h(z)|\), and we are done.
6. FUNCTIONS OF BOUNDED VARIATION

In this section $X$ is the unit ball of $K$, and $BA(X) := \{ f : X \to K : \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x-y} \right| < \infty \}$. Let us define

$$\|f\|_\Delta := \sup_{x,y \in X, x \neq y} \left| \frac{f(x) - f(y)}{x-y} \right| (f \in BA(X)).$$

It will turn out that, in a natural way, $BA(X)$ can be regarded as the space of functions of bounded variation, and that $\| \|_\Delta$ plays the role of the total variation.

**Theorem 6.1** Let $f : X \to K$. Then the following are equivalent

(a) $f \in BA(X)$.

(b) $f$ is a linear combination of two increasing functions.

If $|K|$ is discrete (a), (b) are equivalent to

(y) $f$ is the difference of two bounded monotone functions of some type $\sigma$.

(\delta) $f \in [M_{\Delta S}(X)]$.

If $K$ is a local field then (a)-(\delta) are equivalent to

(c) $f \in [M_D(X)]$.

(\eta) $f \in [M_S(X)]$.

**Proof.** We only prove (a) $\Rightarrow$ (b). The rest follows from (5.10), (5.4).

So let $f \in BA(X)$ and choose $\lambda \in K$ such that $|f(x) - f(y)| < |\lambda| |x-y|$ $(x,y \in X, x \neq y)$. Then $\lambda^{-1} f$ is a pseudocontraction $f(x) = \lambda x + \lambda(\lambda^{-1} f(x) - x)$ $(x \in X)$, where $x \to x$ and $x \to \lambda^{-1} f(x) - x$ are increasing.

In the real case, we can define for a function $[0,1] \to \mathbb{R}$, of bounded variation
V(f) := inf \{\var{g} + \var{h} : f = g + h, g, h monotone\}.

It is an easy exercise to show that $f \mapsto V(f)$ is a seminorm on the space of all functions of bounded variation and that $V$ is equivalent to the total variation $\var{}$, defined via

$$\var{f} = \sup \left\{ \left| \sum f(x_k) - f(x_{k-1}) \right| : x_0 < x_1 < \ldots < x_n \text{ is a partition of } [0,1] \right\}.$$ 

So in the non-archimedean situation we define for $f : X \to K$

$$J(f) = \sup \{ |f(x) - f(y)| : x, y \in X \}.$$ 

(If $f$ is considered to be "monotone" then $J(f)$ can be interpreted as the "total variation" of $f$.) We are led to the following definitions for $f \in \text{BA}(X)$:

$$\var{f} := \inf \{ \max(J(g), J(h)) : f = g + h, g, h \text{ are scalar multiples of increasing functions} \}.$$

(If $|K|$ is discrete): $\var_1 f := \inf \{ \max J(g), J(h) : f = g + h, g, h \text{ are in } M(x) \}).$

(If $K$ is local): $\var_2 f := \inf \{ \max J(g), J(h) : f = g + h, g, h \in M_b(X) \}$

$$\var_3 f := \inf \{ \max J(g), J(h) : f = g + h, g, h \in M_{g}(X) \}.$$

Let us first compare $\var{f}$ and $\|f\|_{\Delta}$. If $f = g + h$ and $g, h$ are scalar multiples of increasing functions we have for $x, y \in X, x \neq y$

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq \max \left( \left| \frac{g(x) - g(y)}{x - y} \right|, \left| \frac{h(x) - h(y)}{x - y} \right| \right) \leq \max (J(g), J(h))$$

so $\|f\|_{\Delta} \leq \var{f}$. Conversely, if $|\lambda| > \sup \left| \frac{f(x) - f(y)}{x - y} \right|$ then

$$f(x) = \lambda x + \lambda^{-1}(\lambda f(x) - x) \quad (x \in X)$$

whence

$$\var{f} \leq |\lambda|$$
So, if $|K|$ is dense we have $\text{Var}\, f = \|f\|_\Delta (f \cdot \mathcal{B}(X))$. Otherwise we have at least

$$\|f\|_\Delta \leq \text{Var}\, f \leq c\|f\|_\Delta \quad (f \in \mathcal{B}(X))$$

(where $c$ is the smallest value $> 1$).

If $|K|$ is discrete we clearly have $\text{Var}_1 f \leq \text{Var}\, f$. Conversely, let $f = g+h$, where $g, h \in M_{bs}(X)$. It follows from the proof of 5.10 that

$$|g(x) - g(y)| \leq M|x-y| \quad (x, y \in X)$$

$$|h(x) - h(y)| \leq N|x-y|$$

where $M = \sup |g(x) - g(y)| = J(g)$ and $N = J(h)$.

So

$$\left|\frac{f(x) - f(y)}{x-y}\right| \leq \max(J(g), J(h)), \text{ whence}$$

$$\|f\|_\Delta \leq \text{Var}_1 f.$$ 

Similar proofs work for $\text{Var}_2 f, \text{Var}_3 f$. We have

**THEOREM 6.2** The seminorms $\text{Var}, \text{Var}_1, \text{Var}_2, \text{Var}_3$ on $\mathcal{B}(X)$ (whenever defined) are all equivalent to $\|\|_\Delta$. 
REFERENCES

