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NON-ARCHIMEDEAN MONOTONE FUNCTIONS

by

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INTRODUCTION

In the sequel, $K$ is a non-archimedean valued field, complete, with residue class field $k$. Our aim is to present reasonable definitions for a function $f : X \rightarrow K$ to be "monotone". ($X$ is a subset of $K$). Since $K$ admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \rightarrow \mathbb{R}$, equivalent to "$f$ is monotone", and such that these statements have translations in $K$ that make sense. This way we obtain several definitions of "$f : X \rightarrow K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of $K"", "the sign of a nonzero element of $K".

In Section 2 we define several notions of monotony. E.g., $f \in M^d(X)$ if $x$ between $y$ and $z$ implies $f(x)$ between $f(y)$ and $f(z)$ and $f \in M^s_s(X)$ if $f(x)$ between $f(y)$ and $f(z)$ implies $x$ between $y$ and $z$. Also monotone functions of type $\sigma$ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M^d(X)$ (or $f \in M^s_s(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions. In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
mation on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of K and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: f' > 0 \iff f increasing.

The notion of pseudo-ordening ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

**Notations.** Let p be a prime. By \( \mathbb{F}_p \) we mean the field of p elements. By \( \mathbb{Q}_p \) the non-archimedean valued field of the p-adic numbers. For a field L we denote its characteristic by \( \chi(L) \). Let E be a vector space over K and \( S \subset E \). By \([S]\) we denote the smallest K-linear subspace of E that contains S.
1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let \( x, y \in K \). Then the smallest ball in \( K \) containing \( x \) and \( y \) is denoted by \( [x, y] \). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \( [x, y] \subset C \).

Sometimes we use a more geometric terminology. Instead of \( z \in [x, y] \) we will say that \( z \) is between \( x \) and \( y \) and instead of \( z \notin [x, y] \) we use the expression: \( x \) and \( y \) are at the same side of \( z \).

Notice that \( [x, y] = [y, x] \) for all \( x, y \in K \) and that \( z \in [x, y] \iff |z-x| \leq |z-y| \iff |x-y| \leq |z-y| \iff z = \lambda x + (1-\lambda)y \) for some \( \lambda \in K \), \( |\lambda| \leq 1 \). If \( x \neq y \) then the \( \lambda \) in this last expression is unique (viz. \( \lambda = \frac{z-y}{x-y} \)).

Examples of convex sets are: the empty set, singletons, balls, \( K \).

It is an easy exercise to show that these are the only convex subsets of \( K \). So formally we may write each convex subset of \( K \) as

\[
\{ x \in K : |x-a| < r \} \quad (a \in K, \ 0 \leq r \leq \infty)
\]

or as

\[
\{ x \in K : |x-a| \leq r \} \quad (a \in K, \ 0 \leq r \leq \infty)
\]

Notice that the only unbounded convex subset of \( K \) is \( K \) itself.

Sometimes we need the notion of convexity with respect to a subset \( X \) of \( K \). A subset \( C \subset X \) is called convex in \( X \) if \( x, y \in C \) implies \( [x, y] \cap X \subset C \) or, equivalently, if \( C \) is the intersection of \( X \) with a convex subset of \( K \).

Let \( x, y, z \in K \). By the strong triangle inequality we have that the "triangle" \( x, y, z \) is isosceles, say \( |x-y| = |y-z| \). Then \( |x-z| \leq |x-y| \), so \( z \) is between \( x \) and \( y \) and \( x \) is between \( y \) and \( z \). If also \( |x-y| = |x-z| \)
then \( y \) is between \( x \) and \( z. \) Otherwise, \( x \) and \( z \) are at the same side of \( y. \)

The relation \( \sim \) defined on \( K^* := K \setminus \{0\} \) by

\[
x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x,y \in K^*)
\]
is an equivalence relation. We have \( x \sim y \iff 0 \notin [x,y] \) i.e. \( \iff |x-y| < |x| \ (= |y|) \) i.e. \( \iff |xy^{-1}-1| < 1. \) Define

\[
K^+ := \{x \in K : |1-x| < 1\}
\]
Then \( K^+ \) is a multiplicative subgroup of \( K^*, \) \( K^+ = \{x \in K^* : x \sim 1\} \) and is called the set of the positive elements of \( K. \) The relation \( \sim \) is also induced by the canonical group homomorphism

\[
\pi : K^* \to K^*/K^+.
\]
Thus, \( x \sim y \) if and only if \( \pi(x) = \pi(y) \ (x,y \in K^*). \) Therefore it is natural to view the group \( \Sigma := K^*/K^+ \) as being the group of signs of elements of \( K^*, \) and we call \( \pi(x) \) the sign of the element \( x \in K^*. \) If \( x \in K^* \) then \( \pi(x) = \{y : |y-x| < |x|\} = xK^+. \) For \( x \in K^*, \ a \in \Sigma \) we sometimes write \( xa \) to indicate the element \( \pi(x).a \) of \( \Sigma. \) In particular, for \( a \in \Sigma \) the opposite sign of \( a, -a, \) is defined as \((-1)a. \) Then \( -a = \{-x : x \in a}\). (Notice that in case \( \chi(K) = 2 \) we have \( a = -a. \))

Let \( a \in \Sigma. \) Then for \( x,y \in a \) we have \( |x| = |y| \) so we can define the absolute value of \( a, |a| \) as follows

\[
|a| := |x| \quad (x \in \pi^{-1}(a)).
\]

In the sequel we also need addition between elements of \( \Sigma. \) Let us first observe that for any \( a, \beta \in \Sigma \) the sum

\[
a + \beta := \{x+y : x \in a, \ y \in \beta\}
\]
is always a ball with radius \( \max(|a|, |\beta|). \) (I.e., of the form
\{x : |x-b| < \max(|a|,|b|)\}. Now \(a+b\) contains 0 if and only if \(a = -b\). Otherwise \(a+b\) is again an element of \(\Sigma\). (Proof: Let \(a \in \alpha\), \(b \in \beta\). Then \(|a+b| = \max(|a|,|b|)\). If also \(x \in \alpha\), \(y \in \beta\) then \(|x+y-(a+b)| \leq \max(|x-a|,|y-b|) < \max(|a|,|b|) = |a+b|\). Thus \(a+b\) contains the ball with center \(a+b\) and radius \(\max(|a|,|b|)\), so \(a+b\) is equal to this ball.)

Let us define

\[\alpha \oplus \beta := a+b = \{x+y : x \in \alpha, y \in \beta\} \quad (a, \beta \in \Sigma, a \neq -\beta).\]

We have

**THEOREM 1.2** Let \(\Sigma, |\cdot| : \Sigma \to \mathbb{R}, \oplus : \Sigma \times \Sigma \\setminus\{(a,-a) : a \in \Sigma\} \to \Sigma\) be as above. Let \(\alpha, \beta, \gamma \in \Sigma\). Then

(i) \(|a\beta| = |a| \cdot |\beta|, |a^{-1}| = |a|^{-1}|.\)

(ii) If \(\alpha \oplus \beta\) is defined then so is \(\beta \oplus \alpha\) and \(\alpha \oplus \beta = \beta \oplus \alpha\).

(iii) If \((\alpha \oplus \beta) \ominus \gamma\) and \(\alpha \ominus (\beta \ominus \gamma)\) are defined then

\((\alpha \oplus \beta) \ominus \gamma = \alpha \ominus (\beta \ominus \gamma)\).

(iv) If \(\alpha \oplus \beta\) or \(\gamma \ominus \gamma\beta\) is defined then so is the other and \(\gamma(\alpha \ominus \beta) = \gamma \ominus \gamma\beta\).

(v) If \(\alpha \ominus \beta\) is defined then \(|\alpha \ominus \beta| = \max(|a|,|\beta|)\). Conversely if \(|s| = \max(|a|,|\beta|)\) for some \(s \in \alpha+\beta\) then \(\alpha \ominus \beta\) is defined.

(vi) \(|\alpha| < |\beta|\) if and only if \(\alpha \ominus \beta = \beta\).

(vii) Let \(n \in \{1,2,\ldots,\chi(k)-1\}\) if \(\chi(k) \neq 0\), let \(n \in \mathbb{N}\) otherwise. Then we define \(\oplus_n\) inductively as follows.

\(\oplus_1 \alpha = \alpha, \oplus_k \alpha := \oplus_{k-1} \alpha \ominus a (k \leq n)\). Then

\(\oplus_n \alpha = n \alpha\).

**Proof.** (i), (ii) are clear. (iii) is almost trivial: if \(x \in \alpha, y \in \beta, z \in \gamma\) then \(x+y+z \in \alpha+\beta+\gamma\) and the latter set can be regarded as
(a © y) © y or as a © (b © y). (It is worth noticing that (a © b) © y may be defined whereas a © (b © y) is not. Choose β = -y and |a| > |β|. Then (a © b) © y = a © y = a, β © y is not defined.)

(iv) is clear. If a © b is defined then for x:α, y:β we have |x+y| ≥ max(|x|, |y|) whence |x+y| = max(|x|, |y|). So |a © b| = max(|a|, |β|). Conversely, if a © b is not defined, then (we saw earlier that) a+β is a ball with center zero and radius max(|a|^-1, |β|^-1).

Thus we have (v). We prove (vi). If |a| < |β| then a+β = β so a © b = β. Conversely, if a © b = β then choose a ∈ α, b ∈ β. Then a+b ∈ β hence a+b ~ b i.e., ab^{-1}+1 ∈ K^+ implying |ab^{-1}| < 1 or |a| < |b|. Hence |a| < |β|. (Note: from (vi) it follows that a © b = a' © b does not imply a = a'). To prove (vii) let a:α and observe that for any k ≤ n, if a is defined, (k-1)a ∈ α. Hence |(k-1)a+a| = |ka| = |a| = |a|, k-1 k-1

so a+a does not contain 0, hence a © a is defined.

Now na is by definition π(n)α. So na ∈ na and na © a. Since both na and a are signs they must coincide.

We now define relations that resemble "ordering".

DEFINITION 1.3 Let α ∈ Σ and x,y ∈ K. Then we say that x is greater than y in the sense of α, notation x >_α y, if x-y ∈ α.

We have the following rules

THEOREM 1.4 (i) If x,y ∈ K, x ≠ y then there is exactly one α ∈ Σ for which x >_α y.

(ii) x >_α x for no α.

(iii) If x >_α y then for all s ∈ K: x+s >_α y+s (x,y ∈ K, α ∈ Σ)

(iv) If x >_α y and s >_β 0 then xs >_α β ys (x,y,s ∈ K, α,β ∈ Σ)
(In particular \( x >_\alpha y \) implies \(-x >_{-\alpha} -y\)).

If \( x >_\alpha y, y >_\beta z \) and if \( \alpha \oplus \beta \) is defined then \( x >_{\alpha \oplus \beta} z \).

Proof. Easy.

The group \( \Sigma_1 := \{ \alpha \in \Sigma : |\alpha| = 1 \} \) is a subgroup of \( \Sigma \), isomorphic to multiplicative group \( k^* \). If \( K \) has discrete valuation and if \( s \in k \) and \(|s|\) is the largest value that is smaller than 1, then for each \( \alpha \in \Sigma \) there is \( x \in \mathbb{Z} \) such that \( \alpha = s^n \alpha_1 \) where \( \alpha_1 \in \Sigma_1 \). It follows easily that the map \((n, \alpha) \mapsto s^n \alpha \) \((n \in \mathbb{Z}, \alpha \in \Sigma_1 \) is an isomorphism of \( \mathbb{Z} \times \Sigma_1 \) onto \( \Sigma \). Thus, in case \( K \) has discrete valuation, \( \Sigma \) is isomorphic to \( \mathbb{Z} \times \Sigma_1 \), or, for that matter, to \( |k^*| \times k^* \).

If \( K \) is a local field we can even define a group embedding \( \rho : \Sigma \to K^* \) such that \( \pi \rho \) is the identity. (Thus, we can pick an element in every \( \alpha \) \((\alpha \in \Sigma) \) such that the resulting set is a subgroup of \( K^* \)). Let \( s \in k, |s| < 1 \) such that \(|s|\) generates the value group and let \( q \) be the cardinality of \( k \). Let \( x \in K^* \). Then there is a unique \( n \in \mathbb{Z} \) such that \( x = s^n x_1 \) where \(|x_1| = 1\).

Define

\[
\nu(x) = s^n \lim_{n \to \infty} x_1^n
\]

It is easy to verify that \( \nu \) is a homomorphism of \( K^* \) into \( K^* \), that \( \pi(\nu(x)) = \pi(x) \) for all \( x \in K^* \) and that \( \nu(x) = 1 \) if and only if \( x \in K^+ \).

Therefore the map \( \rho \) making the diagram

\[
\begin{array}{ccc}
K^* & \xrightarrow{\nu} & K^* \\
\pi \downarrow & & \downarrow \pi \\
\Sigma & \xrightarrow{\rho} & \Sigma
\end{array}
\]

commute solves the problem.

**EXAMPLE 1.5** The signs of \( \Theta \). Let \( \Theta \) be a primitive \((p-1)\th \) root of
unity. Then \( \{ \mathbb{Q}_p^n : i \in \{0, 1, \ldots, p-2\}, n \in \mathbb{Z} \} \) is a subgroup of \( \mathbb{Q}_p^* \) isomorphic to \( \mathbb{Z} \). If
\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0, 1, \ldots, p^{p-2}\}, a_n \neq 0)
\]
is an element of \( \mathbb{Q}_p \), its sign, interpreted as an element of \( \mathbb{Q}_p \) is
\[
\pi(x) = a_n p^n.
\]
For a function $f : [0,1] \to \mathbb{R}$ the following statements are equivalent.

(a) $f$ is monotone (i.e., either $x > y$ implies $f(x) \geq f(y)$ for all $x, y$ or $x > y$ implies $f(x) \leq f(y)$ for all $x, y$).

(b) If $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$ ($x, y, z \in [0, 1]$).

($\gamma$) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex.

Thus we define

**DEFINITION 2.1** Let $X \subseteq K$, $f : X \to K$. We say that $f \in M_b(X)$ if for all $x, y, z \in X$, $x$ between $y$ and $z$ implies $f(x)$ is between $f(y)$ and $f(z)$. In other words, $f \in M_b(X)$ if and only if for all $x, y, z$

$$|x-y| \leq |y-z| \Rightarrow |f(x)-f(y)| \leq |f(y)-f(z)|.$$ 

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

**THEOREM 2.2** Let $X \subset K$ and let $f : X \to K$. Then the following statements are equivalent.

(a) $f \in M_b(X)$.

(b) For each convex $C \subset K$, $f^{-1}(C)$ is convex in $X$.

($\gamma$) For all $x, y, z \in X$: $|x-y| = |x-z| \Rightarrow |f(x)-f(y)| = |f(x)-f(z)|$.

($\delta$) For all $x, y, z \in X$: $|f(x)-f(y)| > |f(x)-f(z)| \Rightarrow |x-y| > |x-z|$.

($\epsilon$) For all $x, y, z \in X$: $|f(x)-f(y)| \neq |f(x)-f(z)| \Rightarrow |x-y| \neq |x-z|$. 
Proof. (a) ⇒ (β). Let \( x, y \in f^{-1}(C) \) and let \( z \in [x, y] \cap X \). Then \(|z-x| \leq |x-y|\), so \(|f(z) - f(x)| \leq |f(x) - f(y)|\), i.e., \( f(z) \in [f(x), f(y)] \subset C \). Hence \( z \in f^{-1}(C) \).

(β) ⇒ (α). Let \( x, y, z \in X \) and \(|x-y| \leq |x-z|\). The set \([f(x), f(z)]\) is convex, hence \( f^{-1}([f(x), f(z)]) \) is convex in \( X \) and contains \( x \) and \( z \), so it must contain \( y \). Thus \( f(y) \in [f(x), f(z)] \).

Clearly, (α) ⇔ (δ) and (γ) ⇔ (ε). We prove (α) ⇒ (γ). Now (α) ⇒ (γ) is clear by symmetry. Suppose (γ) and let \(|x-y| \leq |x-z|\). It suffices to consider the case \(|x-y| < |x-z|\). Then \(|y-z| = |x-z|\), so by (γ) we have \(|f(y) - f(z)| = |f(x) - f(z)|\). Then \(|f(x) - f(y)| \leq \max(|f(x) - f(z)|, |f(z) - f(y)|) = |f(x) - f(z)|\).

THEOREM 2.3 Let \( X \subset X \). Then

(i) For each \( a, b \in K \) the map \( x \mapsto ax+b \) is in \( M_b(X) \).
(ii) If \( f \in M_b(X) \), \( \lambda \in K \) then \( \lambda f \in M_b(X) \).
(iii) \( M_b(X) \) is closed under pointwise limits.
(iv) If \( f \in M_b(X) \) and \( g : f(X) \to K \) is in \( M_b(f(X)) \), then \( g \circ f \in M_b(X) \).
(v) If \( f \in M_b(X) \) and \( f(a) = f(b) \) for some \( a, b \in X \), then \( f \) is constant on \([a, b] \cap X \).

Proof. Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of \( M_b \)-functions. For more, see the sequel.

(1) The constant functions.
(2) Isometries (e.g., \( \exp \)).

(3) Choose in every \( \alpha \in \Lambda \) an element \( x_\alpha \). Define \( \phi : K \to K \) as follows

\[
\phi(x) = \begin{cases} 
0 & \text{if } x = 0 \\
x_\alpha & \text{if } x \neq 0 \quad (\alpha \in \Lambda)
\end{cases}
\]
(Essentially, $\phi|K^*$ is the sign function $\pi$ of section 1).

We prove that $\phi \in M_b(K)$. Since $\phi$ is continuous it suffices to check that $\phi|K^*$ is in $M_b(K^*)$. Now for all $x, y \in K^*$ we have $\phi(x) - \phi(y) = 0$ if $|xy^{-1} - 1| < 1$ and $|\phi(x) - \phi(y)| = |x - y|$ if $|x - y| = \max(|x|, |y|)$. Now take $x, y, z \in K^*$ such that $|x - y| \leq |x - z|$. If $\phi(x) = \phi(z)$ then $|1 - x^{-1}y| \leq |1 - x^{-1}z| < 1$ so $\phi(x) = \phi(y)$.

If $\phi(x) \neq \phi(z)$ then $|\phi(x) - \phi(y)| \leq |x - y| \leq |x - z| = |\phi(x) - \phi(z)|$.

(4) Let $r > 0$ and choose in every ball $B$ of radius $r$ a center $x_B$.

The function defined via

$$\psi(x) = x_B \quad (x \in B)$$

is in $M_b(K)$. The proof is easy.

(5) (A nowhere continuous $M_b$-function). Let $K$ be a field such that $\#K = \#k$ (e.g., a discretely valued field where $\#k$ has the power of the continuum). Let $\sigma : K \to k$ be a bijection and let $\tau : k \to K$ such that $|\tau x - \tau y| = 1$ whenever $x \neq y$. Then $f : \tau \circ \sigma$ satisfies: $|f(x) - f(y)| = 1$ ($x, y \in K$, $x \neq y$).

Clearly $f$ is everywhere discontinuous, $f \in M_b(K)$.

(6) Let $X \subseteq K$. We can strengthen the definition of an $M_b$-function into

if $|x - y| \leq |z - t|$ then $|f(x) - f(y)| \leq |f(z) - f(t)|$ \quad ($x, y, z, t \in X$)

(some "uniform" $M_b$-condition) and we obtain a space, called $M_{ub}(X)$.

Clearly, the examples mentioned in (1), (2), (4), (5) are in $M_{ub}(K)$, whereas the example in (3) is not. (Choose $x, y \in K$ with $|x| > 1$, $|x - y| = 1$. Then $|1 - 0| \leq |x - y|$, but $1 = |\phi(1) - \phi(0)| > |\phi(x) - \phi(y)| = 0$.)

Notice that $\phi$ is locally constant on $K^*$, but not on $K$.

(7) The discontinuous function $f$ of (5) has the property that $f(K)$ consists only of isolated points. This is not accidental. If $f \in M_b(K)$
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing
\( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points,
so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that
the image of an everywhere discontinuous \( M_b \)-function consists only of
isolated points.

(8) For each \( n \), let \( \sigma_n : K \to K \) be the example of 2.4, (4) above where
\( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b (X) \) we have
\( \sigma_n \circ f \in M_b (X) \) and \( \lim \sigma_n \circ f = f \) uniformly. Hence, if \( f \) is continuous
then it can uniformly be approximated by locally constant \( M_b \)-functions.

A monotone function \( f : [0,1] \to \mathbb{R} \) maps convex sets into sets
that are relatively convex in \( f([0,1]) \). In our situation we do not
have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If
\( f : [0,1] \to \mathbb{R} \) maps convex sets into (relatively) convex sets then
\( f \) need not be monotone: any Darboux continuous function has the above
property. In fact a function \( f : [0,1] \to \mathbb{R} \) is Darboux continuous
if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We
define

**DEFINITION 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \to K \). Then \( f \) is
called weakly Darboux continuous if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex in \( f(X) \).

\( f \) is called Darboux continuous if for every relatively convex set \( C \subset X \) the set \( f(C) \) is convex (in \( K \)).

We have the following obvious remarks.

1) \( f : X \to K \) is Darboux continuous if and only if \( f \) is weakly Darboux
continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) such that for every open ball \( B \subset \mathbb{Z}_p \), 
\[ f(B) = \mathbb{Z}_p. \]
Let \( A \subset \mathbb{Z}_p \) be defined as follows, 
\[ x = \prod_{n=0}^{p-1} (a_n \in \{0, 1, \ldots, p-1\}) \]
is in \( A \) if \( a_{2n} = a_{2n+2} = \ldots = 0 \) for some \( n \). Define \( f : \mathbb{Z}_p \to \mathbb{Z}_p \) via
\[
 f(x) = \begin{cases} 
 a_{2N+1} + a_{2N+3}p + a_{2N+5}p^2 + \ldots & \text{if } x \in A \text{ and } N = \min(n : a_{2n} = a_{2n+2} = \ldots = 0) \\
 0 & \text{if } x \notin A
\end{cases}
\]
Then \( f \) maps every non empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.
(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on \( \mathbb{Z}_p \) is not Darboux continuous. Even, a continuous function need not be weakly Darboux continuous. To see this, observe that all open compact subsets of \( \mathbb{Z}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p \) sending 
\( \{x : |x| < 1\} \) into \( \{x : |x| = 1\} \) and \( \{x : |x| = 1\} \) into \( \{x : |x| < 1\} \).
If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( M_p \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4,(7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \to \mathbb{R} \) the following conditions are equivalent.
(a) \( f \) is strictly monotone (i.e., injective and monotone).
(b) \( f \) is injective. For a convex set \( C \subset [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all \( x, y, z \in [0,1] \): if \( f(x) \) is between \( f(y) \) and \( f(z) \) then \( x \) is between \( y \) and \( z \).

(δ) For all \( x, y, z \in [0,1] \): \( f(x) \) is between \( f(z) \) if and only if \( x \) is between \( y \) and \( z \).

Translating (α) – (δ) into the non-archimedean situation we arrive at the following conditions. Let \( X \subset K \) and \( f : X \to K \)

(α') \( f \in M_d(X) \) and \( f \) is injective.

(β') \( f \) is weakly Darboux continuous and injective.

(γ') for all \( x, y, z \in X \), \( |x - y| < |x - z| \) implies \( |f(x) - f(y)| < |f(x) - f(z)| \).

(δ') \( f \in M_d(X) \) and \( f \) satisfies (γ').

It will turn out that the conditions (α') – (γ') although not equivalent are closely related. We start with (γ'):

**DEFINITION 2.6** Let \( X \subset K \), \( f : X \to K \). We say that \( f \in M_s(X) \) if for all \( x, y, z \in X \), \( f(x) \in [f(y), f(z)] \) implies \( x \in [y, z] \).

**THEOREM 2.8** Let \( X \subset K \), \( f : X \to K \). Then the following statements are equivalent:

- (a) \( f \in M_s(X) \).
- (b) \( f \) is injective and weakly Darboux continuous.
- (γ) \( f \) is injective and \( f^{-1} \in M_d(f(X)) \).
- (δ) For all \( x, y, z \in X \), \( |f(x) - f(y)| = |f(x) - f(z)| \) \( \Rightarrow \) \( |x - y| = |x - z| \).
- (ε) For all \( x, y, z \in X \), \( |x - y| < |x - z| \) \( \Rightarrow \) \( |f(x) - f(y)| < |f(x) - f(z)| \).
- (ζ) For all \( x, y, z \in X \), \( |x - y| \neq |x - z| \) \( \Rightarrow \) \( |f(x) - f(y)| \neq |f(x) - f(z)| \).
Proof. The implications $\alpha \rightarrow \beta \rightarrow \gamma \rightarrow \zeta \rightarrow \delta$ are clear from the definitions.

$\delta \rightarrow \gamma$: injectivity follows from $|f(x)-f(y)| = |f(x)-f(y)| + |x-x| = |x-y|$. Use 2.2.($\gamma$).

$\gamma \rightarrow \beta$: Let $g : f(X) \rightarrow X$ be the inverse of $f$. Let $C \subseteq X$ be convex in $X$. Then since $g \in M_b$, $g^{-1}(C)$ is convex in $f(X)$. But $g^{-1}(C) = f(C)$.

Finally, we prove $\beta \rightarrow \alpha$. Let $f(x) \in [f(y), f(z)]$. By ($\beta$) the set $f([y,z] \cap X)$ is convex in $f(X)$ and it contains $f(y), f(z)$, hence $f(x) \in [f(y), f(z)] \cap X \subseteq f([y,z] \cap X)$. Since $f$ is injective, $x \in [y,z] \cap X$ and we are done.

We also have (compare 2.3)

**Theorem 2.9** Let $X \subseteq K$. Then

1. For $a,b \in K$, $a \neq 0$ the map $x \mapsto ax+b$ is in $M_s(X)$.
2. If $f \in M_s(X)$, $\lambda \in K$, $\lambda \neq 0$ then $\lambda f \in M_s(X)$.
3. If $f_1, f_2, \ldots \in M_s(X)$, $\lim f_n = f$ pointwise, $f$ injective then $f \in M_s(X)$.
4. If $f \in M_s(X)$, $g \in M_s(f(X))$ then $g \circ f \in M_s(X)$.

Proof. Obvious verifications.

Returning to our conditions $\alpha' \rightarrow \beta'$ we see that $\beta'$ is equivalent to $\gamma'$, that $\alpha'$ means $f^{-1} \in M_s(f(X))$ and that $\delta'$ means $f \in M_b(X) \cap M_s(X)$.

Our $f$ of example 2.4 (5) is in $M_b$, injective but not in $M_s$. Its inverse yields an example of an $M_s$-function that is not in $M_b$. Thus, in general, we have neither one of the implications $\alpha' \rightarrow \gamma'$, $\gamma' \rightarrow \alpha'$, $\beta' \rightarrow \delta'$, $\alpha' \rightarrow \delta'$. But our counterexample is
rather weird (f is nowhere continuous and the domain of f⁻¹ is discrete). We can do better.

EXAMPLE 2.10 Let K have discrete valuation and let k be infinite.

Then there exists a homeomorphism of the unit ball of K that is in M_d but not in M_s. (The inverse map is in M_s but not in M_d).

Proof. Set X = {a ∈ K : |a| ≤ 1} and let R be a full set of representatives of the equivalence relation x ∼ y iff |x−y| < 1 in X. Then R is infinite. Let π ∈ K be such that |π| is the largest value that is smaller than 1. The map

\[(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{∞} a_n \pi^n \quad (a_i \in R \text{ for each } i)\]

is a bijection of \(R^\mathbb{N}\) onto X. We may suppose that 0 ∈ R.

Since R is infinite we can define injections

\[\tau_1 : R \setminus \{0\} \to R\]
\[\tau_2 : R \to R\]

such that \(\text{im } \tau_1 \cap \text{im } \tau_2 = \emptyset\), \(\text{im } \tau_1 \cup \text{im } \tau_2 = R\).

For \(x = \sum_{n=0}^{∞} a_n \pi^n \in X (a_n \in R \text{ for each } n)\) set

\[f(x) := \begin{cases} \tau_1(a_0) + a_1 \pi + \ldots = x-a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\ \tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi}-a_1 + \tau_2(a_1) & \text{if } a_0 = 0 \end{cases}\]

A simple inspection of the definition shows that f is a bijection of X onto X. If a,b ∈ R, a ≠ b then |aπ−bπ| < |bπ−a|, whereas

|f(aπ)−f(bπ)| = |τ_2(a)−τ_2(b)| = 1 and |f(bπ)−f(a)| = |τ_2(b)−τ_1(a)| = 1,

so f ∉ M_s(X). Finally, let x,y,z ∈ X and |x−y| ≤ |x−z|. We prove that |f(x)−f(y)| ≤ |f(x)−f(z)|. If |f(x)−f(z)| = 1 there is nothing to prove, so suppose |f(x)−f(z)| < 1. Set x = Σa_n \pi^n, y = Σb_n \pi^n, z = Σc_n \pi^n.
If $a_0 = 0$ then also $c_0 = 0$ and $\tau_2(a_1) = \tau_2(c_1)$ so $a_1 = c_1$, hence $|x-z| \leq |z|^2$. Since $|x-y| \leq |x-z|$ we have also $b_0 = 0$, $b_1 = a_1$.

So, $f(x)-f(y) = \frac{x-y}{|x|}$, $f(x)-f(z) = \frac{y-z}{|y|}$ whence $|f(x)-f(y)| \leq |f(x)-f(z)|$.

If $a_0 \neq 0$ then $\tau_1(a_0) = \tau_1(c_0)$ so $a_0 = c_0$. Then also $c_0 = a_0 = b_0$.

Then $f(x)-f(y) = x-y$, $f(x)-f(z) = x-z$ whence $|f(x)-f(y)| \leq |f(x)-f(z)|$.

Let $X \subseteq K$. If $f \in M_S(X)$ then $f^{-1} \in M_S(f(X))$. Conversely, if $f \in M_B(X)$ and $g : f(X) \rightarrow X$ is such that $f \circ g$ is the identity on $f(X)$ then $g \in M_B(f(X))$. This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let $X \subseteq K$ and $f : X \rightarrow K$. $f$ is called **weakly monotone** $(f \in M_W(X))$ if for all $x, y, z \in X$

$$|x-y| < |x-z| + |f(x)-f(y)| \leq |f(x)-f(z)|$$

$f$ is called **strongly monotone** $(f \in M_S(X))$ if $f \in M_S(X) \cap M_B(X)$.

Clearly, $f \in M_{BS}(X)$ if and only if $f^{-1} \in M_{BS}(f(X))$. Also, if $f \in M_W(X)$ and $g : f(X) \rightarrow X$ is such that $f \circ g$ is the identity on $f(X)$ we have $g \in M_W(f(X))$.

Obviously we have $M_B(X) \cup M_S(X) \subseteq M_W(X)$ and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of $M_W$-functions, not for the sake of $M_W$ itself but in order to get results that are valid for $M_B$, $M_S$ simultaneously. The functions in $M_{BS}$ behave reasonable and they may be viewed as the non-archimedean equivalents of strict monotone functions in the real case.
THEOREM 2.12 Let $X \subset K$ and $f : X \to K$. Then the following conditions are equivalent.

(a) $f \in M_{bs}(X)$.

(b) $f$ is injective and $C \mapsto f(C)$ is a 1-1 correspondence between the relatively convex subsets of $X$ and those of $f(X)$.

(c) For all $x, y, z \in X$:
   - $|x-y| < |x-z| \iff |f(x)-f(y)| < |f(x)-f(z)|$.
   - $|x-y| = |x-z| \iff |f(x)-f(y)| = |f(x)-f(z)|$.

(d) For all $x, y, z \in X$:
   - $|x-y| \leq |x-z| \iff |f(x)-f(y)| \leq |f(x)-f(z)|$.

(e) $f \in M_s(X)$, $f^{-1} \in M_s(f(X))$.

Proof. Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An $M_w$-function that is not in $M_s \cup M_b$). Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be any function, constant on the cosets of $\{x \in \mathbb{Z}_p : |x| < 1\}$. Then $f \in M_w(\mathbb{Z}_p)$. Clearly $f \not\in M_s(\mathbb{Z}_p)$, $f \in M_b(\mathbb{Z}_p)$ if and only if the points of $f(\mathbb{Z}_p)$ are equidistant.

(2) (Continuity of monotone functions). Let $X \subset K$.

(a) Let $f \in M_w(X)$. If $f(X)$ has no isolated points, then $f$ is continuous.

Proof. Let $a \in X$ and $\varepsilon > 0$. Then there is $z \in X$ such that $z \neq a$, $|f(z)-f(a)| < \varepsilon$. Let $\delta := |z-a|$. Then for all $x \in X$ with $|x-a| < \delta$ we have, by the weak monotony of $f$, $|f(x)-f(a)| \leq |f(z)-f(a)| < \varepsilon)$. It follows that if $X$ and $Y$ do not have isolated points and if $f$ is an $M_w$-bijection of $X$ onto $Y$, then $f$ is a homeomorphism of $X$ onto $Y$. 
Conversely, it is easy to construct homeomorphisms of \( \mathbb{A}_p \) that are not in \( M_w(\mathbb{A}_p) \).

(b) If \( K \) is a local field then every \( f \in M_w(X) \) is continuous. (See 5.1 (i)).

(c) If \( K \) has discrete valuation then every \( f \in M_s(X) \) is continuous.

(Example 2.4 (5) shows that such a statement is not true for \( f \in M_b(X) \).)

(Proof. If \( f \) were not continuous at some \( a \in X \) then there would be an \( \varepsilon > 0 \) such that for some sequence converging to \( a \) we had \( |f(x_n)-f(a)| \geq \varepsilon \). We may suppose that \( |x_1-a| > |x_2-a| > \ldots \). Since the valuation is discrete we have \( \lim_{n \to \infty} |f(x_n)-f(a)| = 0 \), a contradiction.)

(d) In 5.14 we shall give an example of a function in \( M_{bs}(X) \) that is not continuous. (Of course, \( K \) will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" \( M_w \),..., -conditions.

Thus, by definition, \( f \in M_{uw}(X) \) if for all \( x,y,z,t \in X \)

\[ |x-y| < |z-t| \Rightarrow |f(x)-f(y)| \leq |f(z)-f(t)| \]

\( f \in M_{us}(X) \) if for all \( x,y,z,t \in X \)

\[ |x-y| < |z-t| \Rightarrow |f(x)-f(y)| < |f(z)-f(t)| \]

\( f \in M_{ubs}(X) \) if for all \( x,y,z,t \in X \)

\[ |x-y| < |z-t| \iff |f(x)-f(y)| < |f(z)-f(t)|. \]

Notice that \( f \in M_{ubs}(X) \) means that \( |f(x)-f(y)| \) is a strictly increasing function of \( |x-y| \). Examples of such functions are isometries, but also the function \( f : \mathbb{A}_p \to \mathbb{A}_p \) defined via

\[ \Sigma a_n p^n \mapsto \Sigma a_n 2^{n} \quad (\Sigma a_n p^n \in \mathbb{A}_p) \]

\( |f(x)-f(y)| = |x-y|^2 \) for all \( x,y \in \mathbb{A}_p \).

Monotone functions : \( \mathbb{R} \to \mathbb{R} \) are divided into two classes: the
increasing functions and the decreasing functions. For the non-archimedean case we may ask for a similar classification. First we try to express the situation in the real case in such a way that it can be translated. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be strictly monotone. If $x$ runs through some side of $a$ then $f(x)$ runs through some fixed side of $f(a)$. So there is a map $\sigma : [-1,1] \to [-1,1]$ such that $\sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a))$ ($x \neq a$). Apparently, the only $\sigma$'s that can occur are the identity and $\sigma(x) = -x$ ($x \in \{1,-1\}$). Moreover it turns out that the map $\sigma$ is independent of the choice of $a$.

The two maps $\sigma$ that can occur can be interpreted as multiplication maps (with 1 and $-1$ respectively) or as the bijections $\{1,1\} \to \{-1,1\}$ and there seems to be no philosophical reason to make any decision of preference.

As an example, let us consider a function $f \in M_s(K)$. Let $a \in K$, let $\alpha \in \Sigma$. If $x \in a+\alpha$ and $y \in a+\alpha$ ("$x,y$ are at the same side of $a$") then $x-a, y-a \in a$, so $|x-y| < |y-a|$. Since $f \in M_s(K)$ we have $|f(x)-f(y)| < |f(y)-f(a)|$, whence $|f(x)-f(a)-f(y)-f(a)| < |f(y)-f(a)|$, so $f(x)-f(a)$ and $f(y)-f(a)$ have the same sign. We may conclude that there is a map $\alpha : \Sigma \to \Sigma$ such that for all $x \in K$

$$x \in a + \alpha + f(x) c f(a) + \sigma(a) \quad (\alpha \in \Sigma).$$

Unfortunately, it turns out that in general $\sigma_a$ may be different from $\sigma_b$ if $a \neq b$, even for isometrical maps. For example, let $p \neq 2$ and let $\tau$ be a permutation of $\{0,1,2,\ldots,p-1\}$ and define $f : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$\Sigma a_n p^n \to \Sigma \tau(a_n) p^n \quad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n).$$

Suppose we had a $\sigma : \Sigma \to \Sigma$ such that for all $x,y \in \mathbb{Z}_p, x-y \in a$ implies $f(x)-f(y) \in \sigma(a)$. Let $a = 0^n p^n$ (see 1.5). Then $x-y \in a$ means
\[ x = a_0 + a_1 p + \ldots + a_n p^n \ldots \]
\[ y = b_0 + b_1 p + \ldots + b_n p^n \ldots \]

where \( a_0 = b_0, \ldots, a_{n-1} = b_{n-1}, a_n - b_n = \theta \) modulo \( p \).

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n))p^n + \ldots \), so \( \sigma(a) = \theta \) where \( \tau(a_n) - \tau(b_n) = \theta \) mod \( p \) (\( j \) depending on \( i \) and \( n \)).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = \ldots = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \in K, f : X \rightarrow K \) and let \( \sigma : \Sigma \rightarrow \Sigma \). We say that

\[ \text{f is monotone of type } \sigma \text{ if for all } \alpha \in \Sigma \text{ and all } x, y \in X \]

\[ x - y \in \alpha \text{ implies } f(x) - f(y) \in \sigma(\alpha). \]

(In other words if \( x > y \) implies \( f(x) > f(y) \), see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \rightarrow \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( \alpha, x > y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \in K, f : X \rightarrow K, \beta \in \Sigma \). We say that \( f \) is monotone of type \( \beta \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \text{ implies } f(x) - f(y) \in \alpha \beta. \]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \)
where $o : \Sigma \rightarrow \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$ is monotone of type $\beta$ iff the sign of $\frac{f(x)-f(y)}{x-y}$ is constant $\beta$ for all $x,y \in X, x \neq y$. This leads to

**DEFINITION 2.16** Let $X \subset K$, $f : X \rightarrow K$. $f$ is called increasing if $f$ is monotone of type 1. In other words, $f$ is increasing if for all $x,y \in X$, $x \neq y$ the difference quotient $\frac{f(x)-f(y)}{x-y}$ is positive, i.e., if

$$\left|\frac{f(x)-f(y)}{x-y} - 1\right| < 1.$$ 

In the next section we shall study the monotone functions of type $o$ and we will give a partial answer to the question for which maps $\sigma : \Sigma \rightarrow \Sigma$ there exists an $f : K \rightarrow K$ that is monotone of type $\sigma$. 
3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $\sigma$.

DEFINITION 3.1. Let $X \subset K$, $f: X \to K$. Let $f(x,y) = \frac{f(x) - f(y)}{x - y}$ $(x, y \in X, x \neq y)$. $f$ is called

positive if $f(X) \subset K^+$

strictly positive if $\sup_{x \in X} |f(x) - 1| < 1$

increasing if $f(x,y) \in K^+$ for all $x, y \in X, x \neq y$

strictly increasing if $\sup \{|1 - f(x,y)| : x, y \in X, x \neq y\} < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. Let $X \subset K$.

(i) If $f: X \to K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a, b \in K$ the function $x \mapsto ax + b$ is increasing if and

only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly in­
creasing).

(iii) If $f: X \to K$ is (strictly) increasing and $f$ is (strictly)

positive then $-\frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \to K$ form a convex set.

(v) If $f: X \to K$ and $g: f(X) \to K$ are (strictly) increasing then

so is $g \circ f$.

(vi) If $f: X \to K$ is (strictly) increasing then so is $f^{-1}: f(X) \to K$.

(vii) If $f_1, f_2, \ldots : X \to K$ are increasing and $f := \lim f_n$ pointwise

then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly posi­
tive" and "increasing", "strictly increasing" are equivalent,

respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function
\[ \exp x = 1 + x + \frac{x^2}{2!} + \ldots \]
defined on \( \{ x \in \mathbb{R} : |x| < p^{-1} \} \) if \( x(k) = p \), \( \lambda(k) = 0 \) and on \( \{ x \in \mathbb{R} : |x| < 1 \} \) if \( x(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f: X \to K \) be a \( C^\infty \)-function (i.e., \( \Phi f \) can continuously be extended to a function on \( X \times X \), assume that \( X \subset K \) has no isolated points. See [2]) and suppose \( f'(a) \in K^+ \) for some \( a \in X \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta, |y-a| < \delta, x \neq y \) implies
\[ \frac{|f(x)-f(y)|}{x-y} = f'(a) \leq \delta. \]
For such \( x, y \) we have
\[ \frac{|f(x)-f(y)|}{x-y} - 1 \leq \frac{|f(x)-f(y)|}{x-y} - f'(a) \leq \max(1, |f'(a)-1|) < 1.) \]

(3) The space \( C\mathbb{R}^\mathbb{P} \) of all continuous functions \( \mathbb{R}^\mathbb{P} = \mathbb{R} \), is a Banach space with respect to the sup norm \( || . ||_\infty \). Let \( e_0 := e_0^\mathbb{P} \) and for \( n \geq 1 \) let \( e_n := e_n^\mathbb{P} \) where \( B_n = \{ x \in \mathbb{R}^\mathbb{P} : |x-n| < \frac{1}{n} \} \). It is proved in [1] that \( e_0^\mathbb{P}, e_1^\mathbb{P}, \ldots \) is an orthonormal base of \( C\mathbb{R}^\mathbb{P} \) i.e., for each \( f \in C\mathbb{R}^\mathbb{P} \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that
\[ f = \sum_{n=0}^\infty \lambda_n e_n. \]
The coefficients $\lambda_n$ can be reconstructed from $f$ via

$$\lambda_0 = f(0)$$
$$\lambda_n = f(n) - f(n_-) \quad (n \in \mathbb{N})$$

where $n_-$ is defined as $a_0 + a_1 p + \ldots + a_{s-1} p^{s-1}$ if $n = a_0 + a_1 p + \ldots + a_s p^s$ ($a_s \neq 0$) in base $p$.

Our aim is here to describe a necessary and sufficient condition for the $\lambda_n$ in order that $f = \sum \lambda_n e_n$ is increasing. We show

$$f = \sum \lambda_n e_n \text{ is increasing if and only if for all } n \in \mathbb{N}$$
$$|\lambda_{n_-(n-n_-)}| < |n-n_-|.$$ 

Proof. First observe that $f$ is increasing if and only if for all $x \in \mathbb{Z}_p$

$$f(x) = x + g(x)$$

where $|g(x,y)| < 1$ for all $x, y \in \mathbb{Z}_p$, $x \neq y$.

As

$$x = \sum_{n=1}^{\infty} (n-n_-) e_n(x) \quad (x \in \mathbb{Z}_p)$$

it suffices to show that for $g = \sum \lambda_n e_n \in C(\mathbb{Z}_p)$ we have $|\Phi(g)| < 1$ if and only if $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Suppose first $|\Phi(g)| < 1$. Then for all $n \in \mathbb{N}$, $|\frac{f(n)-f(n_-)}{n-n_-}| < 1$, so

$$|\lambda_n| = |\frac{f(n)-f(n_-)}{n-n_-}| < |n-n_-|.$$ 

Conversely, let $g = \sum \lambda_n e_n$ and let $|\lambda_n| < |n-n_-|$ for all $n \in \mathbb{N}$.

Let $x, y \in \mathbb{Z}_p$ and let $|x-y| = p^{-k}$ for some $k \in \{0, 1, 2, \ldots\}$. Since

$$e_n(a) = e_n(b) \text{ if and only if } |a-b| < \frac{1}{n}$$

we have

$$e_n(x) = e_n(y) \quad \text{for } n \leq p^k.$$
Therefore
\[ |g(x) - g(y)| = |\sum_{n \geq 1} \lambda_n (e_n(x) - e_n(y))| = |\sum_{n \geq k} \lambda_n (e_n(x) - e_n(y))| \]
\[ \leq \max_{n \geq p} |\lambda_n| \leq \max_{n \geq p} |n - n| = p^{-k} = |x - y| \]
so \(\|g\| < 1\).

(4) Let \(K\) have dense valuation and let \(k\) be (countably) infinite. Let \(X\) be the unit ball of \(K\) and let \(B_i\) (\(i \in \mathbb{N}\)) be the balls in \(X\) with radius \(1\). Choose \(c_1, c_2, \ldots \in K\) such that \(|c_1| < |c_2| < \ldots\), \(\lim |c_n| = 1\). For \(n \in \mathbb{N}\) define a function \(f_n : X \to K\) via
\[
f_n(x) = \begin{cases} x + c_i & \text{if } x \in B_i \quad (1 \leq i \leq n) \\ x & \text{elsewhere} \end{cases}
\]
Then each \(f_n\) is strictly increasing (\(\|f_n(x, y) - 1\| \leq \max_{1 \leq i, j \leq n} |c_i - c_j| \leq |c_n| < 1\)). The sequence \(f_1, f_2, \ldots\) converges pointwise to an increasing function \(f\). But \(f\) is not strictly increasing:
\[
\sup_{x \neq y} |f(x, y) - 1| = \sup_{i,j} |c_i - c_j| = 1.
\]
(Compare 3.2, (vii) and (viii).)

Increasing functions are closely related to functions \(g\) for which
\[ |g(x) - g(y)| < |x - y| \quad (x \neq y) \quad (\text{if } f \text{ is increasing, set } g(x) := f(x) - x) \]

**DEFINITION 3.4.** Let \((X, \rho)\) be an ultrametric space. A map \(g : X \to X\)

is called a pseudocontraction if \(\rho(f(x), f(y)) < \rho(x, y)\)

\((x, y \in X, x \neq y)\).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X \to X$ has a fix point. We have

**Lemma 3.5.** Let $(X, \rho)$ be an ultrametric space. Then the following conditions are equivalent.

(a) $X$ is spherically complete.

(b) Each pseudocontraction $X \to X$ has a fix point.

(c) Each pseudocontraction $X \to X$ has a unique fix point.

**Proof.** If $\sigma: X \to X$ is a pseudocontraction and if $x, y$ are fix points and $x \neq y$, then $\rho(x, y) = \rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction. Thus, we have (b) $\Rightarrow$ (c). We prove (a) $\Rightarrow$ (b). Let $B \subset X$ be a ball (i.e., either $B = \{x \in X: \rho(x, a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X: \rho(x, a) < r\}$ for some $a \in X$, $r > 0$). We call $B$ **invariant** if $\sigma(B) \subset B$. Now we observe two facts

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X: \rho(x, \sigma(a)) < \rho(a, \sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x), \sigma(a)) < \rho(x, a) \leq \max(\rho(x, \sigma(a)), \rho(\sigma(a), a)) = \rho(a, \sigma(a))$, hence $\sigma(x) \in V$.) Notice that $a \not\in V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x, y)$ does not depend on $x, y$, since for $z \in B_1$, $u \in B_2$, $\rho(x, z) < \rho(x, y)$ and $\rho(y, u) < \rho(x, y)$, so $\rho(z, u) = \rho(x, y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a smal-
lest invariant ball $S$. If $a \in S$, $\sigma(a) \neq a$ then \{x \in S : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\} is invariant and does not contain $a$, a contradiction. Hence, $\sigma$ has a fix point (actually, $S$ is a singleton).

We prove (β) → (α). If $X$ were not spherically complete, there exist balls $B_1 \supsetneq B_2 \supsetneq \ldots$ such that $\bigcap_{n=0}^{\infty} B_n = \emptyset$. Choose $x_n \in B_n \setminus B_{n+1}$ \((n \in \mathbb{N})\), set $B_0 := X$ and define
\[
\sigma(x) := \begin{cases} 
\frac{x}{n+1} & \text{if } x \in B_n \setminus B_{n+1} \\
\frac{y}{n+1} & \text{if } x \notin B_n \setminus B_{n+1} (n \in \{0, 1, 2, \ldots\}).
\end{cases}
\]

Then $\sigma$ has obviously no fix point. Let $x \in B_n \setminus B_{n+1}$ and $y \in B_m \setminus B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in $B_{m+1}$, whereas $x \in B_n \subseteq B_{m+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$. Then $\sigma$ is a pseudocontraction without a fix point. Contradiction.

**COROLLARY 3.6.** The following conditions are equivalent.

(α) $K$ is spherically complete.

(β) If $C \subset K$ is convex, $f : C \rightarrow C$ is increasing then $f$ is surjective.

(γ) If $C \subset K$ is convex, $f : C \rightarrow K$ is increasing then $f(C)$ is convex.

(δ) An increasing $f : K \rightarrow K$ is surjective.

**Proof.** (α) → (β). Choose $a \in C$ and consider the map $\sigma : x \mapsto x - f(x) + a$ \((x \in C)\). Then $\sigma : C \rightarrow C$. $C$ is spherically complete, $\sigma$ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: $f$ is surjective.

(β) → (γ). For a suitable $s \in K$, $f-s$ sends $C$ into $C$. (γ) → (δ) is clear.

(δ) → (α). Let $\sigma : K \rightarrow K$ be a pseudocontraction. Then $x \mapsto x - \sigma(x)$
is increasing hence is surjective. So then is \( x \in K \) for which \( x-\sigma(x) = 0 \), i.e., \( \sigma \) has a fix point. By 3.5, \( K \) is spherically complete.

In case \( f \) is strictly increasing we do not have to require that \( K \) is spherically complete:

**THEOREM 3.7.** Let \( C \subset K \) be convex and let \( f : C \to K \) be strictly increasing. Then \( f(C) \) is convex. If \( f(C) \subset C \), then \( f(C) = C \).

**Proof.** Reread the proof of (a) + (β), (β) + (γ) above. \( \sigma \) now is a contraction. \( C \) is complete. Apply the Banach contraction theorem.

Let \( X \) be a subset of \( \mathbb{R} \) and let \( f : X \to \mathbb{R} \) be a bounded increasing function. Then \( f \) can be extended to an increasing function \( \mathbb{R} \to \mathbb{R} \) by setting \( f(x) := \inf f \) if \( x \leq y \) for all \( y \in X \) and \( f(x) := \sup \{ f(y) : y \leq x, y \in X \} \) for all other \( x \in \mathbb{R} \). In our situation we can prove

**THEOREM 3.8.** The following conditions are equivalent.

(a) \( K \) is spherically complete.

(β) For every \( X \subset K \) an increasing function \( f : X \to K \) can be extended to an increasing \( \overline{f} : K \to K \).

(γ) Let \( X \subset K \), and let \( f : X \to K \) be a strictly increasing function. Then \( f \) can be extended to a strictly increasing function \( \overline{f} : K \to K \) such that

\[
\sup_{x \neq y} |\overline{f}(x) - \overline{f}(y)| = \sup_{x \neq y} |\overline{f}(x) - f(y)|.
\]

**Proof.** (a) + (β). Let \( a \not\in X \). By Zorn's Lemma it suffices to define \( \overline{f} \) such that \( \overline{f} \) is increasing on \( X \cup \{a\} \). We are done if we can find \( a \in K \) such that for \( x \in X \)
\[
\left| \frac{a - f(x)}{a - x} - 1 \right| < 1
\]
i.e., \( a \in B(x) = B_x(f(x) - (a - x)) \) (x \( \in X \)).

Now \( B_x \cap B_y \neq \emptyset \) (x, y \( \in X \)) since the distance of their centers is
\[
|f(x) - (a - x) - f(y) - (a - y)| = |f(x) - f(y) - (x - y)| = |\Phi(x, y) - 1||x - y| < \\
< \max(|x - a|, |a - y|).
\]
So if, say, \(|x - a| \leq |y - a|\) we see that \(|f(x) - (a - x) - f(y) - (a - y)| < |y - a|\) whence \( f(x) - (a - x) \in B_y \). By the spherical completeness of \( K \) we have \( \bigcap_{x \in X} B_x \neq \emptyset \). Choose \( \alpha \in \bigcap_{x \in X} B_x \).

(\( \beta \) \( \rightarrow \) (\( \alpha \)). Suppose \( K \) is not spherically complete. By 3.6, (\( \delta \)) \( \rightarrow \) (\( \alpha \)) there is a non surjective increasing function \( f: K \to K \). Then its inverse \( g: f(K) \to K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K \to K \).

(\( \beta \) \( \leftrightarrow \) (\( \gamma \)) follows from the fact that (with \( \Phi(x) = x \) for all \( x \))
\[
f \mapsto (1 - c)\Phi + cf \quad (c \in K, |c| < 1)
\]
is a \( 1-1 \) correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \( |1 - \Phi(g)| < |c| \).

We will now investigate the relation between increasingness of \( f \) and positivity of \( f \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by \([2] \) Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \to K \). Let \( \lambda \in K, 0 < |\lambda| < 1 \). Then \( x \mapsto x - \lambda \sigma(x) \) is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \) of \( K \) without isolated points and if \( f \) is differentiable then for each
\[ x \in X, f'(x) = \lim_{y \to x} f(x,y) \in \mathbb{K}. \text{ So } f' \text{ is positive. If, addition, } f \]

is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f: X \to K \) be a (strictly) positive Baire class 1 function. Then does \( f \) have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subseteq K \) and let \( f: X \to K \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots: X \to K \) such that \( |g_n| \leq 1 - \frac{1}{n} \) for each \( n \) and

\[
\frac{f}{\sum g_n} \quad \text{(pointwise)}.
\]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots: X \to K \) such that \( f = \lim_n f_n \) pointwise. There exist locally constant functions \( h_1, h_2, \ldots: X \to K \) such that \( |f_n - h_n| \leq 2^{-n} \), hence \( f = \lim_n h_n \) pointwise. Define \( t_1, t_2, \ldots: X \to K \) as follows

\[
t_n(x) := \begin{cases} h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\ 0 & \text{if } |h_n(x)| > 1 - \frac{1}{n} \end{cases}
\]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N} \). (\( \{x \in X: |h_n(x)| \leq 1 - \frac{1}{n} \} \)

is closed and open in \( X \). |\( t_n | \leq 1 - \frac{1}{n} \) and \( \lim_n t_n = f \). Now let \( g_1 := t_1 \), and \( g_n := t_n - t_{n-1} \) \( (n \geq 2) \). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n} \), \( f = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} g_n \).
LEMMA 3.10. Let \( X \subset K \) have no isolated points and let \( f: X \to K \) be a Baire class 1 function, \(|f(x)| < 1\) for all \( x \in X \). Then \( f \) has an antiderivative \( F \) for which

\[
\frac{|F(x)-F(y)|}{|x-y|} < 1 \quad (x, y \in X, x \neq y).
\]

**Proof.** By Lemma 3.9, \( f = \sum_{n=1}^{\infty} f_n \), where each \( f_n \) is locally constant, \( |f_n| \leq 1 - \frac{1}{n} \). By [2] 3.9 each \( f_n \) has an antiderivative \( F_n \) for which

\[
|F_n(x)-F_n(y)| \leq \max(|f_n(x)|, \frac{1}{2n})|x-y| \quad (x,y \in X).
\]

By [2] 3.7, \( F:= \sum F_n \) is an antiderivative of \( \sum f_n = f \). Now for \( x,y \in X, x \neq y \):

\[
|F(x)-F(y)| \leq \sup_n |F_n(x)-F_n(y)| \leq \sup_n \max(|f_n(x)|, \frac{1}{2n})|x-y| 
\]

\[
\leq |x-y| \max(|f_n(x)|, \frac{1}{2}).\text{ Now for each } x \in X, |f_n(x)| < 1 \text{ for each } n \text{ and } \lim_{n} |f_n(x)| = 0 < 1. \text{ Hence } \max_n |f_n(x)| < 1. \text{ It follows that } |F(x)-F(y)| < |x-y|.
\]

THEOREM 3.11. Let \( X \subset K \) have no isolated points and let \( f: X \to K \) be (strictly) positive. Then \( f \) has a (strictly) increasing antiderivative.

**Proof.** The function \( x \mapsto f(x)-1 \) has, by 3.10, an antiderivative \( H \) such that \(|\phi(H)| < 1\). Let \( F(x) = x + H(x) \) \((x \in X)\). Then \( F' = f \) and \( \phi(F) = 1 + \phi(H) \).

Thus, if \( f \) is positive then \( F \) is increasing. If \( f \) is strictly positive then \(|f(x)-1| < r < 1\) for all \( x \in X \) and, by a trivial extension of 3.10, we may choose \( H \) such that \(|\phi(H)| < r\). It follows that \(|\phi(F)-1| < r\), so \( F \) is strictly increasing.
We collect the results in

COROLLARY 3.12. Let $X \subset K$ have no isolated points. Then

(i) If $f : X + K$ is differentiable and (strictly) increasing
then $f'$ is a (strictly) positive Baire class 1 function.

(ii) If $g : X + K$ is a (strictly) positive Baire class 1 func-
tion then $g$ has a (strictly) increasing antiderivative.

(iii) If $f : X + K$ is differentiable and if $f'$ is (strictly) po-
sitive then $f = g + h$ where $g$ is (strictly) increasing and
where $h' = 0$.

Note. We cannot strengthen 3.12 (iii) by replacing "$h' = 0$" by "$h$ is
locally constant". In fact, if $X = \mathbb{Z}_p$ then every locally constant fun-
tion has bounded difference quotients. If our statements were true,
then every differentiable $f : \mathbb{Z}_p \to \mathbb{Q}_p$ for which $f'$ is positive would
have bounded difference quotients.

But consider the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined via

$$f(x) = \begin{cases} x - p^{2n} & \text{if } |x - p^n| < p^{-2n} \ (n \in \{0,1,2,\ldots\}) \\ x & \text{elsewhere} \end{cases}$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. Let $x := p^n$ and $y_n := p^n + p^{3n} \ (n \in \mathbb{N})$. Then

$$f(x_n) = p^n - p^{2n}, \ f(y_n) = y_n = p^n + p^{3n},$$

so $|f(x_n) - f(y_n)| = |p^{2n}| = p^{-2n}$,

whereas $|x_n - y_n| = |p^{3n}| = p^{-3n}$. So

$$\lim_{n \to \infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n \to \infty} \frac{p^{2n}}{p^{3n}} = p^{-1}.$$  

We now study the connection between increasing $C^1$-functions and
continuous positive functions.

If $f$ is a (strictly) increasing $C^1$-function then clearly $f'$ is a con-
tinuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f : X \to K$ be continuous and positive. Let $P : C(X) \to C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X).$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_n = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in \mathbb{N}$, let $x_n = \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x) - x| < r_n$, $\sigma_n(x) \in R_n$.

See [2] 5.3, 5.4.)

From [2] 5.4, it follows that $Pf$ is a $C^1$-antiderivative of $f$. It suffices to prove that $Pf$ is (strictly) increasing. Let $x, y \in X$, $x \neq y$,

$|x-y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x-y| < r_s$. We have

$x_1 = y_1, \ldots, x_s = y_s$, $x_{s+1} \neq y_{s+1}$. Further $|x_{n+1} - x_n| \leq |x-y|$ ($n > s$),

$|y_{n+1} - y_n| \leq |x-y|$ ($n > s$), $|x_{s+1} - y_{s+1}| \leq |x-y|$. Hence (using the identity

$x = \sum (x_{n+1} - x_n) + x_1$, $y = \sum (y_{n+1} - y_n) + y_1$, $x_1 = y_1$)

$$|Pf(x) - Pf(y) - (x-y)| =$$

$$= \left|\sum_{n=s+1}^{\infty} (f(x_n) - 1)(x_{n+1} - x_n) - \sum_{n>s} (f(x_n) - 1)(x_{n+1} - x_n)
+ \sum_{n>s} (f(y_n) - 1)(y_{n+1} - y_n)\right|.$$
\[
|(Pf)(x)-(Pf)(y)-(x-y)| = \sum_{n=1}^{\infty} (f(x_n)-1)(x_{n+1}-x_n) - (f(y_n)-1)(y_{n+1}-y_n) | < |x| |y| |x-y|.
\]

We have proved:

**THEOREM 3.13.** Let \( X \subset K \) have no isolated points. Then the map \( P \) defined via

\[
(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (f \in C(X), \; x \in X)
\]

maps (strictly) positive functions into (strictly) increasing functions.

**COROLLARY 3.14.** Let \( X \subset K \) have no isolated points. Then if \( f \in C^1(X) \) and \( f' \) is (strictly) positive, then \( f = j+h \) where \( j \) is (strictly) increasing and \( h \) is locally constant.

**Proof.** By 3.12 we have \( f = j+h \) where \( j \) is (strictly) increasing and where \( h' = 0 \). Now by [2] Cor. 5.2 bis there is a locally constant function \( l: X \to K \) with \( |N(h-l)|_\infty < \frac{1}{2} \). Then \( s := j+(h-l) \) is (strictly) increasing, so we have \( t = s+l \), where \( s \) is (strictly) increasing and \( l \) is locally constant.

**Note.** We may also define convex functions. Let \( X \subset K \). A function \( f: X \to K \) is called convex if the second order difference quotient is positive. I.e., if for all \( x, y, z \in X \) \((x \neq y, y \neq z, x \neq z)\) we have

\[
\frac{\delta^2 f(x,y,z)}{y-z} = \frac{\delta f(x,y)-\delta f(x,z)}{y-z} = \frac{f(x)-f(y)}{x-y} - \frac{f(x)-f(z)}{x-z} \in K^+
\]

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function $f$ the function $x \mapsto \Phi_f(x,y)$ defined on $X \setminus \{y\}$ is an isometry, hence can be continuously extended to the whole of $X$. Define $\Phi_f(y,y) = \lim_{x \to y} \Phi_f(x,y)$ ($y \in X$). Thus, $f$ is differentiable. For all $x,y,z,t \in X$ we have

$$|\Phi_f(x,y) - \Phi_f(z,t)| \leq \max(|\Phi_f(x,y) - \Phi_f(z,y)|, |\Phi_f(z,y) - \Phi_f(z,t)|) \leq \max(|x-z|, |y-t|).$$

Hence, $\Phi_f$ is uniformly continuous on $X$, i.e., $f$ is strongly uniformly differentiable in the sense of page 67.

For each $y \in X$ the function $x \mapsto \Phi_f(x,y)$ is increasing on $X$.

If $\chi(K) \neq 2$ then convexity of $f$ implies increasingness of $\Phi_f$.

(Proof. \( \inf \frac{\Phi_f(x,y) - \Phi_f(x',y)}{x-x'} \leq \frac{\Phi_f(x,x') - \Phi_f(x',x)}{x-x'} \leq K^+ (x \neq x') \))

so \( \frac{f'(x) - f'(x')}{x-x'} \leq 2K^+ (x \neq x') \), whence $\Phi(f')(x,x') \leq K^+$ if $x \neq x'$.

Of course, if $f \in C^2(X)$ (see [2] 8.1) then convexity of $f$ implies positivity of $D^2 \Phi_f$ ([2] 8.4). So if $\chi(K) \neq 2$ then $f'' = D^2 \Phi_f$ ([2] 8.14) is positive. If $\chi(K) = 2$ then $f'' = 0$ for all $C^2$-functions.

Note. The functions that are monotone of type $\beta$ ($\beta \in \Sigma$), see Def. 2.15, are easy to describe: $f$ is monotone of type $\beta$ if and only if $b^{-1} f$ is increasing for any $b \in \beta$.

We now turn to the functions $X \to K$ that are of type $\sigma$ where $\sigma : \Sigma \to \Sigma$. (2.14). For examples of such $f$, where $\sigma$ is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this section we will assume that $X$ is an open convex subset of $K$. This implies that the set \{a \in \Sigma : \text{there is } x \in X \text{ such that } x > a\} is independent of $a \in X$. Thus, $X$ is homogeneous in the sense that $(a + \alpha) \cap X \neq \emptyset$ for some $a \in X$, $\alpha \in \Sigma$ then for each $b \in X$, $(b + \alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y\}$. Then for each $a \in X$

$$\Sigma(X) = \{a \in \Sigma : x > a \text{ for some } x \in X\}.$$

Either $\Sigma(X) = K$ or $\Sigma(X) = \{a \in \Sigma : |a| < r\} \text{ for some } r > 0$ or $\Sigma(X) = \{a \in \Sigma : |a| \leq r\} \text{ for some } r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see 1.2) i.e., if $a, \beta \in \Sigma(X)$ and $a \oplus \beta$ is defined then $a \oplus \beta \in \Sigma(X)$.

To be sure that $f$ is monotone both of type $\sigma$ and type $\tau$ implies $\sigma = \tau$ we define

**DEFINITION 3.15.** (Let $X \subseteq K$ be open, convex and) let $\sigma : \Sigma(X) \to \Sigma$.

$f : X \to K$ is called monotone of type $\sigma$ if for all $x, y \in X$ and $a \in \Sigma(X)$

$$x > y \Rightarrow f(x) > f(y).$$

Let $f : X \to K$ be monotone of type $\sigma : \Sigma(X) \to \Sigma$. Then

(i) $\sigma(-a) = -\sigma(a)$ ($a \in \Sigma(X)$).

(ii) Let $a, \beta \in \Sigma(X)$. If $\sigma(a) \Theta \sigma(\beta)$ is defined then so is $a \Theta \beta$ and $\sigma(a \Theta \beta) = \sigma(a) \Theta \sigma(\beta)$.

(iii) Let $a, \beta \in \Sigma(X)$. If $|a| < |\beta|$ then $|\sigma(a)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then

$$\sigma(sa) = s\sigma(a) \quad (a \in \Sigma(X)).$$

(v) If $\beta \in \Sigma(X)$, $|\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $a \in \Sigma(X)$. 

**THEOREM 3.16.** Let $f : X \to K$ be monotone of type $\sigma : \Sigma(X) \to \Sigma$. Then
(vi) $f \in M_{su}(X)$ (i.e., for all $x,y,z,t \in X$, $|x-y| < |z-t|$
implies $|f(x)-f(y)| < |f(z)-f(t)|$).

(vii) $f$ is either nowhere continuous or uniformly continuous
on $X$.

Proof.

(i) Let $x,y \in X$ such that $x > y$. Then $f(x)-f(y) \in \sigma(\alpha); f(y)-f(x) \in -\sigma(\alpha)$. But also $y > x$, hence $f(y)-f(x) \in \sigma(-\alpha)$. So $-\sigma(\alpha)$ and $\sigma(-\alpha)$ are not disjoint and they must coincide.

(ii) Suppose $\sigma(\alpha) \oplus \sigma(\beta)$ is defined. If $\alpha \odot \beta$ were not, then $\beta = -\alpha$ so, by (i), $\sigma(\beta) = \sigma(-\alpha) = -\sigma(\alpha)$. Hence also $\alpha \odot \beta$ is defined. Choose $x,y \in X$ with $x > y$. There is $z \in X$ such that $y > z$. Then $x-y \in \alpha$, $y-z \in \beta$, so $x-z \in -\alpha \odot \beta$. Further $f(x)-f(y) \in \sigma(\alpha), f(y)-f(z) \in \sigma(\beta)$ so $f(x)-f(z) \in \sigma(\alpha) \oplus \sigma(\beta)$. Also $x-z \in \alpha \odot \beta$, so $f(x)-f(z) \in \sigma(\alpha \odot \beta)$.

The signs $\sigma(\alpha) \oplus \sigma(\beta)$ and $\sigma(\alpha \odot \beta)$ are not disjoint and they must coincide.

(iii) Let $|\alpha| < |\beta|$. Choose $x,y,z$ such that $x-y \in \alpha$, $y-z \in \beta$. Then (see 1.2 and preamble) $f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(\alpha) + \sigma(\beta)$, $x-z \in \alpha + \beta = \alpha \odot \beta = \beta$, so $f(x)-f(z) \in \sigma(\beta)$. Thus $[\sigma(\alpha) + \sigma(\beta)] \cap \sigma(\beta)$ is not empty. If $\sigma(\alpha) \oplus \sigma(\beta)$ were not defined then $\sigma(\alpha) = -\sigma(\beta)$ and $\sigma(\alpha) + \sigma(\beta)$ would be a ball with center 0 and radius $|\sigma(\beta)|$, but then $[\sigma(\alpha) + \sigma(\beta)] \cap \sigma(\beta)$ would be empty. Hence $\sigma(\alpha) \oplus \sigma(\beta)$ is defined and by (ii) we have $\sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta)$. By (1.2) (vi), $|\sigma(\alpha)| < |\sigma(\beta)|$.

(iv) Let $\chi(K) \neq 0$. Then $s = n \cdot 1$ for some $n \in \{1,2,\ldots,\chi(K)-1\}$, so by 1.2 (vii), $sa = n\alpha = \sigma a$, $s\sigma(a) = n\sigma(\alpha) = \sigma a$. By a repeated application of (ii), we see $\sigma(\sigma a) = \sigma(a)$. Hence $\sigma(sa) = s\sigma(a)$.

Let $\chi(k) = 0$. Let $s$ be of the form $n \cdot 1$ for some $n \in \mathbb{N}$. By a similar reasoning as above, $\sigma(sa) = s\sigma(a)$. We may identify the prime field of $K$ with $\mathbb{Q}$. 


Now observe that \( \{ s \in K^* : \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) (\( n \in \mathbb{N} \)), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^* : \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1, 1, 2, \ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N} \), \( n = s \mod p \) (\( 1 \leq s < p \)) then \( \sigma(n) = \sigma(s) = n\sigma(s) \) for all \( n \in \mathbb{Z} \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). By (iii) \( |\sigma(\alpha)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( z-t \in \alpha \) for some \( \alpha \) hence \( f(z)-f(t) \in \sigma(\alpha) \). Thus, \( f(z) \neq f(t) \).

(vii) Let \( \rho := \inf |f(x)-f(y)| \). If \( \rho > 0 \) then clearly \( f \) is nowhere continuous. If \( \rho = 0 \), let \( \varepsilon > 0 \). There is \( a,b \in X, a \neq b \) such that \( |f(a)-f(b)| < \varepsilon \). By (vi), for all \( x,y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \varepsilon \) is uniformly continuous.

A natural question that can be raised is the following. If \( f : X \to K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f : X \to K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
\[(\beta) \; f \in M_B(X).\]
\[(\gamma) \; f \in M_{\text{ubs}}(X).\]
\[(\delta) \text{ If, for } \alpha, \beta \in \Sigma(X), \; \alpha \oplus \beta \text{ is defined then so is } \sigma(\alpha) \oplus \sigma(\beta)\]
\[\text{(and } \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\alpha \oplus \beta)).\]
\[(\varepsilon) \text{ If } \alpha, \beta \in \Sigma(X), \; |\sigma(\alpha)| < |\sigma(\beta)| \text{ then } |\alpha| < |\beta|.\]

**Proof.** We prove \((\alpha) \rightarrow (\varepsilon) \rightarrow (\gamma) \rightarrow (\beta) \rightarrow (\delta) \rightarrow (\alpha).\)

\((\alpha) \rightarrow (\varepsilon). \text{ Let } |\sigma(\alpha)| < |\sigma(\beta)| \text{ then } \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta) \text{ (1.2. (vi)). By 3.16, (iii), } \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta). \text{ Since } \sigma \text{ is injective, } \alpha \oplus \beta = \beta \text{ so (again 1.2. (vi)) } |\alpha| < |\beta|.\]

\((\varepsilon) \rightarrow (\gamma). \text{ Let } |x-y| \leq |z-t| \; (x,y,z,t \in X). \text{ We prove } |f(x) - f(y)| \leq |f(z) - f(t)|. \text{ If } z = t \text{ there is nothing to prove. Assume } z \neq t \text{ and } |f(x) - f(y)| > |f(z) - f(t)|. \text{ Then (}f \text{ is injective), supposing } x-y \in \alpha, \; z-t \in \beta \text{ for some } \alpha, \beta \in \Sigma(X), \text{ we have } f(x) - f(y) \in \sigma(\alpha), \; f(z) - f(t) \in \sigma(\beta) \text{ and } |\sigma(\alpha)| > |\sigma(\beta)|. \text{ By (}\varepsilon\text{), } |\alpha| > |\beta| \text{ i.e., } |x-y| > |z-t|. \text{ Contradiction.}\)

\((\gamma) \rightarrow (\beta). \text{ Trivial.}\)

\((\beta) \rightarrow (\delta). \text{ Suppose } \sigma(\alpha) \oplus \sigma(\beta) \text{ is not defined. Then } |\sigma(\alpha)| = |\sigma(\beta)| \text{ and, by 3.16 (iii), } |\alpha| = |\beta|. \text{ Choose } x,y,z \text{ such that } x-y \in \alpha, \; y-z \in \beta. \text{ Then } f(x) - f(z) \in \sigma(\alpha) + \sigma(\beta) \text{ so } |f(x) - f(z)| < |\sigma(\beta)| = |f(x) - f(y)|. \text{ Since } f \in M_B(X), \; |x-z| < |x-y| \text{ hence, since } x-z \in \alpha \oplus \beta, \; x-y \in \alpha: |\alpha \oplus \beta| < |\alpha|. \text{ But } |\alpha \oplus \beta| = \max(|\alpha|, |\beta|), \text{ a contradiction.}\)

\((\delta) \rightarrow (\alpha). \text{ Suppose } \sigma(\alpha) = \sigma(\beta) \text{ and } \alpha \neq \beta. \text{ Then } \alpha \oplus (-\beta) \text{ is defined. By (}\delta\text{), also } \sigma(\alpha) \oplus \sigma(-\beta) \text{ is defined. But } \sigma(-\beta) = -\sigma(\beta) = -\sigma(\alpha), \text{ so } \sigma(\alpha) \oplus -\sigma(\alpha) \text{ is defined, a contradiction.}\)

**THEOREM 3.18.** Let \(k\) be a prime field. Then, if \(f : X \rightarrow K\) is monotone of type \(\sigma\) then \(\sigma\) is injective.
Proof. Suppose \( \sigma(\alpha) = \sigma(\beta) \) for some \( \alpha, \beta \in \Sigma(X) \). Then \( |\sigma(\alpha)| = |\sigma(\beta)| \) so, by 3.16 (iii), \( |\alpha| = |\beta| \). There is \( t \in K \), \( |t| = 1 \) such that \( \beta = ta \). Since \( k \) is a prime field we may suppose \( t \in \{1, 2, \ldots, p-1\} \) if \( k \cong \mathbb{F}_p \) and \( t \in \mathbb{Q}^* \) if \( k \cong \mathbb{Q} \). So, by 3.16 (iv), \( \sigma(\beta) = \sigma(ta) = t\sigma(\alpha) = t\sigma(\beta) \). For \( x \in \sigma(\beta) \) we have \( tx \in \sigma(\beta) \), so \( tx \cdot x^{-1} \in K^+ \) i.e., \( |t-1| < 1 \). It follows easily that \( t = 1 \). Hence, \( \alpha = \beta \).

We now like to determine all \( \sigma : \Sigma \to \Sigma \) that "can occur" as the type of a monotone function in case \( K = \mathbb{Q}_p \). We use the fact that \( \Sigma \) can be identified with the following subgroup of \( \mathbb{Q}_p^* \)

\[
\{ \theta^i \cdot n : i \in \{0, 1, 2, \ldots, p-2\}, n \in \mathbb{Z} \}
\]

where \( \theta \) is a primitive \((p-1)\)th root of 1. (See 1.5.)

First, let \( f : \mathbb{Q}_p \to \mathbb{Q}_p \) be monotone of some type \( \sigma : \Sigma \to \Sigma \). By 3.18, \( \sigma \) is injective. By 3.17, (e), 3.16 (iii) we have \( |\alpha| < |\beta| \Leftrightarrow |\sigma(\alpha)| < |\sigma(\beta)| \) and \( |\alpha| = |\beta| \Leftrightarrow |\sigma(\alpha)| = |\sigma(\beta)| \), so \( |\sigma(\alpha)| \) is a strictly increasing function of \( |\alpha| \).

Set

\[
\sigma(\theta^i \cdot n) = \theta^s(i, n) \cdot \lambda(n) \quad (\theta^i \cdot n \in \Sigma)
\]

Where \( s : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \to \{0, 1, 2, \ldots, p-2\} \) and \( \lambda : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \to \mathbb{Z} \). We see that \( |\sigma(\theta^i \cdot n)| = |\sigma(\theta^j \cdot n)| \) for all \( i, j \in \{0, 1, 2, \ldots, p-2\} \) hence \( \lambda(i, n) = \lambda(j, n) \) for all \( i, j \in \{0, 1, 2, \ldots, p-2\} \). Then

\[
\sigma(\theta^i \cdot n) = \theta^s(i, n) \cdot \lambda(n)
\]

where \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is a strictly increasing function (in the classical sense).

By 3.16 (v), \( \sigma(\theta^i \cdot n) = \theta^i \cdot \sigma(n) = \theta^i \cdot \theta^s(0, n) \cdot \lambda(n) \).
Thus, $\sigma$ is of the form

\[(*) \quad \theta^n \mapsto \theta^s(n) \lambda(n)\]

where $s : \mathbb{N} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

Conversely, if we are given a map $\sigma$ of the form $(*)$ then it is easy to construct an $f : \mathbb{Q} \to \mathbb{Q}$, monotone of type $\sigma$. In fact, let $x \in \mathbb{Q}$,

$x = \sum_{n \in \mathbb{Z}} a_n p^n$, where $a_n \in \{0,1,\ldots,p^{D-2}\}$ for each $n$ and $a_{-n} = 0$ for large $n$. Then set

$$f(x) := \sum_{n \in \mathbb{Z}} a_n \theta^s(n) p^n \lambda(n).$$

Now let $x = \sum_{n \in \mathbb{Z}} a_n p^n$, $y = \sum_{n \in \mathbb{Z}} b_n p^n$ and $\pi(x-y) = \theta^m$ for some $i \in \{0,1,\ldots,p-2\}$, $m \in \mathbb{Z}$. Then $a_n = b_n$ for $n < m$ and $a_n - b_n = \theta^n \mod p$. So the sign of $a_m - b_m$ is $\theta^i$. $f(x) - f(y) = \sum_{n \geq m} (a_n - b_n) \theta^s(n) p^n \lambda(n) = (a_m - b_m) \theta^s(m) p^m \lambda(m) + r$, where $|r| < |f(x) - f(y)|$. The sign of $f(x) - f(y)$ is the sign of $(a_m - b_m) \theta^s(m) p^m \lambda(m)$ which is $\theta^i \theta^s(m) p^m \lambda(m)$. So $\pi(f(x) - f(y)) = \theta^i \theta^s(m) p^m \lambda(m) = \sigma(\theta^i p^m)$. Thus, $f$ is monotone of type $\sigma$. We have found

**Theorem 3.19.** The set \{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q} \to \mathbb{Q}, \text{ monotone of type } \sigma\} is equal to the set of all $\sigma : \Sigma \to \Sigma$ of the form

$$\theta^n \mapsto \theta^s(n) \lambda(n)$$

where $s : \mathbb{Z} \to \{0,1,2,\ldots,p-2\}$ and $\lambda : \mathbb{Z} \to \mathbb{Z}$ is strictly increasing.

**Remark.** With the notations as in 3.19, let $\mu(n) := \lambda(n) - n$. Then

$\mu : \mathbb{Z} \to \mathbb{Z}$ is increasing ($\mu(n+1) = \lambda(n+1) - (n+1) \geq \lambda(n) + 1 - (n+1) = \mu(n)$).

We then have two possibilities for a function $f : \mathbb{Q} \to \mathbb{Q}$, monotone of type $\sigma$. 
(a) \( \lim_{n \to \infty} u(n) = 0 \). Then \( |\sigma(a)| = |a| p^{u(n)} \), \( a = \nu \, p^n \), so \( \lim \frac{|\sigma(a)|}{|a|} = 0 \).

Thus \( \lim_{x-y \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0 \).

(b) \( u \) is bounded above. Then \( u(n) \) is constant, \( c \), for \( n \geq n_0 \). (For example, if \( \sigma \) is bijective then we have even \( u(n) = c \) for all \( n \).)

Thus, for sufficiently small \( |a| \) \( (a = \nu \, p^n \in \Sigma) \) we have

\[ |\sigma(a)| = |p^{\lambda(n)}| = |p^{n+c}| = |p^c| |a|. \]

So, there is \( r \) such that \( |x-y| < r \) implies \( |f(x)-f(y)| = |p^c||x-y| \).

In this case we then have: there is \( r > 0 \) and \( \lambda \in \mathbb{Q}_p \) such that on each ball in \( \mathbb{Q}_p \) of radius \( r \), \( \lambda^{-1} f \) is an isometry.

We now construct an example of a function \( f \) monotone of type \( \sigma \), where \( \sigma \) is not injective. Let \( p = 3 \) mod 4 and let \( K := \mathbb{Q}_p (\sqrt{1}) \). The elements of \( K \) can be written as \( a+bi \) \( (a,b \in \mathbb{Q}_p) \) and \( |a+bi| = \max(|a|,|b|) \).

The value group of \( K \) is the same as the one of \( \mathbb{Q}_p \), the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.17). Let \( X \) be the unit ball of \( K \), let

\[ S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}. \]

For each \( x \in X \) there is a unique \( \bar{x} \in S \) such that \( |x-\bar{x}| < 1 \). As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s,t \in S^*, s \neq t \).

Define a function \( h : S \to K \) as follows

\[ h(a+bi) = \frac{1}{p} a \quad (a+bi \in S). \]
and let \( f : X \to K \) be defined via
\[
f(x) = x + h(x) \quad (x \in X).
\]

We claim that \( f \) is monotone of type \( \sigma \) where
\[
\sigma(\pi(a+bi)) = \pi\left(\frac{1}{p} a\right) \text{ if } a+bi \in S, \ a \neq 0
\]
\[
\sigma(a) = a \quad \text{ elsewhere.}
\]

(Clearly, \( \sigma \) is a well defined map \( \Sigma(X) \to K \), \( \sigma \) is not injective since, for example, \( \sigma(\pi(1)) = \sigma(\pi(1+i)) \).

**Proof.** Let \( |a| < 1 \) and \( x-y \in \alpha \), then \( |x-y| < 1 \) so \( \overline{x} = \overline{y}, h(x) = h(y) \).

It follows that \( f(x)-f(y) = x-y \in \alpha = \sigma(\alpha) \).

Now let \( |a| = 1 \) be of the form \( \pi(bi), b \in \{1,2,\ldots,p-1\} \) and let \( x-y \in \alpha \). Say, \( \overline{x} = r+si, \overline{y} = t+ui \) \( (r,s,t,u \in \{0,1,2,\ldots,p-1\}) \). Then also \( \overline{x-y} \in \alpha \), so \( |r+si-t+ui-bi| < 1 \) hence \( r = t \). Thus, \( h(x) = \frac{1}{p} r = h(y) \),

and we have \( f(x)-f(y) = x-y \in \alpha = \sigma(\alpha) \).

Finally, let \( |a| = 1, a = \pi(a+bi), \) where \( a \neq 0 \) \( (a,b \in \{0,1,2,\ldots,p-1\}) \)

and let \( x-y \in \alpha \). Set \( \overline{x} = r+si, \overline{y} = t+ui \). Then \( \overline{x-y} \in \alpha \), so \( r-t = a \mod p \).

We find \( h(x) = \frac{1}{p} r, h(y) = \frac{1}{p} t \), so \( |h(x)-h(y)| = \frac{1}{p} a | < \frac{1}{|p||a|} \) i.e. \( h(x)-h(y) \in \pi\left(\frac{1}{p} a\right) \). Since \( |\pi(x-y)| < 1 \), we find \( f(x)-f(y) = x-y-(h(x)-h(y)) \in \pi\left(\frac{1}{p} a\right) = \sigma(\pi(a+bi)) = \sigma(\alpha) \).

Concluding:

**EXAMPLE 3.20.** Let \( p = 3 \mod 4 \) and \( K = \mathbb{Q}_p (\sqrt{-1}) \). Then there exists a function
\[
f : \{x \in K : |x| \leq 1\} \to K, \text{ monotone of some type } \sigma, \text{ where } \sigma \text{ is not injective.}
\]

In case \( K \) has discrete valuation we have some extra information.
THEOREM 3.21. Let $K$ have discrete valuation and let $f : X \to K$ be monotone of type $\sigma \in \Sigma(X)$. Then

(i) $f$ is continuous.

(ii) If $\sigma$ is injective, then $f$ and $\phi(f)$ are bounded on bounded sets.

(iii) If $\sigma$ is injective and if $f(X)$ is convex then there is

$\lambda \in K$ such that $\lambda f$ is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2)(c). If $\sigma$ is injective
then by 3.16 (iii) and 3.17 (c), $|\sigma(a)|$ is a strictly increasing function of $|a|$. Suppose $X$ is bounded. Then $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r \in K^*$. Let $|\sigma(a)| = s$ whenever $|a| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|a| = |\pi|^r$. Then $|a| < r$ so $|\sigma(a)| < s$, hence $|\sigma(a)| \leq |\pi|^s$.

By induction, it follows that $|\sigma(a)| \leq |\pi|^n s$ whenever $|a| = |\pi|^n r$, so $|\sigma(a)| \leq |a| \cdot \frac{s}{r}$. So the difference quotients of $f$ are bounded by $sr^{-1}$. Then clearly $f$ is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $\Sigma(f(X))$ has the form $\{a \in \Sigma : |a| \leq s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then $\sigma$ induces an injection of $\{p \in |K^*| : p \leq r\}$ onto $\{a \in |K^*| : |a| \leq s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(a)| = c|a|$ for some $c \in IR$ i.e., $|f(x) - f(y)| = |c||x-y|$ for all $x,y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type $\sigma$, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) $K$ is a local field ($f$ is continuous).

(2) $K$ is finite and $X = \{x \in K : |x| \leq 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be
representatives modulo \( \{ x \in K : |x| < 1 \} \), let \( M = \max |f(a_i) - f(a_j)| \). For each \( x, y \in X \) we have \( i, j \) for which \( |x - a_i| < 1, |y - a_j| < 1 \). Since \( f \in M_s(X) \), we have \( |f(x) - f(a_i)| < M, |f(y) - f(a_j)| < M \) whence \( |f(x) - f(y)| \leq M : f \) is bounded.)

(3) \( K \) is discrete, \( \sigma \) is injective (this is 3.21 (ii)).

On the other hand we have the following

**EXAMPLE 3.22.** Let \( k \) be isomorphic to the algebraic closure of \( \mathbb{F}_p \). Let \( X \) be the unit ball of \( K \). Then there exists a function \( f : X \to K \), monotone of type \( \sigma \), for some \( \sigma : \Sigma(X) \to \Sigma \) such that

(i) \( \sigma \) is not injective.

(ii) \( f, \Phi(f) \) are unbounded.

**Proof.**

As an \( \mathbb{F}_p \)-vector space, \( k \) has a countable base \( e_1, e_2, \ldots \). For any \( \lambda \in \mathbb{F}_p \), \( \lambda = n \xi \), for some \( n \in \{0, 1, 2, \ldots, p-1\} \). (Here for a field \( L \), \( 1_L \) is the unit element of \( L \).) Define \( \lambda := n \xi \). Choose \( c_1, c_2, \ldots \in K \) such that

\[ 1 < |c_1| < |c_2| < \ldots, \lim_{n \to \infty} |c_n| = \infty, \]

and define a map \( h : k \to K \) via

\[ h(\lambda_n e_n) = \lambda_n c_n \] \( (\lambda_n e_n \in k) \)

Define \( f : X \to K \) by

\[ f(x) = x + h(x) \] \( (x \in X) \)

(Here \( \bar{x} \) is the image of \( x \) under the canonical map \( X \to k \)).

Then clearly \( f \) is unbounded and so is \( \Phi(f) \).

Let us identify \( \{ \alpha \in \Sigma : |\alpha| = 1 \} \) with \( k^* \) in the obvious way. We claim that \( f \) is monotone of type \( \sigma \) where
where $\alpha = \alpha$ if $|\alpha| < 1$

$$a (\alpha) = \frac{1}{n} \sum_{i} \alpha_i \lambda_i e_i \text{ if } \alpha = \sum \lambda_i e_i \text{, } n = \max \{m : \lambda_m \neq 0\}.$$

In fact, let $x - y \in \alpha$ and $|\alpha| < 1$. Then $h(x) = h(y)$ so $f(x) - f(y) = x - y \in \sigma(\alpha)$. Now let $x - y \in \alpha$ where $|\alpha| = 1$. Then set $\tilde{x} = \sum \lambda_i e_i \tilde{y} = \sum \mu_i e_i$.

Let $r = \max \{n : \lambda_n \neq \mu_n\}$. Then $x - y = \sum (\lambda_n - \mu_n) e_n \alpha = \alpha$, so $\sigma(\alpha) = \pi((\lambda_n - \mu_n) e_r)$. On the other hand, $f(x) - f(y) = x - y = \sum (\lambda_n - \mu_n) e_r$.

Thus $\pi(f(x) - f(y)) = \pi((\lambda_n - \mu_n) e_r)$. Now we have $\sum (\lambda_n - \mu_n) = \sum (\lambda_n - \mu_n) \mod p$, so $\pi((\lambda_n - \mu_n) e_r)$. It follows that $f(x) - f(y) \in \sigma(\alpha)$.

Obviously, $\sigma$ is not injective.

We will consider briefly the differentiable functions, monotone of type $\sigma$. Let $f : X \to K$ be such a function. If $f'(a) = 0$ for some $a \in X$, then $\lim_{|a| \to 0} \frac{|\sigma(\alpha)|}{|a|} = 0$, so $f' = 0$ uniformly on $X$. Now let $f'(a) \neq 0$. Then if $x$ is sufficiently close to $a$, we have $|\frac{f(x) - f(a)}{x - a} - f'(a)| < |f'(a)|$.

Thus for $|a|$ small enough we have $f'(a) \in \pi(\lambda_n - \mu_n) e_r$ i.e. $\sigma(\alpha) = \frac{\sigma(\alpha)}{\alpha}$ is constant. This implies that $\pi(f'(x))$ does not depend on $x$ ($f'$ has constant sign) and that locally $f$ is monotone of type $\beta$ for some $\beta \in \Sigma$.

We end this section with a discussion on the question which maps $\sigma : \Sigma \to \Sigma$ do occur as a type of a monotone function.

**Lemma 3.23.** Let $\sigma : \Sigma \to \Sigma$. Suppose $\sigma$ satisfies

- if $\alpha \oplus \beta$ is defined then $\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$, $\alpha, \beta \in \Sigma$.

Then

- $\sigma(-\alpha) = -\sigma(\alpha)$, $\alpha \in \Sigma$. 

(ii) If \( \sigma(a) \) is defined then so is \( a \oplus \beta \).

(iii) If \( |a| < |\beta| \) then \( |\sigma(a)| < |\sigma(\beta)| \).

\( \sigma \) is injective.

(v) If \( |a| = |\beta| \) then \( |\sigma(a)| = |\sigma(\beta)| \).

Proof. (i) is trivial if \( \chi(k) = 2 \), so suppose \( \chi(k) \neq 2 \) and let \( -\sigma(a) \neq \sigma(-a) \) for some \( a \in \Sigma \). Then we have the identity \( (a \oplus a) \oplus (-a) = a \), so
\[
\sigma(a \oplus a) \oplus \sigma(-a) = \sigma(a),
\]
whence \( (\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a) = \sigma(a) \). Now by
1.2 (iii) \( \sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a)) \) (this last expression is defined.

If not, then \( -\sigma(a) = \sigma(a) \oplus \sigma(-a) \). Now \( \sigma(a) \oplus \gamma = \sigma(a) \) has only one solution namely \( \gamma = -2\sigma(a) \). So we then would have \( \sigma(-a) = -2\sigma(a) = -(\sigma(a) \oplus \sigma(a)) \), but this contradicts the existence of \( (\sigma(a) \oplus \sigma(a)) \oplus \sigma(-a)) \).

From \( \sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus \sigma(-a)) \) we obtain by 1.2 (vi): \( |\sigma(a) \oplus \sigma(-a)| < |\sigma(a)| \). On the other hand, by 1.2 (v), \( |\sigma(a) \oplus \sigma(-a)| = |\sigma(a)| \vee |\sigma(-a)| \). Contradiction. (i) follows.

Now (ii) follows easily from (i): if \( a \oplus \beta \) were not defined then \( \beta = -a \) so, by (i), \( \sigma(a) \oplus \sigma(\beta) = \sigma(a) \oplus -\sigma(a) \), a contradiction.

Let \( |a| < |\beta| \), then \( a \oplus \beta = \beta \), so \( \sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta) = \sigma(\beta) \). By 1.2 (vi) we find
\[
|\sigma(a)| < |\sigma(\beta)| .
\]
We proved (iii).

If \( \sigma(a) = \sigma(\beta) \) and \( a \neq \beta \) then \( \sigma(a \oplus (-\beta)) = \sigma(a) \oplus \sigma(-\beta) = \sigma(a) \oplus -\sigma(a) \), an absurdity. So \( \sigma \) is injective (iv). Finally, let \( |a| = |\beta| \) and \( |\sigma(a)| > |\sigma(\beta)| \). Then \( \sigma(a) = \sigma(a) \oplus \sigma(\beta) = (by \ (ii)) = \sigma(a \oplus \beta) \). By injectivity of \( \sigma \), \( a = a \oplus \beta \), and by 1.2 (vi), we find \( |\beta| < |a| \).

Now we have
LEMMA 3.24. Let \( K \) be spherically complete, let \( Y \subset K \) (not necessarily convex) and let \( \tau : \Sigma(Y) (= \{ \pi(x-y) : x, y \in Y, x \neq y \}) \to \Sigma \) such that \( f \) is monotone of type \( \tau \) (i.e., \( x, y \in Y, x-y \in \alpha \in \Sigma(Y) \) then \( f(x)-f(y) \in \tau(\alpha) \).

Suppose \( \tau \) can be extended to a \( \sigma : \Sigma \to \Sigma \) satisfying the condition of Lemma 3.23. Then \( f \) can be extended to a monotone function \( \overline{f} : K \to K \) of type \( \sigma \).

Proof. By Zorn’s lemma, it suffices to extend \( f \) to \( Y \cup \{a\} \) (\( a \neq Y \)) such that \( f(x)-f(a) \in \sigma(\pi(x-a)) \), \( f(a)-f(x) \in \sigma(\pi(a-x)) \) for all \( x \in Y \). By 3.23 (i) it suffices to consider only the second case. Let \( B_x := f(x) + \sigma(\pi(a-x)) \) \( (x \in Y) \). Each \( B_x \) is a ball with radius \( |\sigma(\pi(a-x))|^{-1} \). By the spherical completeness of \( K \), we are done if we can show that \( B_x \cap B_y \neq \emptyset \) \( (x \neq y, x, y \in Y) \).

Set \( \alpha := \pi(a-x) \) and \( \beta := \pi(a-y) \). Let \( b \in \sigma(\alpha) \); \( c \in \sigma(\beta) \). We prove:

\[
|f(x)+b-f(y)+c| < |\alpha| \vee |\beta|.
\]

We have two cases:

1) \( \alpha = \beta \). Then \( a-x \in \alpha \), \( a-y \in \alpha \) implies \( |x-y| < |a-x| = |\alpha| \), so

\[
|\pi(x-y)| < |\alpha| \text{ whence } |\pi(f(x)-f(y))| = |\sigma(\pi(x-y))| < |\sigma(\alpha)| \text{ (by 3.23 (iii))},
\]

so \( |f(x)-f(y)| < |\sigma(\alpha)| \). Further, \( b \in \sigma(\alpha) \), \( c \in \sigma(\alpha) \) implies \( |b-c| < |\sigma(\alpha)| \), hence \( |f(x)+b-f(y)-c| < |\sigma(\alpha)| \).

2) \( \alpha \neq \beta \). Then \( x-y = a-y-(a-x) \in \beta \oplus (-\alpha) \), so \( f(x)-f(y)+b-c \in \sigma(\beta \oplus (-\alpha)) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus (-\alpha)) + \sigma(-\beta) = \sigma(\beta \oplus (-\alpha)) - \sigma(\beta \oplus -\alpha) \), hence \( |f(x)-f(y)+b-c| < |\sigma(\beta \oplus -\alpha)| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(\alpha)|, |\sigma(\beta)|) \).

THEOREM 3.25. Let \( K \) be spherically complete and let \( \sigma : \Sigma \to \Sigma \). Suppose

\[
\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta).
\]

Then there exists a function \( f : K \to K \), monotone of type \( \sigma \).
Proof. Choose \( Y := \{0\} \) and let \( g : Y \to K \) be defined via \( g(0) = 0 \). Then \( g \) satisfies the conditions of Lemma 3.24 so it can be extended to a function \( f \) of type \( \sigma \).

We now give a description of the maps \( \sigma : \Sigma \to \Sigma \) mentioned in 3.23. For each \( r \in |K^*| \) choose \( \alpha_r \in \Sigma \) such that \( |\alpha_r| = r \). Further, there is a natural isomorphism of multiplicative groups between \( k^* \) and \( \{\alpha \in \Sigma : |\alpha| = 1\} \), denoted by \( l \mapsto \alpha_l \) \((l \in k^*)\). Of course, if \( l+1 \neq 0 \) then \( \alpha_{l+1} = \alpha_l \cdot \alpha_1 \). Each element of \( \Sigma \) can be written in only one way as \( \alpha^\lambda \) \((r \in |K^*|, l \in k^*)\). Now if \( \sigma \) is as in 3.23 we get

\[
\sigma(\alpha^\lambda) = \alpha^\lambda r n(r,l)
\]

where \( \lambda : |K^*| \to |K^*| \) is strictly increasing and \( l \mapsto n(r,l) \) is an injective group endomorphism of the additive group \( k \). Conversely, if \( \lambda : |K^*| \to |K^*| \) is strictly increasing and for each \( r, l \mapsto n(r,l) \) is an injective group homomorphism \( k \to k \) then

\[
\alpha^\lambda r n(r,l) \mapsto \lambda r n(r,l) \quad (\alpha^\lambda r n(r,l) \in \Sigma)
\]

satisfies the condition of 3.23. So we get

THEOREM 3.26. Let \( K \) be spherically complete and let \(|K| = [0,\infty)\). Then there exist a nowhere continuous \( f : K \to K \), monotone of some type \( \sigma : \Sigma \to \Sigma \).

Proof. With the notations as above, let \( \sigma : \Sigma \to \Sigma \) be defined as follows

\[
\sigma(\alpha^\lambda r n(r,l)) = \alpha^\lambda r+1 n(r,l).
\]

By 3.25 there is an \( f : K \to K \) monotone of type \( \sigma \). Clearly \(|f(x) - f(y)| \geq 1\) if \( x \neq y \) so \( f \) is nowhere continuous.
4. MONOTONE FUNCTIONS, GENERAL THEOREMS

In this section we study $M_w(X), M_b(X), M_s(X), \ldots$. To avoid unnecessary complications we **assume throughout this section that $X$ is a closed subset of $K$ without isolated points.** We collect here the results on monotone functions that are valid for general $K$. In the next section we will see what happens if we put some extra conditions on $K$ (e.g., $|K|$ discrete, ...).

First two elementary lemmas.

**Lemma 4.1** Let $f : X \to K$. Then the following conditions are equivalent

(a) $f \in M_w(X)$ (see Def. 2.11).
(b) For all $x,y,z \in X$, $|x-y| < |x-z|$ implies $|f(x)-f(z)| = |f(y)-f(z)|$.
(c) For all $x,y,z \in X$, $|f(x)-f(z)| \neq |f(y)-f(z)|$ implies $|x-y| = \max(|x-z|,|y-z|)$.

**Proof.** (a) $\Rightarrow$ (b). $|x-y| < |x-z|$ implies $|y-z| = |x-z| > |x-y|$, so

$$|f(x)-f(y)| \leq \min(|f(x)-f(z)|,|f(y)-f(z)|).$$

It follows that $|f(x)-f(z)| = |f(y)-f(z)|$.

(b) $\Rightarrow$ (c). (b) says that $|f(x)-f(z)| \neq |f(y)-f(z)|$ implies $|x-y| \geq |x-z|$. By symmetry, also $|x-y| \geq |y-z|$ where $|x-y| \geq \max(|x-z|,|y-z|)$. The opposite inequality is trivial.

(c) $\Rightarrow$ (a). Let $|x-y| < |x-z|$. Then $|x-y| \neq \max(|x-z|,|z-y|)$ so, by (c), $|f(x)-f(z)| \neq |f(y)-f(z)|$. Then $|f(x)-f(y)| \leq \max(|f(x)-f(z)|,|f(y)-f(z)|) = |f(x)-f(z)|$.

**Lemma 4.2** (i) If $f \in M_w(X), \lambda \in K$ then $\lambda f \in M_w(X)$. 

(ii) If \( f_1, f_2, \ldots \in M_w(X) \) and \( f := \lim_{n \to \infty} f_n \) pointwise then 
\[ f \in M_w(X). \]

(iii) If \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \), then \( g \in M_w(f(X)) \). In particular, if \( f \) is injective and weakly monotone then so is \( f^{-1} \).

(Notice that \( f(X) \) need not be closed and may have isolated points.)

Proof. Obvious.

Remark. Lemmas, similar to 4.1 and 4.3, but now for \( M_b(X), M_s(X), M_{bs}(X) \) have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an \( M_w \)-function need not be continuous (see 2.4(5), 3.26) we will derive properties of \( M_w \)-functions that are closely related to continuity.

**Lemma 4.3** Let \( f \in M_w(X) \). Then \( f \) is bounded on precompact subsets of \( X \).

Proof. Let \( Y \subset X \) be precompact. Assume that \( Y \) is not a singleton. Then \( Y \) is bounded and has a positive diameter \( r = \max\{|x-y| : x, y \in Y\} \).

The equivalence relation \( x \sim y \) iff \( |x-y| < r \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) (\( n \geq 2 \)). Choose \( a_i \in Y_i \) for each \( i \), and let \( M := \max_{1 \leq i \leq n} |f(a_i)| \). We prove: \( |f| \leq M \). In fact, let \( x \in Y \). Then there is \( i \) such that \( |x-a_i| < r \). Choose \( j \neq i \). We have \( |x-a_i| < |a_i-a_j| \) whence \( |f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M \). So \( |f(x)| \leq M \).

The following lemma shows that an \( f \in M_w(X) \) at \( a \in X \) is either continuous or "very discontinuous".

**Lemma 4.4** Let \( f \in M_w(X) \) and let \( a \in X \). Then we have the following alternative.
Either $f$ is continuous at $a$, or for each sequence $x_1, x_2, \ldots \in X$ ($x_n \neq a$ for all $n$) with $\lim x_n = a$ the sequence $f(x_1), f(x_2), \ldots$ is bounded and has no convergent subsequence.

Proof. Since $(x_1, x_2, \ldots)$ is precompact the set $\{f(x_1), f(x_2), \ldots\}$ is bounded by Lemma 4.3. We are done if we can prove the following. If $x_1, x_2, \ldots, \lim x_n = a, x_n \neq a$ for all $n$, $\lim f(x_n)$ exists, then $f$ is continuous at $a$. Now set $a := \lim f(x_n)$. Let $y_1, y_2, \ldots \in X, \lim y_n = a$.

We prove $\lim f(y_n) = a$. (Then it follows that $a = f(a)$ since we may choose $y_n := a$ for all $n$.) Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ for which $|f(x_k) - a| < \varepsilon$. For $n$ sufficiently large we have $|y_n - a| < |x_k - a|$, so for large $m$ (depending on $n$) we have $|y_n - x_m| < |x_k - x_m|$, whence $|f(y_n) - f(x_m)| \leq |f(x_k) - f(x_m)|$. Since $\lim f(x_m) = a$ we find $|f(y_n) - a| \leq |f(x_k) - a| < \varepsilon$, so $\lim f(y_n) = a$.

COROLLARY 4.5 Let $f \in M^w(X)$. Then the graph of $f$

$$\Gamma_f := \{(x, y) \in X \times K : y = f(x)\}$$

is closed in $K^2$.

Proof. Let $(x_n, f(x_n)) \in \Gamma_f$ and let $\lim x_n = x, \lim f(x_n) = a$. If $x_n = x$ for infinitely many $n$ then $a = f(x)$, so $(x, a) \in \Gamma_f$. If not then by the alternative of lemma 4.4, $f$ is continuous at $x$, so $a = f(x)$ and $(x, a) \in \Gamma_f$.

COROLLARY 4.6 Let $f \in M^w(X)$. If each bounded subset of $f(X)$ is precompact then $f$ is continuous.

Proof. Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let $f \in M_\omega(X)$ and let $Y \subset f(X)$ be precompact. Then either 

$f$ is constant on $f^{-1}(Y)$ or $f^{-1}(Y)$ is bounded.

Proof. It suffices to prove: if $Z \subset X$ is unbounded and $f(Z)$ is precompact then $f$ is constant on $Z$. Let $a, b \in Z$. Since $Z$ is unbounded there are $x_1, x_2, \ldots \in Z$ such that

(*) $|a - b| < |x_1 - a| < |x_2 - a| < \ldots$

Since $f(Z)$ is precompact we may assume (by taking a suitable subsequence) that $a = \lim f(x_n)$ exists. From (*) we obtain

$|x_1 - x_2| = |x_2 - a|$, $|x_2 - x_3| = |x_3 - a|$, \ldots, so

$|a - b| < |x_1 - a| < |x_1 - x_2| < |x_2 - x_3| < \ldots$

hence

$|f(a) - f(b)| \leq |f(x_1) - f(a)| \leq |f(x_1) - f(x_2)| \leq \ldots$

it follows that $|f(a) - f(b)| = \lim_{n \to \infty} |f(x_n) - f(x_{n+1})| = 0$ i.e., $f(a) = f(b)$.

LEMMA 4.8 Let $f \in M_\omega(X)$ and let $a \in f(X)$ be a non-isolated point of $f(X)$.

Then we have the following alternative. Either

I. There is $a \in X$ such that for each sequence $x_1, x_2, \ldots$ in $X$

for which $\lim_{n \to \infty} f(x_n) = a$ we have $\lim_{n \to \infty} x_n = a$, or

II. If $x_1, x_2, \ldots \in X$, $\lim_{n \to \infty} f(x_n) = a$, $f(x_n) \neq a$ for all $n$,

then $x_1, x_2, \ldots$ is bounded and has no convergent subsequence.

Proof. Suppose we are not in case II. Since $a$ is not isolated in $f(X)$ and $f(X)$ is dense in $f(X)$ we have a sequence $x_1, x_2, \ldots$ in $X$ for which $f(x_n) \neq a$ for each $n$, and $\lim_{n \to \infty} f(x_n) = a$. Since $f$ is not constant on $x_1, x_2, \ldots$ it follows by Lemma 4.7 that $x_1, x_2, \ldots$ is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by $x_1, x_2, \ldots$ and set
a := \lim_{n \to \infty} x_n. Then a \in X. Now let \( y_1, y_2, \ldots \) be a sequence in X for which \( \lim_{n \to \infty} f(y_n) = a \). We prove that \( \lim_{n \to \infty} y_n = a \). In fact, let \( \epsilon > 0 \). There is \( k \in \mathbb{N} \) such that \( |x_k - a| < \epsilon \). For large \( n \) we have

\[ |f(y_n) - a| < |f(x_k) - a|, \]

so for large \( m \) (depending on \( n \)) we have

\[ |f(y_n) - f(x_m)| < |f(x_k) - f(x_m)| \]

whence \( |y_n - x_m| \leq |x_k - x_m| \), so

\[ |y_n - a| \leq |x_k - a| < \epsilon. \]

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function \( f : X \to K \) injective at \( a \in X \) if \( f(x) = f(a) \) for some \( x \in X \) implies \( x = a \).

Now suppose that we have \( a \in \overline{f(X)} \), not isolated, for which we are in alternative I. Then for a sequence \( x_1, x_2, \ldots \) with \( \lim f(x_n) = a \) we have \( \lim x_n = a \in X \) so \( (a, a) = \lim (x_n, f(x_n)) \), so by Cor. 4.5 we have \( \alpha = f(a) \). Thus, \( \alpha \in f(X) \). \( f \) is injective at \( a \); if \( f(b) = f(a) \) then since \( \lim f(b) = a \) we must have \( \lim b = a \) i.e. \( b = a \). Further, \( f \) is continuous at \( a \) (see 2.13 (2)(a)).

If each bounded subset of \( X \) is precompact we never can be in case II. This is also true if \( f \in M_b(X) \) and \( |X| \) is discrete i.e. if \( x_1, y_1 \in X \)

\[ |x_1 - y_1| > |x_2 - y_2| > \ldots \]

then \( \lim |x_n - y_n| = 0 \). Proof: let \( a \in \overline{f(X)} \) and let \( \lim f(x'_n) = a \), \( f(x'_n) \neq a \) for all \( n \). Without loss of generality we may assume

\[ |a - f(x'_1)| > |a - f(x'_2)| > \ldots \]

hence

\[ |f(x'_1) - f(x'_2)| > |f(x'_2) - f(x'_3)| > \ldots \]

and, since \( f \in M_b(X) \)

\[ |x'_1 - x'_2| > |x'_2 - x'_3| > \ldots \]

Since \( |X| \) is discrete, the sequence \( x'_1, x'_2, \ldots \) is convergent. So we have case I. We find
THEOREM 4.9 Let either each closed and bounded subset of $X$ be compact and $f \in M_w(X)$, or let $|X|$ be discrete and $f \in M_b(X)$. Then

(i) $f(X)$ is closed (in $K$).

(ii) If $f(a) \in f(X)$ is not isolated then $f$ is injective at $a$, $f$ is continuous at $a$. In particular if $f(X)$ has no isolated points then $f$ is a homeomorphism $X \to f(X)$.

Proof. If $a \in f(X) \setminus f(X)$ then $a$ is not isolated. Since we are in alternative I of Lemma 4.8, $a \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an $f$ satisfying the conditions of 4.9 maps closed subsets of $X$ (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let $f : X \to K$.

(i) If $f \in M_w(X)$ and if $Y \subseteq X$ is closed and the closed and bounded subsets of $Y$ are compact then $f(Y)$ is spherically complete.

(ii) If $f \in M_b(X)$ and if $Y \subseteq X$ is spherically complete then so is $f(Y)$.

(iii) If $f \in M_s(X)$ and if $A \subseteq f(X)$ is spherically complete then so is $f^{-1}(A)$.

Proof. (i) Let $B_1 \supsetneq B_2 \supsetneq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1 \setminus B_2$, $f(y_2) \in B_2 \setminus B_3$, $\ldots$. Then we have

$$|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots$$

and, by the weak monotony of $f$. 
\[ |y_1 - y_2| \geq |y_2 - y_3| \geq \ldots \]

Suppose first that \( \lim |y_n - y_{n+1}| = 0 \). Then \( y := \lim y_n \) and there are infinitely many \( k \) for which

\[ |y_k - y_{k-1}| > |y_{k+1} - y_k|. \]

Now \( |y - y_k| \leq \max(|y_k - y_{k+1}|, |y_{k+1} - y_{k+2}|, \ldots) \leq |y_k - y_{k+1}|. \) So we get for infinitely many \( k \)

\[ |y - y_k| < |y_k - y_{k-1}| \]

whence

\[ |f(y) - f(y_k)| \leq |f(y_k) - f(y_{k-1})| \]

so \( f(y) \in B_{k-1} \) for infinitely many \( k \), i.e., \( f(y) \in \bigcap \mathcal{B}_{k} \).

Next, suppose that \( |y_{k+1} - y_k| \geq \varepsilon > 0 \) for all \( k \). Then since \( y_1, y_2, \ldots \) is bounded it has a convergent subsequence \( y_{n_1}, y_{n_2}, \ldots \). Let \( y := \lim y_{n_1} \).

Then we have for infinitely many \( i \)

\[ |y - y_{n_i}| < \varepsilon \leq |y_{n_i} - y_{n_i+1}| \]

whence

\[ |f(y) - f(y_{n_i})| \leq |f(y_{n_i}) - f(y_{n_i+1})|, \]

so \( f(y) \in B_{n_i} \) for infinitely many \( i \), i.e., \( f(y) \in \bigcap \mathcal{B}_{k} \).

(ii) Let \( B_1 \supset B_2 \supset \ldots \) be balls in \( f(Y) \) and let \( y_1, y_2, \ldots \in Y \) such that \( f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots \). Then we have

\[ |f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots \]

and since \( f \in \mathcal{M}_b(X) \):

\[ |y_1 - y_2| > |y_2 - y_3| > \ldots \]

Since \( Y \) is spherically complete, there is \( y \in Y \) such that

\[ |y - y_n| \leq |y_n - y_{n+1}| \] for all \( n \), hence

\[ |f(y) - f(y_n)| \leq |f(y_n) - f(y_{n+1})| \]

for all \( n \). It follows that \( f(y) \in B_n \) for all \( n \).

(iii) Let \( B_1 \supset B_2 \supset \ldots \) be balls in \( f^{-1}(A) \). Choose \( x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_3, \ldots \). Then

\[ |x_1 - x_2| > |x_2 - x_3| > \ldots \]

whence

\[ |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \]

There is \( x \in f^{-1}(A) \) such that \( |f(x) - f(x_n)| \leq |f(x_n) - f(x_{n+1})| \) for all \( n \).

Hence \( |x - x_n| \leq |x_n - x_{n+1}| \) for all \( n \), i.e., \( x \in \bigcap \mathcal{B}_n \).
DEFINITION 4.11 Let $f : X \to X$. The oscillation function $\omega_f : X \to [0, \infty)$ is defined by

$$\omega_f(a) := \lim_{n \to \infty} \sup \left\{ \left| f(x) - f(y) \right| : \left| x - a \right| \leq \frac{1}{n}, \left| y - a \right| \leq \frac{1}{n}, x, y \in X \right\} (a \in X)$$

and

$$= \lim_{n \to \infty} \sup \left\{ \left| f(x) - f(a) \right| : \left| x - a \right| \leq \frac{1}{n}, x \in X \right\}.$$  

THEOREM 4.12 Let $f \in M_w(X)$. Then

$$\omega_f(a) = \inf_{z \neq a} \left| f(z) - f(a) \right| \quad (a \in X).$$

Proof. For $x \neq a$ we have

$$\left| f(x) - f(a) \right| \leq \inf_{z \neq a} \left| f(z) - f(a) \right|$$

and (since $a$ is not isolated) consequently

$$\omega_f(a) \geq \inf_{z \neq a} \left| f(z) - f(a) \right|.$$  

Conversely, let $z \neq a$. Then for all $x$ such that $\left| x - a \right| < \left| z - a \right|$ we have

$$\left| f(x) - f(a) \right| \leq \left| f(z) - f(a) \right|$$

so

$$\omega_f(a) \leq \left| f(z) - f(a) \right|$$

whence

$$\omega_f(a) \leq \inf_{z \neq a} \left| f(z) - f(a) \right|.$$  

THEOREM 4.13 Let $f \in M_w(X)$, $a \in X$. If $x_1, x_2, \ldots \in X$, $x_n = a$ (for all $n$) then

$$\lim_{n \to \infty} \left| f(x_n) - f(a) \right| = \omega_f(a).$$

Proof. By 4.12 we have

$$\lim_{n \to \infty} \left| f(x_n) - f(a) \right| \geq \omega_f(a).$$

Conversely,

$$\lim_{n \to \infty} \left| f(x_n) - f(a) \right| \leq \omega_f(a)$$

is clear from the definition of $\omega_f$.  

5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: \( K \) is local, \( k \) is finite, \( K \) has discrete valuation. Also we can sometimes say a little more if we assume \( X \) to be convex. For the time being, let \( X \) be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case \( K \) is a local field.

**THEOREM 5.1** Let \( K \) be a local field, and let \( f \in M(\omega)(X) \). Then

(i) \( f \) is continuous.

(ii) If \( Y \subset X \) is closed then \( f(Y) \) is closed.

(iii) If \( f(X) \) is bounded and \( f \) is not constant then \( X \) is bounded.

(iv) Let \( a \in X \). Then the following are equivalent

(a) \( f \) is not injective at \( a \)

(b) \( f \) is locally constant at \( a \)

(c) \( f(a) \) is isolated in \( f(X) \).

(v) The following conditions are equivalent

(a) \( f \) is injective

(b) \( f(X) \) has no isolated points

(c) \( f \) is a homeomorphism of \( X \) onto \( f(X) \).

**Proof.** Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for \( M_b \) and \( M_s \)-functions in case \( X \) is convex and \( K \) is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2 Let the residue class field $k$ of $K$ be finite. Let $X$ be convex and let $f \in M_d(X)$. Then

(i) If $a,b,c \in X$, $|a-b| < |a-c|$, $f(a) \neq f(c)$ then
   \[ |f(a)-f(b)| < |f(a)-f(c)|. \]

(ii) If $C \subset X$ is convex then $f(C)$ is convex in $f(X)$ (f is weakly Darboux continuous, see 2.5).

(iii) If $f$ is injective, then $f \in M_S(X)$.

Proof. (i) Let $B := \{x \in K : |x-a| \leq |a-c| \}$. Then $B \subset X$ and $f(B) \subset [f(a),f(c)]$. Define an equivalence relation on $B$ by: $x \sim y$ if $|f(x)-f(y)| < |f(a)-f(c)|$.

Since $k$ is finite we get finitely many equivalence classes $B_1, B_2, \ldots, B_n$. Since $a \neq c$ we have $n \geq 2$. The diameter of $f(B)$ equals $|f(a)-f(c)|$, the distance between $f(B_i)$ and $f(B_j)$ equals $|f(a)-f(c)|$ ($i \neq j$). Since $[f(a),f(c)]$ can contain at most $q := \chi(k)$ sets having distances $|f(a)-f(c)|$ to one another we have $n \leq q$. Hence $2 \leq n \leq q$.

By 2.2 (β), each $B_i$ is convex. If $x,y \in B_i$ and $|x-y| = |a-c|$ then $B_i = B$, contradicting $n \geq 2$. Thus $B$ is a disjoint union of $n$ balls $B_1, \ldots, B_n$, where $2 \leq n \leq q$ and $|x-y| < |a-c|$ whenever $x,y \in B_i$ ($i = 1, \ldots, n$). It follows that $n = q$ and that each $B_i$ has the form $\{x \in K : |x-b_i| < |a-c| \}$ ($b_i \in B$). Hence, if $|a-b| < |a-c|$ then there is $i$ for which $a, b \in B_i$. So $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) Let $a,b \in C$ and let $a \in f(X)$ with $a \in [f(a),f(b)]$. We show that $a \in f(C)$. If $f(a) = f(b)$ this is clear. If $f(a) \neq f(b)$, set $a = f(x)$ where $x \in X$. Then $|f(x)-f(a)| \leq |f(b)-f(a)|$. If $|x-a|$ were $|b-a|$ then $f(x) \neq f(a)$ (since $f \in M_d(X)$) and by (i) we then had $|f(b)-f(a)| < |f(x)-f(a)|$, a contradiction. Hence $|x-a| \leq |b-a|$ i.e., $x \in [a,b] \subset C$, so $a = f(x) \in f(C)$. 
(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on $k$, see 2.10.

COROLLARY 5.3 Let $K$ be a local field and let $f \in M_b(X)$ and $X$ convex. Then the following conditions are equivalent.

(a) $f \in M_b(X)$.

(b) $f$ is injective.

(c) $f \in M_{bs}(X)$.

(d) $f(X)$ has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 Let $K$ be a local field and let $X$ be the unit ball of $K$ (or any bounded convex set, for that matter). If either

$f \in M_s(X)$ or $f \in M_b(X)$ then $f$ has bounded difference quotients.

Proof. $f$ is bounded, let $M := \sup \{|f(x) - f(y)| : x, y \in X\}$. It suffices to prove that $|f(x) - f(0)| \leq M|x|$ for all $x$. Let $\pi \in K$, $|\pi| < 1$, be a generator of the value group. By induction on $n$ we prove:

if $|x| = |\pi|^n$ then $|f(x) - f(0)| \leq |\pi|^n M$.

The statement is clear for $n = 0$. Now suppose the statement is true for $0, 1, \ldots, n-1$.

Let $x \in X$, $|x| = |\pi|^n$. Then $|x - 0| < |\pi^{n-1}-0|$. If $f(\pi^{n-1}) \neq f(0)$ we have either since $f \in M_s(X)$ or by 5.2(i)

$$|f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1} M$$

hence

$$|f(x) - f(0)| \leq |\pi|^n M$$

If $f(\pi^{n-1}) = f(0)$ then $|f(x) - f(0)| \leq |f(\pi^{n-1}) - f(0)| = 0$, so certainly $|f(x) - f(0)| \leq |\pi|^n M$. 

Notes.

(a) 5.4 cannot be extended to the case $X = K$. In fact, let 

$$f : \mathbb{Q}_p \to \mathbb{Q}_p$$

be the map $\mathbb{Q}_p^n \to \mathbb{Q}_p^{2n}$. ($\mathbb{Q}_p^n \in \mathbb{Q}_p$.) Then 

$$f = M_{bs}(\mathbb{Q}_p)$$

but $|p^n f(p^{-n})| = p^n \to \infty$.

(b) If we loose the condition on $K$, for example by requiring that 

the valuation is discrete then 3.22 and 2.4(5) show that the 

conclusion of 5.4 is false both for $M_b$-functions and $M_s$-functions. 

On the other hand, it is clear from the proof of 5.4 that a 

bounded $M_s$-function on $X$ has bounded difference quotients.

(c) One may wonder how difference quotients of $M_w$-functions behave. 

See the example below.

EXAMPLE 5.5 Let $p \neq 2$. Then there is an $f \in M_w(\mathbb{Z} \to \mathbb{Q}_p)$ that has un-
bounded difference quotients.

**Proof.** Let $a_0, a_1, \ldots$ be defined via $a_{2n} = p^n$ ($n = 0, 1, 2, \ldots$) and 

$$a_{2n+1} = 2p^n$$

($n = 0, 1, 2, \ldots$). Thus $(a_0, a_1, a_2, \ldots) = (1, 2, p, p^2, 2p^2, \ldots)$. 

Then $|a_0| \geq |a_1| \geq |a_2| \geq \ldots$, $\lim a_n = 0$, $|a_n| < m$ ($n \geq m$).

Set 

$$f(x) = \begin{cases} 
- a_n & \text{if } |x| = p^{-n} \\
0 & \text{if } x = 0 
\end{cases} \quad (x \in \mathbb{Z}_p)$$

Then the difference quotients of $f$ are not bounded (for $n \in \mathbb{N}$:

$f(p^{-n}) = p^n$, so $|p^{-2n} f(p^{2n})| = p^n \to \infty$ if $n \to \infty$). We show that 

$f \in M_w(\mathbb{Z})$. Since $f$ is continuous it suffices to show that if $x, y, z$ 

are $\neq 0$, $|x-y| < |x-z|$ then $|f(x)-f(y)| \leq |f(x)-f(z)|$. This is clear 

if $|x| = |y|$. If $|x| < |y|$, then $|x| < |y| < |z|$. If $|x| > |y|$, 

then $|y| < |x| < |z|$. Let $f(x) = a_n$, $f(y) = a_m$, $f(z) = a_t$. Then in 

both cases $n \neq m$, $t < \min(n, m)$; $|f(x)-f(y)| = |a_n-a_m| \leq |a_t|$ and 

$|f(x)-f(z)| = |a_n-a_t| = |a_t|$ and we are done.
On the other hand (how surprising is life!)

**THEOREM 5.6** Let $k$ be the field of two elements. Then $M_w(X) = M_b(X)$.

**Proof.** We prove that $|x - y| = |y - z|$ implies $|f(x) - f(y)| \leq |f(y) - f(z)|$

$(x \neq y, y \neq z, x, y, z \in X)$. There is $a \in K^*$ such that $|a(x - y)| = |a(y - z)| = 1$. So since $k = \mathbb{F}_2$, $\overline{a(x-y)} = \overline{a(y-z)} = 1$, whence $a(x - z) = 0$ or $|a(x - z)| < 1$. Thus, $|x - z| < |x - y| = |y - z|$. Since $f \in M_w(X)$, $|f(x) - f(z)| \leq \min(|f(x) - f(y)|, |f(y) - f(z)|)$. Consequently, $|f(x) - f(y)| \leq \max(|f(x) - f(z)|, |f(z) - f(y)|) \leq |f(y) - f(z)|$.

Of particular interest may be monotone functions mapping convex sets onto convex sets.

**THEOREM 5.7** Let $K$ be a local field, let $X$ be a bounded open convex set, and let $f : X \to X$ be surjective. Then the following are equivalent.

$(a) \ f \in M_b(X)$

$(b) \ f \in M^s(X)$

$(c) \ f \in M_{bs}(X)$

$(d) \ f$ is an isometry.

**Proof.** $(a) \Rightarrow (b)$. Since $f(X)$ has no isolated points, $f$ is a homeomorphism, by 5.1. Then $f \in M^s(X)$, by 5.3. $(b) \Rightarrow (c)$. $f^{-1} \in M_b(X)$.

We just have shown $(a) \Rightarrow (b)$, so $f^{-1} \in M^s(X)$ i.e., $f \in M_b(X)$.

$(c) \Rightarrow (d)$. From the proof of 5.4 we have seen that $|f(x) - f(y)| \leq M|x - y|$, where $M = \sup |f(x) - f(y)| = 1$. Hence $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in X$, but by the same token this also holds for $f^{-1}$. Then $f$ is an isometry. $(d) \Rightarrow (a)$ is obvious.
COROLLARY 5.8 Let $X$ be an open convex subset of $K$ and $f : X \to K$, not constant, $f(X)$ convex. Then the following conditions are equivalent.

(a) $f \in M_b(X)$

(b) $f \in M_s(X)$

(c) $f \in M_{bs}(X)$

($\delta$) $f$ is a scalar multiple of an isometry.

Proof. If $X$ is bounded then $f(X)$ is bounded and by a linear transformation we can arrange that $X = f(X)$. The equivalence follows easily. If $X$ is unbounded, then $X = K$, and from 5.1 (iii) we get $f(X)$ is unbounded, so $f(X) = K$. The equivalence of (a) (b) (c) is now easy. To prove (c) $\Rightarrow$ (d) we may assume $f(0) = 0$, $f(1) = 1$. Let $X_n := \{x \in K : |x| \leq n\}$. Then $f(X_n)$ is convex for each $n \in \mathbb{N}$, so there is $c_n$ such that $|f(x) - f(y)| = c_n |x-y|$ $(x, y \in X_n)$. By substituting $x = 1$, $y = 0$ we see that $c_n = 1$.

Similar to what we did in example 3.3, (3) we try to express the condition $f \in M_{ubs}(Z_p)$ into conditions on the coefficients of $f$ with respect to the orthonormal base $e_0, e_1, \ldots$ of $C(Z_p)$. So let the notations be as in 3.3(3), and suppose first $f \in M_{ubs}(Z_p)$ i.e.

$|x-y| = |s-t| \Rightarrow |f(x) - f(y)| = |f(s) - f(t)|$. Let $n, m \in \mathbb{N}$. If $|n-n_-| = |m-m_-|$ then $|f(n) - f(n_-)| = |f(m) - f(m_-)|$, so if we write $f = \sum_{n} a_n e_n$ we find $|\lambda_n| = |\lambda_m|$. Let $n = a_0 + a_1 p + \ldots + a_k p^k$ $(a_k \neq 0)$ then $|n-n_-| = p^{-k}$ where $k = \frac{\log n}{\log p}$. We find

if $\left[ \frac{\log n}{\log p} \right] < \left[ \frac{\log m}{\log p} \right]$ then $|\lambda_n| > |\lambda_m|$

if $\left[ \frac{\log n}{\log p} \right] = \left[ \frac{\log m}{\log p} \right]$ then $|\lambda_n| = |\lambda_m|$. 


Moreover, if \( \left\lfloor \frac{\log n}{\log p} \right\rfloor = k \) and \( n-m \) is invisible by \( p^k \) i.e., \( n_\sim = m_\sim \) then \( |f(n)-f(m)| = |\lambda_n - \lambda_m| \). If \( n > m \) then \( |f(n-m)-f(0)| = |\lambda_{n-m}| = |\lambda_n| \).

We have found the first half of

**THEOREM 5.9** Let \( f = \sum \lambda_n e_n \in C(\mathbb{Z}_p) \). In order that \( f \in M_{ubs}(\mathbb{Z}_p) \) it is necessary and sufficient that condition (*) below holds

\[
(*) \quad |\lambda_n| \text{ is a strictly decreasing function of } \left\lfloor \frac{\log n}{\log p} \right\rfloor \quad (n \in \mathbb{N})
\]

\[
\left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor, \quad n \neq m, \quad n_\sim = m_\sim \implies |\lambda_n - \lambda_m| = |\lambda_m| \quad (n, m \in \mathbb{N}).
\]

We have shown \( f \in M_{ubs}(\mathbb{Z}_p) \implies (*) \). Now suppose (*) and let \( |x-y| = p^{-k} \). We show that \( |f(x)-f(y)| = p^{-k} \). Now \( e_n(x) = e_n(y) \) for \( n < p^k \), so

\[
f(x)-f(y) = \sum_{n \geq p^k} \lambda_n \left( e_n(x) - e_n(y) \right).
\]

Set

\[
x := a_0 + a_1 p + \ldots + a_k p^k + a_{k+1} p^{k+1} + \ldots
\]

\[
y := a_0 + b_1 p + \ldots + b_k p^k + b_{k+1} p^{k+1} + \ldots
\]

Now either \( a_k \) or \( b_k \) is \( \neq 0 \), say, \( a_k \neq 0 \). If \( b_k = 0 \) then by (*)

\[
\left| \sum_{n \geq p^k} \lambda_n e_n(x) \right| < \lambda_k p^k = \left| \sum_{n \geq p^k} \lambda_n e_n(x) \right|, \quad \text{so } |f(x)-f(y)| = |\lambda_k|.
\]

If \( b_k \neq 0 \) then by (*)

\[
|\lambda_k| = |\lambda_k - \lambda_{p^k}| = |f(x)-f(y)|.
\]

**Note.** Using similar methods, we can prove: \( f = \sum \lambda_n e_n \) is in \( M_{ubs}(\mathbb{Z}_p) \)

if and only if we have \((**)\) for all \( n, m \in \mathbb{N} \):
If we assume only that $K$ has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let $X$ be the unit ball of a discretely valued field. Let $f : X \to X$ be surjective, $f \in M_{bs}(X)$. Then $f$ is an isometry.

**Proof.** It is clear from previous theory that $f$ is a homeomorphism of the unit ball. It suffices to show that $|f(x)-f(y)| \leq |x-y|$ for all $x,y \in X$. (Apply this result also for $f^{-1}$. Then $f$ is an isometry.)

Let $\pi \in K$, $|\pi| < 1$, be a generator of $|K^*|$. We prove by induction

\[
\text{if } |x| = |\pi|^n \text{ then } |f(x)-f(0)| \leq |\pi|^n |f(1)-f(0)|.
\]

For $n = 0$ this is clear. ($|x-0| \leq |1-0|$, so $|f(x)-f(0)| \leq |f(1)-f(0)|$).

Suppose the statement is true for $n-1$. Let $|x| = |\pi|^{n-1}$. Then

\[
|x-0| < |\pi|^{n-1}-0|, \text{so } |f(x)-f(0)| \leq |f(\pi^{n-1})-f(0)| \leq |\pi|^{n-1} |f(1)-f(0)|,
\]

so $|f(x)-f(0)| \leq |\pi|^n |f(1)-f(0)|$ and we are done. (In fact, we have shown that a bounded $M_{bs}$-function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let $K$ have discrete valuation and let $f \in M_b(X)$. Then the following conditions are equivalent.

(a) $f(X)$ has no isolated points.

(b) $f$ is injective and continuous.

(c) $f$ is a homeomorphism $X \sim f(X)$. 
Proof. (a) \(\Rightarrow\) (\(\gamma\)) is 4.9(ii). (\(\gamma\)) \(\Rightarrow\) (\(\beta\)) is clear. (\(\beta\)) \(\Rightarrow\) (\(\gamma\)): if \(f(a)\) were an isolated point of \(f(X)\), then \(\{x : f(x) = f(a)\}\) is open in \(X\). Since \(f\) is injective \(\{a\}\) is open. But \(X\) has no isolated points. Contradiction.

To show that 5.11 may not be true if \(K\) has a dense valuation we construct

\textbf{EXAMPLE 5.12} Let \(|K| = [0,\infty)\). Then we construct an \(M_{\mathbb{S}}\)-homeomorphism sending

\[
\{x \in K : \beta < |x| \leq 1\} \text{ onto } \{x \in K : 0 < |x| \leq 1\}.
\]

Proof. Let \(\phi : [\beta, 1] \rightarrow [0, 1]\) be the map \(x \mapsto 2(x-\beta)\) (\(x \in (\beta, 1]\)). For each \(\nu \in (\beta, 1]\), choose \(\beta_{\nu} \in K\) such that \(|\beta_{\nu}| = \frac{\phi(\nu)}{\nu}\). Define \(f : \{x \in K : \beta < |x| \leq 1\} \rightarrow \{x \in K : 0 < |x| \leq 1\}\) as follows

\[
f(x) = \beta_{|x|} x \quad (\beta < |x| \leq 1)
\]

Clearly, \(|f(x)| = |\beta_{|x|}| \cdot |x| = \phi(|x|) \in (0, 1]|. The inverse of \(f\) is given by \(y \mapsto \beta_{|f^{-1}(|y|)}^{-1}\), so \(f\) is a bijection. Since \(f^{-1}\) can be defined in the same way as \(f\) (only with \(\phi^{-1}\) instead of \(\phi\)) it suffices to show that \(f \in M_{\mathbb{S}}\). Let \(|x-y| < |x-z|\).

Suppose \(|x| > |z|\). Then \(|x-z| = |x|\) and \(|y| = \max(|x-y|, |x|) = |x|\) Then \(\beta_{|x|} = \beta_{|y|}\), so \(|f(x)-f(y)| = \beta_{|x|} |x-y|\) and \(|f(x)-f(z)| = |f(x)| = \beta_{|x|} |x-z|\), so we are done in this case. Suppose \(|x| < |z|\).

Then \(|x-z| = |z|\) and \(|y| = \max(|x-y|, |x|) < |z|\). Then \(|f(x)-f(y)| \leq \max(|f(x)|, |f(y)|) < |f(z)| = |f(z)-f(x)|\).

Suppose \(|x| = |z|\). Then \(|y| \geq \max(|x-y|, |x|) \leq |x|\); if \(|y|\) were < \(|x|\) then \(|x-y| = |x| = |z| < |x-z|\), a contradiction, so \(|y| = |x| = |z|\), and \(|f(x)-f(y)| = \beta_{|x|} |x-y|\), \(|f(x)-f(z)| = \beta_{|x|} |x-z|\) whence

\(|f(x)-f(y)| < |f(x)-f(z)|\).
EXAMPLE 5.13 Extend \( f \) to a surjection \( g \) of \( \{ x \in K : |x| \leq 1 \} \) onto itself by defining \( g(x) = 0 \) if \( |x| \leq \frac{1}{2} \). We claim that \( g \in M_b \). Let \( |x-y| \leq |x-z| \). To check whether \( |g(x)-g(y)| \leq |g(x)-g(z)| \) we only have to consider the cases \( |x| \leq \frac{1}{2} \) and \( |y| > \frac{1}{2} \) and \( |x| > \frac{1}{2} \) and \( |y| \leq \frac{1}{2} \). In the first case, \( |x-y| = |y| \leq |x-z| \), so \( |z| = \max(|z-x|,|x|) = |z-x| \geq |y| \). Then \( |g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)| \). In the second case \( |g(x)-g(y)| = |f(x)| \). If \( |x| < |z| \) then \( |f(x)| < |f(z)| = |f(z)-f(x)| = |g(z)-g(x)| \). If \( |x| > |z| \) then \( |f(x)| = |g(x)-g(z)| \).

If \( |x| = |z| \) then \( |f(x)-f(z)| = \beta |x-z| \geq \beta \cdot |x-y| = \beta |x| = |f(x)| \).

Thus we have found a continuous surjection \( g : \{ x \in K : |x| \leq 1 \} \rightarrow \{ x \in K : |x| \leq 1 \}, \ g \in M_b \), such that \( g = 0 \) on \( \{ x : |x| \leq \frac{1}{2} \} \). (Compare 5.11).

EXAMPLE 5.14 Let \( h : \{ x \in K : |x| \leq 1 \} \rightarrow K \) be defined as

\[
h(x) = \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \quad (f \text{ as in 5.12}) \\ 0 & \text{if } x = 0. \end{cases}
\]

Then \( h \) is a non-continuous \( M_{bs} \)-function.

Proof. That \( h \) is not continuous at 0 is clear. Further, \( h \), restricted \( \{ x : 0 < |x| \leq 1 \} \) is in \( M_{bs} \) (see 5.12). Further, since \( g \circ h \) is the identity (\( g \) as in 5.12), we see that \( h = M_s \). It suffices to show that

\[ |x-y| = |x-z| \text{ implies } |h(x)-h(y)| = |h(x)-h(z)| \text{ in case } 0 \in \{ x,y,z \}. \]

We may suppose \( x \neq y, y \neq z, x \neq z \). Let \( x = 0 \). Then \( |y| = |z| \), so

\[ |f^{-1}(y)| = |f^{-1}(z)| \text{ i.e., } |h(x)-h(y)| = |h(x)-h(z)|. \]

Now let \( y = 0 \). Then \( |x| = |x-z| \). Choose \( 0 < |t| \leq 1 \) such that \( |t| < |x| \). Then

\[ |x-t| = |x-z| \text{ so } |f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x)-f^{-1}(z)| \text{ i.e., } \]

\[ |h(x)| = |h(x)-h(z)|, \text{ and we are done.} \]
6. FUNCTIONS OF BOUNDED VARIATION

In this section X is the unit ball of K, and $\text{BA}(X) := \{ f : X \to K : \sup_{x \neq y} \left| \frac{f(x)-f(y)}{x-y} \right| < \infty \}$. Let us define

$$\|f\|_\Delta := \sup \left\{ \frac{|f(x)-f(y)|}{|x-y|} : x, y \in X, x \neq y \right\} (f \in \text{BA}(X)).$$

It will turn out that, in a natural way, $\text{BA}(X)$ can be regarded as the space of functions of bounded variation, and that $\|\|_\Delta$ plays the role of the total variation.

**THEOREM 6.1** Let $f : X \to K$. Then the following are equivalent

(a) $f \in \text{BA}(X)$.

(b) $f$ is a linear combination of two increasing functions.

If $|K|$ is discrete (a), (b) are equivalent to

(c) $f$ is the difference of two bounded monotone functions of some type $\sigma$.

(d) $f \in [M_{\text{BS}}(X)]$.

If $K$ is a local field then (a)-(d) are equivalent to

(e) $f \in [M_{B}(X)]$.

(f) $f \in [M_{S}(X)]$.

**Proof.** We only prove (a) $\Rightarrow$ (b). The rest follows from (5.10), (5.4).

So let $f \in \text{BA}(X)$ and choose $\lambda \in X$ such that $|f(x)-f(y)| < |\lambda| |x-y|$ $(x, y \in X, x \neq y)$. Then $\lambda^{-1}f$ is a pseudocontraction, $f(x) = \lambda x + \lambda (\lambda^{-1}f(x)-x)$ $(x \in X)$, where $x \to x$ and $x \to \lambda^{-1}f(x)-x$ are increasing.

In the real case, we can define for a function $[0,1] \to \mathbb{R}$, of bounded variation
$V(f) := \inf \{ \text{Var } g + \text{Var } h : f = g+h, g,h \text{ monotone} \}.$

It is an easy exercise to show that $f \mapsto V(f)$ is a seminorm on the space of all functions of bounded variation and that $V$ is equivalent to the total variation $\text{Var}$, defined via

$$\text{Var } f = \sup \{ \sum |f(x_k) - f(x_{k-1})| : x_0 < x_1 < \ldots < x_n \text{ is a partition of } [0,1] \}.$$ 

So in the non-archimedean situation we define for $f : X \to K$

$$J(f) = \sup \{|f(x) - f(y)| : x, y \in X\}.$$ 

(If $f$ is considered to be "monotone" then $J(f)$ can be interpreted as the "total variation" of $f$.) We are led to the following definitions for $f \in \mathcal{B}(X)$:

$$\text{Var } f := \inf \{ \max (J(g), J(h)) : f = g+h, g,h \text{ are scalar multiples of increasing functions} \}.$$ 

(If $|K|$ is discrete) $\text{Var}^f := \inf \{ \max (J(g), J(h)) : f = g+h, g,h \text{ are in } M_{\text{bs}}(X) \}.$

(If $K$ is local) $\text{Var}^f := \inf \{ \max (J(g), J(h)) : f = g+h : g,h \in M_{\text{g}}(X) \}.$

Let us first compare $\text{Var } f$ and $\|f\|_\Delta$. If $f = g+h$ and $g,h$ are scalar multiples of increasing functions we have for $x, y \in X, x \neq y$

$$\frac{|f(x) - f(y)|}{x-y} \leq \max \left( \frac{|g(x) - g(y)|}{x-y}, \frac{|h(x) - h(y)|}{x-y} \right) \leq \max (J(g), J(h))$$

so $\|f\|_\Delta \leq \text{Var } f$. Conversely, if $|\lambda| > \sup \left| \frac{f(x) - f(y)}{x-y} \right|$ then

$$f(x) = \lambda x + \lambda^{-1} f(x_0) \quad (x \in X)$$

whence

$$\text{Var } f \leq |\lambda|$$
So, if \(|K|\) is dense we have \(\Var f = \|f\|_{\Delta} (f \cdot B\Delta(X))\). Otherwise we have at least
\[
\|f\|_{\Delta} \leq \Var f \leq c \|f\|_{\Delta} \quad (f \in B\Delta(X))
\]
(where \(c\) is the smallest value \(> 1\)).

If \(|K|\) is discrete we clearly have \(\Var_1 f \leq \Var f\). Conversely, let \(f = g+h\), where \(g, h \in M_{bs}(X)\). It follows from the proof of 5.10 that
\[
\begin{align*}
|g(x) - g(y)| &\leq M|x - y| \\
|h(x) - h(y)| &\leq N|x - y|
\end{align*}
\]
where \(M = \sup |g(x) - g(y)| = J(g)\) and \(N = J(h)\).

So
\[
\left| \frac{f(x) - f(y)}{x - y} \right| \leq \max(J(g), J(h)), \text{ whence}
\]
\[
\|f\|_{\Delta} \leq \Var_1 f.
\]

Similar proofs work for \(\Var_2 f, \ Var_3 f\). We have

**Theorem 6.2** The seminorms \(\Var, \Var_1, \Var_2, \Var_3\) on \(B\Delta(X)\) (whenever defined) are all equivalent to \(\|\|_{\Delta}\).
REFERENCES


Report 7812, Mathematisch Instituut, Nijmegen, the Netherlands (1978), 1-129.