INTRODUCTION

In the sequel, $K$ is a non-archimedean valued field, complete, with residue class field $k$. Our aim is to present reasonable definitions for a function $f : X \to K$ to be "monotone". ($X$ is a subset of $K$). Since $K$ admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \to \mathbb{R}$, equivalent to "$f$ is monotone", and such that these statements have translations in $K$ that make sense. This way we obtain several definitions of "$f : X \to K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of $K"$, "the sign of a nonzero element of $K"$.

In Section 2 we define several notions of monotony. E.g., $f \in M^c(X)$ if $x$ between $y$ and $z$ implies $f(x)$ between $f(y)$ and $f(z)$ and $f \in M^s(X)$ if $f(x)$ between $f(y)$ and $f(z)$ implies $x$ between $y$ and $z$. Also monotone functions of type $\sigma$ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M^c(X)$ (or $f \in M^s(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
mation on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of \( K \) and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: \( f' > 0 \iff f \) increasing.

The notion of pseudo-ordening ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

**Notations.** Let \( p \) be a prime. By \( \mathbb{F}_p \) we mean the field of \( p \) elements.

By \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers. For a field \( L \) we denote its characteristic by \( \chi(L) \). Let \( E \) be a vector space over \( K \) and \( S \subset E \). By \( [S] \) we denote the smallest \( K \)-linear subspace of \( E \) that contains \( S \).
DEFINITION 1.1 Let \( x, y \in K \). Then the smallest ball in \( K \) containing \( x \) and \( y \) is denoted by \([x, y]\). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \([x, y] \subseteq C\).

Sometimes we use a more geometric terminology. Instead of \( z \in [x, y] \) we will say that \( z \) is between \( x \) and \( y \) and instead of \( z \not\in [x, y] \) we use the expression: \( x \) and \( y \) are at the same side of \( z \).

Notice that \([x, y] = [y, x]\) for all \( x, y \in K \) and that \( z \in [x, y] \iff |z-x| \leq |x-y| \iff |z-y| \leq |x-y| \iff z = \lambda x + (1-\lambda)y \) for some \( \lambda \in K \), \( |\lambda| \leq 1 \). If \( x \neq y \) then the \( \lambda \) in this last expression is unique (viz. \( \lambda = \frac{z-y}{x-y} \)).

Examples of convex sets are: the empty set, singletons, balls, \( K \). It is an easy exercise to show that these are the only convex subsets of \( K \). So formally we may write each convex subset of \( K \) as

\[
\{x \in K : |x-a| < r\} \quad (a \in K, 0 \leq r \leq \infty)
\]

or as

\[
\{x \in K : |x-a| \leq r\} \quad (a \in K, 0 \leq r \leq \infty)
\]

Notice that the only unbounded convex subset of \( K \) is \( K \) itself.

Sometimes we need the notion of convexity with respect to a subset \( X \) of \( K \). A subset \( C \subseteq X \) is called convex in \( X \) if \( x, y \in C \) implies \([x, y] \cap X \subseteq C\) or, equivalently, if \( C \) is the intersection of \( X \) with a convex subset of \( K \).

Let \( x, y, z \in K \). By the strong triangle inequality we have that the "triangle" \( x, y, z \) is isosceles, say \( |x-y| = |y-z| \). Then \( |x-z| \leq |x-y| \), so \( z \) is between \( x \) and \( y \) and \( x \) is between \( y \) and \( z \). If also \( |x-y| = |x-z| \)


then \( y \) is between \( x \) and \( z \). Otherwise, \( x \) and \( z \) are at the same side of \( y \).

The relation \( \sim \) defined on \( K^* := K \setminus \{0\} \) by

\[
x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*)
\]

is an equivalence relation. We have \( x \sim y \) iff \( 0 \notin [x, y] \) i.e. iff \( |x-y| < |x| = |y| \) i.e. iff \( |xy^{-1}| < 1 \). Define

\[
K^+ := \{ x \in K : |1-x| < 1 \}
\]

Then \( K^+ \) is a multiplicative subgroup of \( K^* \), \( K^+ = \{ x \in K^* : x \sim 1 \} \) and is called the set of the positive elements of \( K \). The relation \( \sim \) is also induced by the canonical group homomorphism

\[
\pi : K^* \to K^*/K^+
\]

Thus, \( x \sim y \) if and only if \( \pi(x) = \pi(y) \) \( (x, y \in K^*) \). Therefore it is natural to view the group \( \Sigma := K^*/K^+ \) as being the group of signs of elements of \( K^* \), and we call \( \pi(x) \) the sign of the element \( x \in K^* \). If \( x \in K^* \) then \( \pi(x) = \{ y : |y-x| < |x| \} = xx^+ \). For \( x \in K^* \), \( a \in \Sigma \) we sometimes write \( xa \) to indicate the element \( \pi(x) \cdot a \) of \( \Sigma \). In particular, for \( a \in \Sigma \) the opposite sign of \( a \), \( -a \), is defined as \( (-1)a \). Then \( -a = \{-x : x \in a\} \). (Notice that in case \( \chi(K) = 2 \) we have \( a = -a \).)

Let \( a \in \Sigma \). Then for \( x, y \in a \) we have \( |x| = |y| \) so we can define the absolute value of \( a \), \( |a| \) as follows

\[
|a| := |x| \quad (x \in \pi^{-1}(a)).
\]

In the sequel we also need addition between elements of \( \Sigma \). Let us first observe that for any \( a, \beta \in \Sigma \) the sum

\[
a + \beta := \{ x+y : x \in a, \ y \in \beta \}
\]

is always a ball with radius \( \max(|a^-|, |\beta^-|) \). (I.e., of the form
\{x : |x-b| < \max(|a|,|\beta|)\}. Now \alpha + \beta contains 0 if and only if 
\alpha = -\beta. Otherwise \alpha + \beta is again an element of \Sigma. (Proof: Let \alpha \in \Sigma, \beta \in \Sigma. Then |\alpha + \beta| = \max(|\alpha|,|\beta|). If also \alpha \in \Sigma, \beta \in \Sigma then |x+y-(\alpha + \beta)| \leq \max(|x-\alpha|,|y-\beta|) < \max(|\alpha|,|\beta|) = |\alpha + \beta|. Thus \alpha + \beta contains the ball with center \alpha + \beta and radius \max(|\alpha|,|\beta|), so \alpha + \beta is equal to this ball.)

Let us define

\[ \alpha \oplus \beta := \alpha + \beta = \{x+y : x \in \alpha, y \in \beta\} \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta). \]

We have

**THEOREM 1.2** Let \Sigma, | | : \Sigma \rightarrow \mathbb{R}, \Theta : \Sigma \times \Sigma \setminus \{(-\alpha,-\alpha) : \alpha \in \Sigma\} \rightarrow \Sigma be as above. Let \alpha, \beta, \gamma \in \Sigma. Then

(i) \[ |\alpha \beta| = |\alpha| \cdot |\beta|, |\alpha^{-1}| = |\alpha|^{-1}. \]

(ii) If \alpha \oplus \beta is defined then so is \beta \oplus \alpha and \alpha \oplus \beta = \beta \oplus \alpha.

(iii) If \( (\alpha \oplus \beta) \oplus \gamma \) and \( \alpha \oplus (\beta \oplus \gamma) \) are defined then 
\[ (\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma). \]

(iv) If \alpha \oplus \beta or \gamma \oplus \gamma \beta is defined then so is the other
and \( \gamma (\alpha \oplus \beta) = \gamma \alpha \oplus \gamma \beta. \)

(v) If \alpha \oplus \beta is defined then \[ |\alpha \oplus \beta| = \max(|\alpha|,|\beta|). \] Conversely if \[ |s| = \max(|\alpha|,|\beta|) \] for some \( s \in \alpha + 3 \) then \alpha \oplus \beta is defined.

(vi) \[ |\alpha| < |\beta| \iff \alpha \oplus \beta = \beta. \]

(vii) Let \( n \in \{1,2,\ldots,\chi(k)-1\} \) if \( \chi(k) \neq 0 \), let \( n \in \mathbb{N} \) otherwise. Then we define \( \oplus_n \) \( \alpha \) inductively as follows.
\[ \Theta_1 \alpha : \alpha, \Theta_k \alpha := \Theta_{k-1} \alpha \oplus \alpha \quad (k \leq n). \] Then
\[ \Theta_n \alpha = n \alpha. \]

**Proof.** (i), (ii) are clear. (iii) is almost trivial: if \( x \in \alpha, y \in \beta, \)
\( z \in \gamma \) then \( x+y+z \in \alpha + \beta + \gamma \) and the latter set can be regarded as
\[(a \oplus \beta) \oplus \gamma \text{ or } a \oplus (\beta \oplus \gamma). (\text{It is worth noticing that } (a \oplus \beta) \oplus \gamma \text{ may be defined whereas } a \oplus (\beta \oplus \gamma) \text{ is not. Choose } \beta = -\gamma \text{ and } |a| > |\beta|. \text{ Then } (a \oplus \beta) \oplus \gamma = a \oplus \gamma = \alpha, \beta \oplus \gamma \text{ is not defined.})\]

(iv) is clear. If \(a \oplus \beta\) is defined then for \(x, \alpha, y, \beta\) we have \(|x + y| \geq \max(|x|, |y|)| \text{ whence } |x + y| = \max(|x|, |y|). \text{ So } |a \oplus \beta| = \max(|a|, |\beta|). \text{ Conversely, if } a \oplus \beta \text{ is not defined, then (we saw earlier that) } a + \beta \text{ is a ball with center zero and radius } \max(|a|, |\beta|). \text{ Thus we have (v). We prove (vi). If } |a| < |\beta| \text{ then } a + \beta = \beta \text{ so } a \oplus \beta = \beta. \text{ Conversely, if } a \oplus \beta = \beta \text{ then choose } \alpha \oplus \beta, b \in \beta. \text{ Then } a + b \in \beta \text{ hence } a + b \sim b \text{ i.e., } ab^{-1} + 1 \in K^+ \text{ implying } |ab^{-1}| < 1 \text{ or } |a| < |b|. \text{ Hence } |a| < |b|. \text{ (Note: from (vi) it follows that } a \oplus \beta = a' \oplus \beta \text{ does not imply } a = a'.) \text{ To prove (vii) let } a \in K \text{ and observe that for any } k \leq n, \text{ if } a \in K \text{ is defined, } (k-1)a \in a. \text{ Hence } |(k-1)a + a| = |ka| = |a| = |a|, \text{ so } a + a \text{ does not contain 0, hence } a \oplus a \text{ is defined.} \text{ Now } na \text{ is by definition } \pi(n)a. \text{ So } na \in a \text{ and } na + a \in a. \text{ Since both } na \text{ and } a \in \Sigma, \text{ and } a \oplus a \text{ are signs they must coincide.} \]

We now define relations that resemble "ordering".

**DEFINITION 1.3** Let \(\alpha \in \Sigma\) and \(x, y \in K\). Then we say that \(x\) is greater than \(y\) in the sense of \(\alpha\), notation \(x >_\alpha y\), if \(x - y \in \alpha\).

We have the following rules

**THEOREM 1.4** (i) If \(x, y \in K, x \neq y\) then there is exactly one \(\alpha \in \Sigma\) for which \(x >_\alpha y\).

(ii) \(x >_\alpha x\) for no \(\alpha\).

(iii) If \(x >_\alpha y\) then for all \(s \in K\): \(x + s >_\alpha y + s\) \((x, y \in K, \alpha \in \Sigma)\)

(iv) If \(x >_\alpha y\) and \(s >_\beta 0\) then \(xs >_\alpha ys\) \((x, y, s \in K, \alpha, \beta \in \Sigma)\)
(In particular $x > y$ implies $-x < -y$).

(v) If $x > y$, $y > z$ and if $\alpha \oplus \beta$ is defined then $x > \alpha \oplus \beta z$.

Proof. Easy.

The group $\Sigma_1 := \{\alpha \in \Sigma : |\alpha| = 1\}$ is a subgroup of $\Sigma$, isomorphic to multiplicative group $k^*$. If $K$ has discrete valuation and if $s \in K$ and $|s|$ is the largest value that is smaller than 1, then for each $\alpha \in \Sigma$ there is $x \in \mathbb{Z}$ such that $\alpha = s^n x_1$ where $\alpha_1 \in \Sigma_1$. It follows easily that the map $(n, \alpha) \mapsto s^n \alpha$ $(n \in \mathbb{Z}, \alpha \in \Sigma_1)$ is an isomorphism of $\mathbb{Z} \times \Sigma_1$ onto $\Sigma$. Thus, in case $K$ has discrete valuation, $\Sigma$ is isomorphic to $\mathbb{Z} \times \Sigma_1$, or, for that matter, to $|k^*| \times k^*$.

If $K$ is a local field we can even define a group embedding $\rho : \Sigma \to K^*$ such that $\pi \rho$ is the identity. (Thus, we can pick an element in every $\alpha$ ($\alpha \in \Sigma$) such that the resulting set is a subgroup of $K^*$). Let $s \in K$, $|s| < 1$ such that $|s|$ generates the value group and let $q$ be the cardinality of $k$. Let $x \in K$. Then there is a unique $n \in \mathbb{Z}$ such that $x = s^n x_1$ where $|x_1| = 1$.

Define

$$v(x) = s^n \lim_{n \to \infty} x_1^n$$

It is easy to verify that $v$ is a homomorphism of $K^*$ into $K^*$, that $\pi(v(x)) = \pi(x)$ for all $x \in K^*$ and that $v(x) = 1$ if and only if $x \in K^+$. Therefore the map $\rho$ making the diagram

$$\begin{array}{ccc}
K^* & \xrightarrow{v} & K^* \\
\pi \downarrow & & \downarrow \pi \\
\Sigma & \xrightarrow{\rho} & K^* \\
\end{array}$$

commute solves the problem.

EXAMPLE 1.5 The signs of $\varnothing$. Let $\varnothing$ be a primitive $(p-1)^{th}$ root of
unity. Then \( \{ \Theta^i_p : i \in \{0, 1, \ldots, p-2\}, n \in \mathbb{Z} \} \) is a subgroup of \( \Phi_p^* \) isomorphic to \( \Sigma \). If

\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0, 1, \ldots, \phi(p^2)\}, a_n \neq 0)
\]

is an element of \( \Phi_p \), its sign, interpreted as an element of \( \Phi_p \) is

\[
\pi(x) = a_n p^n.
\]
2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function $f : [0,1] \to \mathbb{R}$ the following statements are equivalent.

(a) $f$ is monotone (i.e., either $x > y$ implies $f(x) \geq f(y)$ for all $x,y$ or $x > y$ implies $f(x) \leq f(y)$ for all $x,y$).

(β) If $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$ $(x,y,z \in [0,1])$.

(γ) If $C \subseteq \mathbb{R}$ is convex then $f^{-1}(C)$ is convex.

Thus we define

**DEFINITION 2.1** Let $X \subseteq K$. We say that $f \in M_b(X)$ if for all $x,y,z \in X$, $x$ between $y$ and $z$ implies $f(x)$ is between $f(y)$ and $f(z)$. In other words, $f \in M_b(X)$ if and only if for all $x,y,z$

$$|x-y| \leq |y-z| \Rightarrow |f(x)-f(y)| \leq |f(y)-f(z)|.$$

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

**THEOREM 2.2** Let $X \subseteq K$ and let $f : X \to K$. Then the following statements are equivalent

(a) $f \in M_b(X)$.

(β) For each convex $C \subseteq K$, $f^{-1}(C)$ is convex in $X$.

(γ) For all $x,y,z \in X$: $|x-y| = |x-z|$ \Rightarrow $|f(x)-f(y)| = |f(x)-f(z)|$.

(δ) For all $x,y,z \in X$: $|f(x)-f(y)| > |f(x)-f(z)| \Rightarrow |x-y| > |x-z|$.

(ε) For all $x,y,z \in X$: $|f(x)-f(y)| \neq |f(x)-f(z)| \Rightarrow |x-y| \neq |x-z|$.
Proof. (a) $\Rightarrow$ (b). Let $x, y \in f^{-1}(C)$ and let $z \in [x, y] \cap X$. Then $|z-x| \leq |x-y|$, so $|f(z)-f(x)| \leq |f(x)-f(y)|$ i.e., $f(z) \in [f(x), f(y)] \subset C$. Hence $z \in f^{-1}(C)$.

(b) $\Rightarrow$ (a). Let $x, y, z \in X$ and $|x-y| \leq |x-z|$. The set $[f(x), f(z)]$ is convex, hence $f^{-1}([f(x), f(z)])$ is convex in $X$ and contains $x$ and $z$, so it must contain $y$. Thus $f(y) \in [f(x), f(z)]$.

Clearly, (a) $\iff$ (b) and (y) $\iff$ (e). We prove (a) $\Rightarrow$ (y). Now (a) $\Rightarrow$ (y) is clear by symmetry. Suppose (y) and let $|x-y| \leq |x-z|$. It suffices to consider the case $|x-y| = |x-z|$. Then $|y-z| = |x-z|$, so by (y) we have $|f(y)-f(z)| = |f(x)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) = |f(x)-f(z)|$.

**THEOREM 2.3** Let $X \subset K$. Then

(i) For each $a, b \in K$ the map $x \mapsto ax+b$ is in $M_b(X)$.

(ii) If $f \in M_b(X)$, $\lambda \in K$ then $\lambda f \in M_b(X)$.

(iii) $M_b(X)$ is closed under pointwise limits.

(iv) If $f \in M_b(X)$ and $g : f(X) \to K$ is in $M_b(f(X))$, then $g \circ f \in M_b(X)$.

(v) If $f \in M_b(X)$ and $f(a) = f(b)$ for some $a, b \in X$, then $f$ is constant on $[a, b] \cap X$.

**Proof.** Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of $M_b$-functions. For more, see the sequel.

(1) The constant functions.

(2) Isometries (e.g., exp.).

(3) Choose in every $a \in \Delta$ an element $x_a$. Define $\phi : K \to K$ as follows

$$
\phi(x) = \begin{cases} 
0 & \text{if } x = 0 \\
x_a & \text{if } x \in a \quad (a \in \Delta)
\end{cases}
$$
(Essentially, $\phi|K^*$ is the sign function $\pi$ of section 1).

We prove that $\phi \in M_d(K)$. Since $\phi$ is continuous it suffices to check that $\phi|K^*$ is in $M_d(K^*)$. Now for all $x,y \in K^*$ we have $\phi(x)-\phi(y) = 0$ if $|x^{-1}y-1| < 1$ and $|\phi(x)-\phi(y)| = |x-y|$ if $|x-y| = \max(|x|,|y|)$. Now take $x,y,z \in K^*$ such that $|x-y| \leq |x-z|$. If $\phi(x) = \phi(z)$ then $|1-x^{-1}y| \leq |1-x^{-1}z| < 1$ so $\phi(x) = \phi(y)$.

If $\phi(x) \neq \phi(z)$ then $|\phi(x)-\phi(y)| \leq |x-y| \leq |x-z| = |\phi(x)-\phi(z)|$.

(4) Let $r > 0$ and choose in every ball $B$ of radius $r$ a center $x_B$.

The function $\psi$ defined via

$$\psi(x) = x_B \quad (x \in B)$$

is in $M_d(K)$. The proof is easy.

(5) (A nowhere continuous $M_d$-function). Let $K$ be a field such that $\#K = \#k$ (e.g., a discretely valued field where $\#k$ has the power of the continuum). Let $\sigma : K \rightarrow k$ be a bijection and let $\tau : k \rightarrow K$ such that $|\tau x - \tau y| = 1$ whenever $x \neq y$. Then $f : \tau \circ \sigma$ satisfies: $|f(x)-f(y)| = 1$ for all $x,y \in K$, $x \neq y$.

Clearly $f$ is everywhere discontinuous, $f \in M_d(K)$.

(6) Let $X \subseteq K$. We can strengthen the definition of an $M_d$-function into

$$\text{if } |x-y| \leq |z-t| \text{ then } |f(x)-f(y)| \leq |f(z)-f(t)| \quad (x,y,z,t \in X)$$

(some "uniform" $M_d$-condition) and we obtain a space, called $M_{ub}(X)$.

Clearly, the examples mentioned in (1), (2), (4), (5) are in $M_{ub}(K)$,

whereas the example in (3) is not. (Choose $x,y \in K$ with $|x| > 1$,

$$|x-y| = 1. \text{ Then } |1-0| \leq |x-y|, \text{ but } 1 = |\phi(1)-\phi(0)| > |\phi(x)-\phi(y)| = 0.$$)

Notice that $\phi$ is locally constant on $K^*$, but not on $K$.

(7) The discontinuous function $f$ of (5) has the property that $f(K)$ consists only of isolated points. This is not accidental. If $f \in M_d(K)$
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing \( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points, so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that the image of an everywhere discontinuous \( M_b \)-function consists only of isolated points.

(8) For each \( n \), let \( \sigma_n : K \rightarrow K \) be the example of 2.4, (4) above where \( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b(X) \) we have
\[
\sigma_n \circ f \in M_b(X) \quad \text{and} \quad \lim \sigma_n \circ f = f \text{ uniformly. Hence, if } f \text{ is continuous then it can uniformly be approximated by locally constant } M_b\text{-functions.}
\]

A monotone function \( f : [0,1] \rightarrow \mathbb{R} \) maps convex sets into sets that are relatively convex in \( f([0,1]) \). In our situation we do not have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If \( f : [0,1] \rightarrow \mathbb{R} \) maps convex sets into (relatively) convex sets then \( f \) need not be monotone: any Darboux continuous function has the above property. In fact a function \( f : [0,1] \rightarrow \mathbb{R} \) is Darboux continuous if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We define

**DEFINITION 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \rightarrow K \). Then \( f \) is called weakly Darboux continuous if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex in \( f(X) \).

\( f \) is called Darboux continuous if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex (in \( K \)).

We have the following obvious remarks.

1) \( f : X \rightarrow K \) is Darboux continuous if and only if \( f \) is weakly Darboux continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) such that for every open ball \( B \subset \mathbb{Z}_p \),

\[
f(B) = \mathbb{Z}_p .
\]

Let \( A \subset \mathbb{Z}_p \) be defined as follows.

\[
x = \sum_{n=0}^{p-1} a_n p^n (a_n \in \{0, 1, \ldots, p-1\})
\]

is in \( A \) if \( a_{2n} = a_{2n+2} = \ldots = 0 \) for some \( n \). Define \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) via

\[
f(x) = \begin{cases} 
    a_{2N+1}a_{2N+3}a_{2N+5}^2 + \ldots & \text{if } x \in A \text{ and } N = \min \{ n : a_{2n} = a_{2n+2} = \ldots = 0 \} \\
    0 & \text{if } x \notin A 
\end{cases}
\]

Then \( f \) maps every non empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.

(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on \( \mathbb{Z}_p \) is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous.

To see this, observe that all open compact subsets of \( \mathbb{Z}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p \) sending

\[
\{ x : |x| < 1 \} \to \{ x : |x| = 1 \} \text{ and } \{ x : |x| = 1 \} \to \{ x : |x| < 1 \}.
\]

If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( M_p \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4.(7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \to \mathbb{R} \) the following conditions are equivalent.

(a) \( f \) is strictly monotone (i.e., injective and monotone).

(\( \beta \)) \( f \) is injective. For a convex set \( C \subset [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all \(x, y, z \in [0, 1]\): if \(f(x)\) is between \(f(y)\) and \(f(z)\) then \(x\) is between \(y\) and \(z\).

(δ) For all \(x, y, z \in [0, 1]\): \(f(x)\) is between \(f(z)\) if and only if \(x\) is between \(y\) and \(z\).

Translating (α) – (δ) into the non-archimedean situation we arrive at the following conditions. Let \(X \subset K\) and \(f : X \to K\)

(α') \(f \in M_\mathbb{D}(X)\) and \(f\) is injective.

(β') \(f\) is weakly Darboux continuous and injective.

(γ') For all \(x, y, z \in X\), \(|x-y| < |x-z|\) implies \(|f(x)-f(y)| < |f(x)-f(z)|\).

(δ') \(f \in M_\mathbb{D}(X)\) and \(f\) satisfies (γ').

It will turn out that the conditions (α') – (γ') although not equivalent are closely related. We start with (γ'):

**DEFINITION 2.6** Let \(X \subset K\), \(f : X \to K\). We say that \(f \in M_\mathbb{S}(X)\) if for all \(x, y, z \in X\), \(f(x) \in [f(y), f(z)]\) implies \(x \subset [y, z]\).

**THEOREM 2.8** Let \(X \subset K\), \(f : X \to K\). Then the following statements are equivalent:

(a) \(f \in M_\mathbb{S}(X)\).

(b) \(f\) is injective and weakly Darboux continuous.

(c) \(f\) is injective and \(f^{-1} \in M_\mathbb{D}(f(X))\).

(δ) For all \(x, y, z \in X\), \(|f(x)-f(y)| = |f(x)-f(z)| \Rightarrow |x-y| = |x-z|\).

(ε) For all \(x, y, z \in X\), \(|x-y| < |x-z| \Rightarrow |f(x)-f(y)| < |f(x)-f(z)|\).

(ζ) For all \(x, y, z \in X\), \(|x-y| \neq |x-z| \Rightarrow |f(x)-f(y)| \neq |f(x)-f(z)|\).
Proof. The implications (a) $\implies$ (c) $\implies$ (ζ) $\implies$ (δ) are clear from the definitions.

(δ) $\implies$ (γ): injectivity follows from $|f(x)-f(y)| = |f(x)-f(y)| + |x-x| = |x-y|$. Use 2.2.(γ).

(γ) $\implies$ (β): Let $g : f(X) \to X$ be the inverse of $f$. Let $C \subset X$ be convex in $X$. Then since $g \in M_b$, $g^{-1}(C)$ is convex in $f(X)$. But $g^{-1}(C) = f(C)$.

Finally, we prove (β) $\implies$ (a). Let $f(x) \in [f(y),f(z)]$. By (β) the set $f([y,z] \cap X)$ is convex in $f(X)$ and it contains $f(y), f(z)$, hence $f(x) \in [f(y),f(z)] \cap X < f([y,z] \cap X)$. Since $f$ is injective, $x \in [y,z] \cap X$ and we are done.

We also have (compare 2.3)

THEOREM 2.9 Let $X \subset K$. Then

(i) For $a,b \in K$, $a \neq 0$ the map $x \mapsto ax+b$ is in $M_s(X)$.

(ii) If $f \in M_s(X)$, $\lambda \in K$, $\lambda \neq 0$ then $\lambda f \in M_s(X)$.

(iii) If $f_1,f_2,\ldots \in M_s(X)$, $\lim f_n = f$ pointwise, $f$ injective then $f \in M_s(X)$.

(iv) If $f \in M_s(X)$, $g \in M_s(f(X))$ then $g \circ f \in M_s(X)$.

Proof. Obvious verifications.

Returning to our conditions (a') $\implies$ (δ') we see that (β') is equivalent to (γ'), that (a') means $f^{-1} \in M_s(f(X))$ and that (δ') means $f \in M_b(X) \cap M_s(X)$.

Our $f$ of example 2.4 (5) is in $M_b$, injective but not in $M_s$. Its inverse yields an example of an $M_s$-function that is not in $M_b$. Thus, in general, we have neither one of the implications (a') $\implies$ (γ'), (γ') $\implies$ (a'), (β') $\implies$ (δ'), (a') $\implies$ (δ'). But our counterexample is
rather weird ($f$ is nowhere continuous and the domain of $f^{-1}$ is discrete). We can do better.

**EXAMPLE 2.10** Let $K$ have discrete valuation and let $k$ be infinite. Then there exists a homeomorphism of the unit ball of $K$ that is in $M_{db}$ but not in $M_s$. (The inverse map is in $M_s$ but not in $M_{db}$).

**Proof.** Set $X = \{a \in K : |a| \leq 1\}$ and let $R$ be a full set of representatives of the equivalence relation $x \sim y$ iff $|x-y| < 1$ in $X$. Then $R$ is infinite. Let $\pi \in K$ be such that $|\pi|$ is the largest value that is smaller than 1. The map

$$
(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \quad (a_i \in R \text{ for each } i)
$$

is a bijection of $R^{\mathbb{N}}$ onto $X$. We may suppose that $0 \in R$.

Since $R$ is infinite we can define injections

$$
\tau_1 : R \setminus \{0\} \to R
$$

$$
\tau_2 : R \to R
$$

such that $\im \tau_1 \cap \im \tau_2 = \emptyset$, $\im \tau_1 \cup \im \tau_2 = R$.

For $x = \sum_{n=0}^{\infty} a_n \pi^n \in X \ (a_n \in R \text{ for each } n)$ set

$$
f(x) := \begin{cases} 
\tau_1(a_0) + a_1 \pi + \ldots = x-a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\
\tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0
\end{cases}
$$

A simple inspection of the definition shows that $f$ is a bijection of $X$ onto $X$. If $a, b \in R$, $a \neq b$ then $|a\pi - b\pi| < |b\pi - a|$, whereas

$$
|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1 \quad \text{and} \quad |f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1,
$$

so $f \notin M_s(X)$. Finally, let $x, y, z \in X$ and $|x-y| \leq |x-z|$. We prove that $|f(x) - f(y)| \leq |f(x) - f(z)|$. If $|f(x) - f(z)| = 1$ there is nothing to prove, so suppose $|f(x) - f(z)| < 1$. Set $x = \sum a_n \pi^n$, $y = \sum b_n \pi^n$, $z = \sum c_n \pi^n$. 


If \( a_0 = 0 \) then also \( c_0 = 0 \) and \( \tau_2(a_1) = \tau_2(c_1) \) so \( a_1 = c_1 \), hence 
\( |x-z| \leq |\pi|^2 \). Since \( |x-y| \leq |x-z| \) we have also \( b_0 = 0, b_1 = a_1 \).
So, 
\[
\frac{x-y}{\pi}, \quad f(x)-f(y) = \frac{y-z}{\pi} \quad \text{whence} \quad |f(x)-f(y)| \leq |f(x)-f(z)|.
\]
If \( a_0 \neq 0 \) then \( \tau_1(a_0) = \tau_1(c_0) \) so \( a_0 = c_0 \). Then also \( c_0 = a_0 = b_0 \).
Then \( f(x)-f(y) = x-y, f(x)-f(z) = x-z \) whence 
\( |f(x)-f(y)| \leq |f(x)-f(z)| \).

Let \( X \subseteq K \). If \( f \in M_s(X) \) then \( f^{-1} \in M_B(f(X)) \). Conversely, if \( f \in M_B(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) then \( g \in M_s(f(X)) \). This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let \( X \subseteq K \) and \( f : X \to K \). \( f \) is called weakly monotone 
\((f \in M_w(X)) \) if for all \( x, y, z \in X \)
\[|x-y| < |x-z| \iff |f(x)-f(y)| \leq |f(x)-f(z)|\]
f is called strongly monotone \((f \in M_{bs}(X)) \) if 
\[f \in M_s(X) \cap M_B(X).\]

Clearly, \( f \in M_{bs}(X) \) if and only if \( f^{-1} \in M_{bs}(f(X)) \). Also, if \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) we have 
\[g \in M_w(f(X)).\]

Obviously we have \( M_B(X) \cup M_s(X) \subseteq M_w(X) \) and we will see from the
examples below that the inclusion may be strict. In section 4 we will
study the properties of \( M_w \)-functions, not for the sake of \( M_w \) itself
but in order to get results that are valid for \( M_B, M_s \) simultaneously.
The functions in \( M_{bs} \) behave reasonable and they may be viewd as the
non-archimedean equivalents of strict monotone functions in the real
case.
THEOREM 2.12 Let \( X \subseteq K \) and \( f : X \to K \). Then the following conditions are equivalent.

(a) \( f \in M_{bs}(X) \).

(b) \( f \) is injective and \( C \mapsto f(C) \) is a 1-1 correspondence between the relatively convex subsets of \( X \) and those of \( f(X) \).

(c) For all \( x, y, z \in X \) such that \( |x-y| < |x-z| \), \( |f(x)-f(y)| < |f(x)-f(z)| < |f(y)-f(z)| \).

(d) For all \( x, y, z \in X \) such that \( |x-y| = |x-z| \), \( |f(x)-f(y)| = |f(x)-f(z)| \).

(e) For all \( x, y, z \in X \) such that \( |x-y| \leq |x-z| \), \( |f(x)-f(y)| \leq |f(x)-f(z)| \).

(f) \( f \in M_s(X) \), \( f^{-1} \in M_s(f(X)) \).

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An \( M_w \)-function that is not in \( M_s \cup M_b \)). Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be any function, constant on the cosets of \( \{ x \in \mathbb{Z}_p : |x| < 1 \} \). Then \( f \in M_w(\mathbb{Z}_p) \). Clearly \( f \notin M_s(\mathbb{Z}_p) \), \( f \in M_b(\mathbb{Z}_p) \) if and only if the points of \( f(\mathbb{Z}_p) \) are equidistant.

(2) (Continuity of monotone functions). Let \( X \subseteq K \).

(a) Let \( f \in M_w(X) \). If \( f(X) \) has no isolated points, then \( f \) is continuous.

Proof. Let \( a \in X \) and \( \varepsilon > 0 \). Then there is \( z \in X \) such that \( z \neq a \), \( |f(z)-f(a)| < \varepsilon \). Let \( \delta := |z-a| \). Then for all \( x \in X \) with \( |x-a| < \delta \) we have, by the weak monotony of \( f \), \( |f(x)-f(a)| \leq |f(z)-f(a)| < \varepsilon \). It follows that if \( X \) and \( Y \) do not have isolated points and if \( f \) is an \( M_w \)-bijection of \( X \) onto \( Y \), then \( f \) is a homeomorphism of \( X \) onto \( Y \).
Conversely, it is easy to construct homeomorphisms of \( \mathbb{R}_p \) that are not in \( M^w_\mathbb{R}_p \).

(b) If \( K \) is a local field then every \( f \in M^w_\mathbb{R}(X) \) is continuous. (See 5.1 (i)).

(c) If \( K \) has discrete valuation then every \( f \in M_\mathbb{S}(X) \) is continuous.

(Example 2.4 (5) shows that such a statement is not true for \( f \in M_\mathbb{B}(X) \).)

(Proof. If \( f \) were not continuous at some \( a \in X \) then there would be an \( \varepsilon > 0 \) such that for some sequence converging to \( a \) we had 
\[ |f(x_n) - f(a)| \geq \varepsilon. \] We may suppose that \( |x_n - a| > |x_{n+1} - a| > \ldots \). Since the valuation is discrete we have \( \lim_{n \to \infty} |f(x_n) - f(a)| = 0 \), a contradiction.)

(d) In 5.14 we shall give an example of a function in \( M_{bs}(X) \) that is not continuous. (Of course, \( K \) will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" \( M^w, \ldots \)-conditions.

Thus, by definition, \( f \in M^w_u(X) \) if for all \( x,y,z,t \in X \)
\[ |x-y| < |z-t| \to |f(x)-f(y)| \leq |f(z)-f(t)|. \]

\( f \in M^w_{us}(X) \) if for all \( x,y,z,t \in X \)
\[ |x-y| < |z-t| \to |f(x)-f(y)| < |f(z)-f(t)|. \]

\( f \in M^w_{ubs}(X) \) if for all \( x,y,z,t \in X \)
\[ |x-y| < |z-t| \leftrightarrow |f(x)-f(y)| < |f(z)-f(t)|. \]

Notice that \( f \in M^w_{ubs}(X) \) means that \( |f(x)-f(y)| \) is a strictly increasing function of \( |x-y| \). Examples of such functions are isometries, but also the function \( f : \mathbb{R}_p \to \mathbb{R}_p \) defined via
\[ \sum a_n p^n \to \sum a_n p^{2n}, \quad (\sum a_n p^n \in \mathbb{R}_p) \]
\[ (|f(x)-f(y)| = |x-y|^2 \text{ for all } x,y \in \mathbb{R}_p. \)

Monotone functions : \( \mathbb{R} \to \mathbb{R} \) are divided into two classes: the
increasing functions and the decreasing functions. For the non-archimedean case we may ask for a similar classification. First we try to express the situation in the real case in such a way that it can be translated. Let \( a \in \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be strictly monotone. If \( x \) runs through some side of \( a \) then \( f(x) \) runs through some fixed side of \( f(a) \). So there is a map \( \sigma : \{-1,1\} \to \{-1,1\} \) such that \( \sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a)) \) (\( x \neq a \)). Apparently, the only \( \sigma \)'s that can occur are the identity and \( \sigma(x) = -x \) (\( x \in \{1,-1\} \)). Moreover it turns out that the map \( \sigma \) is independent of the choice of \( a \).

The two maps \( \sigma \) that can occur can be interpreted as multiplication maps (with 1 and -1 respectively) or as the bijections \( \{1,1\} \to \{-1,1\} \) and there seems to be no philosophical reason to make any decision of preference.

As an example, let us consider a function \( f \in M_{s}(K) \). Let \( a \in K \), let \( x \in a+\alpha \) and \( y \in a+\alpha \) (\( x,y \) are at the same side of \( a \)) then \( x-a, y-a \in \alpha \), so \( |x-y| < |y-a| \). Since \( f \in M_{s}(K) \) we have

\[
|f(x)-f(y)| < |f(y)-f(a)|,
\]
whence

\[
|f(x)-f(a) - (f(y)-f(a))| < |f(y)-f(a)|,
\]
so \( f(x)-f(a) \) and \( f(y)-f(a) \) have the same sign. We may conclude that there is a map \( \sigma_{a} : \Sigma \to \Sigma \) such that for all \( x \in K \)

\[
x \in a+\alpha + f(x) \in f(a)+\sigma_{a}(\alpha) \quad (\alpha \in \Sigma).
\]

Unfortunately, it turns out that in general \( \sigma_{a} \) may be different from \( \sigma_{b} \) if \( a \neq b \), even for isometrical maps. For example, let \( p \neq 2 \) and let \( \tau \) be a permutation of \( \{0,1,2,\ldots,p-1\} \) and define \( f : \mathbb{Z}_{p} \to \mathbb{Z}_{p} \) by

\[
\Sigma_{a_{n}}p^{n} \to \Sigma_{\tau(a_{n})p^{n}} \quad (a_{n} \in \{0,1,2,\ldots,p-1\} \text{ for each } n).
\]

Suppose we had a \( \sigma : \Sigma \to \Sigma \) such that for all \( x,y \in \mathbb{Z}_{p}, x-y \in \alpha \) implies \( f(x)-f(y) \in \sigma(\alpha) \). Let \( \alpha = \Theta_{p} \) (see 1.5). Then \( x-y \in \alpha \) means
\[ x = a_0 + a_1 p + \ldots + a_n p^n \]
\[ y = b_0 + b_1 p + \ldots + b_n p^n \]
where \( a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n - b_n = g^i \) modulo \( p \).

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n)) p^n + \ldots \), so \( \sigma(a) = \theta^j p^n \) where
\[ \tau(a_n) - \tau(b_n) = \theta^j \mod p. \] (\( j \) depending on \( i \) and \( n \)).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = \ldots = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \subset K, f : X \to K \) and let \( \sigma : \Sigma \to \Sigma \). We say that
\[ f \text{ is monotone of type } \sigma \text{ if for all } a \in \Sigma \text{ and all } x, y \in X \]
\[ x - y \in a \text{ implies } f(x) - f(y) \in \sigma(a). \]
(In other words if \( x \alpha y \) implies \( f(x) > \sigma(a) f(y) \), see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \to \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( a, x \alpha y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \subset K, f : X \to K, \beta \in \Sigma \). We say that \( f \) is monotone
of type \( \beta \) if for all \( a \in \Sigma \) and all \( x, y \in X \)
\[ x - y \in a \text{ implies } f(x) - f(y) \in \alpha \beta. \]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \).
where $\sigma : \Sigma \to \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$
is monotone of type $\beta$ iff the sign of $\frac{f(x) - f(y)}{x - y}$ is constant $\beta$ for all
$x, y \in X, x \neq y$. This leads to

**DEFINITION 2.16** Let $X \subset K$, $f : X \to K$. $f$ is called increasing if $f$ is
monotone of type 1. In other words, $f$ is increasing

if for all $x, y \in X, x \neq y$ the difference quotient

$$\frac{f(x) - f(y)}{x - y}$$

is positive, i.e., if

$$\left| \frac{f(x) - f(y)}{x - y} - 1 \right| < 1.$$

In the next section we shall study the monotone functions of type $\sigma$
and we will give a partial answer to the question for which maps

$\sigma : \Sigma \to \Sigma$ there exists an $f : K \to K$ that is monotone of type $\sigma$. 

3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $\sigma$.

DEFINITION 3.1. Let $X \subseteq K$, $f : X \rightarrow K$. Let $\Phi f(x,y) := \frac{f(x) - f(y)}{x-y}$ $(x,y \in X$, $x \neq y)$. $f$ is called
positive if $f(X) \subseteq K^+$
strictly positive if $\sup_{x \in X} |f(x) - 1| < 1$
increasing if $\Phi f(x,y) \in K^+$ for all $x,y \in X$, $x \neq y$
strictly increasing if $\sup_{x,y \in X} |1 - \Phi f(x,y)| : x,y \in X$, $x \neq y < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. Let $X \subseteq K$.

(i) If $f : X \rightarrow K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a, b \in K$ the function $x \mapsto ax + b$ is increasing if and only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly increasing).

(iii) If $f : X \rightarrow K$ is (strictly) increasing and $f$ is (strictly) positive then $- \frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \rightarrow K$ form a convex set.

(v) If $f : X \rightarrow K$ and $g : f(X) \rightarrow K$ are (strictly) increasing then so is $g \circ f$.

(vi) If $f : X \rightarrow K$ is (strictly) increasing then so is $f^{-1} : f(X) \rightarrow K$.

(vii) If $f_1, f_2, \ldots : X \rightarrow K$ are increasing and $f := \lim_{n} f_n$ pointwise
then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function

\[ \exp x = 1 + x + \frac{x^2}{2!} + \ldots \]

deﬁned on \( \{ x \in \mathbb{R} : |x| < p \} \) if \( \chi(k) = p \), \( \chi(x) = 0 \) and on \( \{ x \in \mathbb{R} : |x| < 1 \} \) if \( \chi(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f: \mathbb{R} \to \mathbb{R} \) be a \( C^2 \)-function (i.e., \( \forall \delta \) can continuously be extended to a function on \( \mathbb{R} \times \mathbb{R} \), assume that \( \mathbb{R} \) has no isolated points. See [2]) and suppose \( f'(a) \in \mathbb{R}^+ \) for some \( a \in \mathbb{R} \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta \), \( |y-a| < \delta \), \( x \neq y \) implies

\[ \left| \frac{f(x)-f(y)}{x-y} - f'(a) \right| \leq \delta. \]

For such \( x,y \) we have

\[ \left| \frac{f(x)-f(y)}{x-y} - 1 \right| \leq \left| \frac{f(x)-f(y)}{x-y} - f'(a) \right| + |f'(a) - 1| \leq \max(\delta, |f'(a) - 1|) < 1. \]

(3) The space \( C_0(\mathbb{R}) \) of all continuous functions \( \mathbb{R} \to \mathbb{R} \), is a Banach space with respect to the sup norm \( || \cdot ||_\infty \). Let \( e_0 := \xi_0 \) and for \( n \geq 1 \) let \( e_n := \xi_n \) where \( B_n := \{ x \in \mathbb{R} : |x-n| < 1 \} \). It is proved in [1] that \( e_0, e_1, \ldots \) is an orthonormal base of \( C_0(\mathbb{R}) \) i.e., for each \( f \in C_0(\mathbb{R}) \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that

\[ f = \sum_{n=0}^{\infty} \lambda_n e_n. \]
\[ ||f||_{\infty} = \max|\lambda_n|, \]

The coefficients \( \lambda_n \) can be reconstructed from \( f \) via

\[ \begin{align*}
\lambda_0 &= f(0) \\
\lambda_n &= f(n) - f(n_{-}) \quad (n \in \mathbb{N})
\end{align*} \]

where \( n_{-} \) is defined as \( a_0 + a_1 p + \ldots + a_{s-1} p^{s-1} \) if \( n \neq a_0 + a_1 p + \ldots + a_{s} p^{s} \) \((a_s \neq 0)\) in base \( p \).

Our aim is here to describe a necessary and sufficient condition for the \( \lambda_n \) in order that \( f = \sum \lambda_n e_n \) is increasing. We show

\[ f = \sum \lambda_n e_n \text{ is increasing if and only if for all } n \in \mathbb{N} \]
\[ |\lambda_n - (n-n_{-})| < |n-n_{-}|. \]

Proof. First observe that \( f \) is increasing if and only if for all \( x \in \mathbb{Z}_p \)

\[ f(x) = x + g(x) \]

where \( |\phi(g(x,y)| < 1 \) for all \( x,y \in \mathbb{Z}_p, x \neq y \).

As

\[ x = \sum_{n \geq 1} (n-n_{-}) e_n(x) \quad (x \in \mathbb{Z}) \]

it suffices to show that for \( g = \sum \lambda_n e_n \in \mathcal{C}(\mathbb{Z}) \) we have \( |\phi(g)| < 1 \) if and only if \( |\lambda_n| < |n-n_{-}| \) for all \( n \in \mathbb{N} \).

Suppose first \( |\phi(g)| < 1 \). Then for all \( n \in \mathbb{N} \), \( |f(n) - f(n_{-})| < 1 \), so \( |\lambda_n| = |f(n) - f(n_{-})| < |n-n_{-}|. \)

Conversely, let \( g = \sum \lambda_n e_n \) and let \( |\lambda_n| < |n-n_{-}| \) for all \( n \in \mathbb{N} \).

Let \( x,y \in \mathbb{Z}_p \) and let \( |x-y| = p^{-k} \) for some \( k \in \{0,1,2,\ldots\} \). Since \( e_n(a) = e_n(b) \) if and only if \( |a-b| < \frac{1}{n} \) we have

\[ e_n(x) = e_n(y) \quad \text{for } n < p^k. \]
Therefore

\[ |g(x)-g(y)| = \left| \sum_{n=1}^{\infty} \lambda_n (e_n(x)-e_n(y)) \right| = \left| \sum_{n=1}^{\infty} \lambda_n (e_n(x)-e_n(y)) \right| \leq \max_{n} |\lambda_n| \leq \max_{n} |n-n_n| = p^{-k} = |x-y| \]

so \(|\phi g| < 1\).

(4) Let \(K\) have dense valuation and let \(k\) be (countably) infinite. Let \(X\) be the unit ball of \(K\) and let \(B_i (i \in \mathbb{N})\) be the balls in \(X\) with radius \(1^i\). Choose \(c_1, c_2, \ldots \in K\) such that \(|c_1| < |c_2| < \ldots\), \(\lim |c_n| = 1\). For \(n \in \mathbb{N}\) define a function \(f_n : X \to K\) via

\[ f_n(x) = \begin{cases} x + c_i & \text{if } x \in B_i \quad (1 \leq i \leq n) \\ x & \text{elsewhere} \end{cases} \]

Then each \(f_n\) is strictly increasing (\(|\phi f_n(x,y) - 1| \leq \max_{1 \leq i, j \leq n} |c_i - c_j| \leq |c_n| < 1\)). The sequence \(f_1, f_2, \ldots\) converges pointwise to an increasing function \(f\). But \(f\) is not strictly increasing:

\[ \sup_{x \neq y} |\phi f(x,y) - 1| = \sup_{i,j} |c_i - c_j| = 1. \]

(Compare 3.2, (vii) and (viii).)

Increasing functions are closely related to functions \(g\) for which

\[ |g(x)-g(y)| < |x-y| \quad (x \neq y) \quad (\text{if } f \text{ is increasing, set } g(x) := f(x) - x). \]

**DEFINITION 3.4.** Let \((X, \rho)\) be an ultrametric space. A map \(g : X \to X\)

is called a pseudocontraction if \(\rho(f(x), f(y)) < \rho(x, y)\)

\((x, y \in X, x \neq y)\).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X 	imes X$ has a fix point. We have

**Lemma 3.5.** Let $(X, \rho)$ be an ultrametric space. Then the following conditions are equivalent.

(a) $X$ is spherically complete.

(b) Each pseudocontraction $X \times X$ has a fix point.

(c) Each pseudocontraction $X \times X$ has a unique fix point.

**Proof.** If $\sigma: X \times X$ is a pseudocontraction and if $x, y$ are fix points and $x \neq y$, then $\rho(x, y) = \rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction. Thus, we have (b) $\Rightarrow$ (c). We prove (a) $\Rightarrow$ (b). Let $B \subset X$ be a ball (i.e., either $B = \{x \in X: \rho(x, a) \leq r\}$ for some $a \in X, r \geq 0$ or $B = \{x \in X: \rho(x, a) < r\}$ for some $a \in X, r > 0$). We call $B$ invariant if $\sigma(B) \subset B$.

Now we observe two facts

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X: \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x), \sigma(a)) < \rho(x, a) \leq \max(\rho(x, \sigma(a)), \rho(\sigma(a), a)) = \rho(a, \sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \notin V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1, y \in B_2, \rho(x, y)$ does not depend on $x, y$, since for $z \in B_1, u \in B_2, \rho(x, z) < \rho(x, y)$ and $\rho(y, u) < \rho(x, y)$, so $\rho(z, u) = \rho(x, y)$. On the other hand, if $x \in B_1, y \in B_2$ then $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a smal-
lest invariant ball \( S \). If \( a \in S \), \( \sigma(a) \neq a \) then \( \{ x \in S : \rho(x, \sigma(a)) < \rho(a, \sigma(a)) \} \) is invariant and does not contain \( a \), a contradiction. Hence, \( \sigma \) has a fix point (actually, \( S \) is a singleton).

We prove (\( \beta \)) \( \Rightarrow \) (\( \alpha \)). If \( X \) were not spherically complete, there exist balls \( B_1 \supseteq B_2 \supseteq \ldots \) such that \( \cap B_n = \emptyset \). Choose \( x_n \in B_n \setminus B_{n+1} \) (\( n \in \mathbb{N} \)), set \( B_0 := X \) and define

\[
\sigma(x) := \begin{cases} 
 x & \text{if } x \in B_0 \setminus B_{n+1} \\
 x_{n+1} & \text{if } x \in B_n \setminus B_{n+1} \quad (n \in \{0,1,2,\ldots\}).
\end{cases}
\]

Then \( \sigma \) has obviously no fix point. Let \( x \in B_n \setminus B_{n+1} \) and \( y \in B_m \setminus B_{m+1} \), \( x \neq y \). If \( n = m \) then \( \sigma(x) = \sigma(y) \), so suppose \( n > m \). Then \( \sigma(x), \sigma(y) \) are both in \( B_{m+1} \), whereas \( x \in B_n \subseteq B_m \) and \( y \notin B_{m+1} \). Hence \( \rho(\sigma(x), \sigma(y)) < \rho(x, y) \). Then \( \sigma \) is a pseudocontraction without a fix point. Contradiction.

**Corollary 3.6.** The following conditions are equivalent.

1. (\( \alpha \)) \( K \) is spherically complete.
2. (\( \beta \)) If \( C \subseteq K \) is convex, \( f : C \to C \) is increasing then \( f \) is surjective.
3. (\( \gamma \)) If \( C \subseteq K \) is convex, \( f : C \to K \) is increasing then \( f(C) \) is convex.
4. (\( \delta \)) An increasing \( f : K \to K \) is surjective.

**Proof.** (\( \alpha \)) \( \Rightarrow \) (\( \beta \)). Choose \( a \in C \) and consider the map \( \sigma : x \mapsto x - f(x) + a \) (\( x \in C \)). Then \( \sigma : C \to C \). \( C \) is spherically complete, \( \sigma \) is a pseudocontraction. Hence, there is by 3.5 a \( c \in C \) for which \( \sigma(c) = c \) i.e., \( f(c) = a : f \) is surjective.

(\( \beta \)) \( \Rightarrow \) (\( \gamma \)). For a suitable \( s \in K \), \( f-s \) sends \( C \) into \( C \). (\( \gamma \)) \( \Rightarrow \) (\( \delta \)) is clear.

(\( \delta \)) \( \Rightarrow \) (\( \alpha \)). Let \( \sigma : K \to K \) be a pseudocontraction. Then \( x \mapsto x - \sigma(x) \)
is increasing hence is surjective. So then is \( x \in K \) for which \( x - \sigma(x) = 0 \), i.e., \( \sigma \) has a fix point. By 3.5, \( K \) is spherically complete.

In case \( f \) is strictly increasing we do not have to require that \( K \) is spherically complete:

**Theorem 3.7.** Let \( C \subseteq K \) be convex and let \( f : C + K \) be strictly increasing. Then \( f(C) \) is convex. If \( f(C) \subseteq C \), then \( f(C) = C \).

**Proof.** Reread the proof of (a) + (β), (β) + (γ) above. \( \sigma \) now is a contraction. \( C \) is complete. Apply the Banach contraction theorem.

Let \( X \) be a subset of \( \mathbb{R} \) and let \( f : X \to \mathbb{R} \) be a bounded increasing function. Then \( f \) can be extended to an increasing function \( \mathbb{R} \to \mathbb{R} \) by setting \( f(x) := \inf f \) if \( x < y \) for all \( y \in X \) and \( f(x) := \sup \{ f(y) : y \leq x, y \in X \} \) for all other \( x \in \mathbb{R} \). In our situation we can prove

**Theorem 3.8.** The following conditions are equivalent.

(a) \( K \) is spherically complete.

(β) For every \( X \subseteq K \) an increasing function \( f : X + K \) can be extended to an increasing \( \overline{f} : K + K \).

(γ) Let \( X \subseteq K \), and let \( f : X + K \) be a strictly increasing function. Then \( f \) can be extended to a strictly increasing function \( \overline{f} : K + K \) such that

\[
\sup_{x, y \in K} \left| \frac{f(x) - f(y)}{x - y} - 1 \right| = \sup_{x, y \in X} \left| \frac{f(x) - f(y)}{x - y} - 1 \right|
\]

**Proof.** (α) + (β). Let \( a \notin X \). By Zorn's Lemma it suffices to define \( \overline{f} \) such that \( \overline{f} \) is increasing on \( X \cup \{a\} \). We are done if we can find \( a \in K \) such that for \( x \in X \)
\[ \lim_{a \to x} \frac{f(x) - f(\alpha)}{x - \alpha} = l \]

i.e., \( a \in B := \bigcap_{x \in X} B_{x}(|x - \alpha|) \) \( (x \in X) \).

Now \( B \cap B \neq \emptyset \) \( (x, y \in X) \) since the distance of their centers is

\[ |f(x) - f(y) - (x - y)| = |f(x) - f(y) - (x - y)| = |\Phi(x, y) - 1||x - y| <\]

\( < \max(|x - \alpha|, |a - \alpha|) \). So if, say, \( |x - \alpha| \leq |y - \alpha| \), we see that \( |f(x) - (a - x) - f(y) - (a - y)| < |y - \alpha| \), whence \( f(x) - (a - x) \in B_{y} \). By the spherical completeness of \( K \) we have \( \bigcap_{x \in X} B_{x} \neq \emptyset \). Choose \( \alpha \in \bigcap_{x \in X} B_{x} \).

(\( \beta \) \( \Rightarrow \) (\( \alpha \)). Suppose \( K \) is not spherically complete. By 3.6, (\( \delta \) \( \Rightarrow \) (\( \alpha \))

there is a non surjective increasing function \( f: K \to K \). Then its inverse \( g: f(K) \to K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K \to K \).

(\( \beta \) \( \Leftrightarrow \) (\( \gamma \)) follows from the fact that \( (\Phi(x) = x \text{ for all } x) \)

\[ f \mapsto (1 - c)x + cf \quad (c \in K, |c| < 1) \]

is a !-1 correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \( |1 - \Phi(g)| < |c| \).

We will now investigate the relation between increasingness of \( f \)

and positivity of \( f' \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \to K \). Let \( \lambda \in K, 0 < |\lambda| < 1 \). Then \( x \mapsto x - \lambda \sigma(x) \)

is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \)
of \( K \) without isolated points and if \( f \) is differentiable then for each
\[ x \in X, f'(x) = \lim_{y \to x} \frac{f(x,y)}{y-x} \in K^+. \] So \( f' \) is positive. If, addition, \( f \) is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f: X \to K \) be a \( (\text{strictly}) \) positive Baire class 1 function. Then does \( f \) have a \( (\text{strictly}) \) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subset K \) and let \( f: X \to K \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots : X \to K \) such that \( |g_n| \leq 1 - \frac{1}{n} \) for each \( n \) and

\[ f = \sum_{n=1}^{\infty} g_n \quad (\text{pointwise}). \]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots : X \to K \) such that \( f = \lim_{n} f_n \) pointwise. There exist locally constant functions \( h_1, h_2, \ldots : X \to K \) such that \( |f_n - h_n| \leq 2^{-n} \), hence \( f = \lim_{n} h_n \) pointwise. Define \( t_1, t_2, \ldots : X \to K \) as follows

\[
t_n(x) = \begin{cases} 
  h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\
  0 & \text{if } |h_n(x)| > 1 - \frac{1}{n}
\end{cases}
\]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N} \). \( \{x \in X : |h_n(x)| \leq 1 - \frac{1}{n}\} \) is closed and open in \( X \). \( |t_n| \leq 1 - \frac{1}{n} \) and \( \lim_{n} t_n = f \). Now let \( g_1 := t_1 \), and \( g_n := t_n - t_{n-1} \) \( (n \geq 2) \). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n} \), \( f = \sum_{n=1}^{\infty} (t_n - t_{n-1}) = \sum_{n=1}^{\infty} g_n \).
LEMMA 3.10. Let $X \subseteq K$ have no isolated points and let $f : X \to K$ be a Baire class 1 function, $|f(x)| < 1$ for all $x \in X$. Then $f$ has an antiderivative $F$ for which
\[|F(x) - F(y)| < 1 \quad (x, y \in X, x \neq y).\]

Proof. By Lemma 3.9, $f = \sum_{n=1}^{\infty} f_n$, where each $f_n$ is locally constant,
\[|f_n| \leq 1 - \frac{1}{n}.\] By [2] 3.9 each $f_n$ has an antiderivative $F_n$ for which
\[|F_n(x) - F_n(y)| \leq \max \{|f_n(x)|, \frac{1}{2n}\}|x-y| \quad (x, y \in X).\]
By [2] 3.7, $F := \sum F_n$ is an antiderivative of $\sum f_n = f$. Now for $x, y \in X, x \neq y$:
\[|F(x) - F(y)| \leq \sup_n |F_n(x) - F_n(y)| \leq \sup_n \max \{|f_n(x)|, \frac{1}{2n}\}|x-y| \leq |x-y| \max \{|f_n(x)|, \frac{1}{2n}\}.\]
\[\text{and } \lim_{n} |f_n(x)| = 0 < 1. \text{ Hence } \max_n |f_n(x)| < 1. \text{ It follows that}\]
\[|F(x) - F(y)| < |x-y|.\]

THEOREM 3.11. Let $X \subseteq K$ have no isolated points and let $f : X \to K$ be (strictly) positive. Then $f$ has a (strictly) increasing antiderivative.

Proof. The function $x \mapsto f(x)-1$ has, by 3.10, an antiderivative $H$ such that $|\phi(H)| < 1$. Let $F(x) = x + H(x)$ ($x \in X$). Then $F' = f$ and $\phi(F) = 1 + \phi(H)$.
Thus, if $f$ is positive then $F$ is increasing. If $f$ is strictly positive then $|f(x)| - 1 < r < 1$ for all $x \in X$ and, by a trivial extension of 3.10, we may choose $H$ such that $|\phi(H)| < r$. It follows that $|\phi(F)| - 1 < r$, so $F$ is strictly increasing.
We collect the results in

**COROLLARY 3.12.** Let \( X \subset K \) have no isolated points. Then

(i) If \( f: X \to K \) is differentiable and (strictly) increasing then \( f' \) is a (strictly) positive Baire class 1 function.

(ii) If \( g: X \to K \) is a (strictly) positive Baire class 1 function then \( g \) has a (strictly) increasing antiderivative.

(iii) If \( f: X \to K \) is differentiable and if \( f' \) is (strictly) positive then \( f = g + h \) where \( g \) is (strictly) increasing and where \( h' = 0 \).

**Note.** We cannot strengthen 3.12 (iii) by replacing "\( h' = 0 \)" by "\( h \) is locally constant". In fact, if \( X = \mathbb{Z}_p \) then every locally constant function has bounded difference quotients. If our statements were true, then every differentiable \( f: \mathbb{Z}_p \to \Phi_p \) for which \( f' \) is positive would have bounded difference quotients.

But consider the function \( f: \mathbb{Z}_p \to \Phi_p \) defined via

\[
f(x) := \begin{cases} 
 p^n - 2n & \text{if } |x-p^n| < p^{-3n} (n \in \{0,1,2,\ldots\}) \\
 x & \text{elsewhere}
\end{cases}
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z}_p \). Let \( x_n := p^n \) and \( y_n := p^n + 3n \) (\( n \in \mathbb{N} \)). Then

\[
f(x_n) = p^n - 2n, \quad f(y_n) = p^n + 3n,
\]

so \( |f(x_n) - f(y_n)| = |p^{2n}| = p^{-2n} \), whereas \( |x_n - y_n| = |p^{3n}| = p^{-3n} \). So

\[
\lim_{n \to \infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n \to \infty} \frac{p^n}{p^{3n}} = \infty.
\]

We now study the connection between increasing \( C^1 \)-functions and continuous positive functions.

If \( f \) is a (strictly) increasing \( C^1 \)-function then clearly \( f' \) is a continuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f: X \to K$ be continuous and positive. Let $P: C(X) \to C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X).$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_i = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in \mathbb{N}$, let $x_n^\ast = x_n^0(x)$ where $x_n^0(x)$ is characterized by $|x_n^0(x)-x| < r_n$, $x_n^0(x) \in R_n$.

See [2] 5.3, 5.4.)

From [2] 5.4, it follows that $Pf$ is a $C^1$-antiderivative of $f$. It suffices to prove that $Pf$ is (strictly) increasing. Let $x, y \in X$, $x \neq y$,

$|x-y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x-y| < r_s$. We have

$x_1 = y_1, \ldots, x_s = y_s, x_{s+1} \neq y_{s+1}$. Further $|x_{n+1}^0 - x_n^0| \leq |x-y|$ (n>s),

$|y_{n+1}^0 - y_n^0| \leq |x-y|$ (n>s), $|x_{s+1}^0 - y_{s+1}^0| \leq |x-y|$. Hence (using the identity

$x = \sum (x_{n+1}^0 - x_n^0)+ x_1, y = \sum (y_{n+1}^0 - y_n^0)+ y_1, x_1 = y_1$)

$|(Pf(x)-Pf(y)-(x-y)| =$

$$= \sum_{n>s} (f(x_n)-1)(x_{n+1} - x_n) - \sum_{n>s} (f(y_n)-1)(y_{n+1} - y_n).$$

If $|f(x)-1| < \alpha$ for all $x \in X$, we have since $\lim f(x_n)-1$ exists,

$\sup_n |f(x_n)-1| < \alpha, \text{ similarly, } \sup_n |f(y_n)-1| < \alpha.$

So we get $|Pf(x)-Pf(y)-(x-y)| < \alpha |x-y|$.

Now suppose $|x-y| \geq r_1$. Then since for all $n: |x_{n+1}^0 - x_n^0| < r_1$, $|x_1^0 - y_1^0| = |x-y|$ we get (again under the assumption $|f(x)-1| < \alpha$ for all $x \in X$):

$$= \sum_{n>s} (f(x_n)-1)(x_{n+1} - x_n) - \sum_{n>s} (f(y_n)-1)(y_{n+1} - y_n).$$
We have proved:

**THEOREM 3.13.** Let \( X \subset K \) have no isolated points. Then the map \( P \) defined via

\[
(Pf)(x) = x + \sum_{n=1}^{\infty} f(x_n)(x_n - x) \quad (f \in C(X), x \in X)
\]

maps (strictly) positive functions into (strictly) increasing functions.

**COROLLARY 3.14.** Let \( X \subset K \) have no isolated points. Then if \( f \in C^1(X) \) and \( f' \) is (strictly) positive, then \( f = j + h \) where \( j \) is (strictly) increasing and \( h \) is locally constant.

**Proof.** By 3.12 we have \( f = j + h \) where \( j \) is (strictly) increasing and where \( h' = 0 \). Now by [2] Cor. 5.2 bis there is a locally constant function \( l: X \to K \) with \( \|l(h-1)\|_\infty < 1 \). Then \( s = j + (h-1) \) is (strictly) increasing, so we have \( t = s + l \), where \( s \) is (strictly) increasing and \( l \) is locally constant.

**Note.** We may also define convex functions. Let \( X \subset K \). A function \( f: X \to K \) is called **convex** if the second order difference quotient is positive. I.e., if for all \( x,y,z \in X \) \((x \neq y, y \neq z, x \neq z)\) we have

\[
\frac{\frac{f(x) - f(y)}{y-z} - \frac{f(x) - f(z)}{x-z}}{y-z} \in K^+
\]

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function \( f \) the function \( x \mapsto \phi f(x,y) \) defined on \( X \setminus \{y\} \) is an isometry, hence can be continuously extended to the whole of \( X \). Define \( \bar{\phi}(y,y) = \lim_{x \to y} \phi f(x,y) \) (\( y \in X \)). Thus, \( f \) is differentiable. For all \( x,y,z,t \in X \) we have

\[
|\phi f(x,y) - \phi f(z,t)| \leq \max(\max(\phi f(x,y) - \phi f(z,y)), \phi f(z,y) - \phi f(z,t)) \leq \max(|x-z|, |y-t|).
\]

Hence, \( \phi f \) is uniformly continuous on \( X \) i.e., \( f \) is strongly uniformly differentiable in the sense of [2] page 67.

For each \( y \in X \) the function \( x \mapsto \phi f(x,y) \) is increasing on \( X \).

If \( \chi(K) \neq 2 \) then convexity of \( f \) implies increasingness of \( \phi f \).

(Proof.

\[
\lim_{y \to x} \frac{\phi f(x,y) - \phi f(x',y)}{x-x'} = \frac{f'(x) - \phi f(x',x)}{x-x'} \in K^+ (x \neq x')
\]

\[
\lim_{y \to x'} \frac{\phi f(x,y) - \phi f(x',y)}{x-x'} = \frac{\phi f(x,x') - f'(x')}{x-x'} \in K^+ (x \neq x')
\]

so \( \frac{f'(x) - f'(x')}{x-x'} \in 2K^+ (x \neq x') \), whence \( \phi(f(x)) \in K^+ \) if \( x \neq x' \).

Of course, if \( f \in C^2(X) \) (see [2] 8.1) then convexity of \( f \) implies positivity of \( D_2f \) ([2] 8.4). So if \( \chi(K) \neq 2 \) then \( \phi f'' = D_2f \) ([2] 8.14) is positive. If \( \chi(K) = 2 \) then \( f'' = 0 \) for all \( C^2 \)-functions.

Note. The functions that are monotone of type \( \beta \) (\( \beta \in \mathcal{K} \)), see Def. 2.15, are easy to describe: \( f \) is monotone of type \( \beta \) if and only if \( b^{-1}f \) is increasing for any \( b \in \beta \).

We now turn to the functions \( X + K \) that are of type \( \sigma \) where \( \sigma : \Sigma \to \Sigma \). (2.14). For examples of such \( f \), where \( \sigma \) is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this section we will assume that \( X \) is an open convex subset of \( K \). This implies that the set \( \{ a \in \Sigma : \text{there is } x \in X \text{ such that } x > a \} \) is independent of \( a \in X \). Thus, \( X \) is homogeneous in the sense that \( (a + a) \cap X \neq \emptyset \) for some \( a \in X \), \( a \in \Sigma \) then for each \( b \in X \), \( (b + a) \cap X \neq \emptyset \).

Let \( \Sigma(X) := \{ a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y \} \). Then for each \( a \in X \)

\[
\Sigma(X) = \{ a \in \Sigma : x > a \text{ for some } x \in X \}.
\]

Either \( \Sigma(X) = K \) or \( \Sigma(X) = \{ a \in \Sigma : |a| < r \} \) for some \( r > 0 \) or \( \Sigma(X) = \{ a \in \Sigma : |a| \leq r \} \) for some \( r > 0 \). Hence \( \Sigma(X) \) is closed under \( \oplus \) (see 1.2) i.e., if \( a, b \in \Sigma(X) \) and \( a \oplus b \) is defined then \( a \oplus b \in \Sigma(X) \).

To be sure that \( f \) is monotone both of type \( \sigma \) and type \( \tau \) implies \( \sigma = \tau \) we define

**DEFINITION 3.15.** (Let \( X \subset K \) be open, convex and) let \( \sigma : \Sigma(X) \to \Sigma \).

\( f : X \to K \) is called monotone of type \( \sigma \) if for all \( x, y \in X \) and \( a \in \Sigma(X) \)

\[
x > y + f(x) > f(y).
\]

\( a \sigma(a) \)

**THEOREM 3.16.** Let \( f : X \to K \) be monotone of type \( \sigma \); \( \Sigma(X) \to \Sigma \). Then

(i) \( \sigma(-a) = -\sigma(a) \) (\( a \in \Sigma(X) \)).

(ii) Let \( a, b \in \Sigma(X) \). If \( \sigma(a) \oplus \sigma(b) \) is defined then so is \( a \oplus b \) and \( \sigma(a \oplus b) = \sigma(a) \oplus \sigma(b) \).

(iii) Let \( a, b \in \Sigma(X) \). If \( |a| < |b| \) then \( |\sigma(a)| < |\sigma(b)| \).

(iv) Let \( s \) be in the prime field of \( K \) and let \( |s| = 1 \). Then \( \sigma(sa) = s\sigma(a) \) (\( a \in \Sigma(X) \)).

(v) If \( \beta \in \Sigma(X) \), \( |\beta| = 1 \), \( \beta \) contains an element of the prime field of \( K \) then \( \sigma(\beta a) = \beta \sigma(a) \) for all \( a \in \Sigma(X) \).
(vi) \( f \in \mathcal{M}_{\text{us}}(X) \) (i.e., for all \( x, y, z, t \in X \), \( |x-y| < |z-t| \) implies \( |f(x)-f(y)| < |f(z)-f(t)| \)).

(vii) \( f \) is either nowhere continuous or uniformly continuous on \( X \).

Proof.

(i) Let \( x, y \in X \) such that \( x > y \). Then \( f(x)-f(y) \in \sigma(\alpha) \); \( f(y)-f(x) \in -\sigma(\alpha) \).

But also \( y > x \), hence \( f(y)-f(x) \in \sigma(-\alpha) \). So \(-\sigma(\alpha)\) and \( \sigma(-\alpha) \) are not disjoint and they must coincide.

(ii) Suppose \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined. If \( \alpha \oplus \beta \) were not, then \( \beta = -\alpha \) so, by (i), \( \sigma(\beta) = \sigma(-\alpha) = -\sigma(\alpha) \). Hence also \( \alpha \oplus \beta \) is defined. Choose \( x, y \in X \) with \( x > y \). There is \( z \in X \) such that \( y > z \). Then \( x-y \in \alpha \), \( y-z \in \beta \), so \( x-z \in \alpha \oplus \beta \). Further \( f(x)-f(y) \in \sigma(\alpha) \), \( f(y)-f(z) \in \sigma(\beta) \) so \( f(x)-f(z) \in \sigma(\alpha) \oplus \sigma(\beta) \). Also \( x-z \in \alpha \oplus \beta \), so \( f(x)-f(z) \in \sigma(\alpha \oplus \beta) \).

The signs \( \sigma(\alpha) \oplus \sigma(\beta) \) and \( \sigma(\alpha \oplus \beta) \) are not disjoint and they must coincide.

(iii) Let \( |\alpha| < |\beta| \). Choose \( x, y, z \) such that \( x-y \in \alpha \), \( y-z \in \beta \). Then (see 1.2 and preamble) \( f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(\alpha)+\sigma(\beta) \), \( x-z \in \alpha + \beta = \alpha \oplus \beta = \beta \), so \( f(x)-f(z) \in \sigma(\beta) \). Thus \( [\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta) \) is not empty. If \( \sigma(\alpha) \oplus \sigma(\beta) \) were not defined then \( \sigma(\alpha) = -\sigma(\beta) \) and \( \sigma(\alpha) + \sigma(\beta) \) would be a ball with center 0 and radius \( |\sigma(\beta)|^{-} \), but then \( [\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta) \) would be empty. Hence \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined and by (ii) we have \( \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta) \). By (1.2) (vi), \( |\sigma(\alpha)| < |\sigma(\beta)| \).

(iv) Let \( \chi(K) \neq 0 \). Then \( s = n \cdot 1 \) for some \( n \in \{1, 2, \ldots, \chi(K)-1\} \), so by 1.2 (vii), \( sa = \chi(a) = \sigma(a) \), \( sc(\alpha) = n\sigma(\alpha) = \sigma(\alpha) \). By a repeated application of (ii), we see \( \sigma(\alpha) = \sigma(\alpha) \). Hence \( \sigma(sa) = s\sigma(a) \). Let \( \chi(K) = 0 \). Let \( s \) be of the form \( n \cdot 1 \) for some \( n \in \mathbb{N} \). By a similar reasoning as above, \( \sigma(sa) = s\sigma(a) \). We may identify the prime field of \( K \) with \( \mathbb{Q} \).
Now observe that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) (\( n \in \mathbb{N} \)), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1, 1, 2, \ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N} \), \( n = s \mod p \) \((1 \leq s \leq p-1)\) then \( na = sa \) for all \( a \), so \( \sigma(na) = \sigma(sa) = \sigma(a) = n\sigma(a) \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and if \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). By (iii) \( |\sigma(\alpha)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( z-t \in \alpha \) for some \( \alpha \) hence \( f(z)-f(t) \in \sigma(\alpha) \). Thus, \( f(z) \neq f(t) \).

(vii) Let \( \rho := \inf_{x \neq y} |f(x)-f(y)| \). If \( \rho > 0 \) then clearly \( f \) is nowhere continuous. If \( \rho = 0 \), let \( \epsilon > 0 \). There is a, b \( \in X \), a \( \neq b \) such that \( |f(a)-f(b)| < \epsilon \). By (vi), for all \( x, y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \epsilon \). Then \( f \) is uniformly continuous.

A natural question that can be raised is the following. If \( f : X \rightarrow K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f : X \rightarrow K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
\((\beta)\) \(f \in M_d(X)\).

\((\gamma)\) \(f \in M_{\text{ubs}}(X)\).

\((\delta)\) If, for \(\alpha, \beta \in \Sigma(X)\), \(\alpha \oplus \beta\) is defined then so is \(\sigma(\alpha) \oplus \sigma(\beta)\) (and \(\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)\)).

\((\varepsilon)\) If \(\alpha, \beta \in \Sigma(X)\), \(|\sigma(\alpha)| < |\sigma(\beta)|\) then \(|\alpha| < |\beta|\).

**Proof.** We prove \((\alpha) \Rightarrow (\varepsilon) \Rightarrow (\gamma) \Rightarrow (\beta) \Rightarrow (\delta) \Rightarrow (\alpha)\).

\((\alpha) \Rightarrow (\varepsilon).\) Let \(|\sigma(\alpha)| < |\sigma(\beta)|\) then \(\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)\) (1.2. (vi)). By 3.16, (iii), \(\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)\). Since \(\sigma\) is injective, 
\(\alpha \oplus \beta = \beta\) so (again 1.2. (vi)) \(|\alpha| < |\beta|\).

\((\varepsilon) \Rightarrow (\gamma).\) Let \(|x-y| \leq |z-t|\) \((x,y,z,t \in X)\). We prove \(|f(x)-f(y)| \leq |f(z)-f(t)|\). If \(z = t\) there is nothing to prove. Assume \(z \neq t\) and 
\(|f(x)-f(y)| > |f(z)-f(t)|\). Then \((f\) is injective), supposing \(x-y \in \alpha, z-t \in \beta\) for some \(\alpha, \beta \in \Sigma(X)\), we have \(f(x)-f(y) \in \sigma(\alpha), f(z)-f(t) \in \sigma(\beta)\) and \(|\sigma(\alpha)| > |\sigma(\beta)|\). By \((\varepsilon), |\alpha| > |\beta|\) i.e., \(|x-y| > |z-t|\). Contradiction.

\((\gamma) \Rightarrow (\beta).\) Trivial.

\((\beta) \Rightarrow (\delta).\) Suppose \(\sigma(\alpha) \oplus \sigma(\beta)\) is not defined. Then \(|\sigma(\alpha)| = |\sigma(\beta)|\) and, by 3.16 (iii), \(|\alpha| = |\beta|\). Choose \(x,y,z\) such that \(x-y \in \alpha, y-z \in \beta\). Then \(f(x)-f(z) \in \sigma(\alpha) + \sigma(\beta)\) so \(|f(x)-f(z)| < |\sigma(\alpha)| = |f(x)-f(y)|\).

Since \(f \in M_d(X), |x-z| < |x-y|\) hence, since \(x-z \in \alpha \oplus \beta, x-y \in \alpha: \)
\(|\alpha \oplus \beta| < |\alpha|\). But \(|\alpha \oplus \beta| = \max(|\alpha|, |\beta|)\), a contradiction.

\((\delta) \Rightarrow (\alpha).\) Suppose \(\sigma(\alpha) = \sigma(\beta)\) and \(\alpha \neq \beta\). Then \(\alpha \oplus (-\beta)\) is defined. By \((\delta)\), also \(\sigma(\alpha) \oplus \sigma(-\beta)\) is defined. But \(\sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha)\), so 
\(\sigma(\alpha) \oplus -\sigma(\alpha)\) is defined, a contradiction.

**THEOREM 3.18.** Let \(k\) be a prime field. Then, if \(f : X \rightarrow K\) is monotone of type \(\sigma\) then \(\sigma\) is injective.
Proof. Suppose \( \sigma(a) = \sigma(\beta) \) for some \( a, \beta \in \Sigma(X) \). Then \( |\sigma(a)| = |\sigma(\beta)| \) so, by 3.16 (iii), \( |a| = |\beta| \). There is \( t \in K \), \( |t| = 1 \) such that \( \beta = ta \). Since \( k \) is a prime field we may suppose \( t \in \{1, 2, \ldots, p-1\} \) if \( k \approx \mathbb{F}_p \) and \( t \in \mathbb{Q}^* \) if \( k \approx \mathbb{Q} \). So, by 3.16 (iv), \( \sigma(\beta) = \sigma(ta) = t\sigma(a) = t\sigma(\beta) \). For \( x \in \sigma(\beta) \) we have \( tx \in \sigma(\beta) \), so \( tx \cdot x^{-1} \in K \) i.e., \( |t-1| < 1 \). It follows easily that \( t = 1 \). Hence, \( a = \beta \).

We now like to determine all \( \sigma : \Sigma \to \Sigma \) that "can occur" as the type of a monotone function in case \( K = \mathbb{Q}_p \). We use the fact that \( \Sigma \) can be identified with the following subgroup of \( \mathbb{Q}_p^* \)

\[
\{ \theta^{i,n} : i \in \{0, 1, 2, \ldots, p-2\}, n \in \mathbb{Z} \}
\]

where \( \theta \) is a primitive \( (p-1)^{th} \) root of 1. (See 1.5.)

First, let \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) be monotone of some type \( \sigma : \Sigma \to \Sigma \). By 3.18, \( \sigma \) is injective. By 3.17, (v), 3.16 (iii) we have \( |a| < |\beta| \Leftrightarrow |\sigma(a)| < |\sigma(\beta)| \) and \( |a| = |\beta| \Leftrightarrow |\sigma(a)| = |\sigma(\beta)| \), so \( |\sigma(a)| \) is a strictly increasing function of \( |a| \).

Set

\[
\sigma(\theta^{i,n}) = \theta^{s(i,n)} p^{\lambda(i,n)} \quad (\theta^{i,n} \in \Sigma)
\]

Where \( s : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \to \{0, 1, 2, \ldots, p-2\} \) and \( \lambda : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \to \mathbb{Z} \). We see that \( |\sigma(\theta^{i,n})| = |\sigma(\theta^{j,n})| \) for all \( i, j \in \{0, 1, 2, \ldots, p-2\} \) hence \( \lambda(i, n) = \lambda(j, n) \) for all \( i, j \in \{0, 1, 2, \ldots, p-2\} \). Then

\[
\sigma(\theta^{i,n}) = \theta^{s(i,n)} p^{\lambda(n)}
\]

where \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is a strictly increasing function (in the classical sense).

By 3.16 (v), \( \sigma(\theta^{i,n}) = \theta^{-1} \sigma(p^n) = \theta^{-1} \theta(0,n) p^{\lambda(n)} \).


Thus, $\sigma$ is of the form

\[(*) \quad \theta^i p^n \rightarrow \theta^i \theta^{s(n)} p \lambda(n)\]

where $s : \mathbb{N} \rightarrow \{0,1,2,\ldots, p-2\}$ and $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

Conversely, if we are given a map $\sigma$ of the form $(*)$ then it is easy to construct an $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, monotone of type $\sigma$. In fact, let $x \in \mathbb{Q}_p$, $x = \sum n a_n p^n$, where $a_n \in \{0,1,\ldots, p-2\}$ for each $n$ and $a_{-n} = 0$ for large $n$. Then set

\[f(x) = \sum_{n \in \mathbb{Z}} a_n \theta^{s(n)} p \lambda(n).\]

Now let $x = \sum n a_n p^n$, $y = \sum n b_n p^n$ and $\pi(x-y) = \theta^i p^n$ for some $i \in \{0,1,\ldots, p-2\}$, $n \in \mathbb{Z}$. Then $a_n = b_n$ for $n < m$ and $a_m - b_m = \theta^i \mod p$. So the sign of $a_m - b_m$ is $\theta^i$. $f(x) - f(y) = \sum n (a_n - b_n) \theta^{s(n)} p \lambda(n) = (a_m - b_m) \theta^{s(m)} p \lambda(m) + \tau$, where $n \geq m$ and $|\tau| < |f(x) - f(y)|$. The sign of $f(x) - f(y)$ is the sign of $(a_m - b_m) \theta^{s(m)} p \lambda(m)$ which is $\theta^i \theta^{s(m)} p \lambda(m)$. So $\pi(f(x) - f(y)) = \theta^i \theta^{s(m)} p \lambda(m) = \sigma(\theta^i p^n)$. Thus, $f$ is monotone of type $\sigma$. We have found

**Theorem 3.19.** The set \{
\begin{align*}
\sigma : \Sigma \rightarrow \Sigma, &
\text{ there is } f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p, \\
\text{monotone of type } \sigma
\end{align*}
\} is equal to the set of all $\sigma : \Sigma \rightarrow \Sigma$ of the form

\[(*) \quad \theta^i p^n \rightarrow \theta^i \theta^{s(n)} p \lambda(n)\]

where $s : \mathbb{Z} \rightarrow \{0,1,2,\ldots, p-2\}$ and $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is strictly increasing.

**Remark.** With the notations as in 3.19, let $\mu(n) := \lambda(n) - n$. Then $\mu : \mathbb{Z} \rightarrow \mathbb{Z}$ is increasing ($\mu(n+1) = \lambda(n+1) - (n+1) \geq \lambda(n+1) - (n+1) = \mu(n)$). We then have two possibilities for a function $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, monotone of type $\sigma$. 
(a) \( \lim_{n \to \infty} \mu(n) = \infty \). Then \( |\sigma(a)| = |a||p^{\mu(n)}| \), \( (a = \theta^{i_{p^n}}) \), so \( \lim_{|a| \to 0} |\sigma(a)| = 0 \). Thus \( \lim_{x-y \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0 \).

(b) \( \mu \) is bounded above. Then \( \mu(n) \) is constant, \( c \), for \( n \geq n_0 \). (For example, if \( \sigma \) is bijective then we have even \( \mu(n) = c \) for all \( n \).) Thus, for sufficiently small \(|a| \) \( (a = \theta^{i_{p^n}} \in \Sigma) \) we have

\[
|\sigma(a)| = |p^{\lambda(n)}| = |p^{n+c}| = |p^c||a|.
\]

So, there is \( r \) such that \(|x-y| < r \) implies \(|f(x)-f(y)| = |p^c||x-y|\).

In this case we then have: there is \( r > 0 \) and \( \lambda \in \mathbb{Q}_p \) such that on each ball in \( \mathbb{Q}_p \) of radius \( r \), \( \lambda^{-1} f \) is an isometry.

We now construct an example of a function \( f \) monotone of type \( \sigma \), where \( \sigma \) is not injective. Let \( p = 3 \mod 4 \) and let \( K := \mathbb{Q}_p(\sqrt{-1}) \). The elements of \( K \) can be written as \( a+bi \) \( (a,b \in \mathbb{Q}_p) \) and \(|a+bi| = \max(|a|,|b|)|. The value group of \( K \) is the same as the one of \( \mathbb{Q}_p \), the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.18). Let \( X \) be the unit ball of \( K \), let

\[
S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}.
\]

For each \( x \in X \) there is a unique \( \bar{x} \in S \) such that \(|x-\bar{x}| < 1 \). As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s,t \in S^*, s \neq t \).

Define a function \( h : S \to K \) as follows

\[
h(a+bi) = \frac{1}{p^a} a \quad (a+bi \in S)
\]
and let \( f : X \to K \) be defined via
\[
f(x) = x + h(x) \quad (x \in X).
\]

We claim that \( f \) is monotone of type \( \sigma \) where
\[
\sigma(\pi(a+bi)) = \pi \left( \frac{1}{p} a \right) \text{ if } a+bi \in S, \ a \neq 0 \quad \sigma(a) = a \quad \text{ elsewhere.}
\]

(Clearly, \( \sigma \) is a well defined map \( \mathbb{E}(X) \to \mathbb{K} \), \( \sigma \) is not injective since, for example, \( \sigma(\pi(1)) = \sigma(\pi(1+i)) \).

**Proof.** Let \( |a| < 1 \) and \( x-y \in \alpha \), then \( |x-y| < 1 \) so \( \overline{x} = \overline{y} \), \( h(x) = h(y) \).

It follows that \( f(x) - f(y) = x - y \in \alpha = \sigma(\alpha) \).

Now let \( |a| = 1 \) be of the form \( \pi(bi), \ b \in \{1,2,\ldots,p-1\} \) and let \( x-y \in \alpha \). Say, \( \overline{x} = r+si, \overline{y} = t+ui \) \( (r,s,t,u \in \{0,1,2,\ldots,p-1\}) \). Then also \( \overline{x-y} \in \alpha \), so \( |r+si-t+ui| < 1 \) hence \( r = t \). Thus, \( h(x) = \frac{1}{p} r = h(y) \),

and we have \( f(x) - f(y) = x - y \in \alpha = \sigma(\alpha) \).

Finally, let \( |a| = 1, \ a = \pi(a+bi) \), where \( a \neq 0 \ (a,b \in \{0,1,2,\ldots,p-1\}) \)

and let \( x-y \in \alpha \). Set \( \overline{x} = r+si, \overline{y} = t+ui \). Then \( \overline{x-y} \in \alpha \), so \( r-t = a \mod p \).

We find \( h(x) = \frac{1}{p} r, \ h(y) = \frac{1}{p} t \), so \( |h(x)-h(y)| = \frac{1}{p} |a| < \frac{1}{|p||a|} \) i.e.
\[
h(x)-h(y) \in \pi \left( \frac{1}{p} a \right). \quad \text{Since } |\pi(x-y)| < 1, \text{ we find } f(x)-f(y) = x-y-(h(x)-h(y)) \in \pi \left( \frac{1}{p} a \right) = \sigma(\pi(a+bi)) = \sigma(a).
\]

Concluding:

**EXAMPLE 3.20.** Let \( p = 3 \mod 4 \) and \( K = \mathbb{Q}_p (\sqrt{-1}) \). Then there exists a function \( f : \{x \in K : |x| \leq 1\} \to K \), monotone of some type \( \sigma \), where \( \sigma \) is not injective.

In case \( K \) has discrete valuation we have some extra information.
THEOREM 3.21. Let $K$ have discrete valuation and let $f : X \to K$ be monotone of type $\sigma \in \Sigma(X)$. Then

(i) $f$ is continuous.

(ii) If $\sigma$ is injective, then $f$ and $\phi(f)$ are bounded on bounded sets.

(iii) If $\sigma$ is injective and if $f(X)$ is convex then there is $\lambda \in K$ such that $\lambda f$ is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2)(c). If $\sigma$ is injective then by 3.16 (iii) and 3.17 (e), $|\sigma(a)|$ is a strictly increasing function of $|a|$. Suppose $X$ is bounded. Then $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r \in K^*$. Let $|\sigma(a)| = s$ whenever $|a| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|a| = |\pi|r$. Then $|a| < r$ so $|\sigma(a)| < s$, hence $|\sigma(a)| \leq |\pi|s$. By induction, it follows that $|\sigma(a)| \leq |\pi|^n s$ whenever $|a| = |\pi|^n r$, so $|\sigma(a)| \leq |a| \cdot \frac{s}{r}$. So the difference quotients of $f$ are bounded by $sr^{-1}$. Then clearly $f$ is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $\Sigma(f(X))$ has the form $\{a \in \Sigma : |a| \leq s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then $\sigma$ induces an injection of $\{\rho \in |K^*| : \rho \leq r\}$ onto $\{a \in |K^*| : |a| \leq s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(a)| = c|a|$ for some $c \in IR$ i.e., $|f(x) - f(y)| = |c||x-y|$ for all $x,y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type $\sigma$, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) $K$ is a local field ($f$ is continuous).

(2) $k$ is finite and $X = \{x \in K : |x| \leq 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be
representatives modulo \{x \in K:|x| < 1\}, let \(M = \max|f(a_i) - f(a_j)|\). For each \(x, y \in X\) we have \(i, j\) for which \(|x - a_i| < 1, |y - a_j| < 1\). Since \(f \in M_s(X)\), we have \(|f(x) - f(a_i)| < M, |f(y) - f(a_j)| < M\) whence \(|f(x) - f(y)| \leq M: f\) is bounded.\)

(3) \(K\) is discrete, \(\sigma\) is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let \(k\) be isomorphic to the algebraic closure of \(\mathbb{F}_p\). Let \(X\) be the unit ball of \(K\). Then there exists a function \(f : X \to K\), monotone of type \(\sigma\), for some \(\sigma : \Sigma(X) \to \Sigma\) such that

(i) \(\sigma\) is not injective.

(ii) \(f, \Phi(f)\) are unbounded.

Proof.

As an \(\mathbb{F}_p\)-vector space, \(k\) has a countable base \(e_1, e_2, \ldots\). For any \(\lambda \in \mathbb{F}_p\), \(\lambda = n\lambda_1\) for some \(n \in \{0, 1, 2, \ldots, p-1\}\). (Here for a field \(L\), \(1_L\) is the unit element of \(L\).) Define \(\lambda_1 := n\lambda_1\). Choose \(c_1, c_2, \ldots \in K\) such that
\[1 < |c_1| < |c_2| < \ldots, \lim_{n \to \infty} |c_n| = \infty,\]
and define a map \(h : k \to K\) via
\[h(\Sigma \lambda \cdot e_n) = \Sigma \lambda \cdot c_n \quad (\Sigma \lambda \cdot e_n \in k)\]

Define \(f : X \to K\) by
\[f(x) = x + h(\overline{x}) \quad (x \in X)\]
(Here \(\overline{x}\) is the image of \(x\) under the canonical map \(X \to k\)).

Then clearly \(f\) is unbounded and so is \(\Phi(f)\).

Let us identify \(\{a \in \Sigma:|a| = 1\}\) with \(k^*\) in the obvious way. We claim that \(f\) is monotone of type \(\sigma\) where
\( \sigma(a) = \begin{cases} 
\alpha & \text{if } \vert \alpha \vert < 1 \\
\pi(\sum_{n \in C} \lambda) & \text{if } \alpha = \sum_{m \in M} \lambda \cdot e_m, \ n = \max(m : \lambda_m \neq 0). 
\end{cases} \)

In fact, let \( x-y \in \alpha \) and \( \vert \alpha \vert < 1 \). Then \( h(x) = h(y) \) so \( f(x)-f(y) = x-y \in \sigma(\alpha) \). Now let \( x-y \in \alpha \) where \( \vert \alpha \vert = 1 \). Then set \( \bar{x} = \sum_{n \in N} \lambda \cdot e_n, \ \bar{y} = \sum_{n \in N} \mu \cdot e_n \).

Let \( r = \max(n : \lambda_n \neq \mu_n) \). Then \( \bar{x}-\bar{y} = \sum_{n=1}^{r} (\lambda_n-\mu_n) e_n = \alpha \), so \( \sigma(\alpha) = \pi((\sum_{n \in N/r} \lambda_n-\mu_n) c_n) \).

On the other hand, \( f(x)-f(y) = x-y-\bar{h}(x)-h(y) = x-y-\sum_{n=1}^{r} (\lambda_n-\mu_n) c_n = \sum_{n=1}^{r-n} (\lambda_n-\mu_n) c_n \). Thus \( \pi(f(x)-f(y)) = \pi((\sum_{n \in N/r} \lambda_n-\mu_n) c_n) \).

Now we have \( \sum_{n \in N/r} \lambda_n-\mu_n \equiv \lambda_r-\mu_r \mod p \), so \( \pi(\lambda_r-\mu_r) = \pi(\lambda_r-\mu_r) \). It follows that \( f(x)-f(y) \in \sigma(\alpha) \).

Obviously, \( \sigma \) is not injective.

We will consider briefly the differentiable functions, monotone of type \( \sigma \). Let \( f : X \to \mathbb{R} \) be such a function. If \( f'(a) = 0 \) for some \( a \in X \), then \( \lim_{|a| \to 0} \frac{|\sigma(a)|}{|a|} = 0 \), so \( f' = 0 \) uniformly on \( X \). Now let \( f'(a) \neq 0 \). Then if \( x \) is sufficiently close to \( a \), we have \( \frac{f(x)-f(a)}{x-a} = f'(a) \). Thus for \( |a| \) small enough we have \( f'(a) \in \sigma(a) \alpha \) i.e. \( \frac{\sigma(a)}{\alpha} \) is constant.

This implies that \( \pi(f'(x)) \) does not depend on \( x \) (\( f' \) has constant sign) and that locally \( f \) is monotone of type \( \beta \) for some \( \beta \in \Sigma \).

We end this section with a discussion on the question which maps \( \sigma : \Sigma \to \Sigma \) do occur as a type of a monotone function.

**Lemma 3.23.** Let \( \sigma : \Sigma \to \Sigma \). Suppose \( \sigma \) satisfies

if \( \alpha @ \beta \) is defined then \( \sigma(\alpha @ \beta) = \sigma(\alpha) @ \sigma(\beta) \). \((\alpha, \beta \in \Sigma) \).

Then

(i) \( \sigma(-\alpha) = -\sigma(\alpha) \) \((\alpha \in \Sigma) \).
(ii) If \( \sigma(a) \) is defined then so is \( a \oplus \beta \).

(iii) If \( |a| < |\beta| \) then \( |\sigma(a)| < |\sigma(\beta)| \).

\( \sigma \) is injective.

(v) If \( |a| = |\beta| \) then \( |\sigma(a)| = |\sigma(\beta)| \).

Proof. (i) is trivial if \( \chi(k) = 2 \), so suppose \( \chi(k) \neq 2 \) and let \( -\sigma(a) \neq \sigma(-a) \)
for some \( a \in \Sigma \). Then we have the identity \( (a \oplus a) \oplus (-a) = a \), so
\( \sigma(a \oplus a) \oplus \sigma(-a) = \sigma(a) \), whence \( (\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a) = \sigma(a) \). Now by
1.2 (iii) \( \sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a)) \) (this last expression is defined).
If not, then \( -\sigma(a) = \sigma(a) \oplus \sigma(-a) \). Now \( \sigma(a) \oplus \gamma = -\sigma(a) \) has only one
solution namely \( \gamma = -2\sigma(a) \). So we then would have \( \sigma(-a) = -2\sigma(a) = -(\sigma(a)
\oplus \sigma(a)) \), but this contradicts the existence of \( (\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a)) \).
From \( \sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a)) \) we obtain by 1.2 (vi): \( |\sigma(a) \oplus \sigma(-a)| < |\sigma(a)| \). On the other hand, by 1.2 (v), \( |\sigma(a) \oplus \sigma(-a)| = |\sigma(a)| \lor |\sigma(-a)| \).
Contradiction. (i) follows.

Now (ii) follows easily from (i): if \( a \oplus \beta \) were not defined then \( \beta = -a \)
so, by (i), \( \sigma(a) \oplus \sigma(\beta) = \sigma(a) \oplus -\sigma(a) \), a contradiction. Let \( |a| < |\beta| \),
then \( a \oplus \beta = \beta \), so \( \sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta) = \sigma(\beta) \). By 1.2 (vi) we find
\( |\sigma(a)| < |\sigma(\beta)| \). We proved (iii).

If \( \sigma(a) = \sigma(\beta) \) and \( a \neq \beta \) then \( \sigma(a \oplus (-\beta)) = \sigma(a) \oplus \sigma(-\beta) = \sigma(a) \oplus -\sigma(a) \),
an absurdity. So \( \sigma \) is injective (iv). Finally, let \( |a| = |\beta| \) and
\( |\sigma(a)| > |\sigma(\beta)| \). Then \( \sigma(a) = \sigma(a) \oplus \sigma(\beta) \) (by (ii)) = \( \sigma(a \oplus \beta) \). By in­
jectivity of \( \sigma \), \( a = a \oplus \beta \), and by 1.2 (vi), we find \( |\beta| < |a| \).

Now we have
LEMMA 3.24. Let $K$ be spherically complete, let $Y \subseteq K$ (not necessarily convex) and let $\tau : \Sigma(Y) (= \{ \pi(x-y) : x, y \in Y, x \neq y \}) \rightarrow \Sigma$ such that $f$ is monotone of type $\tau$ (i.e., $x, y \in Y, x-y \in \alpha \in \Sigma(Y)$ then $f(x)-f(y) \in \tau(\alpha)$.

Suppose $\tau$ can be extended to a $\sigma : \Sigma \rightarrow \Sigma$ satisfying the condition of Lemma 3.23. Then $f$ can be extended to a monotone function $f : K \rightarrow K$ of type $\sigma$.

Proof. By Zorn's lemma, it suffices to extend $f$ to $Y \cup \{a\} (a \neq Y)$ such that $f(x) - f(a) \in \sigma(\pi(x-a))$, $f(a) - f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x)) (x \in Y)$. Each $B_x$ is a ball with radius $|\pi(a-x)|$. By the spherical completeness of $K$, we are done if we can show that $B_x \cap B_y \neq \emptyset (x \neq y, x, y \in Y)$.

Set $a := \pi(a-x)$ and $b := \pi(a-y)$. Let $a \in \sigma(a); c \in \sigma(b)$. We prove:

$$|f(x)+b-f(y)-c| < |\sigma(a)| \lor |\sigma(b)|.$$  

We have two cases:

1) $a = b$. Then $a-x \in a, a-y \in a$ implies $|x-y| < |a-x| = |a|$, so $|\pi(x-y)| < |a|$ whence $|\pi(f(x)-f(y))| = |\sigma(\pi(x-y))| < |\sigma(a)|$ (by 3.23 (iii)), so $|f(x)-f(y)| < |\sigma(a)|$. Further, $b \in \sigma(a), c \in \sigma(a)$ implies $|b-c| < |\sigma(a)|$, hence $|f(x)+b-f(y)-c| < |\sigma(a)|$.

2) $a \neq b$. Then $x-y = a-y-(a-x) \in b \oplus (-a)$, so $f(x)-f(y)+b-c \in \sigma(b \oplus (-a)) + \sigma(a) + \sigma(-b) = \sigma(b \oplus (-a)) + \sigma(a \oplus -b) = \sigma(b \oplus (-a))-\sigma(b \oplus -a)$, hence $|f(x)-f(y)+b-c| < |\sigma(b \oplus (-a)| = |\sigma(b) \oplus \sigma(-a)| = \max(|\sigma(a)|, |\sigma(b)|)$.

THEOREM 3.25. Let $K$ be spherically complete and let $\sigma : \Sigma \rightarrow \Sigma$. Suppose

$$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).$$

Then there exists a function $f : K \rightarrow K$, monotone of type $\sigma$. 
Proof. Choose $Y := \{0\}$ and let $g : Y \to K$ be defined via $g(0) = 0$. Then $g$ satisfies the conditions of Lemma 3.24 so it can be extended to a function $f$ of type $\sigma$.

We now give a description of the maps $\sigma : \Sigma \to \Sigma$ mentioned in 3.23. For each $r \in |K^*|$ choose $\alpha_r \in \Sigma$ such that $|\alpha_r| = r$. Further, there is a natural isomorphism of multiplicative groups between $k^*$ and $\{\alpha \in \Sigma : |\alpha| = 1\}$, denoted by $l \mapsto \alpha_1$ ($l \in k^*$). Of course, if $l+1' \neq 0$ then $\alpha_{l+1'} = \alpha_l \otimes \alpha_1$. Each element of $\Sigma$ can be written in only one way as $\alpha_r \alpha_1$ ($r \in |K^*|$, $l \in k^*$). Now if $\sigma$ is as in 3.23 we get

$$\sigma(\alpha_r \alpha_1) = \alpha_{\lambda(r)} \alpha(n(r,1))$$

where $\lambda : |K^*| \to |K^*|$ is strictly increasing and $l \mapsto n(r,1)$ is an injective group endomorphism of the additive group $k$. Conversely, if $\lambda : |K^*| \to |K^*|$ is strictly increasing and for each $r$, $l \mapsto n(r,1)$ is an injective group homomorphism $k \to k$ then

$$\alpha_r \alpha_1 \mapsto \alpha_{\lambda(r)} \alpha(n(r,1)) \quad (\alpha_r \alpha_1 \in \Sigma)$$

satisfies the condition of 3.23. So we get

**Theorem 3.26.** Let $K$ be spherically complete and let $|K| = [0,\infty)$. Then there exist a nowhere continuous $f : K \to K$, monotone of some type $\sigma : \Sigma \to \Sigma$.

Proof. With the notations as above, let $\sigma : \Sigma \to \Sigma$ be defined as follows

$$\sigma(\alpha_r \alpha_1) = \alpha_{r+1} \alpha_1.$$  

By 3.25 there is an $f : K \to K$ monotone of type $\sigma$. Clearly $|f(x) - f(y)| \geq 1$ if $x \neq y$ so $f$ is nowhere continuous.
In this section we study $M_w(X)$, $M_b(X)$, $M_s(X)$, etc. To avoid unnecessary complications we assume throughout this section that $X$ is a closed subset of $K$ without isolated points. We collect here the results on monotone functions that are valid for general $K$. In the next section we will see what happens if we put some extra conditions on $K$ (e.g., $|K|$ discrete, ...).

First two elementary lemmas.

**Lemma 4.1** Let $f : X \rightarrow K$. Then the following conditions are equivalent

(a) $f \in M_w(X)$ (see Def. 2.11).

(b) For all $x, y, z \in X$, $|x-y| < |x-z|$ implies $|f(x)-f(z)| = |f(y)-f(z)|$.

(c) For all $x, y, z \in X$, $|f(x)-f(z)| \neq |f(y)-f(z)|$ implies $|x-y| = \max(|x-z|, |y-z|)$.

**Proof.** (a) $\Rightarrow$ (b). $|x-y| < |x-z|$ implies $|y-z| = |x-z| > |x-y|$, so

$$|f(x)-f(y)| \leq \min(|f(x)-f(z)|, |f(y)-f(z)|).$$

It follows that $|f(x)-f(z)| = |f(y)-f(z)|$.

(b) $\Rightarrow$ (c). (b) says that $|f(x)-f(z)| \neq |f(y)-f(z)|$ implies $|x-y| \geq |x-z|$. By symmetry, also $|x-y| \geq |y-z|$ where

$$|x-y| \geq \max(|x-z|, |y-z|).$$

The opposite inequality is trivial.

(c) $\Rightarrow$ (a). Let $|x-y| < |x-z|$. Then $|x-y| \neq \max(|x-z|, |z-y|)$ so, by (c), $|f(x)-f(z)| = |f(y)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(y)-f(z)|) = |f(x)-f(z)|$.

**Lemma 4.2** (i) If $f \in M_w(X)$, $\lambda \in K$ then $\lambda f \in M_w(X)$.
(ii) If \( f_1, f_2, \ldots \in M_w(X) \) and \( f := \lim_{n \to \infty} f_n \) pointwise then \( f \in M_w(X) \).

(iii) If \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \), then \( g \in M_w(f(X)) \). In particular, if \( f \) is injective and weakly monotone then so is \( f^{-1} \).

(Notice that \( f(X) \) need not be closed and may have isolated points.)

Proof. Obvious.

Remark. Lemmas, similar to 4.1 and 4.3, but now for \( M_b(X) \), \( M_s(X) \), \( M_{bs}(X) \) have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an \( M_w \)-function need not be continuous (see 2.4(5), 3.26) we will derive properties of \( M_w \)-functions that are closely related to continuity.

Lemma 4.3 Let \( f \in M_w(X) \). Then \( f \) is bounded on precompact subsets of \( X \).

Proof. Let \( Y \subset X \) be precompact. Assume that \( Y \) is not a singleton. Then \( Y \) is bounded and has a positive diameter \( r = \max \{|x-y| : x, y \in Y \} \).

The equivalence relation \( x \sim y \iff |x-y| < r \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) \((n \geq 2)\). Choose \( a_i \in Y_i \) for each \( i \), and let \( M := \max_{1 \leq i \leq n} |f(a_i)| \). We prove: \( |f| \leq M \). In fact, let \( x \in Y \). Then there is \( i \) such that \( |x-a_i| < r \). Choose \( j \neq i \). We have \( |x-a_i| < |a_i-a_j| \) whence \( |f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M \). So \( |f(x)| \leq M \).

The following lemma shows that an \( f \in M_w(X) \) at \( a \in X \) is either continuous or "very discontinuous".

Lemma 4.4 Let \( f \in M_w(X) \) and let \( a \in X \). Then we have the following alternative.
Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \in X \) (\( x_n \neq a \) for all \( n \)) with \( \lim x_n = a \) the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

Proof. Since \( \{x_1, x_2, \ldots \} \) is precompact the set \( \{f(x_1), f(x_2), \ldots \} \) is bounded by Lemma 4.3. We are done if we can prove the following. If \( x_1, x_2, \ldots, \lim x_n = a, x_n \neq a \) for all \( n \), \( \lim f(x_n) \) exists, then \( f \) is continuous at \( a \). Now set \( \alpha := \lim f(x_n) \). Let \( y_1, y_2, \ldots \in X \), \( \lim y_n = a \).

We prove \( \lim f(y_n) = \alpha \). (Then it follows that \( \alpha = f(a) \) since we may choose \( y_n := a \) for all \( n \).) Let \( \varepsilon > 0 \). There is \( k \in \mathbb{N} \) for which \( |f(x_k) - \alpha| < \varepsilon \). For \( n \) sufficiently large we have \( |y_n - x_n| < |x_k - x_n| \), so for large \( m \) (depending on \( n \)) we have \( |y_m - x_m| < |x_k - x_m| \), whence \( |f(y_n) - f(x_m)| \leq |f(x_k) - f(x_m)| \). Since \( \lim f(x_m) = \alpha \) we find \( |f(y_n) - \alpha| \leq |f(x_k) - \alpha| < \varepsilon \), so \( \lim f(y_n) = \alpha \).

COROLLARY 4.5 Let \( f \in M^w (X) \). Then the graph of \( f \)
\[
\Gamma_f := \{(x, y) \in X \times K : y = f(x)\}
\]
is closed in \( K^2 \).

Proof. Let \( (x_n, f(x_n)) \in \Gamma_f \) and let \( \lim x_n = x, \lim f(x_n) = \alpha \). If \( x_n = x \) for infinitely many \( n \) then \( \alpha = f(x) \), so \( (x, \alpha) \in \Gamma_f \). If not then by the alternative of lemma 4.4, \( f \) is continuous at \( x \), so \( \alpha = f(x) \) and \( (x, \alpha) \in \Gamma_f \).

COROLLARY 4.6 Let \( f \in M^w (X) \). If each bounded subset of \( f(X) \) is precompact then \( f \) is continuous.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let $f \in M_w(X)$ and let $Y \subset f(X)$ be precompact. Then either

$f$ is constant on $f^{-1}(Y)$ or $f^{-1}(Y)$ is bounded.

Proof. It suffices to prove: if $Z \subset X$ is unbounded and $f(Z)$ is precompact then $f$ is constant on $Z$. Let $a, b \in Z$. Since $Z$ is unbounded there are $x_1, x_2, \ldots \in Z$ such that

$$(*) \quad |a-b| < |x_1-a| < |x_2-a| < \ldots$$

Since $f(Z)$ is precompact we may assume (by taking a suitable subsequence) that $a = \lim f(x_n)$ exists. From $(*)$ we obtain

$$|x_1-x_2| = |x_2-a|, \quad |x_2-x_3| = |x_3-a|, \ldots,$$

so

$$|a-b| < |x_1-a| < |x_1-x_2| < |x_2-x_3| < \ldots$$

hence

$$|f(a)-f(b)| \leq |f(x_1)-f(a)| \leq |f(x_1)-f(x_2)| \leq \ldots$$

it follows that $|f(a)-f(b)| = \lim_{n \to \infty} |f(x_n)-f(x_{n+1})| = 0$ i.e., $f(a) = f(b)$.

LEMMA 4.8 Let $f \in M_w(X)$ and let $a \in f(X)$ be a non-isolated point of $f(X)$.

Then we have the following alternative. Either

I. There is $a \in X$ such that for each sequence $x_1, x_2, \ldots$ in $X$ for which $\lim_{n \to \infty} f(x_n) = a$ we have $\lim_{n \to \infty} x_n = a$, or

II. If $x_1, x_2, \ldots \in X$, $\lim_{n \to \infty} f(x_n) = a$, $f(x_n) \neq a$ for all $n$,

then $x_1, x_2, \ldots$ is bounded and has no convergent subsequence.

Proof. Suppose we are not in case II. Since $a$ is not isolated in $f(X)$ and $f(X)$ is dense in $f(X)$ we have a sequence $x_1, x_2, \ldots$ in $X$ for which $f(x_n) \neq a$ for each $n$, and $\lim_{n \to \infty} f(x_n) = a$. Since $f$ is not constant on $x_1, x_2, \ldots$ it follows by Lemma 4.7 that $x_1, x_2, \ldots$ is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by $x_1, x_2, \ldots$ and set
a := \lim_{n} x_n. Then a ∈ X. Now let y_1, y_2, ... be a sequence in X for which \lim f(y_n) = a. We prove that \lim y_n = a. In fact, let ε > 0.

There is k ∈ \mathbb{N} such that |x_k - a| < ε. For large n we have

|f(y_n) - a| < |f(x_k) - a|, so for large m (depending on n) we have

|f(y_n) - f(x_m)| < |f(x_k) - f(x_m)| whence |y_n - x_m| ≤ |x_k - x_m|, so

|y_n - a| ≤ |x_k - a| < ε.

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function f : X → K injective at a ∈ X if f(x) = f(a) for some x ∈ X implies x = a.

Now suppose that we have a ∈ \overline{f(X)}, not isolated, for which we are in alternative I. Then for a sequence x_1, x_2, ... with \lim f(x_n) = a we have lim x_n = a ∈ X so (a, a) = \lim (x_n, f(x_n)), so by Cor.4.5 we have a = f(a). Thus, a ∈ f(X). f is injective at a; if f(b) = f(a) then since \lim f(b) = a we must have \lim b = a i.e. b = a. Further, f is continuous at a (see 2.13 (2)(a)).

If each bounded subset of X is precompact we never can be in case II. This is also true if f ∈ M_b(X) and |X| is discrete i.e. if x_1, y_1 ∈ X |x_1 - y_1| > |x_2 - y_2| > ... then \lim |x_n - y_n| = 0. Proof: let a ∈ \overline{f(X)} and let \lim f(x_n) = a, f(x_n) ≠ a for all n. Without loss of generality we may assume

|a - f(x_1)| > |a - f(x_2)| > ... hence |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > ... and, since f ∈ M_b(X)

|x_1 - x_2| > |x_2 - x_3| > ...

Since |X| is discrete, the sequence x_1, x_2, ... is convergent. So we have case I. We find
THEOREM 4.9 Let either each closed and bounded subset of $X$ be compact and $f \in M_w(X)$, or let $|X|$ be discrete and $f \in M_b(X)$. Then

(i) $f(X)$ is closed (in $K$).

(ii) If $f(a) \in f(X)$ is not isolated then $f$ is injective at $a$, $f$ is continuous at $a$. In particular if $f(X)$ has no isolated points then $f$ is a homeomorphism $X \cong f(X)$.

Proof. If $a \in f(X) \setminus f(X)$ then $a$ is not isolated. Since we are in alternative I of Lemma 4.8, $a \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an $f$ satisfying the conditions of 4.9 maps closed subsets of $X$ (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let $f : X \rightarrow K$.

(i) If $f \in M_w(X)$ and if $Y \subset X$ is closed and the closed and bounded subsets of $Y$ are compact then $f(Y)$ is spherically complete.

(ii) If $f \in M_b(X)$ and if $Y \subset X$ is spherically complete then so is $f(Y)$.

(iii) If $f \in M_s(X)$ and if $A \subset f(X)$ is spherically complete then so is $f^{-1}(A)$.

Proof. (1) Let $B_1 \supsetneq B_2 \supsetneq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1 \setminus B_2$, $f(y_2) \in B_2 \setminus B_3$, $\ldots$ Then we have

$$|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots$$

and, by the weak monotony of $f$
\[ |y_1 - y_2| \geq |y_2 - y_3| \geq \ldots \]

Suppose first that \( \lim |y_n - y_{n+1}| = 0 \). Then \( y := \lim y_n \) and there are infinitely many \( k \) for which

\[ |y_k - y_{k-1}| > |y_{k+1} - y_k| \]

Now \( |y - y_k| \leq \max(|y_k - y_{k+1}|, |y_{k+1} - y_k|, \ldots) \leq |y_k - y_{k+1}| \). So we get for infinitely many \( k \)

\[ |y - y_k| < |y_k - y_{k-1}| \]

whence

\[ |f(y) - f(y_k)| \leq |f(y_k) - f(y_{k-1})| \]

so \( f(y) \in B_{k-1} \) for infinitely many \( k \), i.e., \( f(y) \in \cap B_k \).

Next, suppose that \( |y_{k+1} - y_k| \geq \varepsilon > 0 \) for all \( k \). Then since \( y_1, y_2, \ldots \)

is bounded it has a convergent subsequence \( y_{n_1}, y_{n_2}, \ldots \). Let \( y := \lim y_{n_1} \). Then we have for infinitely many \( i \)

\[ |y - y_{n_i}| < \varepsilon \leq |y_{n_i} - y_{n_i+1}| \]

whence

\[ |f(y) - f(y_{n_i})| \leq |f(y_{n_i}) - f(y_{n_i+1})| \]

so \( f(y) \in B_{n_i} \) for infinitely many \( i \), i.e., \( f(y) \in \cap B_{n_i} \).

(ii) Let \( B_1 \neq B_2 \neq \ldots \) be balls in \( f(Y) \) and let \( y_1, y_2, \ldots \in Y \) such that

\( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots \]

and since \( f \in M_b(X) \):

\[ |y_1 - y_2| > |y_2 - y_3| > \ldots \]

Since \( Y \) is spherically complete, there is \( y \in Y \) such that

\[ |y - y_n| \leq |y_n - y_{n+1}| \text{ for all } n, \text{ hence } |f(y) - f(y_n)| \leq |f(y_n) - f(y_{n+1})| \]

for all \( n \). It follows that \( f(y) \in B_n \) for all \( n \).

(iii) Let \( B_1 \neq B_2 \neq \ldots \) be balls in \( f^{-1}(A) \). Choose \( x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_2, \ldots \).

Then \( |x_1 - x_2| > |x_2 - x_3| > \ldots \) whence

\[ |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \]

There is \( x \in f^{-1}(A) \) such that \( |f(x) - f(x_n)| \leq |f(x_n) - f(x_{n+1})| \) for all \( n \).

Hence \( |x - x_n| \leq |x_n - x_{n+1}| \) for all \( n \) i.e., \( x \in \cap B_n \).
DEFINITION 4.11 Let \( f : X \to K \). The oscillation function \( \omega_f : X \to [0, \infty] \) is defined by

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}, x, y \in X\} \quad (a \in X).
\]

THEOREM 4.12 Let \( f \in M_w(X) \). Then

\[
\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X).
\]

Proof. For \( x \neq a \) we have \( |f(x) - f(a)| \geq \inf_{z \neq a} |f(z) - f(a)| \) and (since \( a \) is not isolated) consequently

\[
\omega_f(a) \geq \inf_{z \neq a} |f(z) - f(a)|.
\]

Conversely, let \( z \neq a \). Then for all \( x \) such that \( |x - a| < |z - a| \) we have

\[
|f(x) - f(a)| \leq |f(z) - f(a)|
\]

so

\[
\omega_f(a) \leq |f(z) - f(a)|
\]

whence

\[
\omega_f(a) \leq \inf_{z \neq a} |f(z) - f(a)|.
\]

THEOREM 4.13 Let \( f \in M_w(X) \), \( a \in X \). If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} x_n = a \) (\( x_n \neq a \) for all \( n \)) then \( \lim_{n \to \infty} |f(x_n) - f(a)| = \omega_f(a) \).

Proof. By 4.12 we have \( \lim_{n \to \infty} |f(x_n) - f(a)| \geq \omega_f(a) \). Conversely, \( \lim_{n \to \infty} |f(x_n) - f(a)| \leq \omega_f(a) \) is clear from the definition of \( \omega_f \).
5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: K is local, k is finite, K has discrete valuation. Also we can sometimes say a little more if we assume X to be convex. For the time being, let X be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case K is a local field.

**THEOREM 5.1** Let K be a local field, and let $f \in M(X)$. Then

(i) $f$ is continuous.

(ii) If $Y \subseteq X$ is closed then $f(Y)$ is closed.

(iii) If $f(X)$ is bounded and $f$ is not constant then $X$ is bounded.

(iv) Let $a \in X$. Then the following are equivalent

(a) $f$ is not injective at $a$

(b) $f$ is locally constant at $a$

(c) $f(a)$ is isolated in $f(X)$.

(v) The following conditions are equivalent

(a) $f$ is injective

(b) $f(X)$ has no isolated points

(c) $f$ is a homeomorphism of $X$ onto $f(X)$.

**Proof.** Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for $M_b$- and $M_s$-functions in case $X$ is convex and K is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2 Let the residue class field \( k \) of \( K \) be finite. Let \( X \) be convex and let \( f \in M_2(X) \). Then

(i) If \( a, b, c \in X, \ |a-b| < |a-c|, \ f(a) \neq f(c) \) then
\[
|f(a) - f(b)| < |f(a) - f(c)|.
\]

(ii) If \( C \subset X \) is convex then \( f(C) \) is convex in \( f(X) \) (\( f \) is weakly Darboux continuous, see 2.5).

(iii) If \( f \) is injective, then \( f \in M_S(X) \).

Proof. (i) Let \( B := \{ x \in K : |x-a| \leq |a-c| \} \). Then \( B \subset X \) and
\[
f(B) \subset [f(a), f(c)].
\]
Define an equivalence relation on \( B \) by: \( x \sim y \) if \( |f(x) - f(y)| < |f(a) - f(c)| \).

Since \( k \) is finite we get finitely many equivalence classes \( B_1, B_2, \ldots, B_n \). Since \( a \neq c \) we have \( n > 2 \). The diameter of \( f(B) \) equals
\[
|f(a) - f(c)|,
\]
the distance between \( f(B_i) \) and \( f(B_j) \) equals \( |f(a) - f(c)| \) \( (i \neq j) \). Since \([f(a), f(c)]\) can contain at most \( q := \chi(k) \) sets having distances \( |f(a) - f(c)| \) to one another we have \( n \leq q \). Hence \( 2 \leq n \leq q \).

By 2.2 (\( \beta \)), each \( B_i \) is convex. If \( x, y \in B_1 \) and \( |x-y| = |a-c| \) then \( B_1 = B \), contradicting \( n > 2 \). Thus \( B \) is a disjoint union of \( n \) balls \( B_1, \ldots, B_n \), where \( 2 \leq n \leq q \) and \( |x-y| < |a-c| \) whenever \( x, y \in B_i \) \( (i = 1, \ldots, n) \). It follows that \( n = q \) and that each \( B_i \) has the form \( \{ x \in K : |x-b_i| < |a-c| \} \) \( (b_i \in B) \). Hence, if \( |a-b| < |a-c| \) then there is \( i \) for which \( a, b \in B_i \).

\[
|f(a) - f(b)| < |f(a) - f(c)|.
\]

(ii) Let \( a, b \in C \) and let \( a \in f(X) \) with \( a \in [f(a), f(b)] \). We show that \( a \in f(C) \). If \( f(a) = f(b) \) this is clear. If \( f(a) \neq f(b) \), set
\[
a = f(x) \text{ where } x \in X.\text{ Then } |f(x) - f(a)| < |f(b) - f(a)|\text{. If } |x-a| > |b-a| \text{ then } f(x) \neq f(a) \text{ (since } f \in M_2(X)) \text{ and by (i) we then had } |f(b) - f(a)| < |f(x) - f(a)|, \text{ a contradiction. Hence } |x-a| \leq |b-a| \text{ i.e., } x \in [a,b] \subset C, \text{ so } a = f(x) \in f(C) \).
(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on \( k \), see 2.10.

**COROLLARY 5.3** Let \( K \) be a local field and let \( f \in M_b(\mathcal{X}) \) and \( \mathcal{X} \) convex. Then the following conditions are equivalent.

(a) \( f \in M_\mathcal{S}(\mathcal{X}) \).

(b) \( f \) is injective.

(c) \( f \in \mathcal{M}_\mathcal{S}(\mathcal{X}) \).

(d) \( f(\mathcal{X}) \) has no isolated points.

**Proof.** 5.2 and 5.1.

**THEOREM 5.4** Let \( K \) be a local field and let \( \mathcal{X} \) be the unit ball of \( K \) (or any bounded convex set, for that matter). If either \( f \in M_s(\mathcal{X}) \) or \( f \in M_b(\mathcal{X}) \), then \( f \) has bounded difference quotients.

**Proof.** \( f \) is bounded, let \( M := \sup\{ |f(x)-f(y)| : x, y \in \mathcal{X} \} \). It suffices to prove that \( |f(x)-f(0)| \leq M|x| \) for all \( x \). Let \( \pi \in \mathcal{K} \), \( |\pi| < 1 \), be a generator of the value group. By induction on \( n \) we prove:

if \( |x| = |\pi|^n \) then \( |f(x)-f(0)| \leq |\pi|^n M \).

The statement is clear for \( n = 0 \). Now suppose the statement is true for \( 0,1,\ldots, n-1 \).

Let \( x \in \mathcal{X} \), \( |a| = |\pi|^n \). Then \( |x-0| < |\pi|^{n-1}-0 \). If \( f(\pi^{n-1}) \neq f(0) \) we have either since \( f \in M_s(\mathcal{X}) \) or by 5.2(1)

\[
|f(x)-f(0)| < |f(\pi^{n-1})-f(0)| \leq |\pi|^n M
\]

hence

\[
|f(x)-f(0)| \leq |\pi|^n M
\]

If \( f(\pi^{n-1}) = f(0) \) then \( |f(x)-f(0)| \leq |f(\pi^{n-1})-f(0)| = 0 \), so certainly

\[
|f(x)-f(0)| \leq |\pi|^n M.
\]
Notes.

(a) 5.4 cannot be extended to the case \( X = K \). In fact, let
\[
f : \mathbb{Q}_p \to \mathbb{Q}_p \text{ be the map } \mathcal{E}_n p^n \mapsto \mathcal{E}_n p^{2n}. \quad (\mathcal{E}_n p^n \in \mathbb{Q}_p)
\]
Then
\[
f \in M_{bs} (\mathbb{Q}_p) \text{ but } |p^n f(p^{-n})| = p^n \to \infty.
\]

(b) If we loose the condition on \( K \), for example by requiring that the valuation is discrete then 3.22 and 2.4(5) show that the conclusion of 5.4 is false both for \( M_b \)-functions and \( M_s \)-functions.

On the other hand, it is clear from the proof of 5.4 that a bounded \( M_s \)-function on \( X \) has bounded difference quotients.

(c) One may wonder how difference quotients of \( M_w \)-functions behave.

See the example below.

**EXAMPLE 5.5** Let \( p \neq 2 \). Then there is an \( f \in M_w (\mathbb{Z}_p \to \mathbb{Q}_p) \) that has unbounded difference quotients.

**Proof.** Let \( a_0, a_1, \ldots \) be defined via \( a_{2n} := p^n \) (\( n = 0, 1, 2, \ldots \)) and
\[
a_{2n+1} := 2p^n \quad (n = 0, 1, 2, \ldots). \quad \text{Thus } (a_0, a_1, a_2, \ldots) = (1, 2, p, p^2, 2p^2, \ldots).
\]
Then \( |a_0| \geq |a_1| \geq |a_2| \geq \ldots, \lim a_n = 0, |a_n - a_m| = |a_m| \) (\( n > m \)).

Set
\[
f(x) := \begin{cases} 
- a_n & \text{if } |x| = p^{-n} \\
0 & \text{if } x = 0 
\end{cases} \quad (x \in \mathbb{Z}_p)
\]
Then the difference quotients of \( f \) are not bounded (for \( n \in \mathbb{N} : f(p^{2n}) = p^n \), so \( |p^{-2n} f(p^{2n})| = p^n \to \infty \) if \( n \to \infty \)). We show that
\[
f \in M_w (\mathbb{Z}_p). \quad \text{Since } f \text{ is continuous it suffices to show that if } x, y, z \text{ are } \neq 0, |x-y| < |x-z| \text{ then } |f(x) - f(y)| \leq |f(x) - f(z)|. \text{ This is clear if } |x| = |y|. \text{ If } |x| < |y|, \text{ then } |x| < |y| < |z|. \text{ If } |x| > |y|,
\]
then \( |y| < |x| < |z| \). Let \( f(x) = a_n, f(y) = a_m, f(z) = a_t \). Then in both cases \( n \neq m, t < \min(n, m) ; |f(x) - f(y)| = |a_n - a_m| \leq |a_t| \) and
\[
|f(x) - f(z)| = |a_n - a_t| = |a_t| \text{ and we are done.}
On the other hand (how surprising is life!)

**THEOREM 5.6** Let $k$ be the field of two elements. Then $M_w(X) = M_b(X)$.

*Proof.* We prove that $|x-y| = |y-z|$ implies $|f(x)-f(y)| \leq |f(y)-f(z)|$  
$(x \neq y, y \neq z, x,y,z \in X)$. There is $a \in K^*$ such that $|a(x-y)| = |a(y-z)| = 1$. So since $k = \mathbb{F}_2$, $a(x-y) = a(y-z) = 1$, whence $a(x-z) = 0$ or $|a(x-z)| < 1$. Thus, $|x-z| < |x-y| = |y-z|$. Since $f \in M_w(X), |f(x)-f(z)| \leq \min(|f(x)-f(y)|, |f(y)-f(z)|)$. Consequently, $|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) \leq |f(y)-f(z)|$.

Of particular interest may be monotone functions mapping convex sets onto convex sets.

**THEOREM 5.7** Let $K$ be a local field, let $X$ be a bounded open convex set, and let $f : X \times X$ be surjective. Then the following are equivalent.

(a) $f \in M_b(X)$

(b) $f \in M_s(X)$

(c) $f \in M_{bs}(X)$

(d) $f$ is an isometry.

*Proof.* (a) $\Rightarrow$ (b). Since $f(X)$ has no isolated points, $f$ is a homeomorphism, by 5.1(v). Then $f \in M_s(X)$, by 5.3. (b) $\Rightarrow$ (c). $f^{-1} \in M_b(X)$.

We just have shown (a) $\Rightarrow$ (b), so $f^{-1} \in M_s(X)$ i.e., $f \in M_b(X)$.

(c) $\Rightarrow$ (d). From the proof of 5.4 we have seen that $|f(x)-f(y)| \leq M|x-y|$, where $M = \sup|f(x)-f(y)| = 1$. Hence $|f(x)-f(y)| \leq |x-y|$ for all $x,y \in X$, but by the same token this also holds for $f^{-1}$. Then $f$ is an isometry. (d) $\Rightarrow$ (a) is obvious.
COROLLARY 5.8 Let $X$ be an open convex subset of $K$ and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.

(a) $f \in M_b(X)$
(b) $f \in M_s(X)$
(c) $f : M_b(X)$
(d) $f$ is a scalar multiple of an isometry.

Proof. If $X$ is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If $X$ is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (a) (b) (c) is now easy. To prove (c) $\implies$ (d) we may assume $f(0) = 0$, $f(1) = 1$. Let $X_n := \{ x \in K : |x| \leq n \}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is $c_n$ such that $|f(x) - f(y)| = c_n|x-y|$ ($x,y \in X_n$). By substituting $x = 1$, $y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in M_{ubs} Z$ into conditions on the coefficients of $f$ with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(Z)$. So let the notations be as in 3.3(3), and suppose first $f \in M_{ubs} (Z)$ i.e.

$|x-y| = |s-t| \iff |f(x)-f(y)| = |f(s)-f(t)|$. Let $n,m \in \mathbb{N}$. If $|n-n_-| = |m-m_-|$ then $|f(n)-f(n_-)| = |f(m)-f(m_-)|$, so if we write $f = \sum_{n} \lambda_n e_n$ we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ ($a_k \neq 0$) then $|n-n_-| = p^{-k}$ where $k = \left\lfloor \frac{\log n}{\log p} \right\rfloor$. We find

if $\left\lfloor \frac{\log n}{\log p} \right\rfloor < \left\lfloor \frac{\log m}{\log p} \right\rfloor$ then $|\lambda_n| > |\lambda_m|$

if $\left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor$ then $|\lambda_n| = |\lambda_m|$. 
Moreover, if \( \frac{\log n}{\log p} = \frac{\log m}{\log p} = k \) and \( n-m \) is indivisible by \( p^k \), i.e., 
\[ \log p_{n_1} = \log p_{m_1} = \log p, \]
\[ \log p_{n_2} = \log p_{m_2} = \log p, \]
\[ \log p_{n_3} = \log p_{m_3} = \log p, \]
\[ \log p_{n_4} = \log p_{m_4} = \log p, \]
\[ \log p_{n_5} = \log p_{m_5} = \log p, \]
\[ \log p_{n_6} = \log p_{m_6} = \log p, \]
\[ \log p_{n_7} = \log p_{m_7} = \log p, \]
\[ \log p_{n_8} = \log p_{m_8} = \log p, \]
\[ \log p_{n_9} = \log p_{m_9} = \log p, \]
\[ \log p_{n_{10}} = \log p_{m_{10}} = \log p, \]
\[ \log p_{n_{11}} = \log p_{m_{11}} = \log p, \]
\[ \log p_{n_{12}} = \log p_{m_{12}} = \log p, \]
\[ \log p_{n_{13}} = \log p_{m_{13}} = \log p, \]
\[ \log p_{n_{14}} = \log p_{m_{14}} = \log p, \]
\[ \log p_{n_{15}} = \log p_{m_{15}} = \log p, \]
\[ \log p_{n_{16}} = \log p_{m_{16}} = \log p, \]
\[ \log p_{n_{17}} = \log p_{m_{17}} = \log p, \]
\[ \log p_{n_{18}} = \log p_{m_{18}} = \log p, \]
\[ \log p_{n_{19}} = \log p_{m_{19}} = \log p, \]
\[ \log p_{n_{20}} = \log p_{m_{20}} = \log p, \]
\[ \log p_{n_{21}} = \log p_{m_{21}} = \log p, \]
\[ \log p_{n_{22}} = \log p_{m_{22}} = \log p, \]
\[ \log p_{n_{23}} = \log p_{m_{23}} = \log p, \]
\[ \log p_{n_{24}} = \log p_{m_{24}} = \log p, \]
\[ \log p_{n_{25}} = \log p_{m_{25}} = \log p, \]
\[ \log p_{n_{26}} = \log p_{m_{26}} = \log p, \]
\[ \log p_{n_{27}} = \log p_{m_{27}} = \log p, \]
\[ \log p_{n_{28}} = \log p_{m_{28}} = \log p, \]
\[ \log p_{n_{29}} = \log p_{m_{29}} = \log p, \]
\[ \log p_{n_{30}} = \log p_{m_{30}} = \log p, \]
\[ \log p_{n_{31}} = \log p_{m_{31}} = \log p, \]
\[ \log p_{n_{32}} = \log p_{m_{32}} = \log p, \]
\[ \log p_{n_{33}} = \log p_{m_{33}} = \log p, \]
\[ \log p_{n_{34}} = \log p_{m_{34}} = \log p, \]
\[ \log p_{n_{35}} = \log p_{m_{35}} = \log p, \]
\[ \log p_{n_{36}} = \log p_{m_{36}} = \log p, \]
\[ \log p_{n_{37}} = \log p_{m_{37}} = \log p, \]
\[ \log p_{n_{38}} = \log p_{m_{38}} = \log p, \]
\[ \log p_{n_{39}} = \log p_{m_{39}} = \log p, \]
\[ \log p_{n_{40}} = \log p_{m_{40}} = \log p, \]
\[ \log p_{n_{41}} = \log p_{m_{41}} = \log p, \]
\[ \log p_{n_{42}} = \log p_{m_{42}} = \log p, \]
\[ \log p_{n_{43}} = \log p_{m_{43}} = \log p, \]
\[ \log p_{n_{44}} = \log p_{m_{44}} = \log p, \]
\[ \log p_{n_{45}} = \log p_{m_{45}} = \log p, \]
\[ \log p_{n_{46}} = \log p_{m_{46}} = \log p, \]
\[ \log p_{n_{47}} = \log p_{m_{47}} = \log p, \]
\[ \log p_{n_{48}} = \log p_{m_{48}} = \log p, \]
\[ \log p_{n_{49}} = \log p_{m_{49}} = \log p, \]
\[ \log p_{n_{50}} = \log p_{m_{50}} = \log p, \]

We have found the first half of

**Theorem 5.9** Let \( f = \sum \lambda_n e_n \in C(\mathbb{Z}_p) \). In order that \( f \in M_{\text{ubs}}(\mathbb{Z}_p) \) it is necessary and sufficient that condition \((*)\) below holds:

\[ (*) \quad \left| \frac{\log n}{\log p} \right| \text{ is a strictly decreasing function of } \left| \frac{\log m}{\log p} \right| \quad (n \in \mathbb{N}) \]

We have shown \( f \in M_{\text{ubs}}(\mathbb{Z}_p) \) \( \rightarrow \) \((*)\). Now suppose \((*)\) and let \(|x-y| = p^{-k}\). We show that \(|f(x) - f(y)| = |\lambda|^{|x-y|}\). Now \( e_n(x) = e_n(y) \) for \( n < p^k \), so

\[ f(x) - f(y) = \sum_{n \geq p^k} \lambda_n (e_n(x) - e_n(y)) \]

Set

\[ x := a_0 + a_1 p + \ldots + a_k p^k + a_{k+1} p^{k+1} + \ldots \]
\[ y := a_0 + b_1 p + \ldots + b_k p^k + b_{k+1} p^{k+1} + \ldots \]

Then

\[ \sum_{n \geq p^k} \lambda_n e_n(x) = \sum_{n \geq p^k} \lambda_n e_n(y) \]

Now either \( a_k \) or \( b_k \) is \( \neq 0 \), say, \( a_k \neq 0 \). If \( b_k = 0 \) then by \((*)\)

\[ |f(x) - f(y)| = |\lambda|^{|x-y|} \cdot \left| a_k p^{k+1} \right| \]

\[ = |\lambda| p^k. \]

If \( b_k \neq 0 \) then by \((*)\)

\[ |f(x) - f(y)| = |\lambda| p^k \cdot \left| a_k p^{k+1} \right| = |f(x) - f(y)|. \]

**Note.** Using similar methods, we can prove: \( f = \sum \lambda_n e_n \in M_{\text{ubs}}(\mathbb{Z}_p) \) if and only if we have \((**\)) for all \( n,m \in \mathbb{N} \):
If we assume only that \( K \) has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let \( X \) be the unit ball of a discretely valued field. Let

\[ f : X \to X \]

be surjective, \( f \in M_{bs}(X) \). Then \( f \) is an isometry.

**Proof.** It is clear from previous theory that \( f \) is a homeomorphism of the unit ball. It suffices to show that \(| f(x) - f(y) | \leq |x-y| \) for all \( x, y \in X \). (Apply this result also for \( f^{-1} \). Then \( f \) is an isometry.)

Let \( \pi \in K \), \( |\pi| < 1 \), be a generator of \( |K^*| \). We prove by induction

if \( |x| = |\pi|^n \) then \( |f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)| \).

For \( n = 0 \) this is clear. (\( |x-0| \leq |1-0| \), so \( |f(x) - f(0)| \leq |f(1) - f(0)| \)).

Suppose the statement is true for \( n = 1 \). Let \( |x| = |\pi|^n \). Then

\( |x-0| < |\pi|^{-1} \), so \( |f(x) - f(0)| < |f(\pi^{-1}) - f(0)| \leq |\pi|^{-1} |f(1) - f(0)| \),

so \( |f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)| \) and we are done. (In fact, we have shown that a bounded \( M_{bs} \)-function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let \( K \) have discrete valuation and let \( f \in M_b(X) \). Then

the following conditions are equivalent.

(a) \( f(X) \) has no isolated points.

(b) \( f \) is injective and continuous.

(c) \( f \) is a homeomorphism \( X \sim f(X) \).
Proof. (a) $\implies (\gamma)$ is 4.9(ii). (\gamma) $\implies (\delta)$ is clear. (\delta) $\implies (\gamma)$: if $f(a)$ were an isolated point of $f(X)$, then $\{x : f(x) = f(a)\}$ is open in $X$. Since $f$ is injective $\{a\}$ is open. But $X$ has no isolated points. Contradiction.

To show that 5.11 may not be true if $K$ has a dense valuation we construct

**EXAMPLE 5.12** Let $|K| = [0, \infty)$. Then we construct an $M_{\text{ds}}$-homeomorphism sending

$\{x \in K : \frac{1}{2} < |x| \leq 1\}$ onto $\{x \in K : 0 < |x| \leq 1\}$.

Proof. Let $\phi : [\frac{1}{2}, 1] \to [0, 1]$ be the map $x \mapsto 2(x - 1)$. For each $v \in (\frac{1}{2}, 1]$, choose $\beta_v \in K$ such that $|\beta_v| = \frac{\phi(v)}{v}$. Define

$f : \{x : K : \frac{1}{2} < |x| \leq 1\} \to \{x \in K : 0 < |x| \leq 1\}$ as follows

$f(x) = \beta_{|x|}$ \quad ($\frac{1}{2} < |x| \leq 1$)

Clearly, $|f(x)| = |\beta_{|x|}| \cdot |x| = \phi(|x|) \in (0, 1]$. The inverse of $f$ is given by $y \mapsto \beta_{|x|}^{-1}(|y|) \cdot y$, so $f$ is a bijection. Since $f^{-1}$ can be defined in the same way as $f$ (only with $\phi^{-1}$ instead of $\phi$) it suffices to show that $f \in M_{\text{ds}}$. Let $|x - y| < |x - z|$.

Suppose $|x| > |z|$. Then $|x - z| = |x|$ and $|y| = \max(|x - y|, |x|) = |x|$. Then $\beta_{|x|} = \beta_{|y|}$, so $|f(x) - f(y)| = \beta_{|x|} |x - y|$ and $|f(x) - f(z)| = |f(x)| = \beta_{|x|} |x - z|$, so we are done in this case. Suppose $|x| < |z|$. Then $|x - z| = |z|$ and $|y| = \max(|x - y|, |x|) < |z|$. Then $|f(x) - f(y)| \leq \max(|f(x)|, |f(y)|) < |f(z)| = |f(z) - f(x)|$.

Suppose $|x| = |z|$. Then $|y| \leq \max(|x - y|, |x|) \leq |x|$; if $|y|$ were $< |x|$ then $|x - y| = |x| = |z| < |x - z|$, a contradiction, so $|y| = |x| = |z|$, and $|f(x) - f(y)| = \beta_{|x|} |x - y|$, $|f(x) - f(z)| = \beta_{|x|} |x - z|$, whence $|f(x) - f(y)| < |f(x) - f(z)|$. 


EXAMPLE 5.13 Extend $f$ to a surjection $g$ of $\{x \in K : |x| \leq 1\}$ onto itself by defining $g(x) = 0$ if $|x| \leq \frac{1}{2}$. We claim that $g \in M_b$. Let $|x-y| \leq |x-z|$. To check whether $|g(x)-g(y)| \leq |g(x)-g(z)|$ we only have to consider the cases $|x| \leq \frac{1}{2}$ and $|y| > \frac{1}{2}$ and $|y| \leq \frac{1}{2}$. In the first case, $|x-y| = |y| \leq |x-z|$, so $|z| = \max(|z-x|,|x|) = |z-x| \geq |y|$. Then $|g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)|$. In the second case $|g(x)-g(y)| = |f(x)|$. If $|x| < |z|$ then $|f(x)| < |f(z)| = |f(x)-f(z)| = |g(z)-g(x)|$. If $|x| > |z|$ then $|f(x)| = |g(x)-g(z)|$. If $|x| = |z|$ then $|f(x)-f(z)| = \beta_{|x|} |x-z| \geq \beta_{|x|} |x-y| = \beta_{|x|} |x| = |f(x)|$.

Thus we have found a continuous surjection $g : \{x \in K : |x| \leq 1\} \rightarrow \{x \in K : |x| \leq 1\}, g \in M_b$, such that $g = 0$ on $\{x : |x| < \frac{1}{2}\}$. (Compare 5.11).

EXAMPLE 5.14 Let $h : \{x \in K : |x| \leq 1\} \rightarrow K$ be defined as

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \ (f \text{ as in 5.12)} \\ 0 & \text{if } x = 0. \end{cases}$$

Then $h$ is a non-continuous $M_{bs}$-function.

Proof. That $h$ is not continuous at 0 is clear. Further, $h$, restricted to $\{x : 0 < |x| \leq 1\}$ is in $M_{bs}$ (see 5.12). Further, since $g \circ h$ is the identity ($g$ as in 5.12), we see that $h \in M_b$. It suffices to show that $|x-y| = |x-z|$ implies $|h(x)-h(y)| = |h(x)-h(z)|$ in case $0 \in \{x, y, z\}$.

We may suppose $x \neq y$, $y \neq z$, $x \neq z$. Let $x = 0$. Then $|y| = |z|$, so $|f^{-1}(y)| = |f^{-1}(z)|$, i.e., $|h(x)-h(y)| = |h(x)-h(z)|$. Now let $y = 0$.

Then $|x| = |x-z|$. Choose $0 < |t| \leq 1$ such that $|t| < |x|$. Then $|x-t| = |x-z|$ so $|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x)-f^{-1}(z)|$, i.e., $|h(x)| = |h(x)-h(z)|$, and we are done.
6. FUNCTIONS OF BOUNDED VARIATION

In this section $X$ is the unit ball of $K$, and $BA(X) := \{f : X \to K : \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right| < \infty \}$. Let us define

$$\|f\|_\Delta := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in X, x \neq y \right\} (f \in BA(X)).$$

It will turn out that, in a natural way, $BA(X)$ can be regarded as the space of functions of bounded variation, and that $\|\|_\Delta$ plays the role of the total variation.

**Theorem 6.1** Let $f : X \to K$. Then the following are equivalent

1. $f \in BA(X)$.
2. $f$ is a linear combination of two increasing functions.
3. $|K|$ is discrete (a), (β) are equivalent to
4. $f$ is the difference of two bounded monotone functions of some type $\alpha$.
5. $f \in [M_D(X)]$.
6. If $K$ is a local field then (α)-(δ) are equivalent to
7. $f \in [M_M(X)]$.
8. $f \in [M_S(X)]$.

**Proof.** We only prove (α) $\Rightarrow$ (β). The rest follows from (5.10), (5.4).

So let $f \in BA(X)$ and choose $\lambda \in K$ such that $|f(x) - f(y)| < |\lambda| |x - y|$ $(x, y \in X, x \neq y)$. Then $\lambda^{-1}f$ is a pseudocontraction, $f(x) = \lambda x + \lambda(\lambda^{-1}f(x) - x)$ $(x \in X)$, where $x \to x$ and $x \to \lambda^{-1}f(x) - x$ are increasing.

In the real case, we can define for a function $[0,1] \to \mathbb{R}$, of bounded variation
$$V(f) := \inf \{ \var{g} + \var{h} : f = g + h, \text{ } g, h \text{ monotone} \}.$$  

It is an easy exercise to show that $f \mapsto V(f)$ is a seminorm on the space of all functions of bounded variation and that $V$ is equivalent to the total variation $\var{}$, defined via

$$\var{f} = \sup \{ \sum |f(x_k) - f(x_{k-1})| : x_0 < x_1 < \ldots < x_n \text{ is a partition of } [0,1] \}.$$  

So in the non-archimedean situation we define for $f : X \to K$

$$J(f) = \sup \{|f(x) - f(y)| : x, y \in X\}.$$  

(If $f$ is considered to be "monotone" then $J(f)$ can be interpreted as the "total variation" of $f$.) We are led to the following definitions for $f \in \text{BA}(X)$:

$$\var{f} := \inf \{ \max (J(g), J(h)) : f = g + h, \text{ } g, h \text{ are scalar multiples of increasing functions} \}.$$  

(If $|K|$ is discrete) $\var^1 f := \inf \{ \max J(g), J(h) : f = g + h, \text{ } g, h \in M^+(X) \}.$  

(If $K$ is local) $\var^2 f := \inf \{ \max J(g), J(h) : f = g + h, \text{ } g, h \in M^+(X) \}$

$$\var^3 f := \inf \{ \max J(g), J(h) : f = g + h, \text{ } g, h \in M^+(X) \}.$$  

Let us first compare $\var{f}$ and $\|f\|_\Delta$. If $f = g + h$ and $g, h$ are scalar multiples of increasing functions we have for $x, y \in X, x \neq y$

$$\left| \frac{f(x) - f(y)}{x-y} \right| \leq \max \left( \left| \frac{g(x) - g(y)}{x-y} \right|, \left| \frac{h(x) - h(y)}{x-y} \right| \right) \leq \max (J(g), J(h))$$

so $\|f\|_\Delta \leq \var{f}$. Conversely, if $|\lambda| > \sup \left| \frac{f(x) - f(y)}{x-y} \right|$ then

$$f(x) = \lambda x + \lambda (\lambda^{-1} f(x) - x) \quad (x \in X)$$  

whence

$$\var{f} \leq |\lambda|.$$
So, if $|K|$ is dense we have $\text{Var} f = \|f\|_\Delta (f \cdot B\Delta(X))$. Otherwise we have at least

$$\|f\|_\Delta \leq \text{Var} f \leq c \|f\|_\Delta \quad (f \in B\Delta(X))$$

(where $c$ is the smallest value $> 1$).

If $|K|$ is discrete we clearly have $\text{Var}_1 f \leq \text{Var} f$. Conversely, let $f = g+h$, where $g,h \in M_{B^X}(X)$. It follows from the proof of 5.10 that

$$|g(x) - g(y)| \leq M|x-y| \quad (x,y \in X)$$
$$|h(x) - h(y)| \leq N|x-y|$$

where $M = \sup |g(x) - g(y)| = J(g)$ and $N = J(h)$.

So

$$\left|\frac{f(x) - f(y)}{x-y}\right| \leq \max(J(g), J(h)), \text{ whence}$$

$$\|f\|_\Delta \leq \text{Var}_1 f.$$ 

Similar proofs work for $\text{Var}_2 f$, $\text{Var}_3 f$. We have

**Theorem 6.2** The seminorms $\text{Var}$, $\text{Var}_1$, $\text{Var}_2$, $\text{Var}_3$, on $B\Delta(X)$ (whenever defined) are all equivalent to $\|\|_\Delta$. 
REFERENCES


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