NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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INTRODUCTION

In the sequel, \( K \) is a non-archimedean valued field, complete, with residue class field \( k \). Our aim is to present reasonable definitions for a function \( f : X \rightarrow K \) to be "monotone". (\( X \) is a subset of \( K \)). Since \( K \) admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for \( f : \mathbb{R} \rightarrow \mathbb{R} \), equivalent to "\( f \) is monotone", and such that these statements have translations in \( K \) that make sense. This way we obtain several definitions of "\( f : X \rightarrow K \) is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of \( K \)", "the sign of a nonzero element of \( K \)."

In Section 2 we define several notions of monotony. E.g., \( f \in M^b(X) \) if \( x \) between \( y \) and \( z \) implies \( f(x) \) between \( f(y) \) and \( f(z) \) and \( f \in M^s(X) \) if \( f(x) \) between \( f(y) \) and \( f(z) \) implies \( x \) between \( y \) and \( z \). Also monotone functions of type \( \sigma \) are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an \( f \in M^b(X) \) (or \( f \in M^s(X) \)) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
tion on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of \( K \) and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: \( f' > 0 \iff f \) increasing.

The notion of pseudo-ordining ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

**Notations.** Let \( p \) be a prime. By \( \mathbb{F}_p \) we mean the field of \( p \) elements. By \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers. For a field \( L \) we denote its characteristic by \( \chi(L) \). Let \( E \) be a vector space over \( K \) and \( S \subset E \). By \( [S] \) we denote the smallest \( K \)-linear subspace of \( E \) that contains \( S \).
1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let \( x, y \in K \). Then the smallest ball in \( K \) containing \( x \) and \( y \) is denoted by \( [x, y] \). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \( [x, y] \subseteq C \).

Sometimes we use a more geometric terminology. Instead of \( z \in [x, y] \) we will say that \( z \) is between \( x \) and \( y \) and instead of \( z \notin [x, y] \) we use the expression: \( x \) and \( y \) are at the same side of \( z \).

Notice that \( [x, y] = [y, x] \) for all \( x, y \in K \) and that \( z \in [x, y] \iff |z-x| \leq |x-y| \iff |z-y| \leq |x-y| \iff z = \lambda x + (1-\lambda)y \) for some \( \lambda \in K \), \( |\lambda| \leq 1 \). If \( x \neq y \) then the \( \lambda \) in this last expression is unique (viz. \( \lambda = \frac{z-y}{x-y} \)).

Examples of convex sets are: the empty set, singletons, balls, \( K \).

It is an easy exercise to show that these are the only convex subsets of \( K \). So formally we may write each convex subset of \( K \) as

\[
\{x \in K : |x-a| < r\} \quad (a \in K, 0 \leq r \leq \infty)
\]

or as

\[
\{x \in K : |x-a| \leq r\} \quad (a \in K, 0 \leq r \leq \infty)
\]

Notice that the only unbounded convex subset of \( K \) is \( K \) itself.

Sometimes we need the notion of convexity with respect to a subset \( X \) of \( K \). A subset \( C \subseteq X \) is called convex in \( X \) if \( x, y \in C \) implies \( [x, y] \cap X \subseteq C \) or, equivalently, if \( C \) is the intersection of \( X \) with a convex subset of \( K \).

Let \( x, y, z \in K \). By the strong triangle inequality we have that the "triangle" \( x, y, z \) is isosceles, say \( |x-y| = |y-z| \). Then \( |x-z| \leq |x-y| \), so \( z \) is between \( x \) and \( y \) and \( x \) is between \( y \) and \( z \). If also \( |x-y| = |x-z| \)
then $y$ is between $x$ and $z$. Otherwise, $x$ and $z$ are at the same side of $y$.

The relation $\sim$ defined on $K^* := K \setminus \{0\}$ by

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in K^*)$$

is an equivalence relation. We have $x \sim y$ iff $0 \not\in [x, y]$ i.e. iff

$$|x-y| < |x| \quad (= |y|) \quad \text{i.e. iff } \frac{|xy^{-1}| - 1} < 1.$$

Define

$$K^+ := \{x \in K : |1-x| < 1\}$$

Then $K^+$ is a multiplicative subgroup of $K^*$, $K^+ = \{x \in K^* : x \sim 1\}$

and is called the set of the positive elements of $K$. The relation

$\sim$ is also induced by the canonical group homomorphism

$$\pi : K^* \rightarrow K^*/K^+.$$ 

Thus, $x \sim y$ if and only if $\pi(x) = \pi(y) \quad (x, y \in K^*)$. Therefore it is natural to view the group $\Sigma := K^*/K^+$ as being the group of signs of elements of $K^*$, and we call $\pi(x)$ the sign of the element $x \in K^*$. If $x \in K^*$ then $\pi(x) = \{y : |y-x| < |x|\} = xK^+$. For $x \in K^*$, $a \in \Sigma$ we sometimes write $xa$ to indicate the element $\pi(x).a$ of $\Sigma$. In particular, for $a \in \Sigma$ the opposite sign of $a$, $-a$, is defined as $(-1)a$. Then $-a = \{-x : x \in a\}$. (Notice that in case $\chi(K) = 2$ we have $a = -a$.)

Let $a \in \Sigma$. Then for $x, y \in a$ we have $|x| = |y|$ so we can define the absolute value of $a$, $|a|$ as follows

$$|a| := |x| \quad (x \in \pi^{-1}(a)).$$

In the sequel we also need addition between elements of $\Sigma$. Let us first observe that for any $a, b \in \Sigma$ the sum

$$a+b := \{x+y : x \in a, y \in b\}$$

is always a ball with radius $\max(|a|, |b|)$. (i.e., of the form
\{x : |x-b| < \max(|a|,|\beta|)\}. Now a+\beta contains 0 if and only if
\alpha = -\beta. Otherwise a+\beta is again an element of \Sigma. (Proof: Let a, b \in \beta. Then |a+b| = \max(|a|,|b|). If also x \in a, \gamma \in \beta then |x+(\gamma-\alpha)| \leq \max(|x-a|,|\gamma-b|) < \max(|a|,|b|) = |a+b|. Thus a+\beta contains the ball with center a+b and radius \max(|a|,|\beta|), so a+\beta is equal to this ball.)

Let us define
\alpha \oplus \beta := a+\beta = \{x+y : x \in a, \gamma \in \beta\} \quad (a, \beta \in \Sigma, \alpha \neq -\beta).

We have

THEOREM 1.2 Let \Sigma, | | : \Sigma \to \mathbb{R}, \Theta : \Sigma \times \Sigma \setminus \{(a,-a) : a \in \Sigma\} \to \Sigma be as above. Let a, b, \gamma \in \Sigma. Then

(i) |a\beta| = |a||\beta||a^{-1}| = |a|^{-1}.

(ii) If a \oplus \beta is defined then so is \beta \oplus a and a \oplus \beta = \beta \oplus a.

(iii) If (a \oplus \beta) \oplus \gamma and a \oplus (\beta \oplus \gamma) are defined then
(a \oplus \beta) \oplus \gamma = a \oplus (\beta \oplus \gamma).

(iv) If a \oplus \beta or \gamma a \oplus \beta is defined then so is the other
and \gamma(a \oplus \beta) = \gamma a \oplus \beta.

(v) If a \oplus \beta is defined then |a \oplus \beta| = \max(|a|,|\beta|). Conversely if |s| = \max(|a|,|\beta|) for some s \in a+\beta then a \oplus \beta
is defined.

(vi) |a| < |\beta| if and only if a \oplus \beta = \beta.

(vii) Let n \in \{1,2,...,\chi(k)-1\} if \chi(k) \neq 0, let n \in \mathbb{N} otherwise. Then we define \Theta_n a inductively as follows.
\Theta_1 a = a, \Theta_k a = \Theta_{k-1} a \oplus a (k \leq n). Then
\Theta_n a = na.

Proof. (i), (ii) are clear. (iii) is almost trivial: if x \in a, \gamma \in \beta,
z \in \gamma then x+y+z \in a+\beta+\gamma and the latter set can be regarded as
(a ⊕ β) ⊕ γ or as a ⊕ (β ⊕ γ). (It is worth noticing that (a ⊕ β) ⊕ γ may be defined whereas a ⊕ (β ⊕ γ) is not. Choose β = −γ and |a| > |β|. Then (a ⊕ β) ⊕ γ = a ⊕ γ = a, β ⊕ γ is not defined.)

(iv) is clear. If a ⊕ β is defined then for x⊙α, y ⊕ β we have |x+y| ≥ max(|x|,|y|) whence |x+y| = max(|x|,|y|). So |a ⊕ β| = max(|a|,|β|). Conversely, if a ⊕ β is not defined, then (we saw earlier that) α+β is a ball with center zero and radius max(|a|−,|β|−).

Thus we have (v). We prove (vi). If |a| < |β| then α+β = β so a ⊕ β = β.

Conversely, if a ⊕ β = β then choose a ⊕ α, b ⊕ β. Then a+b ⊕ β hence a+b ∼ b i.e., ab⁻¹+1 ∈ K⁺ implying |ab⁻¹| < 1 or |a| < |b|. Hence |a| < |β|. (Note: from (vi) it follows that α ⊕ β = α' ⊕ β does not imply α = α'). To prove (vii) let a ⊕ α and observe that for any k ≤ n,

if α is defined, (k-1)a ⊕ α. Hence |(k-1)a+a| = |ka| = |a| = |a|, k-1
so α+a does not contain 0, hence α ⊕ α is defined.

Now na is by definition π(n)α. So na c na and na ⊕ α. Since both na

and ⊕ α are signs they must coincide.

We now define relations that resemble "ordering".

DEFINITION 1.3 Let α ∈ Σ and x,y ∈ K. Then we say that x is greater than y in the sense of α, notation x a y, if x−y ∈ α.

We have the following rules

THEOREM 1.4 (i) If x,y ∈ K, x ≠ y then there is exactly one α ∈ Σ for which x a y.

(ii) x a x for no α.

(iii) If x a y then for all s ∈ K: x+s a y+s (x,y ∈ K, α ∈ Σ)

(iv) If x a y and s a 0 then xs a β ys (x,y,s ∈ K, α,β ∈ Σ)
(In particular $x >_a y$ implies $-x <_a -y$).

(v) If $x >_a y$, $y >_b z$ and if $a \oplus b$ is defined then $x >_{a \oplus b} z$.

Proof. Easy.

The group $\Sigma_1 := \{\alpha \in \Sigma : |\alpha| = 1\}$ is a subgroup of $\Sigma$, isomorphic to multiplicative group $k^\times$. If $K$ has discrete valuation and if $s \in K$ and $|s|$ is the largest value that is smaller than $1$, then for each $\alpha \in \Sigma$ there is $x \in \mathbb{Z}$ such that $\alpha = s^n \alpha_1$ where $\alpha_1 \in \Sigma_1$. It follows easily that the map $(n, \alpha) \mapsto s^n \alpha$ $(n \in \mathbb{Z}, \alpha \in \Sigma_1)$ is an isomorphism of $\mathbb{Z} \times \Sigma_1$ onto $\Sigma$. Thus, in case $K$ has discrete valuation, $\Sigma$ is isomorphic to $\mathbb{Z} \times \Sigma_1$, or, for that matter, to $|k^\times| \times k^\times$.

If $K$ is a local field we can even define a group embedding $\rho : \Sigma \rightarrow K^\times$ such that $\pi \rho$ is the identity. (Thus, we can pick an element in every $\alpha$ $(\alpha \in \Sigma)$ such that the resulting set is a subgroup of $K^\times$). Let $s \in K$, $|s| < 1$ such that $|s|$ generates the value group and let $q$ be the cardinality of $k$. Let $x \in K$. Then there is a unique $n \in \mathbb{Z}$ such that $x = s^n x_1$ where $|x_1| = 1$.

Define $\nu(x) = s^n \lim_{n \to \infty} x_1^n$.

It is easy to verify that $\nu$ is a homomorphism of $K^\times$ into $K^\times$, that $\pi(\nu(x)) = \pi(x)$ for all $x \in K^\times$ and that $\nu(x) = 1$ if and only if $x \in K^+$. Therefore the map $\rho$ making the diagram

\[
\begin{array}{ccc}
K^\times & \xrightarrow{\nu} & K^\times \\
\downarrow{\pi} & & \downarrow{\pi} \\
\Sigma & \xrightarrow{\rho} & K^\times
\end{array}
\]

commute solves the problem.

**EXAMPLE 1.5** The signs of $\Phi_p$. Let $\theta$ be a primitive $(p-1)^{th}$ root of
unity. Then \( \{p^n \cdot i : i \in \{0, 1, \ldots, p-2\}, n \in \mathbb{Z} \} \) is a subgroup of \( \mathbb{Q}_p^* \) isomorphic to \( \mathbb{Z} \). If

\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0, 1, \ldots, p^{p-2}\}, \ a_n \neq 0)
\]

is an element of \( \mathbb{Q}_p \), its sign, interpreted as an element of \( \mathbb{Q}_p \) is

\[
\pi(x) = a_n p^n.
\]
2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function \( f : [0,1] \to \mathbb{R} \) the following statements are equivalent.

(a) \( f \) is monotone (i.e., either \( x > y \) implies \( f(x) \geq f(y) \) for all \( x,y \)
or \( x > y \) implies \( f(x) \leq f(y) \) for all \( x,y \)).

(b) If \( x \) is between \( y \) and \( z \) then \( f(x) \) is between \( f(y) \) and \( f(z) \)
\((x,y,z \in [0,1])\)

(c) If \( C \subset \mathbb{R} \) is convex then \( f^{-1}(C) \) is convex.

Thus we define

**DEFINITION 2.1** Let \( X \subset K \). We say that \( f \in M_b(X) \) if for all \( x,y,z \in X \), \( x \) between \( y \) and \( z \) implies \( f(x) \) is between \( f(y) \) and \( f(z) \). In other words, \( f \in M_b(X) \) if and only if for all \( x,y,z \)
\[ |x-y| \leq |y-z| \Rightarrow |f(x)-f(y)| \leq |f(y)-f(z)|. \]

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

**THEOREM 2.2** Let \( X \subset K \) and let \( f : X \to K \). Then the following statements are equivalent

(a) \( f \in M_b(X) \).

(b) For each convex \( C \subset K \), \( f^{-1}(C) \) is convex in \( X \).

(c) For all \( x,y,z \in X \): \( |x-y| = |x-z| \Rightarrow |f(x)-f(y)| = |f(x)-f(z)| \).

(d) For all \( x,y,z \in X \): \( |f(x)-f(y)| > |f(x)-f(z)| \Rightarrow |x-y| > |x-z| \).

(e) For all \( x,y,z \in X \): \( |f(x)-f(y)| \neq |f(x)-f(z)| \Rightarrow |x-y| \neq |x-z| \).
Proof. (a) $\Rightarrow$ (b). Let $x, y \in f^{-1}(C)$ and let $z \in [x, y] \cap X$. Then $|z-x| \leq |x-y|$, so $|f(z)-f(x)| \leq |f(x)-f(y)|$ i.e., $f(z) \in [f(x), f(y)] \subset C$. Hence $z \in f^{-1}(C)$.

(b) $\Rightarrow$ (a). Let $x, y, z \in X$ and $|x-y| \leq |x-z|$. The set $[f(x), f(z)]$ is convex, hence $f^{-1}([f(x), f(z)])$ is convex in $X$ and contains $x$ and $z$, so it must contain $y$. Thus $f(y) \in [f(x), f(z)]$.

Clearly, (a) $\leftrightarrow$ (b) and (γ) $\leftrightarrow$ (ε). We prove (a) $\Rightarrow$ (γ). Now (a) $\Rightarrow$ (γ) is clear by symmetry. Suppose (γ) and let $|x-y| \leq |x-z|$. It suffices to consider the case $|x-y| < |x-z|$. Then $|y-z| = |x-z|$, so by (γ) we have $|f(y)-f(z)| = |f(x)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) = |f(x)-f(z)|$.

**Theorem 2.3** Let $X \subset X$. Then

(i) For each $a, b \in K$ the map $x \mapsto ax+b$ is in $M_b(X)$.

(ii) If $f \in M_b(X)$, $\lambda \in K$ then $\lambda f \in M_b(X)$.

(iii) $M_b(X)$ is closed under pointwise limits.

(iv) If $f \in M_b(X)$ and $g : f(X) \to K$ is in $M_b(f(X))$, then $g \circ f \in M_b(X)$.

(v) If $f \in M_b(X)$ and $f(a) = f(b)$ for some $a, b \in X$, then $f$ is constant on $[a, b] \cap X$.

**Proof.** Obvious.

**2.4 Examples and Remarks.**

We mention a few examples of $M_b$-functions. For more, see the sequel.

(1) The constant functions.

(2) Isometries (e.g., exp.).

(3) Choose in every $a \in \Sigma$ an element $x_a$. Define $\phi : K \to K$ as follows

$$
\phi(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\lambda_a & \text{if } x \in a \quad (a \in \Sigma)
\end{cases}
$$
(Essentially, \( \phi|K^* \) is the sign function \( \pi \) of section 1).}

We prove that \( \phi \in M_b(K) \). Since \( \phi \) is continuous it suffices to check that \( \phi|K^* \) is in \( M_b(K^*) \). Now for all \( x,y \in K^* \) we have \( \phi(x) - \phi(y) = 0 \) if \( |xy^{-1} - 1| < 1 \) and \( |\phi(x) - \phi(y)| = |x-y| \) if \( |x-y| = \max(|x|,|y|) \). Now take \( x,y,z \in K^* \) such that \( |x-y| \leq |x-z| \). If \( \phi(x) = \phi(z) \) then \( |1-x^{-1}y| \leq |1-x^{-1}z| < 1 \) so \( \phi(x) = \phi(y) \).

If \( \phi(x) \neq \phi(z) \) then \( |\phi(x) - \phi(y)| \leq |x-y| \leq |x-z| = |\phi(x) - \phi(z)| \).

(4) Let \( r > 0 \) and choose in every ball \( B \) of radius \( r \) a center \( x_B \).

The function defined via

\[
\psi(x) = x_B \quad (x \in B)
\]

is in \( M_b(K) \). The proof is easy.

(5) (A nowhere continuous \( M_b \)-function). Let \( K \) be a field such that \( \#K = \#k \) (e.g., a discretely valued field where \( \#k \) has the power of the continuum). Let \( \sigma : K \to k \) be a bijection and let \( \tau : k \to K \) such that \( |\tau x - \tau y| = 1 \) whenever \( x \neq y \). Then \( f : \tau \circ \sigma \) satisfies: \( |f(x) - f(y)| = 1 \) \( (x,y \in K, x \neq y) \).

Clearly \( f \) is everywhere discontinuous, \( f \in M_b(K) \).

(6) Let \( X \subset K \). We can strengthen the definition of an \( M_b \)-function into

\[ \text{if } |x-y| \leq |z-t| \text{ then } |f(x) - f(y)| \leq |f(z) - f(t)| \quad (x,y,z,t \in X) \]

(some "uniform" \( M_b \)-condition) and we obtain a space, called \( M_{ub}(X) \).

Clearly, the examples mentioned in (1), (2), (4), (5) are in \( M_{ub}(K) \), whereas the example in (3) is not. (Choose \( x,y \in K \) with \( |x| > 1 \), \( |x-y| = 1 \). Then \( |1-0| \leq |x-y| \), but \( 1 = |\phi(1) - \phi(0)| > |\phi(x) - \phi(y)| = 0 \).)

Notice that \( \phi \) is locally constant on \( K^* \), but not on \( K \).

(7) The discontinuous function \( f \) of (5) has the property that \( f(K) \) consists only of isolated points. This is not accidental. If \( f \in M_b(K) \)
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing \( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points, so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that the image of an everywhere discontinuous \( M_b \)-function consists only of isolated points.

(8) For each \( n \), let \( \sigma_n : K \to K \) be the example of 2.4, (4) above where \( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b(X) \) we have \( \sigma_n \circ f \in M_b(X) \) and \( \lim_n \sigma_n \circ f = f \) uniformly. Hence, if \( f \) is continuous then it can uniformly be approximated by locally constant \( M_b \)-functions.

A monotone function \( f : [0,1] \to \mathbb{R} \) maps convex sets into sets that are relatively convex in \( f([0,1]) \). In our situation we do not have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If \( f : [0,1] \to \mathbb{R} \) maps convex sets into (relatively) convex sets then \( f \) need not be monotone: any Darboux continuous function has the above property. In fact a function \( f : [0,1] \to \mathbb{R} \) is Darboux continuous if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We define

**DEFINITION 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \to K \). Then \( f \) is called **weakly Darboux continuous** if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex in \( f(X) \).

\( f \) is called **Darboux continuous** if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex (in \( K \)).

We have the following obvious remarks.

1) \( f : X \to K \) is Darboux continuous if and only if \( f \) is weakly Darboux continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) such that for every open ball \( B \subset \mathbb{Z}_p \),

\[
f(B) = \mathbb{Z}_p.
\]

Let \( A \subset \mathbb{Z}_p \) be defined as follows. \( x = \sum_{n=0}^{\infty} a_n p^n (a_n \in \{0,1,\ldots,p-1\}) \) is in \( A \) if \( a_{2n} = a_{2n+2} = \ldots = 0 \) for some \( n \). Define \( f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) via

\[
f(x) = \begin{cases} a_{2N+1} + a_{2N+3}p + a_{2N+5}p^2 + \ldots & \text{if } x \in A \text{ and } N = \min \{ n : a_{2n} = a_{2n+2} = \ldots = 0 \} \\ 0 & \text{if } x \notin A \end{cases}
\]

Then \( f \) maps every non-empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.

(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non-trivial locally constant function on \( \mathbb{Z}_p \) is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous.

To see this, observe that all open compact subsets of \( \mathbb{Z}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p \) sending \( \{ x : |x| < 1 \} \) into \( \{ x : |x| = 1 \} \) and \( \{ x : |x| = 1 \} \) into \( \{ x : |x| < 1 \} \).

If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( \mathbb{Z}_p \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4,(7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \rightarrow \mathbb{R} \) the following conditions are equivalent.

(a) \( f \) is strictly monotone (i.e., injective and monotone).

(b) \( f \) is injective. For a convex set \( C \subset [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all \(x, y, z \in [0,1]\): if \(f(x)\) is between \(f(y)\) and \(f(z)\) then \(x\) is between \(y\) and \(z\).

(δ) For all \(x, y, z \in [0,1]\): \(f(x)\) is between \(f(z)\) if and only if \(x\) is between \(y\) and \(z\).

Translating (α) - (δ) into the non-archimedean situation we arrive at the following conditions. Let \(X \subseteq K\) and \(f : X \to K\)

(α') \(f \in \mathcal{M}_b(X)\) and \(f\) is injective.

(β') \(f\) is weakly Darboux continuous and injective.

(γ') for all \(x, y, z \in X\), \(|x-y| < |x-z|\) implies \(|f(x)-f(y)| < |f(x)-f(z)|\).

(δ') \(f \in \mathcal{M}_b(X)\) and \(f\) satisfies (γ').

It will turn out that the conditions (α') - (γ') although not equivalent are closely related. We start with (γ'):

**DEFINITION 2.6** Let \(X \subseteq K\), \(f : X \to K\). We say that \(f \in \mathcal{M}_s(X)\) if for all \(x, y, z \in X\), \(f(x) \in [f(y), f(z)]\) implies \(x \in [y, z]\).

**THEOREM 2.8** Let \(X \subseteq K\), \(f : X \to K\). Then the following statements are equivalent:

(a) \(f \in \mathcal{M}_s(X)\).

(b) \(f\) is injective and weakly Darboux continuous.

(c) \(f\) is injective and \(f^{-1} \in \mathcal{M}_b(f(X))\).

(δ) For all \(x, y, z \in X\), \(|f(x)-f(y)| = |f(x)-f(z)| \Rightarrow |x-y| = |x-z|\).

(ε) For all \(x, y, z \in X\), \(|x-y| < |x-z| \Rightarrow |f(x)-f(y)| < |f(x)-f(z)|\).

(ζ) For all \(x, y, z \in X\), \(|x-y| \neq |x-z| \Rightarrow |f(x)-f(y)| \neq |f(x)-f(z)|\).
Proof. The implications \((\alpha) \Rightarrow (\epsilon) \Rightarrow (\zeta) \Rightarrow (\delta)\) are clear from the definitions.

\((\delta) \Rightarrow (\gamma)\): injectivity follows from \(|f(x) - f(x)| = |f(x) - f(y)| \Rightarrow |x - x| = |x - y|\). Use 2.2.(\gamma).

\((\gamma) \Rightarrow (\beta)\): Let \(g : f(X) \to X\) be the inverse of \(f\). Let \(C \subseteq X\) be convex in \(X\). Then since \(g \in M_b\), \(g^{-1}(C)\) is convex in \(f(X)\). But \(g^{-1}(C) = f(C)\).

Finally, we prove \((\beta) \Rightarrow (\alpha)\). Let \(f(x) \in [f(y), f(z)]\). By \((\beta)\) the set \(f([y, z] \cap X)\) is convex in \(f(X)\) and it contains \(f(y), f(z)\), hence \(f(x) \in [f(y), f(z)] \cap X < f([y, z] \cap X)\). Since \(f\) is injective, \(x \in [y, z] \cap X\) and we are done.

We also have (compare 2.3)

**Theorem 2.9** Let \(X \subseteq K\). Then

1. For \(a, b \in K\), \(a \neq 0\) the map \(x \mapsto ax + b\) is in \(M_s(X)\).
2. If \(f \in M_s(X), \lambda \in K, \lambda \neq 0\) then \(\lambda f \in M_s(X)\).
3. If \(f_1, f_2, \ldots \in M_s(X)\), \(\lim f_n = f\) pointwise, \(f\) injective then \(f \in M_s(X)\).
4. If \(f \in M_s(X), g \in M_s(f(X))\) then \(g \circ f \in M_s(X)\).

Proof. Obvious verifications.

Returning to our conditions \((\alpha') \Rightarrow (\delta')\) we see that \((\beta')\) is equivalent to \((\gamma')\), that \((\alpha')\) means \(f^{-1} \in M_s(f(X))\) and that \((\delta')\) means \(f \in M_b(X) \cap M_s(X)\).

Our \(f\) of example 2.4 (5) is in \(M_b\), injective but not in \(M_s\). Its inverse yields an example of an \(M_s\)-function that is not in \(M_b\). Thus, in general, we have neither one of the implications \((\alpha') \Rightarrow (\gamma'),

\((\gamma') \Rightarrow (\alpha'), (\beta') \Rightarrow (\delta'), (\alpha') \Rightarrow (\delta')\). But our counterexample is
rather weird ($f$ is nowhere continuous and the domain of $f^{-1}$ is discrete). We can do better.

**EXAMPLE 2.10** Let $K$ have discrete valuation and let $k$ be infinite. Then there exists a homeomorphism of the unit ball of $K$ that is in $M$ but not in $M_s$. (The inverse map is in $M$ but not in $M_s$).

**Proof.** Set $X = \{a \in K : |a| \leq 1\}$ and let $R$ be a full set of representatives of the equivalence relation $x \sim y$ iff $|x-y| < 1$ in $X$. Then $R$ is infinite. Let $\pi \in K$ be such that $|\pi|$ is the largest value that is smaller than 1. The map

$$
(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \quad (a_i \in R \text{ for each } i)
$$

is a bijection of $R^\infty$ onto $X$. We may suppose that $0 \in R$.

Since $R$ is infinite we can define injections

$$
\tau_1 : R \setminus \{0\} \to R
$$

$$
\tau_2 : R \to R
$$

such that $\text{im } \tau_1 \cap \text{im } \tau_2 = \emptyset$, $\text{im } \tau_1 \cup \text{im } \tau_2 = R$.

For $x = \sum_{n=0}^{\infty} a_n \pi^n \in X$ ($a_n \in R$ for each $n$) set

$$
f(x) := \begin{cases} 
\tau_1(a_0) + a_1 \pi + \ldots = x - a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\
\tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0 
\end{cases}
$$

A simple inspection of the definition shows that $f$ is a bijection of $X$ onto $X$. If $a, b \in R$, $a \neq b$ then $|a\pi - b\pi| < |b\pi - a|$, whereas

$$
|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1 \text{ and } |f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1,
$$

so $f \not\in M_s(X)$. Finally, let $x, y, z \in X$ and $|x-y| \leq |x-z|$. We prove that $|f(x) - f(y)| \leq |f(x) - f(z)|$. If $|f(x) - f(z)| = 1$ there is nothing to prove, so suppose $|f(x) - f(z)| < 1$. Set $x = \sum_{n} a_n \pi^n$, $y = \sum_{n} b_n \pi^n$, $z = \sum_{n} c_n \pi^n$. 
If $a_0 = 0$ then also $c_0 = 0$ and $\tau_2(a_1) = \tau_2(c_1)$ so $a_1 = c_1$, hence $|x-z| \leq |a|^2$. Since $|x-y| \leq |x-z|$ we have also $b_0 = 0$, $b_1 = a_1$.

So, $f(x)-f(y) = \frac{x-y}{\pi}$, $f(x)-f(z) = \frac{y-z}{\pi}$ whence $|f(x)-f(y)| \leq |f(x)-f(z)|$.

If $a_0 \neq 0$ then $\tau_1(a_0) = \tau_1(c_0)$ so $a_0 = c_0$. Then also $c_0 = a_0 = b_0$.

Then $f(x)-f(y) = x-y$, $f(x)-f(z) = x-z$ whence $|f(x)-f(y)| \leq |f(x)-f(z)|$.

Let $X \subseteq K$. If $f \in M_S(X)$ then $f^{-1} \in M_B(f(X))$. Conversely, if $f \in M_B(X)$ and $g : f(X) \to X$ is such that $f \circ g$ is the identity on $f(X)$ then $g \in M_S(f(X))$. This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let $X \subseteq K$ and $f : X \to K$. $f$ is called weakly monotone $(f \in M_W(X))$ if for all $x, y, z \in X$

$$|x-y| < |x-z| \Rightarrow |f(x)-f(y)| \leq |f(x)-f(z)|$$

$f$ is called strongly monotone $(f \in M_{bs}(X))$ if

$$f \in M_S(X) \cap M_B(X).$$

Clearly, $f \in M_{bs}(X)$ if and only if $f^{-1} \in M_{bs}(f(X))$. Also, if $f \in M_W(X)$ and $g : f(X) \to X$ is such that $f \circ g$ is the identity on $f(X)$ we have $g \in M_W(f(X))$.

Obviously we have $M_B(X) \cup M_S(X) \subseteq M_W(X)$ and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of $M_W$-functions, not for the sake of $M_W$ itself but in order to get results that are valid for $M_B$, $M_S$ simultaneously. The functions in $M_{bs}$ behave reasonable and they may be viewed as the non-archimedean equivalents of strict monotone functions in the real case.
THEOREM 2.12 Let \( X \subseteq K \) and \( f : X \to K \). Then the following conditions are equivalent.

(a) \( f \in M_{bs}(X) \).

(b) \( f \) is injective and \( C \mapsto f(C) \) is a 1-1 correspondence between the relatively convex subsets of \( X \) and those of \( f(X) \).

(c) For all \( x, y, z \in X \) \( |x - y| < |x - z| \iff |f(x) - f(y)| < |f(x) - f(z)| \).

(d) For all \( x, y, z \in X \) \( |x - y| = |x - z| \iff |f(x) - f(y)| = |f(x) - f(z)| \).

(e) For all \( x, y, z \in X \) \( |x - y| \leq |x - z| \iff |f(x) - f(y)| \leq |f(x) - f(z)| \).

(f) \( f \in M_s(X), f^{-1} \in M_s(f(X)) \).

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An \( M_w \)-function that is not in \( M_s \cup M_b \)). Let \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) be any function, constant on the cosets of \( \{x \in \mathbb{Z}_p : |x| < 1\} \). Then \( f \in M_w(\mathbb{Z}_p) \).

Clearly \( f \notin M_s(\mathbb{Z}_p) \). \( f \in M_b(\mathbb{Z}_p) \) if and only if the points of \( f(\mathbb{Z}_p) \) are equidistant.

(2) (Continuity of monotone functions). Let \( X \subseteq K \).

(a) Let \( f \in M_w(X) \). If \( f(X) \) has no isolated points, then \( f \) is continuous.

Proof. Let \( a \in X \) and \( \epsilon > 0 \). Then there is \( z \in X \) such that \( z \neq a \), \( |f(z) - f(a)| < \epsilon \). Let \( \delta := |z - a| \). Then for all \( x \in X \) with \( |x - a| < \delta \) we have, by the weak monotony of \( f \), \( |f(x) - f(a)| \leq |f(z) - f(a)| < \epsilon \).

It follows that if \( X \) and \( Y \) do not have isolated points and if \( f \) is an \( M_w \)-bijection of \( X \) onto \( Y \), then \( f \) is a homeomorphism of \( X \) onto \( Y \).
Conversely, it is easy to construct homeomorphisms of \( \mathbb{A}_p \) that are not in \( M_w(\mathbb{A}_p) \).

(b) If \( K \) is a local field then every \( f \in M_w(X) \) is continuous. (See 5.1 \((i)\)).

(c) If \( K \) has discrete valuation then every \( f \in M_S(X) \) is continuous.

(Example 2.4 (5) shows that such a statement is not true for \( f \in M_b(X) \).)

(Proof. If \( f \) were not continuous at some \( a \in X \) then there would be an \( \varepsilon > 0 \) such that for some sequence converging to \( a \) we had \( |f(x_n) - f(a)| \geq \varepsilon \). We may suppose that \( |x_1 - a| > |x_2 - a| > \ldots \). Since the valuation is discrete we have \( \lim_{n \to \infty} |f(x_n) - f(a)| = 0 \), a contradiction.)

(d) In 5.14 we shall give an example of a function in \( M_{bs}(X) \) that is not continuous. (Of course, \( K \) will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" \( M_w, \ldots \)-conditions.

Thus, by definition, \( f \in M_{uw}(X) \) if for all \( x, y, z, t \in X \)

\[
|x-y| < |z-t| \implies |f(x)-f(y)| \leq |f(z)-f(t)|
\]

\( f \in M_{us}(X) \) if for all \( x, y, z, t \in X \)

\[
|x-y| < |z-t| \implies |f(x)-f(y)| < |f(z)-f(t)|
\]

\( f \in M_{ubs}(X) \) if for all \( x, y, z, t \in X \)

\[
|x-y| < |z-t| \iff |f(x)-f(y)| < |f(z)-f(t)|.
\]

Notice that \( f \in M_{ubs}(X) \) means that \( |f(x)-f(y)| \) is a strictly increasing function of \( |x-y| \). Examples of such functions are isometries, but also the function \( f : \mathbb{A}_p \to \mathbb{A}_p \) defined via

\[
\Sigma a_n p^n \mapsto \Sigma a_n p^{2n} \quad (\Sigma a_n p^n \in \mathbb{A}_p)
\]

\( |f(x)-f(y)| = |x-y|^2 \) for all \( x, y \in \mathbb{A}_p \).)

Monotone functions : \( \mathbb{R} \to \mathbb{R} \) are divided into two classes: the
increasing functions and the decreasing functions. For the non-archimedean case we may ask for a similar classification. First we try to express the situation in the real case in such a way that it can be translated. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be strictly monotone. If $x$ runs through some side of $a$ then $f(x)$ runs through some fixed side of $f(a)$. So there is a map $\sigma : \{-1,1\} \to \{-1,1\}$ such that $\sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a))$ ($x \neq a$). Apparently, the only $\sigma$'s that can occur are the identity and $\sigma(x) = -x$ ($x \in \{1,-1\}$). Moreover it turns out that the map $\sigma$ is independent of the choice of $a$.

The two maps $\sigma$ that can occur can be interpreted as multiplication maps (with 1 and -1 respectively) or as the bijections $\{1,1\} \to \{-1,1\}$ and there seems to be no philosophical reason to make any decision of preference.

As an example, let us consider a function $f \in M_\Sigma(K)$. Let $a \in K$, let $a \in \Sigma$. If $x \in a+\alpha$ and $y \in a+\alpha$ ("$x,y$ are at the same side of $a$") then $x-a, y-a \in a$, so $|x-y| < |y-a|$. Since $f \in M_\Sigma(K)$ we have $|f(x)-f(y)| < |f(y)-f(a)|$, whence $|f(x)-f(a)-(f(y)-f(a))| < |f(y)-f(a)|$, so $f(x)-f(a)$ and $f(y)-f(a)$ have the same sign. We may conclude that there is a map $\sigma_a : \Sigma \to \Sigma$ such that for all $x \in K$

$$x \in a+\alpha \Rightarrow f(x) \in f(a)+\sigma_a(a) \quad (\alpha \in \Sigma).$$

Unfortunately, it turns out that in general $\sigma_a$ may be different from $\sigma_b$ if $a \neq b$, even for isometrical maps. For example, let $p \neq 2$ and let $\tau$ be a permutation of $\{0,1,2,\ldots,p-1\}$ and define $f : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$\Sigma a_n n^p \mapsto \Sigma \tau(a_n) n^p \quad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n).$$

Suppose we had a $\sigma : \Sigma \to \Sigma$ such that for all $x,y \in \mathbb{Z}_p$, $x-y \in \alpha$ implies $f(x)-f(y) \in \alpha$. Let $\alpha = 0^p n$ (see 1.5). Then $x-y \in \alpha$ means
x = a_0 + a_1 p + ... + a_n p^n ... 
y = b_0 + b_1 p + ... + b_n p^n ... 
where a_0 = b_0, ..., a_{n-1} = b_{n-1}, a_n - b_n = \theta^i \mod p.

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n))p^n + ... \), so \( \sigma(a) = \theta^j p^n \) where \( \tau(a_n) - \tau(b_n) = \theta^j \mod p \). (\( j \) depending on \( i \) and \( n \)).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = ... = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \subset K \), \( f : X \to K \) and let \( \sigma : \Sigma \to \Sigma \). We say that \( f \) is **monotone of type** \( \sigma \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \implies f(x) - f(y) \in \sigma(\alpha). \]

(In other words if \( x >_\alpha y \) implies \( f(x) >_{\sigma(\alpha)} f(y) \), see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \to \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( \alpha \), \( x >_\alpha y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \subset K \), \( f : X \to K \), \( \beta \in \Sigma \). We say that \( f \) is **monotone of type** \( \beta \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \implies f(x) - f(y) \in \alpha \beta. \]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \)
where $\sigma : \Sigma \to \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$
is monotone of type $\beta$ iff the sign of $\frac{f(x) - f(y)}{x - y}$ is constant $\beta$ for all
$x, y \in X, x \neq y$. This leads to

**DEFINITION 2.16** Let $X \subset K, f : X \to K$. $f$ is called increasing if $f$ is
monotone of type 1. In other words, $f$ is increasing
if for all $x, y \in X, x \neq y$ the difference quotient
$\frac{f(x) - f(y)}{x - y}$ is positive, i.e., if

$\left| \frac{f(x) - f(y)}{x - y} - 1 \right| < 1$.

In the next section we shall study the monotone functions of type $\alpha$
and we will give a partial answer to the question for which maps
$\sigma : \Sigma \to \Sigma$ there exists an $f : K \to K$ that is monotone of type $\alpha$. 
3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $\sigma$.

**DEFINITION 3.1.** Let $X \subseteq K$, $f: X \to K$. Let $f(x, y) = \frac{f(x) - f(y)}{x - y}$ ($x, y \in X$, $x \neq y$). $f$ is called

positive if $f(X) \subseteq K^+$

strictly positive if $\sup_{x \in X} |f(x) - 1| < 1$

increasing if $f(x, y) \in K^+$ for all $x, y \in X$, $x \neq y$

strictly increasing if $\sup_{x, y \in X} |1 - f(x, y)| < 1$.

It follows that an increasing function is an isometry. We collect some facts.

**THEOREM 3.2.** Let $X \subseteq K$.

(i) If $f: X \to K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a, b \in K$ the function $x \mapsto ax + b$ is increasing if and only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly increasing).

(iii) If $f: X \to K$ is (strictly) increasing and $f$ is (strictly) positive then $-\frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \to K$ form a convex set.

(v) If $f: X \to K$ and $g: f(X) \to K$ are (strictly) increasing then $g \circ f$ so is $g \circ f$.

(vi) If $f: X \to K$ is (strictly) increasing then so is $f^{-1}: f(X) \to K$.

(vii) If $f_1, f_2, \ldots: X \to K$ are increasing and $f := \lim_{n \to \infty} f_n$ pointwise then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function

\[ \exp x = 1 + x + \frac{x^2}{2!} + \ldots \]

defined on \( \{ x \in \mathbb{K}: |x| < p^{-p} \} \) if \( \chi(k) = p \), \( \chi(X) = 0 \) and on \( \{ x \in \mathbb{K}: |x| < 1 \} \) if \( \chi(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f: \mathbb{X} \to \mathbb{K} \) be a \( C^r \)-function (i.e., \( \Phi f \) can continuously be extended to a function on \( \mathbb{X} \times \mathbb{X} \), assume that \( \mathbb{X} \subseteq \mathbb{K} \) has no isolated points. See [2]) and suppose \( f'(a) \in \mathbb{K}^+ \) for some \( a \in \mathbb{X} \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta \), \( |y-a| < \delta \), \( x \neq y \) implies

\[ \frac{f(x) - f(y)}{x-y} - f'(a) \leq \frac{\varepsilon}{2}. \]

For such \( x, y \) we have \( \frac{f(x) - f(y)}{x-y} - f'(a) \leq \frac{f(x) - f(y)}{x-y} - f'(a) \vee |f'(a) - 1| \leq \max(\frac{\varepsilon}{2}, |f'(a) - 1|) < 1. \)

(3) The space \( C(\mathbb{R}) \) of all continuous functions \( \mathbb{R} \to \mathbb{K} \), is a Banach space with respect to the sup norm \( ||f||_\infty \). Let \( e_0 := \xi_0 \) and for \( n \geq 1 \) let \( e_n := \xi_n \) where \( B_n := \{ x \in \mathbb{R}: |x-n| < \frac{1}{n} \} \). It is proved in [1] that \( e_0, e_1, \ldots \) is an orthonormal base of \( C(\mathbb{R}) \) i.e., for each \( f \in C(\mathbb{R}) \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that

\[ f = \sum_{n=0}^{\infty} \lambda_n e_n. \]
||f||_{\omega} = \max |\lambda_n|.

The coefficients \( \lambda_n \) can be reconstructed from \( f \) via

\[
\lambda_0 = f(0) \\
\lambda_n = f(n) - f(n_-) \quad (n \in \mathbb{N})
\]

where \( n_- \) is defined as \( a_0 + a_1 p + \ldots + a_s p^s - 1 \) if \( n \neq a_0 + a_1 p + \ldots + a_s p^s \) \((a_s \neq 0)\) in base \( p \).

Our aim is here to describe a necessary and sufficient condition for the \( \lambda_n \) in order that \( f = \sum \lambda_n e_n \) is increasing. We show

\[
f = \sum \lambda \in \mathbb{N}^n \text{ increasing if and only if for all } n \in \mathbb{N}.
\]

\[
|\lambda_n - (n-n_-)| < |n-n_-|.
\]

Proof. First observe that \( f \) is increasing if and only if for all \( x \in \mathbb{Z}_p \)

\[
f(x) = x + g(x)
\]

where \(|g(x,y)| < 1 \) for all \( x, y \in \mathbb{Z}_p, x \neq y \).

As

\[
x = \sum_{n \geq 1} (n-n_-) e_n(x) \quad (x \in \mathbb{Z}_p)
\]

it suffices to show that for \( g = \sum \lambda \in \mathbb{N}^n \in C(\mathbb{Z}_p) \) we have \(|\phi(g)| < 1 \) if and only if \(|\lambda_n| < |n-n_-| \) for all \( n \in \mathbb{N} \).

Suppose first \(|\phi(g)| < 1 \). Then for all \( n \in \mathbb{N}, \frac{f(n) - f(n_-)}{n-n_-} < 1 \), so

\[
|\lambda_n| = |f(n) - f(n_-)| < |n-n_-|.
\]

Conversely, let \( g = \sum \lambda \in \mathbb{N}^n \) and let \(|\lambda_n| < |n-n_-| \) for all \( n \in \mathbb{N} \).

Let \( x, y \in \mathbb{Z}_p \) and let \(|x-y| = p^{-k} \) for some \( k \in \{0,1,2,\ldots\} \). Since

\[
e_n(a) = e_n(b) \quad \text{if and only if } |a-b| < \frac{1}{p}\]

we have

\[
e_n(x) = e_n(y) \quad \text{for } n < p^k.
\]
Therefore
\[ |g(x) - g(y)| = \left| \sum_{n=1}^{\infty} \lambda_n \frac{(e_n(x) - e_n(y))}{n} \right| \leq \max \frac{|\lambda|}{n} \leq \max \frac{|n-n_\infty|}{n \geq p} = p^{-k} \leq |x-y| \]
so \(|\phi g| < 1.

(4) Let \(K\) have dense valuation and let \(k\) be (countably) infinite. Let \(X\) be the unit ball of \(K\) and let \(B_i^x\) \((i \in \mathbb{N})\) be the balls in \(X\) with radius 1. Choose \(c_1, c_2, \ldots \in K\) such that \(|c_1| < |c_2| < \ldots\), \(\lim |c_n| = 1\). For \(n \in \mathbb{N}\) define a function \(f_n : X \to K\) via

\[
f_n(x) = \begin{cases} 
    x + c_i & \text{if } x \in B_i (1 \leq i \leq n) \\
    x & \text{elsewhere}
\end{cases}
\]

Then each \(f_n\) is strictly increasing \((|\phi f_n(x,y) - 1| \leq \max_{1 \leq i, j \leq n} |c_i - c_j| \leq c_n < 1)\). The sequence \(f_1, f_2, \ldots\) converges pointwise to an increasing function \(f\). But \(f\) is not strictly increasing:

\[
\sup_{x \neq y} |\phi f(x,y) - 1| = \sup_{i,j} |c_i - c_j| = 1.
\]
(Compare 3.2, (vii) and (viii).)

Increasing functions are closely related to functions \(g\) for which \(|g(x) - g(y)| < |x-y|\) \((x \neq y)\) (if \(f\) is increasing, set \(g(x) := f(x) - x\)).

DEFINITION 3.4. Let \((X, \rho)\) be an ultrametric space. A map \(g : X \to X\)

is called a pseudocontraction if \(\rho(f(x), f(y)) < \rho(x, y)\)

\((x, y \in X, x \neq y)\).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X \rightarrow X$ has a fix point. We have

**LEMMA 3.5.** Let $(X, \rho)$ be an ultrametric space. Then the following conditions are equivalent.

(a) $X$ is spherically complete.

(\beta) Each pseudocontraction $X \rightarrow X$ has a fix point.

(\gamma) Each pseudocontraction $X \rightarrow X$ has a unique fix point.

**Proof.** If $\sigma: X \rightarrow X$ is a pseudocontraction and if $x, y$ are fix points and $x \neq y$, then $\rho(x, y) = \rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction. Thus, we have (\beta) \Rightarrow (\gamma). We prove (a) \Rightarrow (\beta). Let $B \subset X$ be a ball (i.e., either $B = \{x \in X : \rho(x, a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X : \rho(x, a) < r\}$ for some $a \in X$, $r > 0$). We call $B$ invariant if $\sigma(B) \subset B$.

Now we observe two facts

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x), \sigma(a)) < \rho(x, a) \leq \max(\rho(x, \sigma(a)), \rho(\sigma(a), a)) = \rho(a, \sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \notin V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x, y)$ does not depend on $x, y$, since for $z \in B_1$, $u \in B_2$ $\rho(x, z) < \rho(x, y)$ and $\rho(y, u) < \rho(x, y)$, so $\rho(z, u) = \rho(x, y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a smal-
least invariant ball $S$. If $a \in S$, $\sigma(a) \neq a$ then \{ $x \in S: \rho(x, \sigma(a)) < \\
\rho(a, \sigma(a))$ \} is invariant and does not contain $a$, a contradiction. Hence, $\sigma$ has a fix point (actually, $S$ is a singleton).

We prove ($\beta$) $\Rightarrow$ (a). If $X$ were not spherically complete, there exist balls $B_1 \supset B_2 \supset \ldots$ such that $\cap_{n=1}^{\infty} B_n = \emptyset$. Choose $x_n \in B_n \setminus B_{n+1}$ ($n \in \mathbb{N}$), set $B_0 := X$ and define

\[ \sigma(x) := x_{n+1} \quad \text{if} \quad x \in B_n \setminus B_{n+1} \quad (n \in \{0, 1, 2, \ldots\}). \]

Then $\sigma$ has obviously no fix point. Let $x \in B_n \setminus B_{n+1}$ and $y \in B_m \setminus B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in $B_{m+1}$, whereas $x \in B_n \subset B_{m+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x), \sigma(y)) < \\
\rho(x, y)$. Then $\sigma$ is a pseudocontraction without a fix point. Contradiction.

**COROLLARY 3.6.** The following conditions are equivalent.

(a) $K$ is spherically complete.

(\beta) If $C \subset K$ is convex, $f: C \to C$ is increasing then $f$ is surjective.

(\gamma) If $C \subset K$ is convex, $f: C \to K$ is increasing then $f(C)$ is convex.

(\delta) An increasing $f: K \to K$ is surjective.

**Proof.** (a) $\Rightarrow$ (\beta). Choose $a \in C$ and consider the map $\sigma: x \mapsto x - f(x) + a$ ($x \in C$). Then $\sigma: C \to C$. $C$ is spherically complete, $\sigma$ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: $f$ is surjective.

(\beta) $\Rightarrow$ (\gamma). For a suitable $s \in K$, $f-s$ sends $C$ into $C$. (\gamma) $\Rightarrow$ (\delta) is clear.

(\delta) $\Rightarrow$ (a). Let $\sigma: K \to K$ be a pseudocontraction. Then $x \mapsto x - \sigma(x)$
is increasing hence is surjective. So then is $x \in K$ for which $x-a(x) = 0$, i.e., $a$ has a fix point. By 3.5, $K$ is spherically complete.

In case $f$ is strictly increasing we do not have to require that $K$ is spherically complete:

**THEOREM 3.7.** Let $C \subseteq K$ be convex and let $f: C \rightarrow K$ be strictly increasing. Then $f(C)$ is convex. If $f(C) \subseteq C$, then $f(C) = C$.

**Proof.** Reread the proof of (a) $\rightarrow$ (β), (β) $\rightarrow$ (γ) above. $a$ now is a contraction. $C$ is complete. Apply the Banach contraction theorem.

Let $X$ be a subset of $\mathbb{R}$ and let $f: X \rightarrow \mathbb{R}$ be a bounded increasing function. Then $f$ can be extended to an increasing function $\mathbb{R} \rightarrow \mathbb{R}$ by setting $f(x) := \inf_{y \in X} f(y)$ for all $y \in X$ and $f(x) := \sup_{y \in X \setminus \{x\}} f(y)$ for all other $x \in \mathbb{R}$. In our situation we can prove

**THEOREM 3.8.** The following conditions are equivalent.

(a) $K$ is spherically complete.

(β) For every $X \subseteq K$ an increasing function $f: X \rightarrow K$ can be extended to an increasing $\overline{f}: K \rightarrow K$.

(γ) Let $X \subseteq K$, and let $f: X \rightarrow K$ be a strictly increasing function. Then $f$ can be extended to a strictly increasing function $\overline{f}: K \rightarrow K$ such that

$$
\sup_{x,y \in K} \left| \frac{\overline{f}(x) - \overline{f}(y)}{x-y} - 1 \right| = \sup_{x,y \in X} \left| \frac{f(x) - f(y)}{x-y} - 1 \right| 
$$

**Proof.** (a) $\rightarrow$ (β). Let $a \notin X$. By Zorn's Lemma it suffices to define $\overline{f}$ such that $\overline{f}$ is increasing on $X \cup \{a\}$. We are done if we can find $a \in K$ such that for $x \in X$
\[ \frac{a-f(x)}{a-x} -1 < 1 \]

i.e., \( a \in B_x := B_{f(x)-(a-x)}(\{a-x\}) (x \in X) \).

Now \( B_x \cap B_y \neq \emptyset \) (\( x, y \in X \)) since the distance of their centers is

\[ |f(x)-(a-x)-f(y)-(a-y)| = |f(x)-f(y)-(x-y)| = |\Phi(x,y)|-1||x-y|| < \]

\( \leq \max(|x-a|, |a-y|) \). So if, say, \( |x-a| \leq |y-a| \) we see that \( |f(x)-(a-x)-f(y)-(a-y)| < |y-a| \) whence \( f(x)-(a-x) \in B_y \). By the spherical completeness of \( K \) we have \( \bigcap_{x \in X} B_x \neq \emptyset \). Choose \( a \in \bigcap_{x \in X} B_x \).

\((\beta) \to (\alpha)\). Suppose \( K \) is not spherically complete. By 3.6, \((\delta) \to (\alpha)\) there is a non surjective increasing function \( f: K \to K \). Then its inverse \( g: f(K) \to K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K \to K \).

\((\beta) \leftrightarrow (\gamma)\) follows from the fact that (with \( \Phi(x) = x \) for all \( x \))

\[ f \mapsto (1-c)\Phi + cf \quad (c \in K, |c|<1) \]

is a 1-1 correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \( |1-\Phi(g)| < |c| \).

We will now investigate the relation between increasingness of \( f \) and positivity of \( f \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \to K \). Let \( \lambda \in K, 0 < |\lambda| < 1 \). Then \( x \mapsto x-\lambda \sigma(x) \) is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \) of \( K \) without isolated points and if \( f \) is differentiable then for each
\[ f'(x) = \lim_{y \to x} f(x,y) \in K^+ \]. So \( f' \) is positive. If, addition, \( f \)
is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f : X \to K \) be a (strictly) positive Baire class 1 function. Then does \( f \) have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subseteq K \) and let \( f : X \to K \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots : X \to K \) such that \( |g_n| < 1 - \frac{1}{n} \) for each \( n \) and

\[ f = \sum g_n \quad \text{(pointwise)}. \]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots : X \to K \) such that \( f = \lim_{n} f_n \) pointwise. There exist locally constant functions \( h_1, h_2, \ldots : X \to K \) such that \( |f_n - h_n| \leq 2^{-n} \), hence \( f = \lim_{n} h_n \) pointwise. Define

\[ t_1, t_2, \ldots : X \to K \] as follows:

\[ t_n(x) = \begin{cases} h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\ 0 & \text{if } |h_n(x)| > 1 - \frac{1}{n} \end{cases} \]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N} \). (\( \{(x \in X : |h_n(x)| \leq 1 - \frac{1}{n}\} \) is closed and open in \( X \).) \( |t_n| \leq 1 - \frac{1}{n} \) and \( \lim_{n} t_n = f \). Now let \( g_1 := t_1 \), and \( g_n := t_n - t_{n-1} \) (\( n \geq 2 \)). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n} \), \( f = \sum_{n=1}^{\infty} (t_n - t_{n-1}) = \sum_{n=1}^{\infty} g_n \).
LEMMA 3.10. Let \( X \subset K \) have no isolated points and let \( f: X \to K \) be a Baire class 1 function, \(|f(x)| < 1\) for all \( x \in X \). Then \( f \) has an antiderivative \( F \) for which

\[
\left| \frac{F(x) - F(y)}{x-y} \right| < 1 \quad (x, y \in X, x \neq y).
\]

Proof. By Lemma 3.9, \( f = \sum_{n=1}^{\infty} f_n \), where each \( f_n \) is locally constant, \( |f_n| \leq 1 - \frac{1}{n} \). By [2] 3.9 each \( f_n \) has an antiderivative \( F_n \) for which

\[
|F_n(x) - F_n(y)| \leq \max \left( |f_n(x)|, \frac{|x-y|}{2n} \right) \quad (x, y \in X).
\]

By [2] 3.7, \( F = \sum F_n \) is an antiderivative of \( \sum f_n = f \). Now for \( x, y \in X, x \neq y \):

\[
|F(x) - F(y)| \leq \sup_n |F_n(x) - F_n(y)| \leq \sup_n \max \left( |f_n(x)|, \frac{|x-y|}{2n} \right) \quad (x, y \in X).
\]

\[
\leq |x-y| \max \left( |f_n(x)|, \frac{1}{2} \right). \quad \text{Now for each } x \in X, |f_n(x)| < 1 \text{ for each } n
\]

and \( \lim_{n} |f_n(x)| = 0 < 1 \). Hence \( \max_n |f_n(x)| < 1 \). It follows that

\[
|F(x) - F(y)| < |x-y|.
\]

THEOREM 3.11. Let \( X \subset K \) have no isolated points and let \( f: X \to K \) be (strictly) positive. Then \( f \) has a (strictly) increasing antiderivative.

Proof. The function \( x \mapsto f(x) - 1 \) has, by 3.10, an antiderivative \( H \) such that \( |\Phi(H)| < 1 \). Let \( F(x) = x + H(x) \) \((x \in X)\). Then \( F' = f \) and \( \Phi(F) = 1 + \Phi(H) \).

Thus, if \( f \) is positive then \( F \) is increasing. If \( f \) is strictly positive then \( |f(x)| - 1 < r < 1 \) for all \( x \in X \) and, by a trivial extension of 3.10, we may choose \( H \) such that \( |\Phi(H)| < r \). It follows that \( |\Phi(F)| - 1 < r \), so \( F \) is strictly increasing.
We collect the results in

COROLLARY 3.12. Let \( X \subset K \) have no isolated points. Then

(i) If \( f: X + K \) is differentiable and (strictly) increasing
then \( f' \) is a (strictly) positive Baire class 1 function.

(ii) If \( g: X + K \) is a (strictly) positive Baire class 1 func­tion then \( g \) has a (strictly) increasing antiderivative.

(iii) If \( f: X + K \) is differentiable and if \( f' \) is (strictly) po­
sitive then \( f = g + h \) where \( g \) is (strictly) increasing and
where \( h' = 0 \).

Note. We cannot strengthen 3.12 (iii) by replacing "h' = 0" by "h is
locally constant". In fact, if \( X = \mathbb{Z} \) then every locally constant func­tion has bounded difference quotients. If our statements were true,
then every differentiable \( f: \mathbb{Z} + K \) for which \( f' \) is positive would
have bounded difference quotients.

But consider the function \( f: \mathbb{Z} + K \) defined via

\[
f(x) = \begin{cases} 
  x - p^{2n} & \text{if } |x - p^n| < p^{-3n} \ (n \in \{0,1,2,\ldots\}) \\
  x & \text{elsewhere}
\end{cases}
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z} \). Let \( x := p^n \) and \( y := p^n + 3n \ (n \in \mathbb{N}) \). Then
\[
f(x_n) = p^{n-2n}, \quad f(y_n) = p^n + 3n, \quad \text{so } |f(x_n) - f(y_n)| = |p^{2n}| = p^{-2n},
\]
whereas \( |x_n - y_n| = |p^{3n}| = p^{-3n} \). So
\[
\lim_{n \to \infty} \frac{f(x_n) - f(y_n)}{x_n - y_n} = \lim_{n \to \infty} p^n = \infty.
\]

We now study the connection between increasing \( C^1 \)-functions and
continuous positive functions.

If \( f \) is a (strictly) increasing \( C^1 \)-function then clearly \( f' \) is a con­tinuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f : X \to K$ be continuous and positive. Let $P : C(X) \to C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x - x_n) \quad (x \in X).$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_i = 0$. The equivalence relation "$x \sim y$ iff $|x - y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in \mathbb{N}$, let $x_n := \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x) - x| < r_n$, $\sigma_n(x) \in R_n$.

See [2] 5.3, 5.4.)

From [2] 5.4 it follows that $Pf$ is a $C^1$-antiderivative of $f$. It suffices to prove that $Pf$ is (strictly) increasing. Let $x, y \in X$, $x \neq y$.

$|x - y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x - y| < r_s$. We have

$x_1 = y_1$, \ldots, $x_s = y_s$, $x_{s+1} \neq y_{s+1}$. Further $|x_{n+1} - x_n| \leq |x - y|$ $(n > s)$,

$|y_{n+1} - y_n| \leq |x - y|$ $(n > s)$, $|x_{s+1} - y_{s+1}| \leq |x - y|$. Hence (using the identity $x = \sum (x_{n+1} - x_n) + x_1$, $y = \sum (y_{n+1} - y_n) + y_1$, $x_1 = y_1$) $|Pf(x) - Pf(y) - (x - y)| =$

$$= \sum_{n > s} (f(x_{n+1} - y_{n+1})) + \sum_{n > s} (f(x_n - y_n)) - \sum_{n > s} (f(x_{n+1} - y_{n+1})) |.$$

If $|f(x) - 1| < \alpha$ for all $x \in X$, we have since $\lim|f(x_n) - 1|$ exists,

$\sup_{n > s}|f(x_n) - 1| < \alpha$, similarly, $\sup_{n > s}|f(y_n) - 1| < \alpha$.

So we get $|Pf(x) - Pf(y) - (x - y)| < \alpha |x - y|$.  

Now suppose $|x - y| \geq r_1$. Then since for all $n: |x_{n+1} - x_n| < r_1$, $|x_1 - y_1| = |x - y|$ we get (again under the assumption $|f(x) - 1| < \alpha$ for all $x \in X$):
We have proved:

**THEOREM 3.13.** Let $X \subset K$ have no isolated points. Then the map $P$ defined via

$$Pf(x) = x + \sum_{n=1}^{\infty} f(x_n) (x_n - x) \quad (f \in C(X), x \in X)$$

maps (strictly) positive functions into (strictly) increasing functions.

**COROLLARY 3.14.** Let $X \subset K$ have no isolated points. Then if $f \in C^1(X)$ and $f'$ is (strictly) positive, then $f = j + h$ where $j$ is (strictly) increasing and $h$ is locally constant.

**Proof.** By 3.12 we have $f = j + h$ where $j$ is (strictly) increasing and where $h' = 0$. Now by [2] Cor. 5.2 bis there is a locally constant function $l: X \to K$ with $\|h-l\|_\infty < \frac{1}{2}$. Then $s := j + (h-l)$ is (strictly) increasing, so we have $s = s + 1$, where $s$ is (strictly) increasing and $l$ is locally constant.

**Note.** We may also define convex functions. Let $X \subset K$. A function $f: X \to K$ is called convex if the second order difference quotient is positive. I.e., if for all $x, y, z \in X$ ($x \neq y, y \neq z, x \neq z$) we have

$$\frac{\frac{\frac{f(x)-f(y)}{y-z}}{x-y} - \frac{f(x)-f(z)}{y-z}}{x-z} \in K^+$$

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function $f$ the function $x \mapsto \tilde{f}(x,y)$ defined on $X \setminus \{y\}$ is an isometry, hence can be continuously extended to the whole of $X$. Define $\tilde{f}(y,y) = \lim_{x \to y} \tilde{f}(x,y)$ ($y \in X$). Thus, $f$ is differentiable. For all $x,y,z,t \in X$ we have

$$|\tilde{f}(x,y) - \tilde{f}(z,t)| \leq \max(|\tilde{f}(x,y) - \tilde{f}(z,y)|, |\tilde{f}(z,y) - \tilde{f}(z,t)|) \leq \max(|x-z|, |y-t|).$$

Hence, $\tilde{f}$ is uniformly continuous on $X$ i.e., $f$ is strongly uniformly differentiable in the sense of [2] page 67.

For each $y \in X$ the function $x \mapsto \tilde{f}(x,y)$ is increasing on $X$.

If $\chi(K) \neq 2$ then convexity of $f$ implies increasingness of $\tilde{f}'$.

(Proof.)

$$\lim_{y \to x} \frac{\tilde{f}(x,y) - \tilde{f}(x',y)}{x-x'} = \frac{f'(x) - f'(x',x)}{x-x'} \in K^+ (x \neq x')$$

and

$$\lim_{y \to x'} \frac{\tilde{f}(x,y) - \tilde{f}(x',y)}{x-x'} = \frac{\tilde{f}(x,x') - f'(x')}{x-x'} \in K^+ (x \neq x').$$

so

$$\frac{f'(x)-f'(x')}{x-x'} \in 2K^+ (x \neq x'), \text{ whence } \phi(\tilde{f}')_K(x,x') \in K^+ \text{ if } x \neq x' \text{.}$$

Of course, if $f \in C^2(X)$ (see [2] 8.1) then convexity of $f$ implies positivity of $D_2f$ ([2] 8.4). So if $\chi(K) \neq 2$ then $\tilde{f}'' = D_2f$ ([2] 8.14) is positive. If $\chi(K) = 2$ then $f'' = 0$ for all $C^2$-functions.

Note. The functions that are monotone of type $\beta$ ($\beta \in \Sigma$), see Def. 2.15, are easy to describe: $f$ is monotone of type $\beta$ if and only if $b^{-1}f$ is increasing for any $b \in \beta$.

We now turn to the functions $X + K$ that are of type $\sigma$ where

$$\sigma : \Sigma + \Sigma. (2.14).$$

For examples of such $f$, where $\sigma$ is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this section we will assume that $X$ is an open convex subset of $K$. This implies that the set $\{a \in \Sigma : \text{there is } x \in X \text{ such that } x > a\}$ is independent of $a \in X$. Thus, $X$ is homogeneous in the sense that $(a + \alpha) \cap X \neq \emptyset$ for some $a \in X$, $\alpha \in \Sigma$ then for each $b \in X$, $(b + \alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y\}$. Then for each $a \in X$,

$$\Sigma(X) = \{a \in \Sigma : x > a \text{ for some } x \in X\}.$$

Either $\Sigma(X) = K$ or $\Sigma(X) = \{a \in \Sigma : |a| < r\}$ for some $r > 0$ or $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see 1.2) i.e., if $a, \beta \in \Sigma(X)$ and $a \oplus \beta$ is defined then $a \oplus \beta \in \Sigma(X)$.

To be sure that $f$ is monotone both of type $\sigma$ and type $\tau$ implies $\sigma = \tau$ we define

DEFINITION 3.15. (Let $X \subseteq K$ be open, convex and) let $\sigma : \Sigma(X) \to \Sigma$. $f : X \to K$ is called monotone of type $\sigma$ if for all $x, y \in X$ and $a \in \Sigma(X)$

$$x > y + f(x) > f(y).$$

Theorem 3.16. Let $f : X \to K$ be monotone of type $\sigma : \Sigma(X) \to \Sigma$. Then

(i) $\sigma(-\alpha) = -\sigma(\alpha)$ ($\alpha \in \Sigma(X)$).

(ii) Let $a, \beta \in \Sigma(X)$. If $\sigma(a) \oplus \sigma(\beta)$ is defined then so is $a \oplus \beta$ and $\sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta)$.

(iii) Let $a, \beta \in \Sigma(X)$. If $|a| < |\beta|$ then $|\sigma(a)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then $\sigma(sa) = s\sigma(a)$ ($a \in \Sigma(X)$).

(v) If $\beta \in \Sigma(X)$, $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $a \in \Sigma(X)$. 

(vi) \( f \in M_{us}(X) \) (i.e., for all \( x,y,z,t \in X, |x-y| < |z-t| \) implies \( |f(x)-f(y)| < |f(z)-f(t)| \)).

(vii) \( f \) is either nowhere continuous or uniformly continuous on \( X \).

Proof.

(i) Let \( x,y \in X \) such that \( x > y \). Then \( f(x)-f(y) \in \sigma(\alpha) \); \( f(y)-f(x) \in -\sigma(\alpha) \). But also \( y > x \), hence \( f(y)-f(x) \in \sigma(-\alpha) \). So \( -\sigma(\alpha) \) and \( \sigma(-\alpha) \) are not disjoint and they must coincide.

(ii) Suppose \( \sigma(\alpha) \cap \sigma(\beta) \) is defined. If \( \alpha \cap \beta \) were not, then \( \beta = -\alpha \) so, by (i), \( \sigma(\beta) = \sigma(-\alpha) = -\sigma(\alpha) \). Hence also \( \alpha \cap \beta \) is defined. Choose \( x,y \in X \) with \( x > y \). There is \( z \in X \) such that \( y > z \). Then \( x-y \in \alpha \), \( y-z \in \beta \), so \( x-z \in \alpha \cap \beta \). Further \( f(x)-f(y) \in \sigma(\alpha) \), \( f(y)-f(z) \in \sigma(\beta) \) so \( f(x)-f(z) \in \sigma(\alpha) \cap \sigma(\beta) \). Also \( x-z \in \alpha \cap \beta \), so \( f(x)-f(z) \in \sigma(\alpha \cap \beta) \).

The signs \( \sigma(\alpha) \cap \sigma(\beta) \) and \( \sigma(\alpha \cap \beta) \) are not disjoint and they must coincide.

(iii) Let \( |\alpha| < |\beta| \). Choose \( x,y,z \) such that \( x-y \in \alpha \), \( y-z \in \beta \). Then (see 1.2 and preamble) \( f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(\alpha)+\sigma(\beta) \), \( x-z \in \alpha+\beta = \beta \), so \( f(x)-f(z) \in \sigma(\beta) \). Thus \( \sigma(\alpha)+\sigma(\beta) \cap \sigma(\beta) \) is not empty. If \( \sigma(\alpha) \cap \sigma(\beta) \) were not defined then \( \sigma(\alpha) = -\sigma(\beta) \) and \( \sigma(\alpha)+\sigma(\beta) \) would be a ball with center 0 and radius \( |\sigma(\beta)| \), but then \( \sigma(\alpha)+\sigma(\beta) \cap \sigma(\beta) \) would be empty. Hence \( \sigma(\alpha) \cap \sigma(\beta) \) is defined and by (ii) we have \( \sigma(\alpha) \cap \sigma(\beta) = \sigma(\beta) \). By (1.2) (vi), \( |\sigma(\alpha)| < |\sigma(\beta)| \).

(iv) Let \( \chi(K) \neq 0 \). Then \( s = n \cdot 1 \) for some \( n \in \{1,2,...,\chi(K)-1\} \), so by 1.2 (vii), \( sa = na = \theta a \), \( s\sigma(a) = n\sigma(a) = \theta \sigma(a) \). By a repeated application of (ii), we see \( \sigma(\theta a) = \sigma(\sigma(a)) \). Hence \( \sigma(sa) = s\sigma(a) \).

Let \( \chi(K) = 0 \). Let \( s \) be of the form \( n \cdot 1 \) for some \( n \in \text{IN} \). By a similar reasoning as above, \( \sigma(sa) = s\sigma(a) \). We may identify the prime field of \( K \) with \( \mathbb{Q} \).
Now observe that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) (\( n \in \mathbb{N} \)), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^*: \sigma(sa) = \sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1, 1, 2, \ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N}, n = s \text{ mod } p \) (\( 1 \leq s \leq p-1 \)) then \( na = sa \) for all \( a \), so \( \sigma(na) = \sigma(sa) = \sigma(a) = na(a) \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and if \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). By (iii) \( |\sigma(\alpha)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( z-t \in a \) for some \( a \) hence \( f(z)-f(t) \in a(a) \). Thus, \( f(z) \neq f(t) \).

(vii) \( \inf_{x \neq y} |f(x)-f(y)| = \rho \) if \( \rho > 0 \) then clearly \( f \) is nowhere continuous. If \( \rho = 0 \), let \( \varepsilon > 0 \). There is \( a, b \in X, a \neq b \) such that \( |f(a)-f(b)| < \varepsilon \). By (vi), for all \( x, y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \varepsilon \). Then \( f \) is uniformly continuous.

A natural question that can be raised is the following. If \( f: X \to K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f: X \to K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
(β) $f \in M^*_D(X)$.

(γ) $f \in M^*_ub(X)$.

(δ) If, for $\alpha, \beta \in \Sigma(X)$, $\alpha \oplus \beta$ is defined then so is $\sigma(\alpha) \oplus \sigma(\beta)$ (and $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)$).

(ε) If $\alpha, \beta \in \Sigma(X)$, $|\sigma(\alpha)| < |\sigma(\beta)|$ then $|\alpha| < |\beta|$.

**Proof.** We prove $(\alpha) \implies (\epsilon) \implies (\gamma) \implies (\beta) \implies (\delta) \implies (\alpha)$.

$(\alpha) \implies (\epsilon)$. Let $|\sigma(\alpha)| < |\sigma(\beta)|$ then $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$ (1.2. (vi)). By 3.16, (iii), $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. Since $\sigma$ is injective, $\alpha \oplus \beta = \beta$ so (again 1.2. (vi)) $|\alpha| < |\beta|$.

$(\epsilon) \implies (\gamma)$. Let $|x-y| \leq |z-t|$ $(x,y,z,t \in X)$. We prove $|f(x)-f(y)| \leq |f(z)-f(t)|$. If $z = t$ there is nothing to prove. Assume $z \neq t$ and $|f(x)-f(y)| > |f(z)-f(t)|$. Then (f is injective), supposing $x-y \in \alpha$, $z-t \in \beta$ for some $\alpha, \beta \in \Sigma(X)$, we have $f(x)-f(y) \in \sigma(\alpha)$, $f(z)-f(t) \in \sigma(\beta)$ and $|\sigma(\alpha)| > |\sigma(\beta)|$. By $(\epsilon)$, $|\alpha| > |\beta|$ i.e., $|x-y| > |z-t|$. Contradiction.

$(\gamma) \implies (\beta)$. Trivial.

$(\beta) \implies (\delta)$. Suppose $\sigma(\alpha) \oplus \sigma(\beta)$ is not defined. Then $|\sigma(\alpha)| = |\sigma(\beta)|$ and, by 3.16 (iii), $|\alpha| = |\beta|$. Choose $x,y,z$ such that $x-y \in \alpha$, $y-z \in \beta$. Then $f(x)-f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$ so $|f(x)-f(z)| < |\sigma(\alpha)| = |f(x)-f(y)|$.

Since $f \in M^*_D(X)$, $|x-z| < |x-y|$ hence, since $x-z \in \alpha \oplus \beta$, $x-y \in \alpha$:

$|\alpha \oplus \beta| < |\alpha|$. But $|\alpha \oplus \beta| = \max(|\alpha|,|\beta|)$, a contradiction.

$(\delta) \implies (\alpha)$. Suppose $\sigma(\alpha) = \sigma(\beta)$ and $\alpha \neq \beta$. Then $\alpha \oplus (-\beta)$ is defined. By $(\delta)$, also $\sigma(\alpha) \oplus \sigma(-\beta)$ is defined. But $\sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha)$, so $\sigma(\alpha) \oplus -\sigma(\alpha)$ is defined, a contradiction.

**Theorem 3.18.** Let $k$ be a prime field. Then, if $f : X \to K$ is monotone of type $\sigma$ then $\sigma$ is injective.
Proof. Suppose $\sigma(a) = \sigma(\beta)$ for some $a, \beta \in \Sigma(X)$. Then $|\sigma(a)| = |\sigma(\beta)|$ so, by 3.16 (iii), $|a| = |\beta|$. There is $t \in K$, $|t| = 1$ such that $\beta = ta$. Since $k$ is a prime field we may suppose $t \in \{1, 2, \ldots, p-1\}$ if $k \sim \mathbb{F}_p$ and $t \in \mathbb{Q}^*$ if $k \sim \mathbb{Q}$. So, by 3.16 (iv), $\sigma(\beta) = \sigma(ta) = t\sigma(a) = t\sigma(\beta)$. For $x \in \sigma(\beta)$ we have $tx \in \sigma(\beta)$, so $tx \cdot x^{-1} \in K$ i.e., $|t-1| < 1$. It follows easily that $t = 1$. Hence, $a = \beta$.

We now like to determine all $\sigma : \Sigma \to \Sigma$ that "can occur" as the type of a monotone function in case $K = \mathbb{Q}_p$. We use the fact that $\Sigma$ can be identified with the following subgroup of $\mathbb{Q}_p^*$

$$\{\theta_i^np : i \in \{0,1,2,\ldots,p-2\}, n \in \mathbb{Z}\}$$

where $\theta$ is a primitive $(p-1)^{th}$ root of 1. (See 1.5.)

First, let $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be monotone of some type $\sigma : \Sigma \to \Sigma$. By 3.18, $\sigma$ is injective. By 3.17, (e), 3.16 (iii) we have $|a| < |\beta| \iff |\sigma(a)| < |\sigma(\beta)|$ and $|a| = |\beta| \iff |\sigma(a)| = |\sigma(\beta)|$, so $|\sigma(a)|$ is a strictly increasing function of $|a|$.

Set

$$\sigma(\theta_i^np) = \theta^s(i,n)p^\lambda(i,n)$$

$(\theta_i^np \in \Sigma)$

Where $s : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \{0,1,2,\ldots,p-2\} \times \mathbb{Z} \to \mathbb{Z}$. We see that $|\sigma(\theta_i^np)| = |\sigma(\theta_j^np)|$ for all $i, j \in \{0,1,2,\ldots,p-2\}$ hence $\lambda(i,n) = \lambda(j,n)$ for all $i, j \in \{0,1,2,\ldots,p-2\}$. Then

$$\sigma(\theta_i^np) = \theta^s(i,n)p^\lambda(n)$$

where $\lambda : \mathbb{Z} \to \mathbb{Z}$ is a strictly increasing function (in the classical sense).

By 3.16 (v), $\sigma(\theta_i^np) = \theta_i^s(p^n) = \theta_i^s(0,n)p^\lambda(n)$. 
Thus, $\sigma$ is of the form

\[ (*) \theta^i_p n \to \theta^i_g s(n)_p \lambda(n) \]

where $s : \mathbb{N} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we are given a map $\sigma$ of the form $(*)$ then it is easy to construct an $f : \mathbb{Q}_p \to \mathbb{Q}_p$, monotone of type $\sigma$. In fact, let $x \in \mathbb{Q}_p$, $x = \sum a_p^n$, where $a_n \in \{0,1,\ldots,p^{p-2}\}$ for each $n$ and $a_n = 0$ for large $n$. Then set

\[ f(x) = \sum_{n \in \mathbb{Z}} a_s(n)_p \theta^i g_s(p)_p \lambda(n). \]

Now let $x = \sum a_p^n$, $y = \sum b_p^n$ and $\pi(x-y) = \theta^i_p m$ for some $i \in \{0,1,\ldots,p-2\}$, $m \in \mathbb{Z}$. Then $a_n = b_n$ for $n < m$ and $a_m - b_m = \theta^i \mod p$. So the sign of $a_m - b_m$ is $\theta^i$. $f(x) - f(y) = \sum (a_n - b_n) \theta^i s(n)_p \lambda(n) = (a_m - b_m) \theta^i s(m)_p \lambda(m) + r$, where $|r| < |f(x) - f(y)|$. The sign of $f(x) - f(y)$ is the sign of $(a_m - b_m) \theta^i s(m)_p \lambda(m)$ which is $\theta^i \theta^i s(m)_p \lambda(m)$. So $\pi(f(x) - f(y)) = \theta^i \theta^i s(m)_p \lambda(m) = \sigma(\theta^i p^m)$. Thus, $f$ is monotone of type $\sigma$. We have found

**Theorem 3.19.** The set \( \{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p, \text{monotone of type } \sigma\} \) is equal to the set of all $\sigma : \Sigma \to \Sigma$ of the form

\[ \theta^i_p n \to \theta^i_g s(n)_p \lambda(n) \]

where $s : \mathbb{Z} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

**Remark.** With the notations as in 3.19, let $\mu(n) := \lambda(n) - n$. Then $\mu : \mathbb{Z} \to \mathbb{Z}$ is increasing ($\mu(n+1) = \lambda(n+1) - (n+1) \geq \lambda(n+1) - (n+1) = \mu(n)$).

We then have two possibilities for a function $f : \mathbb{Q}_p \to \mathbb{Q}_p$, monotone of type $\sigma$. 

(a) \( \lim_{n \to \infty} u(n) = \infty. \) Then \( |\sigma(a)| = |a||p^{\mu(n)}|, \) \((a = \theta_{p^n}), \) so \( \lim_{\alpha \to 0} |\sigma(a)| = 0. \)

Thus \( \lim_{x-y \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0. \)

(b) \( \mu \) is bounded above. Then \( \mu(n) \) is constant, \( c, \) for \( n \geq n_0. \) (For example, if \( \sigma \) is bijective then we have even \( \mu(n) = c \) for all \( n. \))

Thus, for sufficiently small \( |a| (a = \theta_{p^n} \in \Sigma) \) we have

\[ |\sigma(a)| = |p^{\lambda(n)}| = |p^{n+c}| = |p^c| |a|. \]

So, there is \( r \) such that \( |x-y| < r \) implies \( |f(x)-f(y)| = |p^c||x-y|. \)

In this case we then have: \textit{there is} \( r > 0 \) and \( \lambda \in \mathbb{Q}_p \) \textit{such that on each ball in} \( \mathbb{Q}_p \) \textit{of radius} \( r, \) \( \lambda^{-1} f \) \textit{is an isometry.}

We now construct an example of a function \( f \) monotone of type \( \sigma, \) where \( \sigma \) is not injective. Let \( p = 3 \) mod \( 4 \) and let \( K := \mathbb{Q}_p(\sqrt{-1}). \) The elements of \( K \) can be written as \( a+bi \) \((a,b \in \mathbb{Q}_p)\) and \( |a+bi| = \max(|a|, |b|). \)

The value group of \( K \) is the same as the one of \( \mathbb{Q}_p, \) the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.18). Let \( X \) be the unit ball of \( K, \) let

\[ S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}. \]

For each \( x \in X \) there is a unique \( \overline{x} \in S \) such that \( |x-\overline{x}| < 1. \) As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s,t \in S^*, \) \( s \neq t. \)

Define a function \( h : S \to K \) as follows

\[ h(a+bi) = \frac{1}{p} a \quad (a+bi \in S) \]
and let \( f : X \to K \) be defined via
\[
f(x) = x + h(x) \quad (x \in X).
\]

We claim that \( f \) is monotone of type \( \sigma \) where
\[
\sigma(\pi(a+b)) = \pi\left(\frac{1}{p} a\right) \text{ if } a+b \in S, a \neq 0
\]
\[
\sigma(a) = a \quad \text{elsewhere.}
\]

(Clearly, \( \sigma \) is a well-defined map \( E(X) \to \mathbb{K} \), \( \sigma \) is not injective since, for example, \( \sigma(\pi(1)) = \sigma(\pi(1+i)) \).

**Proof.** Let \( |a| < 1 \) and \( x-y \in a \), then \( |x-y| < 1 \) so \( x = y \), \( h(x) = h(y) \). It follows that \( f(x) - f(y) = x-y \in a = \sigma(a) \).

Now let \( |a| = 1 \) be of the form \( \pi(bi) \), \( b \in \{1,2,\ldots,p-1\} \) and let \( x-y \in a \). Say, \( x = r+si \), \( y = t+ui \) \((r,s,t,u \in \{0,1,2,\ldots,p-1\})\). Then also \( x-y \in a \), so \( |r+si-t+ui| < 1 \) hence \( r = t \). Thus, \( h(x) = \frac{1}{p} r = h(y) \), and we have \( f(x) - f(y) = x-y \in a = \sigma(a) \).

Finally, let \( |a| = 1 \), \( \alpha = \pi(a+bi) \), where \( a \neq 0 \) \((a,b \in \{0,1,2,\ldots,p-1\})\) and let \( x-y \in a \). Set \( x = r+si \), \( y = t+ui \). Then \( x-y \in a \), so \( r-t = a \mod p \).

We find \( h(x) = \frac{1}{p} r \), \( h(y) = \frac{1}{p} t \), so \( |h(x) - h(y)| - \frac{1}{p} a < \frac{1}{|p|} |a| \) i.e. \( h(x) - h(y) \in \{\frac{1}{p} a\} \). Since \( |\pi(x-y)| \leq 1 \), we find \( f(x) - f(y) = x-y - (h(x) - h(y)) \in \{\frac{1}{p} a\} = \sigma(\pi(a+bi)) = \sigma(a) \).

Concluding:

**Example 3.20.** Let \( p = 3 \mod 4 \) and \( K = \mathbb{Q}_p(\sqrt{-1}) \). Then there exists a function \( f : \{x \in K : |x| \leq 1\} \to K \), monotone of some type \( \sigma \), where \( \sigma \) is not injective.

In case \( K \) has discrete valuation we have some extra information.
THEOREM 3.21. Let K have discrete valuation and let f : X → K be mono-
tone of type σ ∈ ℂ(X). Then

(i) f is continuous.

(ii) If σ is injective, then f and φ(f) are bounded on bounded
sets.

(iii) If σ is injective and if f(X) is convex then there is a λ ∈ K such that λf is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2) (c). If σ is injective
then by 3.16 (iii) and 3.17 (c), |σ(a)| is a strictly increasing func-
tion of |a|. Suppose X is bounded. Then Σ(X) = {a ∈ ℂ: |a| ≤ r} for some
r ∈ K∗. Let |σ(a)| = s whenever |a| = r. Let |π| < 1 be the generator
of |K∗|. Let |a| = |π|r. Then |a| < r so |σ(a)| < s, hence |σ(a)| ≤ |π|s.
By induction, it follows that |σ(a)| ≤ |π|n for whenever |a| = |π|nr, so
|σ(a)| ≤ |a|·rn. So the difference quotients of f are bounded by sr−1.
Then clearly f is bounded. So we have (ii). We prove (iii). If f(X) is
convex then Σ(f(X)) has the form {a ∈ ℂ: |a| ≤ s} for some s ∈ |K∗| ∪ {∞}.
Then σ induces an injection of {p ∈ |K∗| : p ≤ r} onto {a ∈ |K∗| : |a| ≤ s}
that is (strictly) increasing. It follows easily that this map is a mul-
tiplier. So |σ(a)| = c|a| for some c ∈ ℝ i.e., |f(x) - f(y)| = c||x - y||
for all x, y ∈ X.

3.21. (ii) induces the question under what conditions an f : X → K,
monotone of type σ, is bounded on bounded sets. We have an affirmative
answer in each of the following cases.
(1) K is a local field (f is continuous).
(2) K is finite and X = {x ∈ K : |x| ≤ 1}. (Proof let a₁, a₂, ..., aₙ ∈ X be
representatives modulo \{x \in K:|x| < 1\}, let \( M = \max|f(a_i) - f(a_j)|. \) For each \( x, y \in X \) we have \( i, j \) for which \(|x - a_i| < 1, |y - a_j| < 1. \) Since \( f \in M(X), \) we have \(|f(x) - f(a_i)| < M, |f(y) - f(a_j)| < M \) whence \(|f(x) - f(y)| \leq M; f \) is bounded.)

(3) \( K \) is discrete, \( \sigma \) is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let \( k \) be isomorphic to the algebraic closure of \( \mathbb{F}_p. \) Let \( X \) be the unit ball of \( K. \) Then there exists a function \( f : X \to K, \) monotone of type \( \sigma, \) for some \( \sigma : \Sigma(X) \to \Sigma \) such that

(i) \( \sigma \) is not injective.

(ii) \( f, \Phi(f) \) are unbounded.

Proof.
As an \( \mathbb{F}_p \)-vector space, \( k \) has a countable base \( e_1, e_2, \ldots. \) For any \( \lambda \in \mathbb{F}_p, \)
\( \lambda = n_1 \) for some \( n \in \{0,1,2,\ldots,p-1\}. \) (Here for a field \( L, \) \( 1_L \) is the unit element of \( L. \)) Define \( \lambda : = n_1 \). Choose \( c_1, c_2, \ldots \in K \) such that
\( 1 < |c_1| < |c_2| < \ldots, \lim_{n \to \infty} |c_n| = \infty, \) and define a map \( h : k \to K \) via
\[
\begin{align*}
    h(\Sigma e_n) &= \Sigma \lambda e_n \\
    & (\Sigma e_n \in k)
\end{align*}
\]
Define \( f : X \to K \) by
\[
f(x) = x + h(\overline{x}) \quad (x \in X)
\]
(Here \( \overline{x} \) is the image of \( x \) under the canonical map \( X \to k). \)
Then clearly \( f \) is unbounded and so is \( \Phi(f). \)
Let us identify \( \{a \in \Sigma:|a| = 1\} \) with \( k^* \) in the obvious way. We claim that \( f \) is monotone of type \( \sigma \) where
\[ \sigma(a) = \begin{cases} 
\alpha & \text{if } |\alpha| < 1 \\
\pi\left(\prod_{n}^\alpha \mathbb{C}_n \right) & \text{if } \alpha = \sum_{m}^\lambda m \mathbb{E}_m, \ n = \max(m : \lambda_m \neq 0). 
\end{cases} \]

In fact, let \( x - y \in \alpha \) and \( |\alpha| < 1 \). Then \( h(x) = h(y) \) so \( f(x) - f(y) = x - y \in \sigma(\alpha) \). Now let \( x - y \in \alpha \) where \( |\alpha| = 1 \). Then set \( \bar{x} = \sum_{n}^\lambda e_n \), \( \bar{y} = \sum_{n}^\mu e_n \). Let \( r = \max\{n : \lambda_n \neq \mu_n\} \). Then \( \bar{x} - \bar{y} = \sum_{n=1}^{r} (\lambda_n - \mu_n) e_n = \alpha \), so \( \sigma(\alpha) = \pi(\prod_{r}^{\lambda_n - \mu_n} \mathbb{C}_n) \).

On the other hand, \( f(x) - f(y) = x - y - (h(x) - h(y)) = x - y - \sum_{n=1}^{r} (\lambda_n - \mu_n) e_n = x - y - \sum_{n=1}^{\lambda_n - \mu_n} e_n \). Thus \( \pi(f(x) - f(y)) = \pi(\prod_{r}^{\lambda_n - \mu_n} \mathbb{C}_n) \).

Now we have \( \sum_{r}^{\lambda_n - \mu_n} = \lambda_n - \mu_n \mod p \), so \( \pi(\lambda_n - \mu_n) = \pi(\lambda_n - \mu_n) \). It follows that \( f(x) - f(y) \in \sigma(\alpha) \).

Obviously, \( \sigma \) is not injective.

We will consider briefly the differentiable functions, monotone of type \( \sigma \). Let \( f : X \to K \) be such a function. If \( f'(a) = 0 \) for some \( a \in X \), then \( \lim_{|a| \to 0} \frac{|\sigma(a)|}{|a|} = 0 \), so \( f' = 0 \) uniformly on \( X \). Now let \( f'(a) \neq 0 \). Then if \( x \) is sufficiently close to \( a \), we have \( |f(x) - f(a) - f'(a)| < |f'(a)| \).

Thus for \( |\alpha| \) small enough we have \( f'(a) \in \frac{\sigma(\alpha)}{\alpha} \) i.e. \( \frac{\sigma(\alpha)}{\alpha} \) is constant.

This implies that \( \pi(f'(x)) \) does not depend on \( x \) (\( f' \) has constant sign) and that locally \( f \) is monotone of type \( \beta \) for some \( \beta \in \Sigma \).

We end this section with a discussion on the question which maps \( \sigma : \Sigma \to \Sigma \) do occur as a type of a monotone function.

**Lemma 3.23.** Let \( \sigma : \Sigma \to \Sigma \). Suppose \( \sigma \) satisfies

- if \( \alpha \in \beta \) is defined then \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \). \( (\alpha, \beta \in \Sigma) \).

Then

(1) \( \sigma(-\alpha) = -\sigma(\alpha) \) \( (\alpha \in \Sigma) \).
(ii) If $\sigma(\alpha)$ is defined then so is $\alpha \oplus \beta$.

(iii) If $|\alpha| < |\beta|$ then $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) $\sigma$ is injective.

(v) If $|\alpha| = |\beta|$ then $|\sigma(\alpha)| = |\sigma(\beta)|$.

Proof. (i) is trivial if $\chi(k) = 2$, so suppose $\chi(k) \neq 2$ and let $-\sigma(\alpha) \neq \sigma(-\alpha)$ for some $\alpha \in \Sigma$. Then we have the identity $(\alpha \oplus \alpha) \oplus (-\alpha) = \alpha$, so $\sigma(\alpha \oplus \alpha) \oplus \sigma(-\alpha) = \sigma(\alpha)$, whence $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha) = \sigma(\alpha)$. Now by 1.2 (iii) $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ (this last expression is defined.

If not, then $-\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(-\alpha)$. Now $\sigma(\alpha) \oplus \gamma = -\sigma(\alpha)$ has only one solution namely $\gamma = -2\sigma(\alpha)$. So we then would have $\sigma(-\alpha) = -2\sigma(\alpha) = -\sigma(\alpha) \oplus \sigma(\alpha)$, but this contradicts the existence of $(\sigma(\alpha) \oplus \sigma(\alpha)) \oplus \sigma(-\alpha))$.

From $\sigma(\alpha) = \sigma(\alpha) \oplus (\sigma(\alpha) \oplus \sigma(-\alpha))$ we obtain by 1.2 (vi): $|\sigma(\alpha) \oplus \sigma(-\alpha)| < |\sigma(\alpha)|$. On the other hand, by 1.2 (v), $|\sigma(\alpha) \oplus \sigma(-\alpha)| = |\sigma(\alpha)| \vee |\sigma(-\alpha)|$.

Contradiction. (i) follows.

Now (ii) follows easily from (i): if $\alpha \oplus \beta$ were not defined then $\beta = -\alpha$ so, by (i), $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, a contradiction. Let $|\alpha| < |\beta|$, then $\alpha \oplus \beta = \beta$, so $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. By 1.2 (vi) we find $|\sigma(\alpha)| < |\sigma(\beta)|$. We proved (iii).

If $\sigma(\alpha) = \sigma(\beta)$ and $\alpha \neq \beta$ then $\sigma(\alpha \oplus (-\beta)) = \sigma(\alpha) \oplus \sigma(-\beta) = \sigma(\alpha) \oplus -\sigma(\alpha)$, an absurdity. So $\sigma$ is injective (iv). Finally, let $|\alpha| = |\beta|$ and $|\sigma(\alpha)| > |\sigma(\beta)|$. Then $\sigma(\alpha) = \sigma(\alpha) \oplus \sigma(\beta) = (by \ (ii)) = \sigma(\alpha \oplus \beta)$. By injectivity of $\sigma$, $\alpha = \alpha \oplus \beta$, and by 1.2 (vi), we find $|\beta| < |\alpha|$. Now we have
LEMMA 3.24. Let $K$ be spherically complete, let $Y \subset K$ (not necessarily convex) and let $\tau : \Sigma(Y) (= \{\pi(x-y) : x,y \in Y, x \neq y\}) \to \Sigma$ such that $f$ is monotone of type $\tau$ (i.e., $x,y \in Y, x-y \in \sigma \in \Sigma(Y)$ then $f(x)-f(y) \in \tau(\sigma)$.

Suppose $\tau$ can be extended to a $\sigma : \Sigma \to \Sigma$ satisfying the condition of Lemma 3.23. Then $f$ can be extended to a monotone function $\tilde{f} : K \to K$ of type $\sigma$.

Proof. By Zorn's lemma, it suffices to extend $f$ to $Y \cup \{a\}$ ($a \neq Y$) such that $f(x)-f(a) \in \sigma(\pi(x-a))$, $f(a)-f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x))$ ($x \in Y$). Each $B_x$ is a ball with radius $|\sigma(\pi(a-x))|$. By the spherical completeness of $K$, we are done if we can show that $B_x \cap B_y \neq \emptyset$ ($x \neq y, x,y \in Y$).

Set $\alpha := \pi(a-x)$ and $\beta := \pi(a-y)$. Let $b \in \sigma(\alpha); c \in \sigma(\beta)$. We prove:

$$|f(x)+b-f(y)-c| < |\sigma(\alpha)| \vee |\sigma(\beta)|.$$ We have two cases:

1) $\alpha = \beta$. Then $a-x \in \alpha$, $a-y \in \alpha$ implies $|x-y| < |a-x| = |\alpha|$, so

$$|\pi(x-y)| < |\alpha| \text{ whence } |\pi(f(x)-f(y))| = |\sigma(\pi(x-y))| < |\sigma(\alpha)| \text{ (by 3.23 (iii))},$$

so $|f(x)-f(y)| < |\sigma(\alpha)|$. Further, $b \in \sigma(\alpha), c \in \sigma(\alpha)$ implies $|b-c| < |\sigma(\alpha)|$, hence $|f(x)+b-f(y)-c| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x-y = a-y-(a-x) \in \beta \oplus (-\alpha)$, so $f(x)-f(y)+b-c \in \sigma(\beta \oplus -\alpha) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus -\alpha) + \sigma(\alpha \oplus -\beta) = \sigma(\beta \oplus (-\alpha)) - \sigma(\beta \oplus -\alpha)$, hence $|f(x)-f(y)+b-c| < |\sigma(\beta \oplus -\alpha)| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(\alpha)|, |\sigma(\beta)|)$.

THEOREM 3.25. Let $K$ be spherically complete and let $\sigma : \Sigma \to \Sigma$. Suppose

$$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).$$

Then there exists a function $f : K \to K$, monotone of type $\sigma$. 

Proof. Choose $Y := \{0\}$ and let $g : Y \to K$ be defined via $g(0) = 0$. Then $g$ satisfies the conditions of Lemma 3.24 so it can be extended to a function $f$ of type $\sigma$.

We now give a description of the maps $\sigma : \Sigma \to \Sigma$ mentioned in 3.23. For each $r \in |K^*|$ choose $a_r \in \Sigma$ such that $|a_r| = r$. Further, there is a natural isomorphism of multiplicative groups between $k^*$ and $\{a \in \Sigma : |a| = 1\}$, denoted by $l \mapsto a_l$ ($l \in k^*$). Of course, if $l + l' \neq 0$ then $a_{l+l'} = a_l \oplus a_{l'}$. Each element of $\Sigma$ can be written in only one way as $a_\lambda a_l$ ($r \in |K^*|, l \in k^*$). Now if $\sigma$ is as in 3.23 we get

$$\sigma(a_\lambda a_l) = a_\lambda a_l n(r,l)$$

where $\lambda : |K^*| \to |K^*|$ is strictly increasing and $l \mapsto n(r,l)$ is an injective group endomorphism of the additive group $k$. Conversely, if $\lambda : |K^*| \to |K^*|$ is strictly increasing and for each $r$, $l \mapsto n(r,l)$ is an injective group homomorphism $k \to k$ then

$$a_\lambda a_l \mapsto a_\lambda a_l n(r,l) (a_\lambda a_l \in \Sigma)$$

satisfies the condition of 3.23. So we get

THEOREM 3.26. Let $K$ be spherically complete and let $|K| = [0,\infty)$. Then there exist a nowhere continuous $f : K \to K$, monotone of some type $\sigma : \Sigma \to \Sigma$.

Proof. With the notations as above, let $\sigma : \Sigma \to \Sigma$ be defined as follows

$$\sigma(a_\lambda a_l) = a_{\lambda+1} a_l.$$ 

By 3.25 there is an $f : K \to K$ monotone of type $\sigma$. Clearly $|f(x)-f(y)| \geq 1$ if $x \neq y$ so $f$ is nowhere continuous.
4. MONOTONE FUNCTIONS, GENERAL THEOREMS

In this section we study \( M_w(X), M_b(X), M_s(X), \ldots \). To avoid unnecessary complications we assume throughout this section that \( X \) is a closed subset of \( K \) without isolated points. We collect here the results on monotone functions that are valid for general \( K \). In the next section we will see what happens if we put some extra conditions on \( K \) (e.g., \(|K|\) discrete, \( \ldots \)).

First two elementary lemmas.

**Lemma 4.1** Let \( f : X \to K \). Then the following conditions are equivalent

(a) \( f \in M_w(X) \) (see Def. 2.11).

(b) For all \( x, y, z \in X \), \(|x-y| < |x-z|\) implies \(|f(x)-f(z)| = |f(y)-f(z)|\).

(c) For all \( x, y, z \in X \), \(|f(x)-f(z)| \neq |f(y)-f(z)|\) implies \(|x-y| = \max(|x-z|, |y-z|)\).

**Proof.** (a) \( \Rightarrow \) (b). \(|x-y| < |x-z|\) implies \(|y-z| = |x-z| > |x-y|\), so

\[ |f(x)-f(y)| \leq \min(|f(x)-f(z)|, |f(y)-f(z)|). \]

It follows that \(|f(x)-f(z)| = |f(y)-f(z)|\).

(b) \( \Rightarrow \) (c). \((b)\) says that \(|f(x)-f(z)| \neq |f(y)-f(z)|\) implies \(|x-y| \geq |x-z|\). By symmetry, also \(|x-y| \geq |y-z|\) where \(|x-y| \geq \max(|x-z|, |y-z|)\). The opposite inequality is trivial.

(c) \( \Rightarrow \) (a). Let \(|x-y| < |x-z|\). Then \(|x-y| \neq \max(|x-z|, |z-y|)\) so, by \((c)\), \(|f(x)-f(z)| = |f(y)-f(z)|\). Then \(|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(y)-f(z)|) = |f(x)-f(z)|\).

**Lemma 4.2** (i) If \( f \in M_w(X), \lambda \in K \) then \( \lambda f \in M_w(X) \).
(ii) If $f_1, f_2, \ldots \in M_w(X)$ and $f := \lim_{n} f_n$ pointwise then $f \in M_w(X)$.

(iii) If $f \in M_w(X)$ and $g : f(X) \to X$ is such that $f \circ g$ is the identity on $f(X)$, then $g \in M_w(f(X))$. In particular, if $f$ is injective and weakly monotone then so is $f^{-1}$.

(Notice that $f(X)$ need not be closed and may have isolated points.)

Proof. Obvious.

Remark. Lemmas, similar to 4.1 and 4.3, but now for $M_b(X), M_s(X), M_{bs}(X)$ have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an $M_w$-function need not be continuous (see 2.4(5), 3.26) we will derive properties of $M_w$-functions that are closely related to continuity.

**Lemma 4.3** Let $f \in M_w(X)$. Then $f$ is bounded on precompact subsets of $X$.

Proof. Let $Y \subset X$ be precompact. Assume that $Y$ is not a singleton. Then $Y$ is bounded and has a positive diameter $r = \max\{|x-y| : x, y \in Y\}$.

The equivalence relation $x \sim y$ iff $|x-y| < r$ divides $Y$ into finitely many classes $Y_1, \ldots, Y_n$ ($n \geq 2$). Choose $a_i \in Y_i$ for each $i$, and let $M := \max |f(a_i)|$. We prove: $|f| \leq M$. In fact, let $x \in Y$. Then there is $i$ such that $|x-a_i| < r$. Choose $j \neq i$. We have $|x-a_i| < |a_i-a_j|$ whence $|f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M$. So $|f(x)| \leq M$.

The following lemma shows that an $f \in M_w(X)$ at $a \in X$ is either continuous or "very discontinuous".

**Lemma 4.4** Let $f \in M_w(X)$ and let $a \in X$. Then we have the following alternative.
Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots \in X$ $(x_n \neq a$ for all $n$) with $\lim x_n = a$ the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Proof. Since $\{x_1, x_2, \ldots\}$ is precompact the set $\{f(x_1), f(x_2), \ldots\}$ is bounded by Lemma 4.3. We are done if we can prove the following. If $x_1, x_2, \ldots, \lim x_n = a$, $x_n \neq a$ for all $n$, $\lim f(x_n)$ exists, then $f$ is continuous at $a$. Now set $a := \lim f(x_n)$. Let $y_1, y_2, \ldots \in X$, $\lim y_n = a$. We prove $\lim f(y_n) = a$. (Then it follows that $a = f(a)$ since we may choose $y_n := a$ for all $n$.) Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|f(x_k) - a| < \varepsilon$. For $n$ sufficiently large we have $|y_n - a| < |x_k - a|$, so for large $m$ (depending on $n$) we have $|y_n - m| < |x_k - m|$, whence $|f(y_n) - f(m)| \leq |f(x_k) - f(m)|$. Since $\lim_{m \to \infty} f(x_k) = a$ we find $|f(y_n) - a| \leq |f(x_k) - a| < \varepsilon$, so $\lim_{n \to \infty} f(y_n) = a$.

COROLLARY 4.5 Let $f \in M_w(X)$. Then the graph of $f$

$$\Gamma_f := \{(x, y) \in X \times K : y = f(x)\}$$

is closed in $K^2$.

Proof. Let $(x_n, f(x_n)) \in \Gamma_f$ and let $\lim x_n = x$, $\lim f(x_n) = a$. If $x_n = x$ for infinitely many $n$ then $a = f(x)$, so $(x, a) \in \Gamma_f$. If not then by the alternative of lemma 4.4, $f$ is continuous at $x$, so $a = f(x)$ and $(x, a) \in \Gamma_f$.

COROLLARY 4.6 Let $f \in M_w(X)$. If each bounded subset of $f(X)$ is precompact then $f$ is continuous.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let \( f \in M_\omega(X) \) and let \( Y \subseteq f(X) \) be precompact. Then either 
f is constant on \( f^{-1}(Y) \) or \( f^{-1}(Y) \) is bounded.

Proof. It suffices to prove: if \( Z \subseteq X \) is unbounded and \( f(Z) \) is precompact then \( f \) is constant on \( Z \). Let \( a, b \in Z \). Since \( Z \) is unbounded there are \( x_1, x_2, \ldots \in Z \) such that

\[
(*) \quad |a - b| < |x_1 - a| < |x_2 - a| < \ldots
\]

Since \( f(Z) \) is precompact we may assume (by taking a suitable subsequence) that \( a' = \lim f(x_n) \) exists. From (*) we obtain

\[
|x_1 - x_2| = |x_2 - a|, \quad |x_2 - x_3| = |x_3 - a|, \ldots,
\]

so

\[
|a - b| < |x_1 - a| < |x_1 - x_2| < |x_2 - x_3| < \ldots
\]

hence

\[
|f(a) - f(b)| \leq |f(x_1) - f(a)| \leq |f(x_1) - f(x_2)| \leq \ldots
\]

it follows that \( |f(a) - f(b)| = \lim_{n \to \infty} |f(x_n) - f(x_{n+1})| = 0 \). i.e., \( f(a) = f(b) \).

LEMMA 4.8 Let \( f \in M_\omega(X) \) and let \( a \in f(X) \) be a non-isolated point of \( f(X) \).

Then we have the following alternative. Either

I. There is \( a \in X \) such that for each sequence \( x_1, x_2, \ldots \) in \( X \n \)

for which \( \lim_{n \to \infty} f(x_n) = a \) we have \( \lim_{n \to \infty} x_n = a \), or

II. If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} f(x_n) = a \), \( f(x_n) \neq a \) for all \( n \),

then \( x_1, x_2, \ldots \) is bounded and has no convergent subsequence.

Proof. Suppose we are not in case II. Since \( a \) is not isolated in \( f(X) \) and \( f(X) \) is dense in \( f(X) \) we have a sequence \( x_1, x_2, \ldots \) in \( X \) for which \( f(x_n) \neq a \) for each \( n \), and \( \lim_{n \to \infty} f(x_n) = a \). Since \( f \) is not constant on \( \{x_1, x_2, \ldots\} \) it follows by Lemma 4.7 that \( \{x_1, x_2, \ldots\} \) is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by \( x_1, x_2, \ldots \) and set
Let \( a := \lim_{n \to \infty} x_n \). Then \( a \in X \). Now let \( y_1, y_2, \ldots \) be a sequence in \( X \) for which \( \lim f(y_n) = a \). We prove that \( \lim y_n = a \). In fact, let \( \epsilon > 0 \).

There is \( k \in \mathbb{N} \) such that \( |x_k - a| < \epsilon \). For large \( n \) we have
\[
|f(y_n) - a| < |f(x_k) - a|,
\]
so for large \( m \) (depending on \( n \)) we have
\[
|f(y_n) - f(x_m)| < |f(x_k) - f(x_m)|
\]
whence \( |y_n - x_m| \leq |x_k - x_m| \), so
\[
|y_n - a| \leq |x_k - a| < \epsilon.
\]

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function \( f : X \to K \) **injective at** \( a \in X \) if \( f(x) = f(a) \) for some \( x \in X \) implies \( x = a \).

Now suppose that we have \( a \in f(X) \), not isolated, for which we are in alternative I. Then for a sequence \( x_1, x_2, \ldots \) with \( \lim f(x_n) = a \) we have \( \lim x_n = a \in X \) so \( (a, a) = \lim (x_n, f(x_n)) \), so by Cor. 4.5 we have \( a = f(a) \). Thus, \( a \in f(X) \). \( f \) is injective at \( a \); if \( f(b) = f(a) \) then since \( \lim f(b) = a \) we must have \( \lim b = a \) i.e. \( b = a \). Further, \( f \) is continuous at \( a \) (see 2.13 (2)(a)).

If each bounded subset of \( X \) is precompact we never can be in case II.

This is also true if \( f \in \mathcal{M}_d(X) \) and \( |X| \) is discrete i.e. if \( x_1, y_1 \in X \)
\[
|x_1 - y_1| > |x_2 - y_2| > \ldots
\]
then \( \lim |x_n - y_n| = 0 \). Proof: let \( a \in f(X) \) and let \( \lim f(x_n) = a \), \( f(x_n) \neq a \) for all \( n \). Without loss of generality we may assume
\[
|a - f(x_1)| > |a - f(x_2)| > \ldots
\]
hence
\[
|f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots
\]
and, since \( f \in \mathcal{M}_d(X) \)
\[
|x_1 - x_2| > |x_2 - x_3| > \ldots
\]
Since \( |X| \) is discrete, the sequence \( x_1, x_2, \ldots \) is convergent. So we have case I. We find
THEOREM 4.9 Let either each closed and bounded subset of $X$ be compact and $f \in \mathcal{M}_w(X)$, or let $|X|$ be discrete and $f \in \mathcal{M}_b(X)$. Then

(i) $f(X)$ is closed (in $K$).

(ii) If $f(a) \notin f(X)$ is not isolated then $f$ is injective at $a$, $f$ is continuous at $a$. In particular if $f(X)$ has no isolated points then $f$ is a homeomorphism $X \sim f(X)$.

Proof. If $a \in f(X) \setminus f(X)$ then $a$ is not isolated. Since we are in alternative I of Lemma 4.8, $a \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an $f$ satisfying the conditions of 4.9 maps closed subsets of $X$ (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let $f : X \rightarrow K$.

(i) If $f \in \mathcal{M}_w(X)$ and if $Y \subseteq X$ is closed and the closed and bounded subsets of $Y$ are compact then $f(Y)$ is spherically complete.

(ii) If $f \in \mathcal{M}_b(X)$ and if $Y \subseteq X$ is spherically complete then so is $f(Y)$.

(iii) If $f \in \mathcal{M}_g(X)$ and if $A \subseteq f(X)$ is spherically complete then so is $f^{-1}(A)$.

Proof. (i) Let $B_1 \supseteq B_2 \supseteq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1 \setminus B_2$, $f(y_2) \in B_2 \setminus B_3$, ... Then we have

$$|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots$$

and, by the weak monotony of $f$
\[ |y_1-y_2| \geq |y_2-y_3| \geq \ldots \]

Suppose first that \( \lim |y_n-y_{n+1}| = 0 \). Then \( y := \lim y_n \) and there are infinitely many \( k \) for which

\[ |y_k-y_{k-1}| > |y_{k+1}-y_k|. \]

Now \( |y-y_k| \leq \max(|y_k-y_{k+1}|,|y_{k+1}-y_{k+2}|,\ldots) \leq |y_k-y_{k+1}| \). So we get for infinitely many \( k \)

\[ |y-y_k| < |y_k-y_{k-1}| \]

whence

\[ |f(y)-f(y_k)| \leq |f(y_k)-f(y_{k-1})| \]

so \( f(y) \in B_{k-1} \) for infinitely many \( k \), i.e., \( f(y) \in \cap B_k \).

Next, suppose that \( |y_{k+1}-y_k| \geq \varepsilon > 0 \) for all \( k \). Then since \( y_1, y_2, \ldots \)
is bounded it has a convergent subsequence \( y_{n_1}, y_{n_2}, \ldots \). Let \( y := \lim y_{n_i} \).

Then we have for infinitely many \( i \)

\[ |y-y_{n_i}| < \varepsilon \leq |y_{n_i}-y_{n_i+1}| \]

whence

\[ |f(y)-f(y_{n_i})| \leq |f(y_{n_i})-f(y_{n_i+1})| \]

so \( f(y) \in B_{n_i} \) for infinitely many \( i \), i.e., \( f(y) \in \cap B_k \).

(ii) Let \( B_1 \not\supseteq B_2 \not\supseteq \ldots \) be balls in \( f(Y) \) and let \( y_1, y_2, \ldots \in Y \) such that \( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1)-f(y_2)| > |f(y_2)-f(y_3)| > \ldots \]

and since \( f \in M_b(X) \):

\[ |y_1-y_2| > |y_2-y_3| > \ldots \]

Since \( Y \) is spherically complete, there is \( y \in Y \) such that

\[ |y-y_n| \leq |y_n-y_{n+1}| \text{ for all } n \text{, hence } |f(y)-f(y_n)| \leq |f(y_n)-f(y_{n+1})| \]

for all \( n \). It follows that \( f(y) \in B_n \) for all \( n \).

(iii) Let \( B_1 \not\supset B_2 \not\supset \ldots \) be balls in \( f^{-1}(A) \). Choose \( x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_2, \ldots \).

Then \( |x_1-x_2| > |x_2-x_3| > \ldots \), whence \( |f(x_1)-f(x_2)| > |f(x_2)-f(x_3)| > \ldots \)

There is \( x \in f^{-1}(A) \) such that \( |f(x)-f(x_n)| \leq |f(x_n)-f(x_{n+1})| \) for all \( n \).

Hence \( |x-x_n| \leq |x_n-x_{n+1}| \) for all \( n \) i.e., \( x \in \cap B_n \).
DEFINITION 4.11 Let \( f : X \to K \). The oscillation function \( \omega_f : X \to [0, \infty] \) is defined by

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{ |f(x) - f(y)| : |x-a| \leq \frac{1}{n}, |y-a| \leq \frac{1}{n}, x, y \in X \} (a \in X)
\]

\[
= \lim_{n \to \infty} \sup \{ |f(x) - f(a)| : |x-a| \leq \frac{1}{n}, x \in X \}.
\]

THEOREM 4.12 Let \( f \in M_w(X) \). Then

\[
\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X).
\]

Proof. For \( x \neq a \) we have \(|f(x) - f(a)| \geq \inf_{z \neq a} |f(z) - f(a)| \) and (since \( a \) is not isolated) consequently

\[
\omega_f(a) \geq \inf_{z \neq a} |f(z) - f(a)|.
\]

Conversely, let \( z \neq a \). Then for all \( x \) such that \(|x-a| < |z-a|\) we have

\[
|f(x) - f(a)| \leq |f(z) - f(a)|
\]

so

\[
\omega_f(a) \leq |f(z) - f(a)|
\]

whence

\[
\omega_f(a) \leq \inf_{z \neq a} |f(z) - f(a)|.
\]

THEOREM 4.13 Let \( f \in M_w(X) \), \( a \in X \). If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} x_n = a \) (\( x_n \neq a \) for all \( n \)) then \( \lim_{n \to \infty} |f(x_n) - f(a)| = \omega_f(a) \).

Proof. By 4.12 we have \( \lim_{n \to \infty} |f(x_n) - f(a)| \geq \omega_f(a) \). Conversely, \( \lim_{n \to \infty} |f(x_n) - f(a)| \leq \omega_f(a) \) is clear from the definition of \( \omega_f \).
5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: $K$ is local, $k$ is finite, $K$ has discrete valuation. Also we can sometimes say a little more if we assume $X$ to be convex. For the time being, let $X$ be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case $K$ is a local field.

THEOREM 5.1 Let $K$ be a local field, and let $f \in M_w(X)$. Then

(i) $f$ is continuous.

(ii) If $Y \subset X$ is closed then $f(Y)$ is closed.

(iii) If $f(X)$ is bounded and $f$ is not constant then $X$ is bounded.

(iv) Let $a \in X$. Then the following are equivalent

(a) $f$ is not injective at $a$

(b) $f$ is locally constant at $a$

(c) $f(a)$ is isolated in $f(X)$.

(v) The following conditions are equivalent

(a) $f$ is injective

(b) $f(X)$ has no isolated points

(c) $f$ is a homeomorphism of $X$ onto $f(X)$.

Proof. Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for $M_b$- and $M_s$-functions in case $X$ is convex and $K$ is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2 Let the residue class field \( k \) of \( K \) be finite. Let \( X \) be convex and let \( f \in M_\mathcal{B}(X) \). Then

(i) If \( a, b, c \in X, |a-b| < |a-c|, f(a) \neq f(c) \) then
\[ |f(a)-f(b)| < |f(a)-f(c)|. \]

(ii) If \( C \subset X \) is convex then \( f(C) \) is convex in \( f(X) \) (\( f \)

(iii) If \( f \) is injective, then \( f \in M_\mathcal{B}(X) \).

Proof. (i) Let \( B := \{ x \in K : |x-a| \leq |a-c| \} \). Then \( B \subset X \) and \( f(B) \subset [f(a),f(c)] \). Define an equivalence relation on \( B \) by: \( x \sim y \)

if \( |f(x)-f(y)| < |f(a)-f(c)| \).

Since \( k \) is finite we get finitely many equivalence classes \( B_1, B_2, \ldots, B_n \). Since \( a \neq c \) we have \( n \geq 2 \). The diameter of \( f(B) \) equals \( |f(a)-f(c)| \), the distance between \( f(B_i) \) and \( f(B_j) \) equals \( |f(a)-f(c)| \) (\( i \neq j \)). Since \( [f(a),f(c)] \) can contain at most \( q := \chi(k) \) sets having distances \( |f(a)-f(c)| \) to one another we have \( n \leq q \). Hence \( 2 \leq n \leq q \). By 2.2 (\( \beta \)), each \( B_i \) is convex. If \( x, y \in B_i \) and \( |x-y| \) were \( |a-c| \) then \( B_i = B \), contradicting \( n \geq 2 \). Thus \( B \) is a disjoint union of \( n \) balls \( B_1, \ldots, B_n \), where \( 2 \leq n \leq q \) and \( |x-y| < |a-c| \) whenever \( x, y \in B_i \) (\( i = 1, \ldots, n \)). It follows that \( n = q \) and that each \( B_i \) has the form \( \{ x \in K : |x-b_i| < |a-c| \} \) \( (b_i \in B) \). Hence, if \( |a-b| < |a-c| \) then there is \( i \) for which \( a, b \in B_i \).

So \( |f(a)-f(b)| < |f(a)-f(c)| \).

(ii) Let \( a, b \in C \) and let \( a \in f(X) \) with \( a \in [f(a),f(b)] \). We show that \( a \in f(C) \). If \( f(a) = f(b) \) this is clear. If \( f(a) \neq f(b) \), set \( a = f(x) \) where \( x \in X \). Then \( |f(x)-f(a)| \leq |f(b)-f(a)| \). If \( |x-a| \) were > \( |b-a| \) then \( f(x) \neq f(a) \) (since \( f \in M_\mathcal{B}(X) \)) and by (i) we then had \( |f(b)-f(a)| < |f(x)-f(a)| \), a contradiction. Hence \( |x-a| \leq |b-a| \) i.e., \( x \in [a,b] \subset C \), so \( a = f(x) \in f(C) \).
(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on k, see 2.10.

COROLLARY 5.3 Let K be a local field and let \( f \in M_b(X) \) and \( X \) convex. Then the following conditions are equivalent.

(a) \( f \in M_s(X) \).

(b) \( f \) is injective.

(c) \( f \in M_{bs}(X) \).

(d) \( f(X) \) has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 Let K be a local field and let \( X \) be the unit ball of \( K \) (or any bounded convex set, for that matter). If either \( f \in M_s(X) \) or \( f \in M_b(X) \) then \( f \) has bounded difference quotients.

Proof. \( f \) is bounded, let \( M := \sup \{ |f(x) - f(y)| : x, y \in X \} \). It suffices to prove that \( |f(x) - f(0)| \leq M|x| \) for all \( x \). Let \( \pi \in K, |\pi| < 1 \), be a generator of the value group. By induction on \( n \) we prove:

if \( |x| = |\pi|^n \) then \( |f(x) - f(0)| \leq |\pi|^n M \).

The statement is clear for \( n = 0 \). Now suppose the statement is true for \( 0, 1, \ldots, n-1 \).

Let \( x \in X, |a| = |\pi|^n \). Then \( |x - 0| < |\pi^{n-1}| - 0 \). If \( f(\pi^{n-1}) \neq f(0) \) we have either since \( f \in M_s(X) \) or by 5.2(i)

\[
|f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1} M
\]

hence

\[
|f(x) - f(0)| \leq |\pi|^n M
\]

If \( f(\pi^{n-1}) = f(0) \) then \( |f(x) - f(0)| \leq |f(\pi^{n-1}) - f(0)| = 0 \), so certainly

\[
|f(x) - f(0)| \leq |\pi|^n M.
\]
Notes.

(a) 5.4 cannot be extended to the case \( X = \mathbb{K} \). In fact, let
\[
f : \mathbb{Q}_p \to \mathbb{Q}_p \text{ be the map } \mathbb{Q}_n p^\mathbb{n} \to \mathbb{Q}_n p^{2\mathbb{n}}. \quad (\mathbb{Q}_n p^\mathbb{n} \in \mathbb{Q}_p.)
\]
Then
\[
f \in M_{sb}(\mathbb{Q}_p) \text{ but } |p^n f(p^{-n})| = p^n \to \infty.
\]

(b) If we loose the condition on \( K \), for example by requiring that
the valuation is discrete then 3.22 and 2.4(5) show that the
conclusion of 5.4 is false both for \( M_b \)-functions and \( M_s \)-functions.
On the other hand, it is clear from the proof of 5.4 that a
bounded \( M_s \)-function on \( X \) has bounded difference quotients.

(c) One may wonder how difference quotients of \( M_w \)-functions behave.
See the example below.

EXAMPLE 5.5 Let \( p \neq 2 \). Then there is an \( f \in M_w(\mathbb{Z}_p \to \mathbb{Q}_p) \) that has un-
bounded difference quotients.

**Proof.** Let \( a_0, a_1, \ldots \) be defined via \( a_{2n} = p^n \) \((n = 0,1,2,\ldots)\) and
\( a_{2n+1} = 2p^n \) \((n = 0,1,2,\ldots)\). Thus \( (a_0, a_1, a_2, \ldots) = (1,2, p, p^2, 2p^2, \ldots) \).
Then \( |a_0| \geq |a_1| \geq |a_2| \geq \ldots, \lim a_n = 0, \ |a_n - a_m| = |a_m| \) \((n > m)\).

Set
\[
f(x) := \begin{cases} a_n & \text{if } |x| = p^{-n} \\
0 & \text{if } x = 0 \end{cases} \quad (x \in \mathbb{Z}_p)
\]

Then the difference quotients of \( f \) are not bounded (for \( n \in \mathbb{N} : \)
\( f(p^{2n}) = p^n \), so \( |p^{-2n} f(p^{2n})| = p^n \to \infty \text{ if } n \to \infty \)). We show that
\( f \in M_w(\mathbb{Z}_p). \) Since \( f \) is continuous it suffices to show that if \( x,y,z \)
are \( \neq 0 \), \( |x-y| < |x-z| \) then \( |f(x)-f(y)| \leq |f(x)-f(z)| \). This is clear
if \( |x| = |y| \). If \( |x| < |y| \), then \( |x| < |y| < |z| \). If \( |x| > |y| \),
then \( |y| < |x| < |z| \). Let \( f(x) = a_n, f(y) = a_m, f(z) = a_t. \) Then in
both cases \( n \neq m, t < \min(n,m) ; \) \( |f(x)-f(y)| = |a_n - a_m| \leq |a_t| \text{ and}
\( |f(x)-f(z)| = |a_n - a_t| = |a_t| \) and we are done.
On the other hand (how surprising is life!)

**THEOREM 5.6** Let \( k \) be the field of two elements. Then \( M^w(X) = M^b(X) \).

**Proof.** We prove that \(|x-y| = |y-z| \) implies \(|f(x)-f(y)| \leq |f(y)-f(z)|\) (\( x \neq y, y \neq z, x, y, z \in X \)). There is \( a \in K^* \) such that \(|a(x-y)| = |a(y-z)| = 1\). So since \( k = F_2 \), \( a(x-y) = a(y-z) = 1 \), whence \( a(x-z) = 0 \) or \(|a(x-z)| < 1\). Thus, \(|x-z| < |x-y| = |y-z|\). Since \( f \in M_w(X) \), \(|f(x)-f(z)| \leq \min(|f(x)-f(y)|, |f(y)-f(z)|)\) Consequently, \(|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) \leq |f(y)-f(z)|\).

Of particular interest may be monotone functions mapping convex sets onto convex sets.

**THEOREM 5.7** Let \( K \) be a local field, let \( X \) be a bounded open convex set, and let \( f : X \to X \) be surjective. Then the following are equivalent.

1. \( f \in M_b(X) \)
2. \( f \in M_s(X) \)
3. \( f \in M_{bs}(X) \)
4. \( f \) is an isometry.

**Proof.** (1) \( \Rightarrow \) (2). Since \( f(X) \) has no isolated points, \( f \) is a homeomorphism, by 5.1(v). Then \( f \in M_s(X) \), by 5.3. (2) \( \Rightarrow \) (3). \( f^{-1} \in M_b(X) \). We just have shown (1) \( \Rightarrow \) (2), so \( f^{-1} \in M_s(X) \) i.e., \( f \in M_b(X) \).

(3) \( \Rightarrow \) (4). From the proof of 5.4 we have seen that \(|f(x)-f(y)| \leq M|x-y|\), where \( M = \sup|f(x)-f(y)| = 1 \). Hence \(|f(x)-f(y)| \leq |x-y| \) for all \( x, y \in X \), but by the same token this also holds for \( f^{-1} \). Then \( f \) is an isometry. (4) \( \Rightarrow \) (1) is obvious.
COROLLARY 5.8 Let $X$ be an open convex subset of $K$ and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.

(a) $f \in M_D(X)$

(b) $f \in M_s(X)$

(c) $f \in M_{bs}(X)$

(δ) $f$ is a scalar multiple of an isometry.

Proof. If $X$ is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If $X$ is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (a) (β) (γ) is now easy. To prove (γ) $\Rightarrow$ (δ) we may assume $f(0) = 0$, $f(1) = 1$. Let $X_n := \{ x \in K : |x| \leq n \}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is $c_n$ such that $|f(x) - f(y)| = c_n|x-y|$ $(x,y \in X_n)$. By substituting $x = 1$, $y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in M_{ubs}(\mathbb{Z}_p)$ into conditions on the coefficients of $f$ with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(\mathbb{Z}_p)$. So let the notations be as in 3.3(3), and suppose first $f \in M_{ubs}(\mathbb{Z}_p)$ i.e. $|x-y| = |s-t| \iff |f(x) - f(y)| = |f(s) - f(t)|$. Let $n,m \in \mathbb{N}$. If $|n-n_-| = |m-m_-|$ then $|f(n) - f(n_-)| = |f(m) - f(m_-)|$, so if we write $f = \sum_{n} \lambda_n e_n$, we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ $(a_k \neq 0)$ then $|n-n_-| = p^{-k}$ where $k = \lfloor \log n \rfloor - \lfloor \log p \rfloor$. We find

$$\text{if } \left\lfloor \frac{\log n}{\log p} \right\rfloor < \left\lfloor \frac{\log m}{\log p} \right\rfloor \text{ then } |\lambda_n| > |\lambda_m|$$

$$\text{if } \left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor \text{ then } |\lambda_n| = |\lambda_m|.$$
Moreover, if \[ \left\lfloor \frac{\log n}{\log p} \right\rfloor = k \text{ and } n = m \text{ is invisible by } p^k \text{ i.e., } n_\gamma = m_\gamma \text{ then } |f(n) - f(m)| = |\lambda_n - \lambda_m|. \]

If \( n > m \) then \( |f(n-m) - f(0)| = |\lambda_{n-m}| = |\lambda_n|. \)

We have found the first half of

\[ \text{THEOREM 5.9} \]

Let \( f = \sum \lambda_n e_n \in C(\mathbb{Z}_p). \) In order that \( f \in M_{ubs}(\mathbb{Z}_p) \) it is necessary and sufficient that condition (*) below holds

\[ (*) \quad |\lambda_n| \text{ is a strictly decreasing function of } \left\lfloor \frac{\log n}{\log p} \right\rfloor (n \in \mathbb{N}) \]

\[ \left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor, \quad n \neq m, \quad n_\gamma = m_\gamma \implies |\lambda_n - \lambda_m| = |\lambda_n| = |\lambda_m| \quad (n, m \in \mathbb{N}). \]

We have shown \( f \in M_{ubs}(\mathbb{Z}_p) \rightarrow (*) \). Now suppose (*) and let \( |x - y| = p^{-k}. \) We show that \( |f(x) - f(y)| = |\lambda_p^k|. \) Now \( e_n(x) = e_n(y) \) for \( n < p^k \), so

\[ f(x) - f(y) = \sum_{n \geq p^k} \lambda_n (e_n(x) - e_n(y)). \]

Set

\[ x := a_0 + a_1 p + \cdots + a_k p^k + a_{k+1} p^{k+1} + \cdots \]

\[ y := a_0 + a_1 p + \cdots + b_k p^k + b_{k+1} p^{k+1} + \cdots \]

Then

\[ \sum_{n \geq p^k} \lambda_n e_n(x) = \sum_{n \geq p^k} \lambda_n e_n(y) = |\lambda_{p^k}^k + \lambda_{p^k}^k + \lambda_{p^k}^{k+1} + \cdots |. \]

Now either \( a_k \) or \( b_k \) is \( \neq 0 \), say, \( a_k \neq 0 \). If \( b_k = 0 \) then by (*)

\[ |\sum_{n \geq p^k} \lambda_n e_n(y)| < |\lambda_{p^k}^k| = |\sum_{n \geq p^k} \lambda_n e_n(x)|, \quad \text{so } |f(x) - f(y)| = |\lambda_{p^k}^k| \cdot |\lambda_{p^k}^k| = |\lambda_{p^k}^k|. \]

If \( b_k \neq 0 \) then by (*)

\[ |\lambda_{p^k}^k| = |\lambda_{p^k}^k - \lambda_{p^k}^k| = |f(x) - f(y)|. \]

\[ |\lambda_{p^k}^k|. \]

Note. Using similar methods, we can prove: \( f = \sum \lambda_n e_n \) is in \( M_{ubs}(\mathbb{Z}_p) \)

if and only if we have (**) for all \( n, m \in \mathbb{N} \):
\[
\begin{bmatrix}
\frac{\log n}{\log p}
\end{bmatrix}
\quad>
\begin{bmatrix}
\frac{\log m}{\log p}
\end{bmatrix}
= k
\Rightarrow
|\lambda_n| < |\lambda_m|^{n-m \text{ divisible by } p^k}
\]

\[\log \begin{bmatrix}
\log n \\
\log m
\end{bmatrix}
= \log \begin{bmatrix}
\log p \\
\log p
\end{bmatrix}
\quad|\lambda_n| = |\lambda_m| = |\lambda_n - \lambda_m|.
\]

If we assume only that \(K\) has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let \(X\) be the unit ball of a discretely valued field. Let \(f : X \rightarrow X\) be surjective, \(f \in M_{bs}(X)\). Then \(f\) is an isometry.

**Proof.** It is clear from previous theory that \(f\) is a homeomorphism of the unit ball. It suffices to show that \(|f(x) - f(y)| \leq |x - y|\) for all \(x, y \in X\). (Apply this result also for \(f^{-1}\). Then \(f\) is an isometry.)

Let \(\pi \in K\), \(|\pi| < 1\), be a generator of \(|K^*|\). We prove by induction

if \(|x| = |\pi|^n\) then \(|f(x) - f(0)| \leq |\pi|^n|f(1) - f(0)|\).

For \(n = 0\) this is clear. (\(|x - 0| \leq |1 - 0|\), so \(|f(x) - f(0)| \leq |f(1) - f(0)|\)).

Suppose the statement is true for \(n = 1\). Let \(|x| = |\pi|^n\). Then

\(|x - 0| < |\pi|^{n-1} - 0|, \text{ so } |f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1}|f(1) - f(0)|,\)

so \(|f(x) - f(0)| \leq |\pi|^n|f(1) - f(0)|\) and we are done. (In fact, we have shown that a bounded \(M_s\)-function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let \(K\) have discrete valuation and let \(f \in M_b(X)\). Then the following conditions are equivalent.

(a) \(f(X)\) has no isolated points.

(b) \(f\) is injective and continuous.

(c) \(f\) is a homeomorphism \(X \sim f(X)\).
Proof. (a) \( (\gamma) \) is 4.9(ii). (\( \gamma \) \( (\delta) \) is clear. (\( \delta \) \( (\gamma) \): if \( f(a) \) were an isolated point of \( f(X) \), then \( \{ x : f(x) = f(a) \} \) is open in \( X \).

Since \( f \) is injective \( \{ a \} \) is open. But \( X \) has no isolated points. Contradiction.

To show that 5.11 may not be true if \( K \) has a dense valuation we construct

EXAMPLE 5.12 Let \( |K| = [0,\infty) \). Then we construct an \( M_\delta \)-homeomorphism sending

\[ \{ x \in K : \frac{1}{2} < |x| \leq 1 \} \text{ onto } \{ x \in K : 0 < |x| \leq 1 \} \]

Proof. Let \( \phi : [\frac{1}{2},1] \rightarrow [0,1] \) be the map \( x \mapsto 2(x-\frac{1}{2}) \ (x \in (\frac{1}{2},1]) \). For each \( v \in (\frac{1}{2},1] \), choose \( \beta_v \in K \) such that \( |\beta_v| = \frac{\phi(v)}{v} \). Define

\[ f : \{ x \in K : \frac{1}{2} < |x| \leq 1 \} \rightarrow \{ x \in K : 0 < |x| \leq 1 \} \]

as follows

\[ f(x) = \beta_{|x|^2} \ x \ (\frac{1}{2} < |x| \leq 1) \]

Clearly, \( |f(x)| = |\beta_{|x|^2}| \ |x| = \phi(|x|) \in (0,1] \). The inverse of \( f \) is given by \( y \mapsto \beta_{\phi^{-1}(|y|)} \), so \( f \) is a bijection. Since \( f^{-1} \) can be defined in the same way as \( f \) (only with \( \phi^{-1} \) instead of \( \phi \)) it suffices to show that \( f \in M_\delta \). Let \( |x-y| < |x-z| \).

Suppose \( |x| > |z| \). Then \( |x-z| = |x| \) and \( |y| = \max(|x-y|,|x|) = |x| \).

Then \( \beta_{|x|} = \beta_{|y|} \), so \( |f(x)-f(y)| = \beta_{|x|} \ |x-y| \) and \( |f(x)-f(z)| = |f(x)| = \beta_{|x|} \ |x-z| \), so we are done in this case. Suppose \( |x| < |z| \).

Then \( |x-z| = |z| \) and \( |y| = \max(|x-y|,|x|) < |z| \). Then \( |f(x)-f(y)| \leq \max(|f(x)|,|f(y)|) < |f(z)| = |f(z)-f(x)| \).

Suppose \( |x| = |z| \). Then \( |y| \leq \max(|x-y|,|x|) \leq |x| \); if \( |y| \) were \( < |x| \) then \( |x-y| = |x| = |z| < |x-z| \), a contradiction, so \( |y| = |x| = |z| \), and \( |f(x)-f(y)| = \beta_{|x|} \ |x-y| \), \( |f(x)-f(z)| = \beta_{|x|} \ |x-z| \) whence

\( |f(x)-f(y)| < |f(x)-f(z)| \).
EXAMPLE 5.13 Extend $f$ to a surjection $g$ of $\{ x \in K : |x| \leq 1 \}$ onto itself by defining $g(x) = 0$ if $|x| \leq \frac{1}{2}$. We claim that $g \in M_b$. Let $|x-y| \leq |x-z|$. To check whether $|g(x)-g(y)| \leq |g(x)-g(z)|$ we only have to consider the cases $|x| \leq \frac{1}{2}$ and $|y| > \frac{1}{2}$ and $|y| \leq \frac{1}{2}$. In the first case, $|x-y| = |y| \leq |x-z|$, so $|z| = \max(|z-x|, |x|) = |z-x| \geq |y|$. Then $|g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)|$. In the second case $|g(x)-g(y)| = |f(x)|$. If $|x| < |z|$ then $|f(x)| < |f(z)| = |f(x)-f(x)| = |g(z)-g(x)|$. If $|x| > |z|$ then $|f(x)| = |g(x)-g(z)|$.

Thus we have found a continuous surjection $g : \{ x \in K : |x| \leq 1 \} \rightarrow \{ x \in K : |x| \leq 1 \}$, $g \in M_b$, such that $g = 0$ on $\{ x : |x| \leq \frac{1}{2} \}$. (Compare 5.11).

EXAMPLE 5.14 Let $h : \{ x \in K : |x| \leq 1 \} \rightarrow K$ be defined as

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \quad (f \text{ as in 5.12}) \\ 0 & \text{if } x = 0. \end{cases}$$

Then $h$ is a non-continuous $M_{bs}$-function.

Proof. That $h$ is not continuous at 0 is clear. Further, $h$, restricted $\{ x : 0 < |x| \leq 1 \}$ is in $M_{bs}$ (see 5.12). Further, since $g \circ h$ is the identity ($g$ as in 5.12), we see that $h \in M_s$. It suffices to show that $|x-y| = |x-z|$ implies $|h(x)-h(y)| = |h(x)-h(z)|$ in case $0 \in \{ x,y,z \}$.

We may suppose $x \neq y$, $y \neq z$, $x \neq z$. Let $x = 0$. Then $|y| = |z|$, so $|f^{-1}(y)| = |f^{-1}(z)|$ i.e., $|h(x)-h(y)| = |h(x)-h(z)|$. Now let $y = 0$.

Then $|x| = |x-z|$. Choose $0 < |t| \leq 1$ such that $|t| < |x|$. Then $|x-t| = |x-z|$ so $|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x) - f^{-1}(z)|$ i.e., $|h(x)| = |h(x)-h(z)|$, and we are done.
6. FUNCTIONS OF BOUNDED VARIATION

In this section X is the unit ball of K, and \( B^\Delta(X) := \{ f : X \to K : \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right| < \infty \} \). Let us define

\[
\| f \|_\Delta := \sup \left\{ \left| \frac{f(x) - f(y)}{x - y} \right| : x, y \in X, x \neq y \right\} (f \in B^\Delta(X)).
\]

It will turn out that, in a natural way, \( B^\Delta(X) \) can be regarded as the space of functions of bounded variation, and that \( \| \cdot \|_\Delta \) plays the role of the total variation.

**THEOREM 6.1** Let \( f : X \to K \). Then the following are equivalent

(a) \( f \in B^\Delta(X) \).

(b) \( f \) is a linear combination of two increasing functions.

If \( |X| \) is discrete (a), (b) are equivalent to

(c) \( f \) is the difference of two bounded monotone functions of some type \( \sigma \).

(d) \( f \in [M^S_{ds}(X)] \).

If \( K \) is a local field then (a)-(d) are equivalent to

(e) \( f \in [M^S_d(X)] \).

(f) \( f \in [M^S_s(X)] \).

**Proof.** We only prove (a) \( \Rightarrow \) (b). The rest follows from (5.10), (5.4).

So let \( f \in B^\Delta(X) \) and choose \( \lambda \in X \) such that \( |f(x) - f(y)| < |\lambda| |x - y| \) \((x, y \in X, x \neq y)\). Then \( \lambda^{-1} f \) is a pseudocontraction, \( f(x) = \lambda x + \lambda (\lambda^{-1} f(x) - x) \) \((x \in X)\), where \( x \to x \) and \( x \to \lambda^{-1} f(x) - x \) are increasing.

In the real case, we can define for a function \([0,1] \to \mathbb{R}\), of bounded variation

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\[ V(f) := \inf \{ \operatorname{Var} g + \operatorname{Var} h : f = g + h, \, g, h \text{ monotone} \}. \]

It is an easy exercise to show that \( f \mapsto V(f) \) is a seminorm on the space of all functions of bounded variation and that \( V \) is equivalent to the total variation \( \operatorname{Var} \), defined via

\[
\operatorname{Var} f = \sup \{ \sum |f(x_k) - f(x_{k-1})| : x_0 < x_1 < \ldots < x_n \text{ is a partition of [0,1]} \}.
\]

So in the non-archimedean setting we define for \( f : X \to K \)

\[ J(f) = \sup \{|f(x) - f(y)| : x, y \in X\}. \]

(If \( f \) is considered to be "monotone" then \( J(f) \) can be interpreted as the "total variation" of \( f \).) We are led to the following definitions for \( f \in BA(X) \):

\[ \operatorname{Var} f := \inf \{ \max(J(g), J(h)) : f = g + h, \, g, h \text{ are scalar multiples of increasing functions} \}. \]

(If \(|K|\) is discrete) \( \operatorname{Var}^f := \inf \{ \max J(g), J(h) : f = g + h, \, g, h \text{ are in } M^s(X) \} \).

(If \( K \) is local) \( \operatorname{Var}^f := \inf \{ \max J(g), J(h) : f = g + h, \, g, h \in M^p(X) \} \).

Let us first compare \( \operatorname{Var} f \) and \( \|f\|_\Lambda \). If \( f = g + h \) and \( g, h \) are scalar multiples of increasing functions we have for \( x, y \in X, \, x \neq y \)

\[
\left| \frac{f(x) - f(y)}{x - y} \right| \leq \max \left( \left| \frac{g(x) - g(y)}{x - y} \right|, \left| \frac{h(x) - h(y)}{x - y} \right| \right) \leq \max(J(g), J(h))
\]

so \( \|f\|_\Lambda \leq \operatorname{Var} f \). Conversely, if \( |\lambda| > \sup \left| \frac{f(x) - f(y)}{x - y} \right| \) then

\[ f(x) = \lambda x + \lambda (\lambda^{-1} f(x) - x) \quad (x \in X) \]

whence

\[ \operatorname{Var} f \leq |\lambda| \]
So, if \(|K|\) is dense we have \(\text{Var} f = \|f\|_\Delta (f \cdot B\Delta(X))\). Otherwise we have at least

\[\|f\|_\Delta \leq \text{Var} f \leq c\|f\|_\Delta \quad (f \in B\Delta(X))\]

(where \(c\) is the smallest value \(> 1\)).

If \(|K|\) is discrete we clearly have \(\text{Var}_1 f \leq \text{Var} f\). Conversely, let \(f = g + h\), where \(g, h \in M_{bs}(X)\). It follows from the proof of 5.10 that

\[|g(x) - g(y)| \leq M|x-y| \quad (x, y \in X)\]
\[|h(x) - h(y)| \leq N|x-y|\]

where \(M = \sup |g(x) - g(y)| = J(g)\) and \(N = J(h)\).

So

\[\frac{|f(x) - f(y)|}{x-y} \leq \max(J(g), J(h)), \text{ whence}\]

\[\|f\|_\Delta \leq \text{Var}_1 f\]

Similar proofs work for \(\text{Var}_2 f, \text{Var}_3 f\). We have

**Theorem 6.2** The seminorms \(\text{Var}, \text{Var}_1, \text{Var}_2, \text{Var}_3\) on \(B\Delta(X)\) (whenever defined) are all equivalent to \(\|\|_\Delta\).
REFERENCES

