NON-ARCHIMEDEAN MONOTONE FUNCTIONS

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INTRODUCTION

In the sequel, $K$ is a non-archimedean valued field, complete, with residue class field $k$. Our aim is to present reasonable definitions for a function $f : X \to K$ to be "monotone". ($X$ is a subset of $K$). Since $K$ admits no ordering in the usual sense it is not possible to just formally take over the classical definitions. Thus, we consider statements for $f : \mathbb{R} \to \mathbb{R}$, equivalent to "$f$ is monotone", and such that these statements have translations in $K$ that make sense. This way we obtain several definitions of "$f : X \to K$ is monotone", which are closely related (although not equivalent).

In Section 1 we introduce notions that are needed later for defining monotony. Thus we define "convex subset of $K"", "the sign of a nonzero element of $K"."

In Section 2 we define several notions of monotony. E.g.,

$f \in M^D(X)$ if $x$ between $y$ and $z$ implies $f(x)$ between $f(y)$ and $f(z)$ and $f \in M^S(X)$ if $f(x)$ between $f(y)$ and $f(z)$ implies $x$ between $y$ and $z$.

Also monotone functions of type $\sigma$ are defined (a translation of the "type" of a real monotone function: decreasing/increasing). It turns out in the non-archimedean case we may have infinitely many "types" and that an $f \in M^D(X)$ (or $f \in M^S(X)$) is in general not of a certain "type".

Sections 3, 4, 5 deal with the properties of monotone functions.

In Section 6 it turns out that in reasonable situations the linear span of the space of the monotone functions is just the space of functions satisfying a Lipschitz condition.

The proofs are mostly quite elementary. For background-infor-
mation on non-archimedean (functional) analysis, see [1].

The connection of this theory with the other parts of non-archimedean analysis are yet not very tight. But we have 3.6 (relationship between spherical completeness of \( K \) and surjectivity of increasing functions) and 3.12 that is a translation of the classical theorem: \( f' > 0 \iff f \) increasing.

The notion of pseudo-ordening ("at the same side of") goes back to an idea of Monna ([3], Chapter VII, 4.7).

**Notations.** Let \( p \) be a prime. By \( \mathbb{F}_p \) we mean the field of \( p \) elements. By \( \mathbb{Q}_p \) the non-archimedean valued field of the \( p \)-adic numbers. For a field \( L \) we denote its characteristic by \( \chi(L) \). Let \( E \) be a vector space over \( K \) and \( S \subseteq E \). By \([S]\) we denote the smallest \( K \)-linear subspace of \( E \) that contains \( S \).
1. SUBSTITUTES FOR ORDERING

DEFINITION 1.1 Let \( x, y \in K \). Then the smallest ball in \( K \) containing \( x \) and \( y \) is denoted by \( [x, y] \). A subset \( C \) of \( K \) is called convex if \( x, y \in C \) implies \( [x, y] \subseteq C \).

Sometimes we use a more geometric terminology. Instead of \( z \in [x, y] \) we will say that \( z \) is between \( x \) and \( y \) and instead of \( z \notin [x, y] \) we use the expression: \( x \) and \( y \) are at the same side of \( z \).

Notice that \( [x, y] = [y, x] \) for all \( x, y \in K \) and that \( z \in [x, y] \iff |z-x| \leq |x-y| \iff |z-y| \leq |x-y| \iff z = \lambda x + (1-\lambda)y \) for some \( \lambda \in K \), \( |\lambda| \leq 1 \). If \( x \neq y \) then the \( \lambda \) in this last expression is unique (viz. \( \lambda = \frac{z-y}{x-y} \)).

Examples of convex sets are: the empty set, singletons, balls, \( K \). It is an easy exercise to show that these are the only convex subsets of \( K \). So formally we may write each convex subset of \( K \) as:

\[
\left\{ x \in K : |x-a| < r \right\} \quad (a \in K, 0 \leq r \leq \infty)
\]

or as

\[
\left\{ x \in K : |x-a| \leq r \right\} \quad (a \in K, 0 \leq r \leq \infty)
\]

Notice that the only unbounded convex subset of \( K \) is \( K \) itself.

Sometimes we need the notion of convexity with respect to a subset \( X \) of \( K \). A subset \( C \subseteq X \) is called convex in \( X \) if \( x, y \in C \) implies \( [x, y] \cap X \subseteq C \) or, equivalently, if \( C \) is the intersection of \( X \) with a convex subset of \( K \).

Let \( x, y, z \in K \). By the strong triangle inequality we have that the "triangle" \( x, y, z \) is isosceles, say \( |x-y| = |y-z| \). Then \( |x-z| \leq |x-y| \), so \( z \) is between \( x \) and \( y \) and \( x \) is between \( y \) and \( z \). If also \( |x-y| = |x-z| \)
then $y$ is between $x$ and $z$. Otherwise, $x$ and $z$ are at the same side of $y$.

The relation $\sim$ defined on $\mathbb{K}^* := \mathbb{K}\setminus\{0\}$ by

$$x \sim y \text{ if } x \text{ and } y \text{ are at the same side of } 0 \quad (x, y \in \mathbb{K}^*)$$

is an equivalence relation. We have $x \sim y$ iff $0 \not\in [x, y]$ i.e. iff $|x-y| < |x| (= |y|)$ i.e. iff $|xy^{-1}| < 1$. Define

$$\mathbb{K}^+ := \{ x \in \mathbb{K} : |1-x| < 1 \}$$

Then $\mathbb{K}^+$ is a multiplicative subgroup of $\mathbb{K}^*$, $\mathbb{K}^+ = \{ x \in \mathbb{K}^* : x \sim 1 \}$
and is called the set of the positive elements of $\mathbb{K}$. The relation $\sim$ is also induced by the canonical group homomorphism

$$\pi : \mathbb{K}^* \rightarrow \mathbb{K}^*/\mathbb{K}^+.$$ Thus, $x \sim y$ if and only if $\pi(x) = \pi(y)$ $(x, y \in \mathbb{K}^*)$. Therefore it is natural to view the group $\Sigma := \mathbb{K}^*/\mathbb{K}^+$ as being the group of signs of elements of $\mathbb{K}^*$, and we call $\pi(x)$ the sign of the element $x \in \mathbb{K}^*$. If $x \in \mathbb{K}^*$ then $\pi(x) = \{ y : |y-x| < |x| \} = x \mathbb{K}^+$. For $x \in \mathbb{K}^*$, $a \in \Sigma$ we sometimes write $xa$ to indicate the element $\pi(x).a$ of $\Sigma$. In particular, for $a \in \Sigma$ the opposite sign of $a$, $-a$, is defined as $(-1)a$. Then $-a = \{-x : x \in a\}$. (Notice that in case $\chi(\mathbb{K}) = 2$ we have $a = -a$.)

Let $a \in \Sigma$. Then for $x, y \in a$ we have $|x| = |y|$ so we can define the absolute value of $a$, $|a|$ as follows

$$|a| := |x| \quad (x \in \pi^{-1}(a)).$$ In the sequel we also need addition between elements of $\Sigma$. Let us first observe that for any $a, \beta \in \Sigma$ the sum

$$a+\beta := \{ x+y : x \in a, \ y \in \beta \}$$
is always a ball with radius $\max(|a|, |\beta|)$. (I.e., of the form
\{x : |x-b| < \max(|a|,|b|)\}. Now \(a+\beta\) contains 0 if and only if 
\(\alpha = -\beta\). Otherwise \(a+\beta\) is again an element of \(\Sigma\). (Proof: Let \(a \in \alpha\),  
\(b \in \beta\). Then \(|a+b| = \max(|a|,|b|)\). If also \(x \in \alpha\), \(y \in \beta\) then \(|x+y-(a+b)| \leq  
\max(|x-a|,|y-b|) < \max(|a|,|b|) = |a+b|\). Thus \(a+\beta\) contains the ball 
with center \(a+b\) and radius \(\max(|a|,|\beta|)\), so \(a+\beta\) is equal to this 
ball.)

Let us define

\[\alpha \oplus \beta := a+\beta = \{x+y : x \in \alpha, y \in \beta\} \quad (\alpha, \beta \in \Sigma, \alpha \neq -\beta)\]

We have

THEOREM 1.2 Let \(\Sigma, \|\| : \Sigma \to \mathbb{R}, \oplus : \Sigma \times \Sigma \setminus \{(\alpha, -\alpha) : \alpha \in \Sigma\} \to \Sigma\) be as 
above. Let \(\alpha, \beta, \gamma \in \Sigma\). Then

(i) \(|a\beta| = |a| |\beta|, |a^{-1}| = |a|^{-1}\).

(ii) If \(\alpha \oplus \beta\) is defined then so is \(\beta \oplus \alpha\) and \(\alpha \oplus \beta = \beta \oplus \alpha\).

(iii) If \((\alpha \oplus \beta) \oplus \gamma\) and \(\alpha \oplus (\beta \oplus \gamma)\) are defined then 
\((\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)\).

(iv) If \(\alpha \oplus \beta\) or \(\gamma \alpha \oplus \gamma \beta\) is defined then so is the other 
and \(\gamma(\alpha \oplus \beta) = \gamma \alpha \oplus \gamma \beta\).

(v) If \(\alpha \oplus \beta\) is defined then \(|\alpha \oplus \beta| = \max(|a|,|\beta|)\). Converse-
ely if \(|s| = \max(|a|,|\beta|)\) for some \(s \in a+\beta\) then \(\alpha \oplus \beta\) 
is defined.

(vi) \(|a| < |\beta|\) if and only if \(\alpha \oplus \beta = \beta\).

(vii) Let \(n \in \{1, 2, \ldots, \chi(k)-1\}\) if \(\chi(k) \neq 0\), let \(n \in \mathbb{N}\) other-
wise. Then we define \(\oplus_n \alpha\) inductively as follows.

\[\oplus_1 \alpha = \alpha, \oplus_k \alpha := \oplus_{k-1} \alpha \oplus \alpha (k \leq n)\]. Then 
\[\oplus_n \alpha = n\alpha\].

Proof. (i), (ii) are clear. (iii) is almost trivial: if \(x \in \alpha, y \in \beta,\)  
\(z \in \gamma\) then \(x+y+z \in a+\beta+\gamma\) and the latter set can be regarded as
(a © β) © γ or as a © (β © γ). (It is worth noticing that (a © β) © γ may be defined whereas a © (β © γ) is not. Choose β = −γ and |α| > |β|. Then (a © β) © γ = a © γ = α, β © γ is not defined.)

(iv) is clear. If a © β is defined then for x © α, y © β we have |x+y| ≥ max(|x|, |y|) whence |x+y| = max(|x|, |y|). So |α © β| = max(|α|, |β|). Conversely, if α © β is not defined, then (we saw earlier that) α+β is a ball with center zero and radius max(|α|, |β|).

Thus we have (v). We prove (vi). If |α| < |β| then α+β = β so α © β = β. Conversely, if α © β = β then choose α © α, β © β. Then α+β © β hence a+b © b i.e., ab−1+1 ∈ K° implying |ab−1| < 1 or |α| < |β|. Hence |α| < |β|. (Note: from (vi) it follows that α © β = α' © β does not imply α = α'). To prove (vii) let α © α and observe that for any k ≤ n, if α © α is defined, (k−1)α © α. Hence |(k−1)α+α| = |kα| = |α| = |α|,

so © α+α does not contain 0, hence © α © α is defined.

Now nα is by definition π(n)α. So nα © nα and nα © α. Since both nα and © α are signs they must coincide.

We now define relations that resemble "ordering".

**DEFINITION 1.3** Let α © I and x, y © K. Then we say that x is greater than y in the sense of α, notation x > α y, if x−y © α.

We have the following rules

**THEOREM 1.4** (i) If x, y © K, x ≠ y then there is exactly one α © I for which x > α y.

(ii) x > α x for no α.

(iii) If x > α y then for all s © K: x+s > α y+s (x, y © K, α © I)

(iv) If x > α y and s > β 0 then xs > β ys (x, y, s © K, α, β © I)
If $x > y$, $y > z$ and if $\alpha \oplus \beta$ is defined then $x > \alpha \oplus \beta z$.

**Proof. Easy.**

The group $\Sigma_1 := \{ \alpha \in \Sigma : |\alpha| = 1 \}$ is a subgroup of $\Sigma$, isomorphic to multiplicative group $k^*$. If $K$ has discrete valuation and if $s \in K$ and $|s|$ is the largest value that is smaller than 1, then for each $\alpha \in \Sigma$ there is $x \in \mathbb{Z}$ such that $\alpha = s^n \alpha_1$ where $\alpha_1 \in \Sigma_1$. It follows easily that the map $(n, \alpha) \mapsto s^n \alpha$ ($n \in \mathbb{Z}$, $\alpha \in \Sigma_1$) is an isomorphism of $\mathbb{Z} \times \Sigma_1$ onto $\Sigma$. Thus, in case $K$ has discrete valuation, $\Sigma$ is isomorphic to $\mathbb{Z} \times \Sigma_1$, or, for that matter, to $|K^*| \times k^*$.

If $K$ is a local field we can even define a group embedding $\rho : \Sigma \to K^*$ such that $\pi \rho$ is the identity. (Thus, we can pick an element in every $\alpha$ ($\alpha \in \Sigma$) such that the resulting set is a subgroup of $K^*$). Let $s \in K$, $|s| < 1$ such that $|s|$ generates the value group and let $q$ be the cardinality of $k$. Let $x \in K$. Then there is a unique $n \in \mathbb{Z}$ such that $x = s^n x_1$ where $|x_1| = 1$.

Define

$$v(x) = s^n \lim_{n \to \infty} x_1^n$$

It is easy to verify that $v$ is a homomorphism of $K^*$ into $K^*$, that $\pi(v(x)) = \pi(x)$ for all $x \in K^*$ and that $v(x) = 1$ if and only if $x \in K^*$. Therefore the map $\rho$ making the diagram

$$
\begin{array}{ccc}
K^* & \overset{v}{\longrightarrow} & K^* \\
\pi \downarrow & & \downarrow \rho \\
\Sigma & &
\end{array}
$$

commute solves the problem.

**EXAMPLE 15** The signs of $\mathcal{O}_p$. Let $\theta$ be a primitive $(p-1)^{th}$ root of
unity. Then \( \{ \Theta_p^n : i \in \{0,1,\ldots,p-2\}, n \in \mathbb{Z} \} \) is a subgroup of \( \mathbb{Q}_p^* \) isomorphic to \( \mathbb{Z} \). If

\[
x = \sum_{k \geq n} a_k p^k \quad (a_k \in \{0,1,\ldots,p^{p-2}\}, a_n \neq 0)
\]

is an element of \( \mathbb{Q}_p \), its sign, interpreted as an element of \( \mathbb{Q}_p \) is

\[
\pi(x) = a_n p^n.
\]
2. DEFINITIONS OF MONOTONE FUNCTIONS

For a function $f : [0,1] \rightarrow \mathbb{R}$ the following statements are equivalent.

(a) $f$ is monotone (i.e., either $x > y$ implies $f(x) \geq f(y)$ for all $x,y$ or $x > y$ implies $f(x) \leq f(y)$ for all $x,y$).

(b) If $x$ is between $y$ and $z$ then $f(x)$ is between $f(y)$ and $f(z)$ $(x,y,z \in [0,1])$.

(c) If $C \subset \mathbb{R}$ is convex then $f^{-1}(C)$ is convex.

Thus we define

**DEFINITION 2.1** Let $X \subset K$. We say that $f \in M_b(X)$ if for all $x,y,z \in X$, $x$ between $y$ and $z$ implies $f(x)$ is between $f(y)$ and $f(z)$. In other words, $f \in M_b(X)$ if and only if for all $x,y,z$

$$|x-y| \leq |y-z| \Rightarrow |f(x)-f(y)| \leq |f(y)-f(z)|.$$ 

In the following theorems we state some immediate consequences of the definition. (Compare the properties of real monotone functions.)

**THEOREM 2.2** Let $X \subset K$ and let $f : X \rightarrow K$. Then the following statements are equivalent

(a) $f \in M_b(X)$.

(b) For each convex $C \subset K$, $f^{-1}(C)$ is convex in $X$.

(c) For all $x,y,z \in X$: $|x-y| = |x-z| \Rightarrow |f(x)-f(y)| = |f(x)-f(z)|$.

(d) For all $x,y,z \in X$: $|f(x)-f(y)| > |f(x)-f(z)| \Rightarrow |x-y| > |x-z|$.

(e) For all $x,y,z \in X$: $|f(x)-f(y)| \neq |f(x)-f(z)| \Rightarrow |x-y| \neq |x-z|$.
Proof. \((a) \Rightarrow (b)\). Let \(x, y \in f^{-1}(C)\) and let \(z \in [x, y] \cap X\). Then \(|z-x| \leq |x-y|\), so \(|f(z) - f(x)| \leq |f(x) - f(y)|\), i.e., \(f(z) \in [f(x), f(y)] \subset C\). Hence \(z \in f^{-1}(C)\).

\((b) \Rightarrow (a)\). Let \(x, y, z \in X\) and \(|x-y| \leq |x-z|\). The set \([f(x), f(z)]\) is convex, hence \(f^{-1}([f(x), f(z)])\) is convex in \(X\) and contains \(x\) and \(z\), so it must contain \(y\). Thus \(f(y) \in [f(x), f(z)]\).

Clearly, \((a) \Leftrightarrow (b)\) and \((y) \Leftrightarrow (e)\). We prove \((a) \Rightarrow (y)\). Now \((a) \Rightarrow (y)\) is clear by symmetry. Suppose \((y)\) and let \(|x-y| \leq |x-z|\). It suffices to consider the case \(|x-y| < |x-z|\). Then \(|y-z| = |x-z|\), so by \((y)\) we have \(|f(y) - f(z)| = |f(x) - f(z)|\). Then \(|f(x) - f(y)| \leq \max(|f(x) - f(z)|, |f(z) - f(y)|) = |f(x) - f(z)|\).

THEOREM 2.3 Let \(X \subset X\). Then

\(\text{(i) For each } a, b \in K \text{ the map } x \mapsto ax+b \text{ is in } M^b_b(X).\)

\(\text{(ii) If } f \in M^b_b(X), \lambda \in K \text{ then } \lambda f \in M^b_b(X).\)

\(\text{(iii) } M^b_b(X) \text{ is closed under pointwise limits.}\)

\(\text{(iv) If } f \in M^b_b(X) \text{ and } g : f(X) \to K \text{ is in } M^b_b(f(X)), \text{ then } g \circ f \in M^b_b(X).\)

\(\text{(v) If } f \in M^b_b(X) \text{ and } f(a) = f(b) \text{ for some } a, b \in X, \text{ then } f \text{ is constant on } [a, b] \cap X.\)

Proof. Obvious.

2.4 EXAMPLES AND REMARKS.

We mention a few examples of \(M^b_b\)-functions. For more, see the sequel.

(1) The constant functions.

(2) Isometries (e.g., exp.).

(3) Choose in every \(x \in \Sigma\) an element \(x_\alpha\). Define \(\phi : K \to K\) as follows

\[
\phi(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\alpha x & \text{if } x \neq 0 
\end{cases} \quad (\alpha \in \Sigma)
\]
Essentially, $\phi|K^*$ is the sign function $\pi$ of section 1).

We prove that $\phi \in M_b(K)$. Since $\phi$ is continuous it suffices to check that $\phi|K^*$ is in $M_b(K^*)$. Now for all $x,y \in K^*$ we have $\phi(x) - \phi(y) = 0$ if $|xy^{-1} - 1| < 1$ and $|\phi(x) - \phi(y)| = |x - y|$ if $|x - y| = \max(|x|, |y|)$. Now take $x, y, z \in K^*$ such that $|x - y| \leq |x - z|$. If $\phi(x) = \phi(z)$ then $|1 - x^{-1}y| \leq |1 - x^{-1}z| < 1$ so $\phi(x) = \phi(y)$.

If $\phi(x) \neq \phi(z)$ then $|\phi(x) - \phi(y)| \leq |x - y| \leq |x - z| = |\phi(x) - \phi(z)|$.

(4) Let $r > 0$ and choose in every ball $B$ of radius $r$ a center $x_B$.

The function defined via

$$\psi(x) = x_B \quad (x \in B)$$

is in $M_b(K)$. The proof is easy.

(5) (A nowhere continuous $M_b$-function). Let $K$ be a field such that $\#K = \#k$ (e.g., a discretely valued field where $\#k$ has the power of the continuum). Let $\sigma : K \twoheadrightarrow k$ be a bijection and let $\tau : k \rightarrow K$ such that $|\tau x - \tau y| = 1$ whenever $x \neq y$. Then $f : \tau \circ \sigma$ satisfies: $|f(x) - f(y)| = 1$ ($x, y \in K, x \neq y$).

Clearly $f$ is everywhere discontinuous, $f \in M_b(K)$.

(6) Let $X \subseteq K$. We can strengthen the definition of an $M_b$-function into

if $|x - y| \leq |z - t|$ then $|f(x) - f(y)| \leq |f(z) - f(t)| \quad (x, y, z, t \in X)$

(some "uniform" $M_b$-condition) and we obtain a space, called $M_{ub}(X)$.

Clearly, the examples mentioned in (1), (2), (4), (5) are in $M_{ub}(K)$, whereas the example in (3) is not. (Choose $x, y \in K$ with $|x| > 1$,

$|x - y| = 1$. Then $|1 - 0| \leq |x - y|$, but $1 = |\phi(1) - \phi(0)| > |\phi(x) - \phi(y)| = 0$.)

Notice that $\phi$ is locally constant on $K^*$, but not on $K$.

(7) The discontinuous function $f$ of (5) has the property that $f(K)$ consists only of isolated points. This is not accidental. If $f \in M_b(K)$
if \( f(a) \) is a non-isolated point of \( f(K) \) and \( B \) is a ball containing \( f(a) \) then \( f^{-1}(B) \) is convex (2.2 (d)) and contains at least two points, so \( f^{-1}(B) \) is an open set. Thus, \( f \) is continuous at \( a \). It follows that the image of an everywhere discontinuous \( M_b \)-function consists only of isolated points.

(8) For each \( n \), let \( \sigma_n : K \to K \) be the example of 2.4, (4) above where \( r = \frac{1}{n} \). Then \( \sigma_n \) is locally constant. For each \( f \in M_b(K) \) we have

\[
\sigma_n \circ f \in M_b(K) \quad \text{and} \quad \lim_{n \to \infty} \sigma_n \circ f = f \quad \text{uniformly. Hence, if} \ f \ \text{is continuous then it can uniformly be approximated by locally constant} \ M_b \text{-functions.}
\]

A monotone function \( f : [0,1] \to \mathbb{R} \) maps convex sets into sets that are relatively convex in \( f([0,1]) \). In our situation we do not have such a property for \( M_b \)-functions. See 2.10 but also 5.2. If \( f : [0,1] \to \mathbb{R} \) maps convex sets into (relatively) convex sets then \( f \) need not be monotone: any Darboux continuous function has the above property. In fact a function \( f : [0,1] \to \mathbb{R} \) is Darboux continuous if and only if \( f \) maps convex sets into convex subsets of \( \mathbb{R} \). We define

**DEFINITION 2.5** Let \( X \) be a subset of \( K \), and let \( f : X \to K \). Then \( f \) is called weakly Darboux continuous if for every relatively convex set \( C \subseteq X \) the set \( f(C) \) is convex in \( f(X) \).\[ f \text{ is called Darboux continuous if for every relatively convex set } C \subseteq X \text{ the set } f(C) \text{ is convex (in } K) \text{.}\]

We have the following obvious remarks.

1) \( f : X \to K \) is Darboux continuous if and only if \( f \) is weakly Darboux continuous and \( f(X) \) is convex in \( K \).
2) A Darboux continuous function need not be continuous. In fact we can construct an \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) such that for every open ball \( B \subset \mathbb{Z}_p \),
\[
f(B) = \mathbb{Z}_p.
\]
Let \( A \subset \mathbb{Z}_p \) be defined as follows. \( x = \sum_{n=0}^{p-1} a_n p^n \) (\( a_n \in \{0,1,...,p-1\} \)) is in \( A \) if \( a_{2n} = a_{2n+2} = ... = 0 \) for some \( n \). Define \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) via
\[
f(x) = \begin{cases} a_{2n+1} + a_{2n+3} p + a_{2n+5} p^2 + ... & \text{if } x \in A \text{ and } N = \min \{n : a_{2n} = a_{2n+2} = ... = 0\} \\
0 & \text{if } x \notin A
\end{cases}
\]
Then \( f \) maps every non empty open set onto \( \mathbb{Z}_p \) (so \( f \) is Darboux continuous) but \( f \) is nowhere continuous.
(Constructions, similar to the one above are well known in the real case).

3) Somewhat more surprising: A continuous function need not be Darboux continuous. In fact, a non trivial locally constant function on \( \mathbb{Z}_p \) is not Darboux continuous.

Even, a continuous function need not be weakly Darboux continuous.
To see this, observe that all open compact subsets of \( \mathbb{Z}_p \) are homeomorphic, so we can make a homeomorphism of \( \mathbb{Z}_p \) onto \( \mathbb{Z}_p \) sending
\[
\{x : |x| < 1\} \to \{x : |x| = 1\} \text{ and } \{x : |x| = 1\} \to \{x : |x| < 1\}.
\]
If \( p \neq 2 \) this map is not weakly Darboux continuous.

4) A Darboux continuous \( M \)-function is continuous. (Proof. \( f(X) \) is convex, so either \( f \) is constant or \( f(X) \) has no isolated points. By the remark made in 2.4, (7), \( f \) is continuous.)

Next, we consider translations of "strict monotony". For an \( f : [0,1] \to \mathbb{R} \) the following conditions are equivalent.

(a) \( f \) is strictly monotone (i.e., injective and monotone).

(β) \( f \) is injective. For a convex set \( C \subset [0,1] \), \( f(C) \) is convex in \( f([0,1]) \).
(γ) For all $x, y, z \in [0,1]$: if $f(x)$ is between $f(y)$ and $f(z)$ then $x$ is between $y$ and $z$.

(δ) For all $x, y, z \in [0,1]$: $f(x)$ is between $f(z)$ if and only if $x$ is between $y$ and $z$.

Translating (α) – (δ) into the non-archimedean situation we arrive at the following conditions. Let $X \subseteq K$ and $f : X \rightarrow K$

(α′) $f \in M^s_d(X)$ and $f$ is injective.

(β′) $f$ is weakly Darboux continuous and injective.

(γ′) for all $x, y, z \in X$, $|x - y| < |x - z|$ implies $|f(x) - f(y)| < |f(x) - f(z)|$.

(δ′) $f \in M^d_d(X)$ and $f$ satisfies (γ′).

It will turn out that the conditions (α′) – (γ′) although not equivalent are closely related. We start with (γ′):

**DEFINITION 2.6** Let $X \subseteq K$, $f : X \rightarrow K$. We say that $f \in M^s_s(X)$ if for all $x, y, z \in X$, $f(x) \in [f(y), f(z)]$ implies $x \in [y, z]$.

**THEOREM 2.8** Let $X \subseteq K$, $f : X \rightarrow K$. Then the following statements are equivalent:

(a) $f \in M^s_s(X)$.

(b) $f$ is injective and weakly Darboux continuous.

(c) $f$ is injective and $f^{-1} \in M^d_d(f(X))$.

(δ) For all $x, y, z \in X$ $|f(x) - f(y)| = |f(x) - f(z)| \Rightarrow |x - y| = |x - z|$.

(ε) For all $x, y, z \in X$ $|x - y| < |x - z|$ $\Rightarrow |f(x) - f(y)| < |f(x) - f(z)|$.

(ζ) For all $x, y, z \in X$ $|x - y| \neq |x - z|$ $\Rightarrow |f(x) - f(y)| \neq |f(x) - f(z)|$. 

Proof. The implications (a) $\implies$ (c) $\implies$ (z) $\implies$ (d) are clear from the definitions.

(d) $\implies$ (y): injectivity follows from $|f(x)-f(x)| = |f(x)-f(y)| + |x-x| = |x-y|$. Use 2.2.(y).

(y) $\implies$ (f): Let $g : f(X) \to X$ be the inverse of $f$. Let $C \subset X$ be convex in $X$. Then since $g \in M_b$, $g^{-1}(C)$ is convex in $f(X)$. But $g^{-1}(C) = f(C)$.

Finally, we prove (f) $\implies$ (a). Let $f(x) \in [f(y), f(z)]$. By (f) the set $f([y, z] \cap X)$ is convex in $f(X)$ and it contains $f(y), f(z)$, hence $f(x) \in [f(y), f(z)] \cap X \subset f([y, z] \cap X)$. Since $f$ is injective, $x \in [y, z] \cap X$ and we are done.

We also have (compare 2.3)

**Theorem 2.9** Let $X \subset K$. Then

(i) For $a, b \in K$, $a \neq 0$ the map $x \mapsto ax + b$ is in $M_s(X)$.

(ii) If $f \in M_s(X)$, $\lambda \in K$, $\lambda \neq 0$ then $\lambda f \in M_s(X)$.

(iii) If $f_1, f_2, \ldots \in M_s(X)$, $\lim f_n = f$ pointwise, $f$ injective then $f \in M_s(X)$.

(iv) If $f \in M_s(X)$, $g \in M_s(f(X))$ then $g \circ f \in M_s(X)$.

Proof. Obvious verifications.

Returning to our conditions (a') $\iff$ (d') we see that (b') is equivalent to (y'), that (a') means $f^{-1} \in M_s(f(X))$ and that (d') means $f \in M_b(X) \cap M_s(X)$.

Our $f$ of example 2.4 (5) is in $M_b$, injective but not in $M_s$. Its inverse yields an example of an $M_s$-function that is not in $M_b$. Thus, in general, we have neither one of the implications (a') $\iff$ (y'), (y') $\iff$ (a'), (b') $\iff$ (d'), (a') $\iff$ (d'). But our counterexample is
rather weird (f is nowhere continuous and the domain of $f^{-1}$ is discrete). We can do better.

**EXAMPLE 2.10** Let $K$ have discrete valuation and let $k$ be infinite. Then there exists a homeomorphism of the unit ball of $K$ that is in $M_*$ but not in $M_s$. (The inverse map is in $M_*$ but not in $M_*$.)

**Proof.** Set $X = \{a \in K : |a| \leq 1\}$ and let $R$ be a full set of representatives of the equivalence relation $x \sim y$ iff $|x-y| < 1$ in $X$. Then $R$ is infinite. Let $\pi \in K$ be such that $|\pi|$ is the largest value that is smaller than 1. The map

$$
(a_0, a_1, \ldots) \mapsto \sum_{n=0}^{\infty} a_n \pi^n \quad (a_i \in R \text{ for each } i)
$$

is a bijection of $R^\mathbb{N}$ onto $X$. We may suppose that $0 \in R$.

Since $R$ is infinite we can define injections

$$
\tau_1 : R \setminus \{0\} \to R
$$

$$
\tau_2 : R \to R
$$

such that $\text{im } \tau_1 \cap \text{im } \tau_2 = \emptyset$, $\text{im } \tau_1 \cup \text{im } \tau_2 = R$.

For $x = \sum_{n=0}^{\infty} a_n \pi^n \in X (a_n \in R \text{ for each } n)$ set

$$
f(x) := \begin{cases} 
\tau_1(a_0) + a_1 \pi + \ldots = x - a_0 + \tau_1(a_0) & \text{if } a_0 \neq 0 \\
\tau_2(a_1) + a_2 \pi + \ldots = \frac{x}{\pi} - a_1 + \tau_2(a_1) & \text{if } a_0 = 0
\end{cases}
$$

A simple inspection of the definition shows that $f$ is a bijection of $X$ onto $X$. If $a, b \in R$, $a \neq b$ then $|a\pi - b\pi| < |b\pi - a|$, whereas

$$
|f(a\pi) - f(b\pi)| = |\tau_2(a) - \tau_2(b)| = 1 \text{ and } |f(b\pi) - f(a)| = |\tau_2(b) - \tau_1(a)| = 1,
$$

so $f \notin M_*(X)$. Finally, let $x, y, z \in X$ and $|x-y| \leq |x-z|$. We prove that $|f(x) - f(y)| \leq |f(x) - f(z)|$. If $|f(x) - f(z)| = 1$ there is nothing to prove, so suppose $|f(x) - f(z)| < 1$. Set $x = \sum a_n \pi^n$, $y = \sum b_n \pi^n$, $z = \sum c_n \pi^n$. 

If \( a_0 = 0 \) then also \( c_0 = 0 \) and \( \tau_2(a_1) = \tau_2(c_1) \) so \( a_1 = c_1 \), hence \( |x-z| \leq |\pi|^2 \). Since \( |x-y| \leq |x-z| \) we have also \( b_0 = 0 \), \( b_1 = a_1 \).

So, \( f(x) - f(y) = \frac{x-y}{\pi} \), \( f(x) - f(z) = \frac{y-z}{\pi} \) whence \( |f(x) - f(y)| \leq |f(x) - f(z)| \).

If \( a_0 \neq 0 \) then \( \tau_1(a_0) = \tau_1(c_0) \) so \( a_0 = c_0 \). Then also \( c_0 = a_0 = b_0 \). Then \( f(x) - f(y) = x-y \), \( f(x) - f(z) = x-z \) whence \( |f(x) - f(y)| \leq |f(x) - f(z)| \).

Let \( X \subset K \). If \( f \in M_s(X) \) then \( f^{-1} \in M_b(f(X)) \). Conversely, if \( f \in M_b(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) then \( g \in M_s(f(X)) \). This "asymmetry" can be "solved" in two ways.

**DEFINITION 2.11** Let \( X \subset K \) and \( f : X \to K \). \( f \) is called **weakly monotone** (*\( f \in M_w(X) \*)) if for all \( x, y, z \in X \)

\[ |x-y| < |x-z| + |f(x)-f(y)| \leq |f(x)-f(z)| \]

\( f \) is called **strongly monotone** (*\( f \in M_{bs}(X) \*)) if

\[ f \in M_s(X) \cap M_b(X) \).

Clearly, \( f \in M_{bs}(X) \) if and only if \( f^{-1} \in M_{bs}(f(X)) \). Also, if \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \) we have

\[ g \in M_w(f(X)) \).

Obviously we have \( M_b(X) \cup M_s(X) \subset M_w(X) \) and we will see from the examples below that the inclusion may be strict. In section 4 we will study the properties of \( M_w \)-functions, not for the sake of \( M_w \) itself but in order to get results that are valid for \( M_b, M_s \) simultaneously.

The functions in \( M_{bs} \) behave reasonable and they may be viewed as the non-archimedean equivalents of strict monotone functions in the real case.
THEOREM 2.12 Let $X \subset K$ and $f : X \to K$. Then the following conditions are equivalent.

(a) $f \in M_{bs}(X)$.

(b) $f$ is injective and $C \mapsto f(C)$ is a 1-1 correspondence between the relatively convex subsets of $X$ and those of $f(X)$.

(c) For all $x, y, z \in X$, $|x-y| < |x-z| \iff |f(x)-f(y)| < |f(x)-f(z)|$.

(d) For all $x, y, z \in X$, $|x-y| = |x-z| \iff |f(x)-f(y)| = |f(x)-f(z)|$.

(e) For all $x, y, z \in X$, $|x-y| \leq |x-z| \iff |f(x)-f(y)| \leq |f(x)-f(z)|$.

(f) $f \in M_{s}(X)$, $f^{-1} \in M_{s}(f(X))$.

**Proof.** Clear from 2.2 and 2.8.

2.13 EXAMPLES AND REMARKS.

(1) (An $M_{w}$-function that is not in $M_{s} \cup M_{b}$.) Let $f : \mathbb{Z}_{p} \to \mathbb{Q}_{p}$ be any function, constant on the cosets of $\{x \in \mathbb{Z}_{p} : |x| < 1\}$. Then $f \in M_{w}(\mathbb{Z}_{p})$. Clearly $f \notin M_{s}(\mathbb{Z}_{p})$. $f \in M_{b}(\mathbb{Z}_{p})$ if and only if the points of $f(\mathbb{Z}_{p})$ are equidistant.

(2) (Continuity of monotone functions). Let $X \subset K$.

(a) Let $f \in M_{w}(X)$. If $f(X)$ has no isolated points, then $f$ is continuous.

**Proof.** Let $a \in X$ and $\varepsilon > 0$. Then there is $z \in X$ such that $z \neq a$, $|f(z)-f(a)| < \varepsilon$. Let $\delta := |z-a|$. Then for all $x \in X$ with $|x-a| < \delta$ we have, by the weak monotony of $f$, $|f(x)-f(a)| \leq |f(z)-f(a)| < \varepsilon$.

It follows that if $X$ and $Y$ do not have isolated points and if $f$ is an $M_{w}$-bijection of $X$ onto $Y$, then $f$ is a homeomorphism of $X$ onto $Y$. 


Conversely, it is easy to construct homeomorphisms of $\mathbb{Z}_p$ that are not in $M_w(\mathbb{Z}_p)$.

(b) If $K$ is a local field then every $f \in M_w(X)$ is continuous. (See 5.1 (i)).

(c) If $K$ has discrete valuation then every $f \in M_s(X)$ is continuous. (Example 2.4 (5) shows that such a statement is not true for $f \in M_b(X)$.)

(Proof. If $f$ were not continuous at some $a \in X$ then there would be an $\varepsilon > 0$ such that for some sequence converging to $a$ we had $|f(x_n) - f(a)| \geq \varepsilon$. We may suppose that $|x_n - a| > |x_{n+1} - a| > \ldots$. Since the valuation is discrete we have $\lim_{n \to \infty} |f(x_n) - f(a)| = 0$, a contradiction.)

(d) In 5.14 we shall give an example of a function in $M_{bs}(X)$ that is not continuous. (Of course, $K$ will have a dense valuation.)

(3) As we did in 2.4 we may formulate "uniform" $M_w,\ldots$-conditions.

Thus, by definition, $f \in M_{uw}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \Rightarrow |f(x)-f(y)| \leq |f(z)-f(t)|$$

$f \in M_{us}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \Rightarrow |f(x)-f(y)| < |f(z)-f(t)|$$

$f \in M_{ubs}(X)$ if for all $x,y,z,t \in X$

$$|x-y| < |z-t| \Leftarrow |f(x)-f(y)| < |f(z)-f(t)|.$$  

Notice that $f \in M_{ubs}(X)$ means that $|f(x)-f(y)|$ is a strictly increasing function of $|x-y|$. Examples of such functions are isometries, but also the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined via

$$\sum \alpha_p^n \mapsto \sum \alpha_p^{2n} \quad (\sum \alpha_p^n \in \mathbb{Z}_p)$$

$$|f(x)-f(y)| = |x-y|^2 \quad \text{for all } x,y \in \mathbb{Z}_p.$$

Monotone functions : $\mathbb{R} \to \mathbb{R}$ are divided into two classes: the
increasing functions and the decreasing functions. For the non-ar-chi-
medean case we may ask for a similar classification. First we try to
express the situation in the real case in such a way that it can be
translated. Let \( a \in \mathbb{R} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be strictly monotone. If
\( x \) runs through some side of \( a \) then \( f(x) \) runs through some fixed side
of \( f(a) \). So there is a map \( \sigma : \{-1,1\} \to \{-1,1\} \) such that
\( \sigma(\text{sgn}(x-a)) = \text{sgn}(f(x)-f(a)) \) (\( x \neq a \)). Apparently, the only \( \sigma \)'s that can occur are
the identity and \( \sigma(x) = -x \) (\( x \in \{1,-1\} \)). Moreover it turns out that the
map \( \sigma \) is independent of the choice of \( a \).

The two maps \( \sigma \) that can occur can be interpreted as multiplication
maps (with 1 and -1 respectively) or as the bijections \( \{1,1\} \to \{-1,1\} \)
and there seems to be no philosophical reason to make any decision of
preference.

As an example, let us consider a function \( f \in M_\Sigma(K) \). Let \( a \in K \),
let \( a \in \Sigma \). If \( x \in a+\alpha \) and \( y \in a+\alpha \) ("\( x, y \) are at the same side of \( a \)"
then \( x-a, y-a \in a \), so \( |x-y| < |y-a| \). Since \( f \in M_\Sigma(K) \) we have
\( |f(x)-f(y)| < |f(y)-f(a)| \), whence \( |f(x)-f(a)-(f(y)-f(a))| < |f(y)-f(a)| \),
so \( f(x)-f(a) \) and \( f(y)-f(a) \) have the same sign. We may conclude that
there is a map \( \sigma_a : \Sigma \to \Sigma \) such that for all \( x \in K \)
\[ x \in a+\alpha \Rightarrow f(x) \in f(a)+\sigma_a(\alpha) \quad (\alpha \in \Sigma). \]

Unfortunately, it turns out that in general \( \sigma_a \) may be different
from \( \sigma_b \) if \( a \neq b \), even for isometrical maps. For example, let \( p \neq 2 \nand let \( \tau \) be a permutation of \( \{0,1,2,\ldots,p-1\} \) and define \( f : \mathbb{Z}_P \to \mathbb{Z}_P \)
by
\[ \Sigma a_p^n \to \Sigma \tau(a_n)p^n \quad (a_n \in \{0,1,2,\ldots,p-1\} \text{ for each } n). \]
Suppose we had a \( \sigma : \Sigma \to \Sigma \) such that for all \( x,y \in \mathbb{Z}_P \), \( x-y \in \alpha \) implies
\( f(x)-f(y) \in \sigma(\alpha) \). Let \( \alpha = \theta_i p^n \) (see 1.5). Then \( x-y \in \alpha \) means
x = a_0 + a_1 p + ... + a_n p^n + ...

y = b_0 + b_1 p + ... + b_n p^n + ...

where a_0 = b_0, ..., a_{n-1} = b_{n-1}, a_n - b_n = \theta^i \text{ modulo } p.

Then \( f(x) - f(y) = (\tau(a_n) - \tau(b_n))p^n + \ldots \), so \( \sigma(a) = \theta^j p^n \) where 
\( \tau(a_n) - \tau(b_n) = \theta^j \mod p \). (j depending on i and n).

It turns out that \( \tau(1) - \tau(0) = \tau(2) - \tau(1) = \ldots = \tau(p-1) - \tau(p-2) \) and it is clear that we can choose \( \tau \) such that \( \tau(1) - \tau(0) \neq \tau(2) - \tau(1) \), a contradiction.

Concluding we may say that in general we cannot assign to every monotone function (isometry) a "type" of monotony in the sense suggested above.

We are led to define

**DEFINITION 2.14** Let \( X \subset K, f : X \rightarrow K \) and let \( \sigma : \Sigma \rightarrow \Sigma \). We say that

\( f \) is monotone of type \( \sigma \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \text{ implies } f(x) - f(y) \in \sigma(\alpha). \]

(In other words if \( x >_\alpha y \) implies \( f(x) >_{\sigma(\alpha)} f(y) \),
see 1.3.)

(Notice that if \( X \neq K \) \( f \) can be of type \( \sigma \) and of type \( \tau : \Sigma \rightarrow \Sigma \) where \( \sigma \neq \tau \), due to the fact that for some \( \alpha \), \( x >_\alpha y \) for no \( x, y \in X \), but we leave this technical fact aside for the moment.)

With our previous philosophy about real functions in mind, we define

**DEFINITION 2.15** Let \( X \subset K, f : X \rightarrow K, \beta \in \Sigma \). We say that \( f \) is monotone of type \( \beta \) if for all \( \alpha \in \Sigma \) and all \( x, y \in X \)

\[ x - y \in \alpha \text{ implies } f(x) - f(y) \in \alpha \beta. \]

In other words, \( f \) is monotone of type \( \beta \) iff it is monotone of type \( \sigma \)
where $\sigma : \Sigma \times \Sigma$ is the multiplication with $\beta$. Or, equivalently, $f$ is monotone of type $\beta$ iff the sign of $\frac{f(x) - f(y)}{x - y}$ is constant $\beta$ for all $x, y \in X, x \neq y$. This leads to

DEFINITION 2.16 Let $X \subset K$, $f : X \times K$. $f$ is called increasing if $f$ is monotone of type 1. In other words, $f$ is increasing if for all $x, y \in X, x \neq y$ the difference quotient $\frac{f(x) - f(y)}{x - y}$ is positive, i.e., if

$$\left| \frac{f(x) - f(y)}{x - y} - 1 \right| < 1.$$ 

In the next section we shall study the monotone functions of type $\sigma$ and we will give a partial answer to the question for which maps $\sigma : \Sigma \times \Sigma$ there exists an $f : K \times K$ that is monotone of type $\sigma$. 
3. INCREASING FUNCTIONS, FUNCTIONS OF TYPE $g$.

DEFINITION 3.1. Let $X \subseteq K$, $f : X \to K$. Let $\Phi f(x,y) = \frac{f(x) - f(y)}{x-y} (x,y \in X, x \neq y)$. $f$ is called positive if $f(X) \subseteq K^+$,
strictly positive if $\sup_{x \in X} |f(x) - 1| < 1$
increasing if $\Phi f(x,y) \in K^+$ for all $x,y \in X, x \neq y$
strictly increasing if $\sup \{|1 - \Phi f(x,y)| : x,y \in X, x \neq y\} < 1$.

It follows that an increasing function is an isometry. We collect some facts.

THEOREM 3.2. Let $X \subseteq K$.

(i) If $f : X \to K$ is (strictly) increasing and $a \in K^+$ then $af$ is (strictly) increasing.

(ii) For $a,b \in K$ the function $x \mapsto ax + b$ is increasing if and only if $a \in K^+$ (if and only if $x \mapsto ax + b$ is strictly increasing).

(iii) If $f : X \to K$ is (strictly) increasing and $f$ is (strictly) positive then $-\frac{1}{f}$ is (strictly) increasing.

(iv) The (strictly) increasing functions $X \to K$ form a convex set.

(v) If $f : X \to K$ and $g : f(X) \to K$ are (strictly) increasing then so is $g \circ f$.

(vi) If $f : X \to K$ is (strictly) increasing then so is $f^{-1} : f(X) \to K$.

(vii) If $f_1, f_2, \ldots : X \to K$ are increasing and $f = \lim f_n$ pointwise then $f$ is increasing.

(viii) If $K$ has discrete valuation then "positive", "strictly positive" and "increasing", "strictly increasing" are equivalent, respectively.
Proof. Elementary.

3.3. EXAMPLES.

(1) The exponential function

$$\exp x = 1 + x + \frac{x^2}{2!} + \ldots$$

defined on \( \{ x \in \mathbb{R} : |x| < p \} \) if \( \chi(k) = p \), \( \chi(K) = 0 \) and on \( \{ x \in \mathbb{R} : |x| < 1 \} \) if \( \chi(k) = 0 \), is increasing, as an easy computation may show. In Section 2 we have seen that not every isometry is increasing.

(2) Let \( f : X \to \mathbb{R} \) be a \( C^2 \)-function (i.e., \( \Phi f \) can continuously be extended to a function on \( X \times X \), assume that \( X \subset \mathbb{R} \) has no isolated points. See [2]) and suppose \( f'(a) \in K^+ \) for some \( a \in X \). Then \( f \) is locally (strictly) increasing at \( a \).

(Proof. There is \( \delta > 0 \) such that \( |x-a| < \delta, |y-a| < \delta, x \neq y \) implies

$$\left| \frac{f(x)-f(y)}{x-y} - f'(a) \right| \leq \frac{\delta}{2}.$$ 

For such \( x, y \) we have

$$|\frac{f(x)-f(y)}{x-y} - f'(a)| < \frac{\delta}{2}.$$ 

(3) The space \( C(\mathbb{R}) \) of all continuous functions \( \mathbb{R} \to \mathbb{R} \), is a Banach space with respect to the sup norm \( |.|_\infty \). Let \( e_0 := \xi_p \) and for \( n \geq 1 \) let \( e_n := \xi_p \) where \( B := \{ \xi \in \mathbb{A} : |x-n| < \frac{1}{n} \} \). It is proved in [1] that \( e_0, e_1, \ldots \) is an orthonormal base of \( C(\mathbb{R}) \) i.e., for each \( f \in C(\mathbb{R}) \) there exists a unique null sequence \( \lambda_0, \lambda_1, \ldots \) such that

$$f = \sum_{n=0}^{\infty} \lambda_n e_n.$$
The coefficients \( \lambda_n \) can be reconstructed from \( f \) via

\[
\lambda_0 = f(0) \\
\lambda_n = f(n) - f(n_-) \quad (n \in \mathbb{N})
\]

where \( n_- \) is defined as \( a_0 + a_1 p + \ldots + a_{s-1} p^{s-1} \) if \( n \neq a_0 + a_1 p + \ldots + a_{s-1} p^s \) \((a_s \neq 0)\) in base \( p \).

Our aim is here to describe a necessary and sufficient condition for the \( \lambda_n \) in order that \( f = \sum \lambda_n e_n \) is increasing. We show

\[
f = \sum \lambda_n e_n \text{ is increasing if and only if for all } n \in \mathbb{N} \quad |\lambda_n - (n-n_-)| < |n-n_-|.
\]

Proof. First observe that \( f \) is increasing if and only if for all \( x \in \mathbb{Z}_p \)

\[
f(x) = x + g(x)
\]

where \( |g(x,y)| < 1 \) for all \( x, y \in \mathbb{Z}_p, x \neq y \).

As

\[
x = \sum_{n \geq 1} (n-n_-) e_n(x) \quad (x \in \mathbb{Z}_p)
\]

it suffices to show that for \( g = \sum_{n \in \mathbb{Z}} \lambda_n e_n \in C(\mathbb{Z}_p) \) we have \( |\phi(g)| < 1 \) if and only if \( |\lambda_n| < |n-n_-| \) for all \( n \in \mathbb{N} \).

Suppose first \( |\phi(g)| < 1 \). Then for all \( n \in \mathbb{N} \), \( |\frac{f(n) - f(n_-)}{n-n_-}| < 1 \), so

\[
|\lambda_n| = |f(n) - f(n_-)| < |n-n_-|.
\]

Conversely, let \( g = \sum_{n \in \mathbb{N}} \lambda_n e_n \) and let \( |\lambda_n| < |n-n_-| \) for all \( n \in \mathbb{N} \).

Let \( x, y \in \mathbb{Z}_p \) and let \( |x-y| = p^{-k} \) for some \( k \in \{0, 1, 2, \ldots\} \). Since

\[
e_n(a) = e_n(b) \text{ if and only if } |a-b| < \frac{1}{n}
\]

we have

\[
e_n(x) = e_n(y) \quad \text{for } n < p^k.
\]
Therefore
\[
|g(x) - g(y)| = \left| \sum_{n \geq 1} \lambda_n (e_n(x) - e_n(y)) \right| = \left| \sum_{n \geq K} \lambda_n (e_n(x) - e_n(y)) \right| \\
\leq \max_{k \geq p} |\lambda_k| < \max_{k \geq p} |n - n_k| = p^{-k} = |x - y| \\
\]
so \(|\phi g| < 1\).

(4) Let \(K\) have dense valuation and let \(k\) be (countably) infinite. Let \(X\) be the unit ball of \(K\) and let \(B_i\) \((i \in \mathbb{N})\) be the balls in \(X\) with radius \(1^i\). Choose \(c_1, c_2, \ldots \in K\) such that \(|c_1| < |c_2| < \ldots\), \(\lim |c_n| = 1\). For \(n \in \mathbb{N}\) define a function \(f_n : X \to K\) via
\[
f_n(x) = \begin{cases} 
  x + c_i & \text{if } x \in B_i \quad (1 \leq i \leq n) \\
  x & \text{elsewhere}
\end{cases}
\]
Then each \(f_n\) is strictly increasing (\(|\phi f_n (x, y) - 1| \leq \max_{1 \leq i, j \leq n} |c_i - c_j| \leq |c_n| < 1\)). The sequence \(f_1, f_2, \ldots\) converges pointwise to an increasing function \(f\). But \(f\) is not strictly increasing:
\[
\sup_{x \neq y} |\phi f(x, y) - 1| = \sup_{i, j} |c_i - c_j| = 1.
\]
(Compare 3.2, (vii) and (viii)).

Increasing functions are closely related to functions \(g\) for which
\(|g(x) - g(y)| < |x - y| \quad (x \neq y) \quad (\text{if } f \text{ is increasing, set } g(x) := f(x) - x)\).

**DEFINITION 3.4.** Let \((X, \rho)\) be an ultrametric space. A map \(g : X \to X\)

is called a **pseudocontraction** if \(\rho(f(x), f(y)) < \rho(x, y)\)

\((x, y \in X, x \neq y)\).
The Banach contraction theorem states that $X$ is complete if and only if each contraction $X \to X$ has a fix point. We have

**Lemma 3.5.** Let $(X,\rho)$ be an ultrametric space. Then the following conditions are equivalent.

(a) $X$ is spherically complete.

(b) Each pseudocontraction $X \to X$ has a fix point.

(c) Each pseudocontraction $X \to X$ has a unique fix point.

**Proof.** If $\sigma: X \to X$ is a pseudocontraction and if $x,y$ are fix points and $x \neq y$, then $\rho(x,y) = \rho(\sigma(x),\sigma(y)) < \rho(x,y)$, a contradiction. Thus, we have (b) $\Rightarrow$ (c). We prove (a) $\Rightarrow$ (b). Let $B \subset X$ be a ball (i.e., either $B = \{x \in X: \rho(x,a) \leq r\}$ for some $a \in X$, $r \geq 0$ or $B = \{x \in X: \rho(x,a) < r\}$ for some $a \in X$, $r > 0$). We call $B$ **invariant** if $\sigma(B) \subset B$.

Now we observe two facts:

a) $X$ has invariant balls. In fact, let $a \in X$ such that $\sigma(a) \neq a$. Then

$$V := \{x \in X: \rho(x,\sigma(a)) < \rho(a,\sigma(a))\}$$

is invariant. (If $x \in V$ then $x \neq a$, so $\rho(\sigma(x),\sigma(a)) < \rho(x,a) \leq \max(\rho(x,\sigma(a)),\rho(\sigma(a),a)) = \rho(a,\sigma(a))$, hence $\sigma(x) \notin V$.)

Notice that $a \notin V$.

b) If $B_1$ and $B_2$ are invariant balls then $B_1 \cap B_2 \neq \emptyset$ (so either $B_1 \subset B_2$ or $B_2 \subset B_1$). (Suppose $B_1 \cap B_2 = \emptyset$. Then for $x \in B_1$, $y \in B_2$, $\rho(x,y)$ does not depend on $x,y$, since for $z \in B_1$, $u \in B_2$, $\rho(x,z) < \rho(x,y)$ and $\rho(y,u) < \rho(x,y)$, so $\rho(z,u) = \rho(x,y)$. On the other hand, if $x \in B_1$, $y \in B_2$ then $\rho(\sigma(x),\sigma(y)) < \rho(x,y)$, a contradiction).

It follows that the collection of invariant balls in $X$ form a non-empty nest and by the spherical completeness of $X$ there is a smal-
least invariant ball $S$. If $a \in S$, $\sigma(a) \neq a$ then \{\(x \in S : \rho(x, \sigma(a)) < \rho(a, \sigma(a))\)\} is invariant and does not contain $a$, a contradiction. Hence, $\sigma$ has a fix point (actually, $S$ is a singleton).

We prove (β) → (α). If $X$ were not spherically complete, there exist balls $B_1 \supsetneq B_2 \supsetneq \ldots$ such that $\cap_{n=1}^{\infty} B_n = \emptyset$. Choose $x_n \in B \setminus B_{n+1}$ $(n \in \mathbb{N})$, set $B_0 := X$ and define

$$\sigma(x) := \begin{cases} x_{n+1} & \text{if } x \in B_n \setminus B_{n+1} (n \in \{0, 1, 2, \ldots\}) \end{cases}.$$

Then $\sigma$ has obviously no fix point. Let $x \in B_n \setminus B_{n+1}$ and $y \in B_m \setminus B_{m+1}$, $x \neq y$. If $n = m$ then $\sigma(x) = \sigma(y)$, so suppose $n > m$. Then $\sigma(x), \sigma(y)$ are both in $B_{m+1}$, whereas $x \in B_n \subset B_{m+1}$ and $y \notin B_{m+1}$. Hence $\rho(\sigma(x), \sigma(y)) < \rho(x, y)$. Then $\sigma$ is a pseudocontraction without a fix point. Contradiction.

**COROLLARY 3.6.** The following conditions are equivalent.

(a) $K$ is spherically complete.

(β) If $C \subset K$ is convex, $f : C \to C$ is increasing then $f$ is surjective.

(γ) If $C \subset K$ is convex, $f : C \to K$ is increasing then $f(C)$ is convex.

(δ) An increasing $f : K \to K$ is surjective.

**Proof.** (α) → (β). Choose $a \in C$ and consider the map $\sigma : x \mapsto x - f(x) + a$ $(x \in C)$. Then $\sigma : C \to C$. $C$ is spherically complete, $\sigma$ is a pseudocontraction. Hence, there is by 3.5 a $c \in C$ for which $\sigma(c) = c$ i.e., $f(c) = a$: $f$ is surjective.

(β) → (γ). For a suitable $s \in K$, $f-s$ sends $C$ into $C$. (γ) → (δ) is clear.

(δ) → (α). Let $\sigma : K \to K$ be a pseudocontraction. Then $x \mapsto x - \sigma(x)$
is increasing hence is surjective. So then is $x \in K$ for which $x - \sigma(x) = 0$, i.e., $\sigma$ has a fix point. By 3.5, $K$ is spherically complete.

In case $f$ is strictly increasing we do not have to require that $K$ is spherically complete:

THEOREM 3.7. Let $C \subset K$ be convex and let $f: C \to K$ be strictly increasing.

Then $f(C)$ is convex. If $f(C) \subset C$, then $f(C) = C$.

Proof. Reread the proof of $(\alpha) + (\beta)$, $(\beta) + (\gamma)$ above. $\sigma$ now is a contraction. $C$ is complete. Apply the Banach contraction theorem.

Let $X$ be a subset of $\mathbb{R}$ and let $f: X \to \mathbb{R}$ be a bounded increasing function. Then $f$ can be extended to an increasing function $\mathbb{R} \to \mathbb{R}$ by setting $f(x) = \inf f$ if $x < y$ for all $y \in X$ and $f(x) = \sup \{f(y): y \leq x, y \in X\}$ for all other $x \in \mathbb{R}$. In our situation we can prove

THEOREM 3.8. The following conditions are equivalent.

(a) $K$ is spherically complete.

(\beta) For every $X \subset K$ an increasing function $f: X \to K$ can be extended to an increasing $\overline{f}: K \to K$.

(\gamma) Let $X \subset K$, and let $f: X \to K$ be a strictly increasing function. Then $f$ can be extended to a strictly increasing function $\overline{f}: K \to K$ such that

$$\sup_{x, y \in K} \frac{|\overline{f}(x) - \overline{f}(y)|}{x - y} - 1 = \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|}{x - y} - 1.$$ 

Proof. $(\alpha) + (\beta)$. Let a $\notin X$. By Zorn's Lemma it suffices to define $\overline{f}$ such that $\overline{f}$ is increasing on $X \cup \{a\}$. We are done if we can find $a \in K$ such that for $x \in X$
\[
\frac{|a-f(x)|}{a-x} - 1 < 1
\]

i.e., \( a \in B := B_x := B_f(x) - (a-x) \) for \( x \in X \).

Now \( B_x \cap B_y \neq \emptyset \) for \( x, y \in X \) since the distance of their centers is

\[
|f(x) - f(y) - (x-y)| = |f(x) - f(y) - (a-x) - (a-y)| = |f(x,y) - 1||x-y| < \\
\]

\( \max(|x-a|, |y-a|) \). So if, say, \( |x-a| \leq |y-a| \) we see that \( |f(x) - f(y) - (a-x) - (a-y)| < |y-a| \) whence \( f(x) - (a-x) \in B_y \). By the spherical completeness of \( K \) we have \( \bigcap_{x \in X} B_x \neq \emptyset \). Choose \( a \in \bigcap_{x \in X} B_x \).

(\( \beta \)) + (\( \alpha \)). Suppose \( K \) is not spherically complete. By 3.6, (\( \delta \)) + (\( \alpha \)) there is a non surjective increasing function \( f: K \rightarrow K \). Then its inverse \( g: f(K) \rightarrow K \) is increasing, surjective, and can obviously not be extended to an increasing \( g: K \rightarrow K \).

(\( \beta \)) + (\( \gamma \)) follows from the fact that (with \( \Phi(x) = x \) for all \( x \))

\[
f \mapsto (1-c)\Phi + cf \quad (c \in K, \ |c| < 1)
\]

is a 1-1 correspondence between the collection of all increasing functions on a set \( X \) and the collection of all strictly increasing functions \( g \) for which \( |1-\Phi(g)| < |c| \).

We will now investigate the relation between increasingness of \( f \) and positivity of \( f \). First we observe that there exist nowhere differentiable increasing functions (in contrast to the theorem of Lebesgue in the real case). In fact, by [2] Cor. 2.6, there exists a nowhere differentiable isometry \( \sigma: K \rightarrow K \). Let \( \lambda \in K, \ 0 < |\lambda| < 1 \). Then \( x \mapsto x - \lambda \sigma(x) \) is increasing, nowhere differentiable.

Clearly, if \( f \) is an increasing function, defined on some subset \( X \) of \( K \) without isolated points and if \( f \) is differentiable then for each
\( x \in X, f'(x) = \lim_{y \to x} f(x, y) \in \mathbb{K}^+ \). So \( f' \) is positive. If, addition, \( f \) is strictly increasing, then \( f' \) is strictly positive.

By [2] 3.6 any derivative is of Baire class 1 and by [2] 3.10 every Baire class 1 function has an antiderivative.

So a reasonable question is: let \( f: X \to \mathbb{K} \) be a (strictly) positive Baire class 1 function. Then does \( f \) have a (strictly) increasing antiderivative? We proceed to prove that the answer is "yes".

We first need a refinement of [2], 3.4, (c).

**Lemma 3.9.** Let \( X \subset \mathbb{K} \) and let \( f: X \to \mathbb{K} \) be a Baire class 1 function such that \( |f(x)| < 1 \) for all \( x \in X \). Then there exist locally constant functions \( g_1, g_2, \ldots : X \to \mathbb{K} \) such that \( |g_n| \leq 1 - \frac{1}{n} \) for each \( n \) and

\[
 f = \sum g_n \quad \text{(pointwise)}.
\]

**Proof.** There exist continuous functions \( f_1, f_2, \ldots : X \to \mathbb{K} \) such that

\[
 f = \lim_{n} f_n \quad \text{pointwise.}
\]

There exist locally constant functions \( h_1, h_2, \ldots : X \to \mathbb{K} \) such that \( |f_n - h_n| \leq 2^{-n}, \) hence \( f = \lim_{n} h_n \) pointwise. Define \( t_1, t_2, \ldots : X \to \mathbb{K} \) as follows

\[
 t_n(x) := \begin{cases}
 h_n(x) & \text{if } |h_n(x)| \leq 1 - \frac{1}{n} \\
 0 & \text{if } |h_n(x)| > 1 - \frac{1}{n}.
\end{cases}
\]

Then \( t_n \) is locally constant for each \( n \in \mathbb{N} \). \( \{x \in X : |h_n(x)| \leq 1 - \frac{1}{n} \} \) is closed and open in \( X \). \( |t_n| \leq 1 - \frac{1}{n} \) and \( \lim_{n} t_n = f \). Now let \( g_1 := t_1, \) and \( g_n := t_n - t_{n-1} \) \((n \geq 2)\). Then \( |g_1| = |t_1| = 0 \), each \( g_n \) is locally constant, \( |g_n| \leq \max(|t_n|, |t_{n-1}|) \leq \max(1 - \frac{1}{n}, 1 - \frac{1}{n-1}) = 1 - \frac{1}{n} \), \( f = \infty \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} g_n \).
LEMMA 3.10. Let $X \subset K$ have no isolated points and let $f : X \to K$ be a Baire class 1 function, $|f(x)| < 1$ for all $x \in X$. Then $f$ has an antiderivative $F$ for which

$$\frac{|F(x) - F(y)|}{|x - y|} < 1 \quad (x, y \in X, x \neq y).$$

Proof. By Lemma 3.9, $f = \sum_{n=1}^{\infty} f_n$, where each $f_n$ is locally constant, $|f_n| \leq 1 - \frac{1}{n}$. By [2] 3.9 each $f_n$ has an antiderivative $F_n$ for which

$$|F_n(x) - F_n(y)| \leq \max \left( \frac{|f_n(x)|}{2n}, \frac{1}{2n} \right) |x - y| \quad (x, y \in X).$$

By [2] 3.7, $F := \sum F_n$ is an antiderivative of $\sum f_n = f$. Now for $x, y \in X, x \neq y$:

$$|F(x) - F(y)| \leq \sup_n |F_n(x) - F_n(y)| \leq \sup_n \max \left( \frac{|f_n(x)|}{2n}, \frac{1}{2n} \right) |x - y| \leq |x - y| \max \left( \frac{|f_n(x)|}{2}, \frac{1}{2} \right).$$

Now for each $x \in X, |f_n(x)| < 1$ for each $n$ and $\lim_{n} |f_n(x)| = 0 < 1$. Hence $\max_n |f_n(x)| < 1$. It follows that

$$|F(x) - F(y)| < |x - y|.$$

THEOREM 3.11. Let $X \subset K$ have no isolated points and let $f : X \to K$ be (strictly) positive. Then $f$ has a (strictly) increasing antiderivative.

Proof. The function $x \mapsto f(x) - 1$ has, by 3.10, an antiderivative $H$ such that $|\phi(H)| < 1$. Let $F(x) := x + H(x) \quad (x \in X).$ Then $F' = f$ and $\phi(F) = 1 + \phi(H)$. Thus, if $f$ is positive then $F$ is increasing. If $f$ is strictly positive then $|f(x) - 1| < r$ for all $x \in X$ and, by a trivial extension of 3.10, we may choose $H$ such that $|\phi(H)| < r$. It follows that $|\phi(F) - 1| < r$, so $F$ is strictly increasing.
We collect the results in

**COROLLARY 3.12.** Let \( X \subset K \) have no isolated points. Then

(i) If \( f: X \to K \) is differentiable and (strictly) increasing
then \( f' \) is a (strictly) positive Baire class 1 function.

(ii) If \( g: X \to K \) is a (strictly) positive Baire class 1 function
then \( g \) has a (strictly) increasing antiderivative.

(iii) If \( f: X \to K \) is differentiable and if \( f' \) is (strictly) positive
then \( f = g + h \) where \( g \) is (strictly) increasing and
where \( h' = 0 \).

**Note.** We cannot strengthen 3.12 (iii) by replacing "\( h' = 0 \)" by "\( h \) is locally constant". In fact, if \( X = \mathbb{Z} \), then every locally constant function has bounded difference quotients. If our statements were true, then every differentiable \( f: \mathbb{Z} \to \mathbb{Q} \) for which \( f' \) is positive would have bounded difference quotients.

But consider the function \( f: \mathbb{Z} \to \mathbb{Q} \) defined via

\[
f(x) := \begin{cases} \frac{x-p^n}{2^n} & \text{if } |x-p^n| < p^{-3n} \quad (n \in \{0,1,2,\ldots\}) \\ x & \text{elsewhere} \end{cases}
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z} \). Let \( x := p^n \) and \( y_n := p^n + p^{3n} \quad (n \in \mathbb{N}) \). Then

\[
f(x_n) = \frac{p^n - 2^n}{p} \quad \text{and} \quad f(y_n) = p^n + p^{3n}, \quad \text{so } |f(x_n) - f(y_n)| = |2^n| = p^{-2n},
\]

whereas \( |x_n - y_n| = |p^{3n}| = p^{-3n} \). So

\[
\lim_{n \to \infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n \to \infty} \frac{|2^n|}{p^{3n}} = \infty.
\]

We now study the connection between increasing \( C^1 \)-functions and continuous positive functions.

If \( f \) is a (strictly) increasing \( C^1 \)-function then clearly \( f' \) is a continuous (strictly) positive function.
Conversely, let $X \subset K$ have no isolated points and let $f : X \rightarrow K$ be continuous and positive. Let $P : C(X) \rightarrow C^1(X)$ a map similar to the one defined in [2] page 46, e.g.,

$$(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (x \in X).$$

(Here the $x_n$ are defined in the following way. Let $1 > r_1 > r_2 > \ldots$, $\lim r_n = 0$. The equivalence relation "$x \sim y$ iff $|x-y| < r_n$" yields a partition of $X$ into balls. Choose a center in each such ball. They form a collection $R_n$. We can arrange that $R_n \subset R_{n+1}$ for each $n$. For $n \in \mathbb{N}$, let $x_n := \sigma_n(x)$ where $\sigma_n(x)$ is characterized by $|\sigma_n(x)-x| < r_n$, $\sigma_n(x) \in R_n$.)

From [2] 5.4, it follows that $Pf$ is a $C^1$-antiderivative of $f$. It suffices to prove that $Pf$ is (strictly) increasing. Let $x, y \in X$, $x \neq y$,

$|x-y| < r_1$. Then there is $s$ such that $r_{s+1} \leq |x-y| < r_s$. We have

$x_1 = y_1$, $\ldots$, $x_s = y_s$, $x_{s+1} \neq y_{s+1}$. Further $|x_{n+1} - x_n| \leq |x-y|$ (n>s),

$|y_{n+1} - y_n| \leq |x-y|$ (n>s), $|x_{s+1} - y_{s+1}| \leq |x-y|$. Hence (using the identity $x = \sum (x_{n+1} - x_n) + x_1$, $y = \sum (y_{n+1} - y_n) + y_1$, $x_1 = y_1$) $|Pf(x) - Pf(y) - (x-y)| =$

$$|f(x_1) - 1)(x_{s+1} - y_{s+1}) + \sum_{n=s+1}^{\infty} (f(x_n) - 1)(x_{n+1} - x_n) -$$

$$- \sum_{n=s+1}^{\infty} (f(y_n) - 1)(y_{n+1} - y_n)|.$$

If $|f(x) - 1| < \alpha$ for all $x \in X$, we have since $\lim |f(x_n) - 1|$ exists,

$\sup |f(x_n) - 1| < \alpha$, similarly, $\sup |f(y_n) - 1| < \alpha$.

So we get $|Pf(x) - Pf(y) - (x-y)| < \alpha |x-y|$.

Now suppose $|x-y| \geq r_1$. Then since for all $n$: $|x_{n+1} - x_n| < r_1$, $|x_1 - y_1| = |x-y|$ we get (again under the assumption $|f(x) - 1| < \alpha$ for all $x \in X$):
\[
(Pf)(x) - (Pf)(y) - (x - y) = \sum_{n=1}^{\infty} (f(x_n) - 1)(x_{n+1} - x_n) - (f(y_n) - 1)(y_{n+1} - y_n) < \alpha |x_1| + |x| |x - y|.
\]

We have proved:

**THEOREM 3.13.** Let \( X \subset K \) have no isolated points. Then the map \( P \) defined via

\[
(Pf)(x) = x_1 + \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (f \in C(X), \ x \in X)
\]

maps (strictly) positive functions into (strictly) increasing functions.

**COROLLARY 3.14.** Let \( X \subset K \) have no isolated points. Then if \( f \in C^1(X) \) and \( f' \) is (strictly) positive, then \( f = j + h \) where \( j \) is (strictly) increasing and \( h \) is locally constant.

**Proof.** By 3.12 we have \( f = j + h \) where \( j \) is (strictly) increasing and where \( h' = 0 \). Now by [2] Cor. 5.2 bis there is a locally constant function \( l: X \rightarrow K \) with \( |l(h-1)|_\infty < \frac{1}{2} \). Then \( s = j + (h-1) \) is (strictly) increasing, so we have \( t = s + 1 \), where \( s \) is (strictly) increasing and \( l \) is locally constant.

**Note.** We may also define convex functions. Let \( X \subset K \). A function \( f: X \rightarrow K \) is called convex if the second order difference quotient is positive. I.e., if for all \( x, y, z \in X \) (\( x \neq y, y \neq z, x \neq z \)) we have

\[
\phi_2 f(x,y,z) := \frac{\phi f(x,y) - \phi f(x,z)}{y - z} = \frac{f(x) - f(y)}{x - y} \quad \frac{f(x) - f(z)}{y - z} \in K^+
\]

(Just as we did for increasing function we may distinguish between convex and strictly convex.)
It follows that for a convex function $f$ the function $x \mapsto \Phi f(x,y)$
defined on $X \setminus \{y\}$ is an isometry, hence can be continuously extended to
the whole of $X$. Define $\Phi f(y,y) = \lim_{x \to y} \Phi f(x,y)$ ($y \in X$). Thus, $f$ is differen-
tiable. For all $x, y, z, t \in X$ we have

$$|\Phi f(x,y) - \Phi f(x,t)| \leq \max\{|\Phi f(x,y) - \Phi f(z,y)|, |\Phi f(z,y) - \Phi f(z,t)|\} \leq \max(|x-z|, |y-t|).$$

Hence, $\Phi f$ is uniformly continuous on $X$ i.e., $f$ is strongly uniformly differentiable in the sense of [2] page 67.

For each $y \in X$ the function $x \mapsto \Phi f(x,y)$ is increasing on $X$.

If $\chi(K) \neq 2$ then convexity of $f$ implies increasingness of $\frac{d^2 f}{dx^2}$.

(Proof.

$$\lim_{y \to x} \frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} = \frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} \in K^+(x \neq x')$$

$$\lim_{y \to x'} \frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} = \frac{\Phi f(x,x') - \Phi f(x',x')}{x-x'} \in K^+(x \neq x')$$

so

$$\frac{\Phi f(x,y) - \Phi f(x',y)}{x-x'} \in 2K^+(x \neq x'), \text{ whence } \Phi'\frac{d^2 f}{dx^2}(x,x') \in K^+ \text{ if } x \neq x'.$$

Of course, if $f \in C^2(X)$ (see [2] 8.1) then convexity of $f$ implies positivity of $D^2 f$ ([2] 8.4). So if $\chi(K) \neq 2$ then $\Phi f''' = D^2 f$ ([2] 8.14) is positive. If $\chi(K) = 2$ then $f''' = 0$ for all $C^2$-functions.

Note. The functions that are monotone of type $\beta$ ($\beta \in \Sigma$), see Def. 2.15,
are easy to describe: $f$ is monotone of type $\beta$ if and only if $b^{-1} f$ is
increasing for any $b \in \beta$.

We now turn to the functions $X \mapsto K$ that are of type $\sigma$ where

$$\sigma : \Sigma \mapsto \Sigma. (2.14).$$

For examples of such $f$, where $\sigma$ is not a multiplier
see 3.19 and 3.20. To avoid needless complications from now on in this section we will assume that $X$ is an open convex subset of $K$. This implies that the set $\{a \in \Sigma : \text{there is } x \in X \text{ such that } x > a \}$ is independent of $a \in X$. Thus, $X$ is homogeneous in the sense that $(a + \alpha) \cap X \neq \emptyset$ for some $a \in X, \alpha \in \Sigma$ then for each $b \in X, (b + \alpha) \cap X \neq \emptyset$.

Let $\Sigma(X) := \{a \in \Sigma : \text{there is } x, y \in X \text{ such that } x > y \}$. Then for each $a \in X$,

$$\Sigma(X) = \{a \in \Sigma : x > a \text{ for some } x \in X\}.$$ 

Either $\Sigma(X) = K$ or $\Sigma(X) = \{a \in \Sigma : |a| < r\}$ for some $r > 0$ or $\Sigma(X) = \{a \in \Sigma : |a| \leq r\}$ for some $r > 0$. Hence $\Sigma(X)$ is closed under $\oplus$ (see 1.2) i.e., if $a, \beta \in \Sigma(X)$ and $a \oplus \beta$ is defined then $a \oplus \beta \in \Sigma(X)$.

To be sure that $f$ is monotone both of type $\sigma$ and type $\tau$ implies $\sigma = \tau$ we define

**DEFINITION 3.15.** (Let $X \subseteq K$ be open, convex and) let $f : \Sigma(X) \rightarrow \Sigma$. $f : X \rightarrow K$ is called monotone of type $\sigma$ if for all $x, y \in X$ and $a \in \Sigma(X)$

$$x > y + f(x) > f(y).$$

$a \sigma(a)$

**THEOREM 3.16.** Let $f : X \rightarrow K$ be monotone of type $\sigma : \Sigma(X) \rightarrow \Sigma$. Then

(i) $\sigma(-a) = -\sigma(a)$ $(a \in \Sigma(X))$.

(ii) Let $a, \beta \in \Sigma(X)$. If $\sigma(a) \oplus \sigma(\beta)$ is defined then so is $a \oplus \beta$ and $\sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta)$.

(iii) Let $a, \beta \in \Sigma(X)$. If $|a| < |\beta|$ then $|\sigma(a)| < |\sigma(\beta)|$.

(iv) Let $s$ be in the prime field of $K$ and let $|s| = 1$. Then

$$\sigma(sa) = s \sigma(a) \quad (a \in \Sigma(X)).$$

(v) If $\beta \in \Sigma(X), |\beta| = 1$, $\beta$ contains an element of the prime field of $K$ then $\sigma(\beta a) = \beta \sigma(a)$ for all $a \in \Sigma(X)$. 
(vi) \( f \in M_{us}(X) \) \( (i.e., \text{for all} \ x,y,z,t \in X, |x-y| < |z-t| \)
implies \(|f(x)-f(y)| < |f(z)-f(t)|\).

(vii) \( f \) is either nowhere continuous or uniformly continuous
on \( X \).

Proof.

(i) Let \( x,y \in X \) such that \( x>y \). Then \( f(x)-f(y) \in \sigma(\alpha) \); \( f(y)-f(x) \in -\sigma(\alpha) \).
But also \( y>x \), hence \( f(y)-f(x) \in \sigma(-\alpha) \). So \( -\sigma(\alpha) \) and \( \sigma(-\alpha) \) are not
disjoint and they must coincide.

(ii) Suppose \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined. If \( \alpha \oplus \beta \) were not, then \( \beta = -\alpha \) so,
by (i), \( \sigma(\beta) = \sigma(-\alpha) = -\sigma(\alpha) \). Hence also \( \alpha \oplus \beta \) is defined. Choose
\( x,y \in X \) with \( x>y \). There is \( z \in X \) such that \( y>z \). Then \( x-y \in \alpha, y-z \in \beta \). Further \( f(x)-f(y) \in \sigma(\alpha), f(y)-f(z) \in \sigma(\beta) \) so
\( f(x)-f(z) \in \sigma(\alpha) \oplus \sigma(\beta) \). Also \( x-z \in \alpha \oplus \beta \), so \( f(x)-f(z) \in \sigma(\alpha \oplus \beta) \).
The signs \( \sigma(\alpha) \oplus \sigma(\beta) \) and \( \sigma(\alpha \oplus \beta) \) are not disjoint and they must coincide.

(iii) Let \( |\alpha| < |\beta| \). Choose \( x,y,z \) such that \( x-y \in \alpha, y-z \in \beta \). Then (see
1.2 and preamble) \( f(x)-f(z) = f(x)-f(y)+f(y)-f(z) \in \sigma(\alpha)+\sigma(\beta), x-z \in \alpha + \beta = \alpha \oplus \beta = \beta, \) so \( f(x)-f(z) \in \sigma(\beta) \). Thus \([\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta) \) is not
empty. If \( \sigma(\alpha) \oplus \sigma(\beta) \) were not defined then \( \sigma(\alpha) = -\sigma(\beta) \) and \( \sigma(\alpha) + \sigma(\beta) \)
would be a ball with center 0 and radius \(|\sigma(\beta)|^{-}\), but then \([\sigma(\alpha)+\sigma(\beta)] \cap \sigma(\beta) \) would be empty. Hence \( \sigma(\alpha) \oplus \sigma(\beta) \) is defined and by (ii) we have
\( \sigma(\alpha) \oplus \sigma(\beta) = \sigma(\beta) \). By (1.2) (vi), \(|\sigma(\alpha)| < |\sigma(\beta)| \).

(iv) Let \( \chi(K) \neq 0 \). Then \( s = n \cdot 1 \) for some \( n \in \{1,2,...,\chi(K)-1\} \), so by
1.2 (vii), \( sa = na = \oplus a, s\sigma(a) = n\sigma(a) = \oplus \sigma(a) \). By a repeated appli-
cation of (ii), we see \( \sigma(\oplus a) = \oplus \sigma(a) \). Hence \( \sigma(sa) = sa(\sigma(a)) \).

Let \( \chi(K) = 0 \). Let \( s \) be of the form \( n \cdot 1 \) for some \( n \in \mathbb{N} \). By a simi-
lar reasoning as above, \( \sigma(sa) = sa(\sigma(a)) \). We may identify the prime field
of \( K \) with \( \mathbb{Q} \).
Now observe that \( \{ s \in K^* : \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing all elements of the form \( n \cdot 1 \) (\( n \in \mathbb{N} \)), and \(-1\) (by (i)), hence it must contain \( \mathbb{Q}^* \).

Let \( \chi(K) = 0, \chi(k) = p \neq 0 \). By a similar reasoning as above, we arrive at the fact that \( \{ s \in K^* : \sigma(sa) = s\sigma(a) \text{ for all } a \in \Sigma \} \) is a multiplicative subgroup of \( K^* \) containing \(-1, 1, 2, \ldots, p-1\). We may identify the prime field of \( K \) with \( \mathbb{Q} \). If \( n \in \mathbb{N} \), \( n = s \mod p \) (\( 1 \leq s \leq p-1 \)) then \( na = sa \) for all \( a \), so \( \sigma(na) = \sigma(sa) = s\sigma(a) = na(a) \). Thus our multiplicative group contains all \( n \in \mathbb{Z} \) that are not divisible by \( p \), so it contains all \( s \in \mathbb{Q} \), having absolute value 1.

(v) is an easy consequence of (iv).

(vi) If \( |x-y| < |z-t| \) and \( x-y \neq 0 \) then \( x-y \in \alpha, z-t \in \beta \) for some \( \alpha, \beta \in \Sigma, |\alpha| < |\beta| \). By (iii) \( |\sigma(\alpha)| < |\sigma(\beta)| \) so \( |f(x)-f(y)| < |f(z)-f(t)| \).

For the case \( x-y = 0 \) we must prove that \( f \) is injective. Now \( z \neq t \), so \( z-t \in \alpha \) for some \( \alpha \) hence \( f(z)-f(t) \in \sigma(\alpha) \). Thus, \( f(z) \neq f(t) \).

(vii) Let \( \rho := \inf |f(x)-f(y)| \). If \( \rho > 0 \) then clearly \( f \) is nowhere continuous. If \( \rho = 0 \), let \( \varepsilon > 0 \). There is a, b \( \in X \), \( a \neq b \) such that \( |f(a)-f(b)| < \varepsilon \). By (vi), for all \( x, y \in X \) with \( |x-y| < |a-b| \), \( |f(x)-f(y)| < |f(a)-f(b)| < \varepsilon \). Then \( f \) is uniformly continuous.

A natural question that can be raised is the following. If \( f : X \rightarrow K \) is of type \( \sigma \), does it follow that \( \sigma \) is injective? We have (see also 3.18 and 3.20)

**Theorem 3.17.** Let \( f : X \rightarrow K \) be monotone of type \( \sigma \). Then the following conditions are equivalent.

(a) \( \sigma \) is injective.
\(f \in M^*_D(X)\).

(\gamma) \ f \in M^*_\text{ubs}(X).

(\delta) If, for \(a, \beta \in \Sigma(X)\), \(a \oplus \beta\) is defined then so is \(\sigma(a) \oplus \sigma(\beta)\) and \(\sigma(a) \oplus \sigma(\beta) = \sigma(a \oplus \beta)\).

(\epsilon) If \(a, \beta \in \Sigma(X)\), \(|\sigma(a)| < |\sigma(\beta)|\) then \(|a| < |\beta|\).

**Proof.** We prove (\alpha) \(\Rightarrow\) (\epsilon) \(\Rightarrow\) (\gamma) \(\Rightarrow\) (\beta) \(\Rightarrow\) (\delta) \(\Rightarrow\) (\alpha).

(\alpha) \(\Rightarrow\) (\epsilon). Let \(|\sigma(a)| < |\sigma(\beta)|\) then \(\sigma(a) \oplus \sigma(\beta) = \sigma(\beta)\) (1.2.(vi)). By 3.16, (iii), \(\sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta) = \sigma(\beta)\). Since \(\sigma\) is injective, \(a \oplus \beta = \beta\) so (again 1.2.(vi)) \(|a| < |\beta|\).

(\epsilon) \(\Rightarrow\) (\gamma). Let \(|x-y| < |z-t|\) \((x,y,z,t \in X)\). We prove \(|f(x) - f(y)| < |f(z) - f(t)|\). If \(z = t\) there is nothing to prove. Assume \(z \neq t\) and \(|f(x) - f(y)| > |f(z) - f(t)|\). Then \(f\) is injective, supposing \(x-y \in a\), \(z-t \in \beta\) for some \(a, \beta \in \Sigma(X)\), we have \(f(x) - f(y) \in \sigma(a)\), \(f(z) - f(t) \in \sigma(\beta)\) and \(|\sigma(a)| > |\sigma(\beta)|\). By (\epsilon), \(|a| > |\beta|\) i.e., \(|x-y| > |z-t|\). Contradiction.

(\gamma) \(\Rightarrow\) (\beta). Trivial.

(\beta) \(\Rightarrow\) (\delta). Suppose \(\sigma(a) \oplus \sigma(\beta)\) is not defined. Then \(|\sigma(a)| = |\sigma(\beta)|\) and, by 3.16 (iii), \(|a| = |\beta|\). Choose \(x, y, z\) such that \(x-y \in a\), \(y-z \in \beta\). Then \(f(x) - f(z) \in \sigma(a) + \sigma(\beta)\) so \(|f(x) - f(z)| < |\sigma(a)| = |f(x) - f(y)|\).

Since \(f \in M^*_D(X)\), \(|x-z| < |x-y|\) hence, since \(x-z \in a \oplus \beta\), \(x-y \in a\): \(|a \oplus \beta| < |a|\). But \(|a \oplus \beta| = \max(|a|, |\beta|)\), a contradiction.

(\delta) \(\Rightarrow\) (\alpha). Suppose \(\sigma(a) = \sigma(\beta)\) and \(a \neq \beta\). Then \(a \oplus (-\beta)\) is defined. By (\delta), also \(\sigma(a) \oplus \sigma(-\beta)\) is defined. But \(\sigma(-\beta) = -\sigma(\beta) = -\sigma(a)\), so \(\sigma(a) \oplus -\sigma(a)\) is defined, a contradiction.

**Theorem 3.18.** Let \(k\) be a prime field. Then, if \(f : X \to K\) is monotone of type \(\sigma\) then \(\sigma\) is injective.
Proof. Suppose $\sigma(\alpha) = \sigma(\beta)$ for some $\alpha, \beta \in \Sigma(X)$. Then $|\sigma(\alpha)| = |\sigma(\beta)|$ so, by 3.16 (iii), $|a| = |b|$. There is $t \in K$, $|t| = 1$ such that $\beta = ta$. Since $k$ is a prime field we may suppose $t \in \{1, 2, \ldots, p-1\}$ if $k \sim \mathbb{F}_p$ and $t \in \mathbb{Q}^*$ if $k \sim \mathbb{Q}$. So, by 3.16 (iv), $\sigma(\beta) = \sigma(ta) = t\sigma(\alpha) = t\sigma(\beta)$. For $x \in \sigma(\beta)$ we have $tx \in \sigma(\beta)$, so $tx \cdot x^{-1} \in K$ i.e., $|t-1| < 1$. It follows easily that $t = 1$. Hence, $a = \beta$.

We now like to determine all $\sigma : \Sigma \rightarrow \Sigma$ that "can occur" as the type of a monotone function in case $K = \mathbb{Q}_p$. We use the fact that $\Sigma$ can be identified with the following subgroup of $\mathbb{Q}_p^*$

$$\{\theta^i_n : i \in \{0, 1, 2, \ldots, p-2\}, n \in \mathbb{Z}\}$$

where $\theta$ is a primitive $(p-1)^{th}$ root of 1. (See 1.5.)

First, let $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be monotone of some type $\sigma : \Sigma \rightarrow \Sigma$. By 3.18, $\sigma$ is injective. By 3.17, (e), 3.16 (iii) we have $|a| < |b| \leftrightarrow |\sigma(a)| < |\sigma(b)|$ and $|a| = |b| \leftrightarrow |\sigma(a)| = |\sigma(b)|$, so $|\sigma(a)|$ is a strictly increasing function of $|a|$.

Set

$$\sigma^{(i,n)}_p = \theta^i s(i,n)_p \lambda(i,n) \quad ((\theta^i_p)^n \in \Sigma)$$

Where $s : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \rightarrow \{0, 1, 2, \ldots, p-2\}$ and $\lambda : \{0, 1, 2, \ldots, p-2\} \times \mathbb{Z} \rightarrow \mathbb{Z}$. We see that $|\sigma^{(i,n)}_p| = |\sigma^{(j,n)}_p|\lambda(i,n)$ for all $i, j \in \{0, 1, 2, \ldots, p-2\}$ hence $\lambda(i,n) = \lambda(j,n)$ for all $i, j \in \{0, 1, 2, \ldots, p-2\}$. Then

$$\sigma^{(i,n)}_p = \theta^i s(i,n)_p \lambda(n)$$

where $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is a strictly increasing function (in the classical sense).

By 3.16 (v), $\sigma^{(i,n)}_p = \theta^i s(0,n)_p \lambda(n)$.
Thus, \( \sigma \) is of the form

\[
(*) \quad \theta^* p^n \to \theta^* s(n) p^\lambda(n)
\]

where \( s : \mathbb{N} \to \{0,1,2,...,p-2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

Conversely, if we are given a map \( \sigma \) of the form \((*)\) then it is easy to construct an \( f : \mathbb{Q}_p \to \mathbb{Q}_p' \), monotone of type \( \sigma \). In fact, let \( x \in \mathbb{Q}_p' \), \( x = \sum a_n p^n \), where \( a_n \in \{0,1,\ldots,p^{B-2}\} \) for each \( n \) and \( a_{-n} = 0 \) for large \( n \). Then set

\[
f(x)^{n \in \mathbb{Z}} = \sum a_n \theta^s(n) p^\lambda(n).
\]

Now let \( x = \sum a_n p^n, y = \sum b_n p^n \) and \( \pi(x-y) = \theta^a p^m \) for some \( a \in \{0,1,...,p-2\} \), \( m \in \mathbb{Z} \). Then \( a = b_n \) for \( n < m \) and \( a - b_m = \theta^a \mod p \). So the sign of \( a - b_m \) is \( \theta^a \). \( f(x)^{n \in \mathbb{Z}} - f(y)^{n \in \mathbb{Z}} = \sum (a_n - b_n) \theta^s(n) p^\lambda(n) = (a_m - b_m) \theta^s(m) p^\lambda(m) + r \), where \( |r| < |f(x)-f(y)| \). The sign of \( f(x)-f(y) \) is the sign of \( (a_m - b_m) \theta^s(m) p^\lambda(m) \) which is \( \theta^a \theta^s(m) p^\lambda(m) \). So \( \pi(f(x)-f(y)) = \theta^a \theta^s(m) p^\lambda(m) = \sigma(\theta^a p^m) \). Thus, \( f \) is monotone of type \( \sigma \). We have found

**THEOREM 3.19.** The set \( \{\sigma : \Sigma \to \Sigma : \text{there is } f : \mathbb{Q}_p \to \mathbb{Q}_p' \text{, monotone of type } \sigma\} \) is equal to the set of all \( \sigma : \Sigma \to \Sigma \) of the form

\[
\theta^* p^n \to \theta^* s(n) p^\lambda(n)
\]

where \( s : \mathbb{Z} \to \{0,1,2,...,p-2\} \) and \( \lambda : \mathbb{Z} \to \mathbb{Z} \) is strictly increasing.

**Remark.** With the notations as in 3.19, let \( \mu(n) := \lambda(n) - n \). Then \( \mu : \mathbb{Z} \to \mathbb{Z} \) is increasing \( (\mu(n+1) = \lambda(n+1) - (n+1) \geq \lambda(n)+1-(n+1) = \mu(n)) \). We then have two possibilities for a function \( f : \mathbb{Q}_p \to \mathbb{Q}_p' \), monotone of type \( \sigma \).
(a) \( \lim_{n \to \infty} u(n) = \infty \). Then \( |\sigma(a)| = |a| |p^{\mu(n)}| \), \( (a = \theta^{i_n} p^n) \), so \( \lim_{|a| \to 0} |\sigma(a)| = 0 \).

Thus \( \lim_{x-y \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0 \) uniformly: \( f \) is uniformly differentiable and \( f' = 0 \).

(b) \( u \) is bounded above. Then \( u(n) \) is constant, \( c \), for \( n \geq n_0 \). (For example, if \( \sigma \) is bijective then we have even \( u(n) = c \) for all \( n \).)

Thus, for sufficiently small \( |a| \) \( (a = \theta^{i_n} p^n \in \Sigma) \) we have

\[
|\sigma(a)| = |p^{\lambda(n)}| = |p^{n+c}| = |p^c| |a|.
\]

So, there is \( r \) such that \( |x-y| < r \) implies \( |f(x)-f(y)| = |p^c||x-y| \).

In this case we then have: there is \( r > 0 \) and \( \lambda \in \mathbb{Q}_p \) such that on each ball in \( \mathbb{Q}_p \) of radius \( r \), \( \lambda^{-1}f \) is an isometry.

We now construct an example of a function \( f \) monotone of type \( \sigma \), where \( \sigma \) is not injective. Let \( p = 3 \) mod 4 and let \( K := \mathbb{Q}_p(\sqrt{-1}) \). The elements of \( K \) can be written as \( a+bi \) \( (a,b \in \mathbb{Q}_p) \) and \( |a+bi| = \max(|a|,|b|) \).

The value group of \( K \) is the same as the one of \( \mathbb{Q}_p \), the residue class field has \( p^2 \) elements (hence is not a prime field, see 3.18). Let \( X \) be the unit ball of \( K \), let

\[
S := \{a+bi: a,b \in \{0,1,2,\ldots,p-1\}\}.
\]

For each \( x \in X \) there is a unique \( \tilde{x} \in S \) such that \( |x-\tilde{x}| < 1 \). As in section 1, let \( \pi : K^* \to \Sigma \) be the sign map. Notice that \( \pi(s) \neq \pi(t) \) for \( s,t \in S^*, s \neq t \).

Define a function \( h : S \to K \) as follows

\[
h(a+bi) = \frac{1}{p^a} (a+bi \in S)
\]
and let \( f : X \to K \) be defined via
\[
f(x) = x + h(x) \quad (x \in X).
\]

We claim that \( f \) is monotone of type \( \sigma \) where
\[
\sigma(\pi(a+bi)) = \pi\left(\frac{1}{p} a\right) \text{ if } a+bi \in S, \ a \neq 0
\]
\[
\sigma(a) = a \quad \text{elsewhere.}
\]

(Clearly, \( \sigma \) is a well defined map \( \Sigma(X) \to \Sigma, \sigma \) is not injective since, for example, \( \sigma(\pi(1)) = \sigma(\pi(1+i)) \).

**Proof.** Let \( |a| < 1 \) and \( x-y \in \alpha \), then \( |x-y| < 1 \) so \( \bar{x} = \bar{y}, \ h(\bar{x}) = h(\bar{y}) \).

It follows that \( f(x)-f(y) = x-y \in \alpha = \sigma(\alpha) \).

Now let \( |a| = 1 \) be of the form \( \pi(bi), \ b \in \{1,2,\ldots,p-1\} \) and let \( x-y \in \alpha \). Say, \( \bar{x} = r+si, \ \bar{y} = t+ui \ (r,s,t,u \in \{0,1,2,\ldots,p-1\}) \). Then also \( \bar{x}-\bar{y} \in \alpha \), so \( |r+si-t+ui| < 1 \) hence \( r = t \). Thus, \( h(\bar{x}) = \frac{1}{p} r = h(\bar{y}) \), and we have \( f(x)-f(y) = x-y \in \alpha = \sigma(\alpha) \).

Finally, let \( |a| = 1, a = \pi(a+bi) \), where \( a \neq 0 \) \( (a,b \in \{0,1,2,\ldots,p-1\}) \) and \( x-y \in \alpha \). Set \( \bar{x} = r+si, \ \bar{y} = t+ui \). Then \( \bar{x}-\bar{y} \in \alpha \), so \( r-t = a \bmod p \).

We find \( h(\bar{x}) = \frac{1}{p} r, h(\bar{y}) = \frac{1}{p} t \), so \( |h(\bar{x})-h(\bar{y})| < \frac{1}{|p|} |a| \) i.e. \( h(\bar{x})-h(\bar{y}) \in \pi(\frac{1}{p} a) \). Since \( |\pi(x-y)| < 1 \), we find \( f(x)-f(y) = x-y-(h(\bar{x})-h(\bar{y})) \in \pi(\frac{1}{p} a) = \sigma(\pi(a+bi)) = \sigma(\alpha) \).

Concluding:

**EXAMPLE 3.20.** Let \( p = 3 \bmod 4 \) and \( K = \mathbb{Q}_p(\sqrt{-1}) \). Then there exists a function \( f : \{x \in K: |x| \leq 1\} \to K \), monotone of some type \( \sigma \), where \( \sigma \) is not injective.

In case \( K \) has discrete valuation we have some extra information.
THEOREM 3.21. Let $K$ have discrete valuation and let $f : X \to K$ be monotone of type $\sigma \in \Sigma(X)$. Then

(i) $f$ is continuous.

(ii) If $\sigma$ is injective, then $f$ and $\phi(f)$ are bounded on bounded sets.

(iii) If $\sigma$ is injective and if $f(X)$ is convex then there is $\lambda \in K$ such that $\lambda f$ is an isometry.

Proof. (i) Follows from 3.16 (vi) and 2.13 (2) (c). If $\sigma$ is injective then by 3.16 (iii) and 3.17 (e), $|\sigma(a)|$ is a strictly increasing function of $|a|$. Suppose $X$ is bounded. Then $\Sigma(X) = \{a \in E : |a| \leq r\}$ for some $r \in K^*$. Let $|\sigma(a)| = s$ whenever $|a| = r$. Let $|\pi| < 1$ be the generator of $|K^*|$. Let $|a| = |\pi|r$. Then $|a| < r$ so $|\sigma(a)| < s$, hence $|\sigma(a)| \leq |\pi|s$. By induction, it follows that $|\sigma(a)| \leq |\pi|^n s$ whenever $|a| = |\pi|^n r$, so $|\sigma(a)| \leq |a| \cdot \frac{s}{r}$. So the difference quotients of $f$ are bounded by $sr^{-1}$. Then clearly $f$ is bounded. So we have (ii). We prove (iii). If $f(X)$ is convex then $\Sigma(f(X))$ has the form $\{a \in E : |a| \leq s\}$ for some $s \in |K^*| \cup \{\infty\}$. Then $\sigma$ induces an injection of $\{p \in |K^*| : p \leq r\}$ onto $\{a \in |K^*| : |a| \leq s\}$ that is (strictly) increasing. It follows easily that this map is a multiplier. So $|\sigma(a)| = c|a|$ for some $c \in IR$ i.e., $|f(x) - f(y)| = |c||x - y|$ for all $x, y \in X$.

3.21. (ii) induces the question under what conditions an $f : X \to K$, monotone of type $\sigma$, is bounded on bounded sets. We have an affirmative answer in each of the following cases.

(1) $K$ is a local field ($f$ is continuous).

(2) $K$ is finite and $X = \{x \in K : |x| \leq 1\}$. (Proof let $a_1, a_2, \ldots, a_n \in X$ be
representatives modulo \( \{ x \in K : |x| < 1 \} \), let \( M = \max |f(a_i) - f(a_j)| \). For each \( x, y \in X \) we have \( i, j \) for which \( |x - a_i| < 1, |y - a_j| < 1 \). Since \( f \in M(X) \), we have \( |f(x) - f(a_i)| < M, |f(y) - f(a_j)| < M \) whence \( |f(x) - f(y)| \leq M \); \( f \) is bounded.)

(3) \( K \) is discrete, \( \sigma \) is injective (this is 3.21 (ii)).

On the other hand we have the following

EXAMPLE 3.22. Let \( k \) be isomorphic to the algebraic closure of \( \mathbb{F}_p \). Let \( X \) be the unit ball of \( K \). Then there exists a function \( f : X \to K \), monotone of type \( \sigma \), for some \( \sigma : \Sigma(X) \to \Sigma \) such that

(i) \( \sigma \) is not injective.

(ii) \( f, \Phi(f) \) are unbounded.

Proof. As an \( \mathbb{F}_p \)-vector space, \( k \) has a countable base \( e_1, e_2, \ldots \). For any \( \lambda \in \mathbb{F}_p \), \( \lambda = n \lambda^p \) for some \( n \in \{0, 1, 2, \ldots, p-1\} \). (Here for a field \( L \), \( 1_L \) is the unit element of \( L \).) Define \( \lambda_n := n \lambda \). Choose \( c_1, c_2, \ldots \in K \) such that

\[ 0 < |c_1| < |c_2| < \ldots, \lim_{n \to \infty} |c_n| = \infty, \]

and define a map \( h : k \to K \) via

\[ h(\lambda_n e_n) = \lambda_n c_n \quad (\lambda_n e_n \in K) \]

Define \( f : X \to K \) by

\[ f(x) = x + h(x) \quad (x \in X) \]

(Here \( h(x) \) is the image of \( x \) under the canonical map \( X \to k \)).

Then clearly \( f \) is unbounded and so is \( \Phi(f) \).

Let us identify \( \{ \alpha \in \Sigma : |\alpha| = 1 \} \) with \( k^* \) in the obvious way. We claim that \( f \) is monotone of type \( \sigma \) where
\[
\sigma(a) = \begin{cases} 
\alpha & \text{if } |\alpha| < 1 \\
\pi(\lambda_n c_n) & \text{if } \alpha = \sum \lambda_m e_m, \ n = \max(m : \lambda_m \neq 0).
\end{cases}
\]

In fact, let \( x-y \in \alpha \) and \(|\alpha| < 1 \). Then \( h(x) = h(y) \) so \( f(x)-f(y) = x-y \in \sigma(\alpha) \). Now let \( x-y \in \alpha \) where \(|\alpha| = 1 \). Then set \( \overline{x} = \sum \lambda_n e_n, \overline{y} = \sum \mu_n e_n \). Let \( r = \max\{n : \lambda_n \neq \mu_n\} \). Then \( \overline{x-y} = \sum (\lambda_n-\mu_n)e_n = \alpha \), so \( \sigma(\alpha) = \pi((\lambda_n-\mu_n)c_n) \).

On the other hand, \( f(x)-f(y) = x-y-(h(x)-h(y)) = x-y-\sum (\lambda_n-\mu_n)c_n = x-y-\sum (\lambda_n-\mu_n)c_n \). Thus \( \pi(f(x)-f(y)) = \pi((\lambda_n-\mu_n)c_n) \).

Now we have \( \sum_{n=1}^{r} \lambda_n c_n = \sum_{n=1}^{r} \mu_n c_n \mod p \), so \( \pi(\lambda_n-\mu_n) = \pi(\lambda_n-\mu_n) \). It follows that \( f(x)-f(y) \in \sigma(\alpha) \).

Obviously, \( \sigma \) is not injective.

We will consider briefly the differentiable functions, monotone of type \( \sigma \). Let \( f : X \to K \) be such a function. If \( f'(a) = 0 \) for some \( a \in X \), then \( \lim_{|a| \to 0} \frac{|\sigma(a)|}{|a|} = 0 \), so \( f' = 0 \) uniformly on \( X \). Now let \( f'(a) \neq 0 \). Then if \( x \) is sufficiently close to \( a \), we have \( \frac{|f(x)-f(a)|}{x-a} < |f'(a)| \).

Thus for \(|a| \) small enough we have \( f'(a) \in \frac{\sigma(a)}{a} \) i.e. \( \frac{\sigma(a)}{a} \) is constant.

This implies that \( \pi(f'(x)) \) does not depend on \( x \) (\( f' \) has constant sign) and that locally \( f \) is monotone of type \( \beta \) for some \( \beta \in \Sigma \).

We end this section with a discussion on the question which maps \( \sigma : \Sigma \to \Sigma \) do occur as a type of a monotone function.

**Lemma 3.23.** Let \( \sigma : \Sigma \to \Sigma \). Suppose \( \sigma \) satisfies

if \( \alpha \oplus \beta \) is defined then \( \sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta) \). \((\alpha, \beta \in \Sigma)\).

Then

(i) \( \sigma(-\alpha) = -\sigma(\alpha) \). \((\alpha \in \Sigma)\).
(ii) If $\sigma(a)$ is defined then so is $a \oplus \beta$.

(iii) If $|a| < |\beta|$ then $|\sigma(a)| < |\sigma(\beta)|$.

(iv) $\sigma$ is injective.

(v) If $|a| = |\beta|$ then $|\sigma(a)| = |\sigma(\beta)|$.

Proof. (i) is trivial if $\chi(k) = 2$, so suppose $\chi(k) \neq 2$ and let $-\sigma(a) \neq \sigma(-a)$ for some $a \in \Sigma$. Then we have the identity $(a \oplus a) \oplus (-a) = a$, so $\sigma(a \oplus a) \oplus (-a) = \sigma(a)$, whence $(\sigma(a) \oplus \sigma(a)) \oplus (-a) = \sigma(a)$. Now by 1.2 (iii) $\sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus (-a))$ (this last expression is defined).

If not, then $-\sigma(a) = \sigma(a) \oplus \sigma(-a)$. Now $\sigma(a) \oplus \gamma = -\sigma(a)$ has only one solution namely $\gamma = -2\sigma(a)$. So we then would have $\sigma(-a) = -2\sigma(a) = -(\sigma(a) \oplus \sigma(a))$, but this contradicts the existence of $(\sigma(a) \oplus \sigma(a)) \oplus (-a))$.

From $\sigma(a) = \sigma(a) \oplus (\sigma(a) \oplus (-a))$ we obtain by 1.2 (vi): $|\sigma(a) \oplus \sigma(-a)| < |\sigma(a)|$. On the other hand, by 1.2 (v), $|\sigma(a) \oplus \sigma(-a)| = |\sigma(a)| + |\sigma(-a)|$.

Contradiction. (i) follows.

Now (ii) follows easily from (i): if $a \oplus \beta$ were not defined then $\beta = -a$ so, by (i), $\sigma(a) \oplus \sigma(\beta) = \sigma(a) \oplus -\sigma(a)$, a contradiction. Let $|a| < |\beta|$, then $a \oplus \beta = \beta$, so $\sigma(a \oplus \beta) = \sigma(a) \oplus \sigma(\beta) = \sigma(\beta)$. By 1.2 (vi) we find $|\sigma(a)| < |\sigma(\beta)|$. We proved (iii).

If $\sigma(a) = \sigma(\beta)$ and $a \neq \beta$ then $\sigma(a \oplus (-\beta)) = \sigma(a) \oplus \sigma(-\beta) = \sigma(a) \oplus -\sigma(a)$, an absurdity. So $\sigma$ is injective (iv). Finally, let $|a| = |\beta|$ and $|\sigma(a)| > |\sigma(\beta)|$. Then $\sigma(\alpha) = \sigma(a) \oplus \sigma(\beta) = (\text{by (ii)}) = \sigma(a \oplus \beta)$. By injectivity of $\sigma$, $a = a \oplus \beta$, and by 1.2 (vi), we find $|\beta| < |\alpha|$.

Now we have
LEMMA 3.24. Let $K$ be spherically complete, let $Y \subseteq K$ (not necessarily convex) and let $\tau : \Sigma(Y) = \{\pi(x-y) : x, y \in Y, x \neq y\} \to \Sigma$ such that $f$ is monotone of type $\tau$ (i.e., $x, y \in Y, x-y \in \alpha \in \Sigma(Y)$ then $f(x)-f(y) \in \tau(\alpha)$.

Suppose $\tau$ can be extended to a $\sigma : \Sigma \to \Sigma$ satisfying the condition of Lemma 3.23. Then $f$ can be extended to a monotone function $f : K \to K$ of type $\sigma$.

Proof. By Zorn's lemma, it suffices to extend $f$ to $Y \cup \{a\}$ ($a \notin Y$) such that $f(x)-f(a) \in \sigma(\pi(x-a))$, $f(a)-f(x) \in \sigma(\pi(a-x))$ for all $x \in Y$. By 3.23 (i) it suffices to consider only the second case. Let $B_x := f(x) + \sigma(\pi(a-x))$ ($x \in Y$). Each $B_x$ is a ball with radius $|\sigma(\pi(a-x))|$. By the spherical completeness of $K$, we are done if we can show that $B_x \cap B_y \neq \emptyset$ ($x \neq y, x, y \in Y$).

Set $a := \pi(a-x)$ and $\beta := \pi(a-y)$. Let $b \in \sigma(a)$; $c \in \sigma(\beta)$. We prove:

$|f(x)+b-f(y)-c| < |\sigma(a)| \wedge |\sigma(\beta)|$. We have two cases:

1) $\alpha = \beta$. Then $a-x \in a, a-y \in \alpha$ implies $|x-y| < |a-x| = |a|$, so $|\pi(x-y)| < |a|$ whence $|\pi(f(x)-f(y))| = |\sigma(\pi(x-y))| < |\sigma(a)|$ (by 3.23 (iii)), so $|f(x)-f(y)| < |\sigma(a)|$. Further, $b \in \sigma(\alpha), c \in \sigma(\alpha)$ implies $|b-c| < |\sigma(\alpha)|$, hence $|f(x)+b-f(y)-c| < |\sigma(\alpha)|$.

2) $\alpha \neq \beta$. Then $x-y = a-y-(a-x) \in \beta \oplus (-\alpha)$, so $f(x)-f(y)+b-c \in \sigma(\beta \oplus -\alpha) + \sigma(\alpha) + \sigma(-\beta) = \sigma(\beta \oplus -\alpha) + \sigma(\alpha \oplus -\beta) = \sigma(\beta \oplus (-\alpha)) - \sigma(\beta \oplus -\alpha)$, hence $|f(x)-f(y)+b-c| < |\sigma(\beta \oplus -\alpha)| = |\sigma(\beta) \oplus \sigma(-\alpha)| = \max(|\sigma(a)|, |\sigma(\beta)|)$.

THEOREM 3.25. Let $K$ be spherically complete and let $\sigma : \Sigma \to \Sigma$. Suppose

$\sigma(\alpha \oplus \beta) = \sigma(\alpha) \oplus \sigma(\beta)$ ($\alpha, \beta \in \Sigma, \alpha \neq -\beta$).

Then there exists a function $f : K \to K$, monotone of type $\sigma$. 
Proof. Choose \( Y := \{0\} \) and let \( g : Y \to K \) be defined via \( g(0) = 0 \). Then \( g \) satisfies the conditions of Lemma 3.24 so it can be extended to a function \( f \) of type \( \sigma \).

We now give a description of the maps \( \sigma : \Sigma \to \Sigma \) mentioned in 3.23. For each \( r \in |K^*| \) choose \( a_r \in \Sigma \) such that \( |a_r| = r \). Further, there is a natural isomorphism of multiplicative groups between \( k^* \) and \( \{a \in \Sigma : |a| = 1\} \), denoted by \( l \mapsto a_l \) (\( l \in k^* \)). Of course, if \( 1+1' \neq 0 \) then \( a_{l+1'} = a_l \otimes a_{1'} \). Each element of \( \Sigma \) can be written in only one way as \( a_r a_l \) (\( r \in |K^*|, l \in k^* \)). Now if \( \sigma \) is as in 3.23 we get

\[
\sigma(a_r a_l) = a_{\lambda(r)} n(r,l)
\]

where \( \lambda : |K^*| \to |K^*| \) is strictly increasing and \( l \mapsto n(r,l) \) is an injective group endomorphism of the additive group \( k \). Conversely, if \( \lambda : |K^*| \to |K^*| \) is strictly increasing and for each \( r, l \mapsto n(r,l) \) is an injective group homomorphism \( k \to k \) then

\[
\sigma(a_r a_l) = a_{\lambda(r)} n(r,l) \quad (a_r a_l \in \Sigma)
\]

satisfies the condition of 3.23. So we get

**THEOREM 3.26.** Let \( K \) be spherically complete and let \( |K| = [0, \infty) \). Then there exist a nowhere continuous \( f : K \to K \), monotone of some type \( \sigma : \Sigma \to \Sigma \).

**Proof.** With the notations as above, let \( \sigma : \Sigma \to \Sigma \) be defined as follows

\[
\sigma(a_r a_l) = a_{r+1} a_l.
\]

By 3.25 there is an \( f : K \to K \) monotone of type \( \sigma \). Clearly \( |f(x) - f(y)| \geq 1 \) if \( x \neq y \) so \( f \) is nowhere continuous.
In this section we study \( M_w(X), M_d(X), M_s(X) \). To avoid unnecessary complications we ASSUME THROUGHOUT THIS SECTION THAT \( X \) IS A CLOSED SUBSET OF \( K \) WITHOUT ISOLATED POINTS. We collect here the results on monotone functions that are valid for general \( K \). In the next section we will see what happens if we put some extra conditions on \( K \) (e.g., \(|K| \) discrete, ...).

First two elementary lemmas.

**LEMMA 4.1** Let \( f : X \to K \). Then the following conditions are equivalent

1. \( f \in M_w(X) \) (see Def. 2.11).
2. For all \( x, y, z \in X \), \(|x-y| < |x-z| \) implies \(|f(x)-f(z)| \leq |f(y)-f(z)|\).
3. For all \( x, y, z \in X \), \(|f(x)-f(z)| \neq |f(y)-f(z)| \) implies \(|x-y| = \max(|x-z|, |y-z|)\).

**Proof.** (a) \( \Rightarrow \) (b). \(|x-y| < |x-z| \) implies \(|y-z| = |x-z| > |x-y|\), so

\[
|f(x)-f(y)| \leq \min(|f(x)-f(z)|, |f(y)-f(z)|).
\]

It follows that \(|f(x)-f(z)| = |f(y)-f(z)|\).

(b) \( \Rightarrow \) (γ). (b) says that \(|f(x)-f(z)| \neq |f(y)-f(z)| \) implies \(|x-y| \geq |x-z|\). By symmetry, also \(|x-y| \geq |y-z|\) where \(|x-y| \geq \max(|x-z|, |y-z|)\). The opposite inequality is trivial.

(γ) \( \Rightarrow \) (a). Let \(|x-y| < |x-z|\). Then \(|x-y| \neq \max(|x-z|, |z-y|)\), so, by (γ), \(|f(x)-f(z)| \neq |f(y)-f(z)|\). Then \(|f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(y)-f(z)|) = |f(x)-f(z)|\).

**LEMMA 4.2** (i) If \( f \in M_w(X) \), \( \lambda \in K \) then \( \lambda f \in M_w(X) \).
(ii) If \( f_1, f_2, \ldots \in M_w(X) \) and \( f := \lim_{n \to \infty} f_n \) pointwise then \( f \in M_w(X) \).

(iii) If \( f \in M_w(X) \) and \( g : f(X) \to X \) is such that \( f \circ g \) is the identity on \( f(X) \), then \( g \in M_w(f(X)) \). In particular, if \( f \) is injective and weakly monotone then so is \( f^{-1} \).

(Notice that \( f(X) \) need not be closed and may have isolated points.)

Proof. Obvious.

Remark. Lemmas, similar to 4.1 and 4.3, but now for \( M_b(X), M_s(X), M_{bs}(X) \) have been formulated in 2.2, 2.3, 2.8, 2.9, 2.12.

Although an \( M_w \)-function need not be continuous (see 2.4(5), 3.26) we will derive properties of \( M_w \)-functions that are closely related to continuity.

**Lemma 4.3** Let \( f \in M_w(X) \). Then \( f \) is bounded on precompact subsets of \( X \).

*Proof.* Let \( Y \subset X \) be precompact. Assume that \( Y \) is not a singleton. Then \( Y \) is bounded and has a positive diameter \( r = \max(\{|x-y| : x, y \in Y\}) \).

The equivalence relation \( x \sim y \) iff \( |x-y| < r \) divides \( Y \) into finitely many classes \( Y_1, \ldots, Y_n \) \((n \geq 2)\). Choose \( a_i \in Y_i \) for each \( i \), and let \( M := \max |f(a_i)| \). We prove: \( |f| \leq M \). In fact, let \( x \in Y \). Then there is \( i \) such that \( |x-a_i| < r \). Choose \( j \neq i \). We have \( |x-a_i| < |a_i-a_j| \) whence \( |f(x)-f(a_i)| \leq |f(a_i)-f(a_j)| \leq M \). So \( |f(x)| \leq M \).

The following lemma shows that an \( f \in M_w(X) \) at \( a \in X \) is either continuous or "very discontinuous".

**Lemma 4.4** Let \( f \in M_w(X) \) and let \( a \in X \). Then we have the following alternative.
Either \( f \) is continuous at \( a \), or for each sequence \( x_1, x_2, \ldots \in X \) (\( x_n \neq a \) for all \( n \)) with \( \lim x_n = a \) the sequence \( f(x_1), f(x_2), \ldots \) is bounded and has no convergent subsequence.

*Proof.* Since \( \{x_1, x_2, \ldots \} \) is precompact the set \( \{f(x_1), f(x_2), \ldots \} \) is bounded by Lemma 4.3. We are done if we can prove the following. If \( x_1, x_2, \ldots, \lim x_n = a, x_n \neq a \) for all \( n \), \( \lim f(x_n) \) exists, then \( f \) is continuous at \( a \). Now set \( a := \lim f(x_n) \). Let \( y_1, y_2, \ldots \in X \), \( \lim y_n = a \).

We prove \( \lim f(y_n) = a \). (Then it follows that \( a = f(a) \) since we may choose \( y_n := a \) for all \( n \).) Let \( \varepsilon > 0 \). There is \( k \in \mathbb{N} \) for which
\[
|f(x_k) - a| < \varepsilon.
\]
For \( n \) sufficiently large we have \( |y_n - a| < |x_k - a| \), so for large \( m \) (depending on \( n \)) we have \( |y_n - x_m| < |x_k - x_m| \), whence
\[
|f(y_n) - f(x_m)| \leq |f(x_k) - f(x_m)|.
\]
Since \( \lim_{m \to \infty} f(x_m) = a \) we find
\[
|f(y_n) - a| \leq |f(x_k) - a| < \varepsilon,
\]
so \( \lim f(y_n) = a \).

**Corollary 4.5** Let \( f \in \mathcal{M}_w(X) \). Then the graph of \( f \)
\[
\Gamma_f := \{(x, y) \in X \times K : y = f(x)\}
\]
is closed in \( K^2 \).

*Proof.* Let \( (x_n, f(x_n)) \in \Gamma_f \) and let \( \lim x_n = x, \lim f(x_n) = a \). If \( x_n = x \) for infinitely many \( n \) then \( a = f(x) \), so \( (x, a) \in \Gamma_f \). If not then by the alternative of lemma 4.4, \( f \) is continuous at \( x \), so \( a = f(x) \) and \( (x, a) \in \Gamma_f \).

**Corollary 4.6** Let \( f \in \mathcal{M}_w(X) \). If each bounded subset of \( f(X) \) is precompact then \( f \) is continuous.

*Proof.* Direct consequence of 4.4.

To prove a statement dual to 4.4 (see 4.8) we first formulate a dual form of 4.3.
LEMMA 4.7 Let \( f \in M_w(X) \) and let \( Y \subset f(X) \) be precompact. Then either

\[ f \text{ is constant on } f^{-1}(Y) \text{ or } f^{-1}(Y) \text{ is bounded.} \]

Proof. It suffices to prove: if \( Z \subset X \) is unbounded and \( f(Z) \) is precompact then \( f \) is constant on \( Z \). Let \( a, b \in Z \). Since \( Z \) is unbounded there are \( x_1, x_2, \ldots \in Z \) such that

\[ (*) \quad |a-b| < |x_1-a| < |x_2-a| < \ldots \]

Since \( f(Z) \) is precompact we may assume (by taking a suitable subsequence) that \( a' = \lim f(x_n) \) exists. From \( (*) \) we obtain

\[ |x_1-x_2| = |x_2-a|, \quad |x_2-x_3| = |x_3-a|, \ldots, \]

so

\[ |a-b| < |x_1-a| < |x_1-x_2| < |x_2-x_3| < \ldots \]

hence

\[ |f(a)-f(b)| \leq |f(x_1)-f(a)| \leq |f(x_1)-f(x_2)| \leq \ldots \]

it follows that \[ |f(a)-f(b)| = \lim_{n \to \infty} |f(x_n)-f(x_{n+1})| = 0 \] i.e., \( f(a) = f(b) \).

LEMMA 4.8 Let \( f \in M_w(X) \) and let \( a \in f(X) \) be a non-isolated point of \( f(X) \).

Then we have the following alternative. Either

I. There is \( a \in X \) such that for each sequence \( x_1, x_2, \ldots \) in \( X \)

for which \( \lim_{n \to \infty} f(x_n) = a \) we have \( \lim_{n \to \infty} x_n = a \), or

II. If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} f(x_n) = a, f(x_n) \neq a \) for all \( n \),

then \( x_1, x_2, \ldots \) is bounded and has no convergent subsequence.

Proof. Suppose we are not in case II. Since \( a \) is not isolated in \( f(X) \) and \( f(X) \) is dense in \( f(X) \) we have a sequence \( x_1, x_2, \ldots \) in \( X \) for which

\[ f(x_n) \neq a \] for each \( n \), and \( \lim_{n \to \infty} f(x_n) = a \). Since \( f \) is not constant on \( \{x_1, x_2, \ldots\} \) it follows by Lemma 4.7 that \( \{x_1, x_2, \ldots\} \) is bounded. Hence, since we are not in II, we may assume it has a convergent subsequence. Let us denote this sequence again by \( x_1, x_2, \ldots \) and set
a := \lim x_n. \text{ Then } a \in X. \text{ Now let } y_1, y_2, \ldots \text{ be a sequence in } X \text{ for which } \lim f(y_n) = a. \text{ We prove that } \lim y_n = a. \text{ In fact, let } \varepsilon > 0.

There is } k \in \mathbb{N} \text{ such that } |x_k - a| < \varepsilon. \text{ For large } n \text{ we have } |f(y_n) - a| < |f(x_k) - a|, \text{ so for large } m \text{ (depending on } n) \text{ we have } |f(y_n) - f(x_m)| < |f(x_k) - f(x_m)| \text{ whence } |y_n - x_m| \leq |x_k - x_m|, \text{ so } |y_n - a| \leq |x_k - a| < \varepsilon.

It is worth stating in detail what is at stake if we are in alternative I of above.

Let us call a function } f : X \to K \text{ injective at } a \in X \text{ if } f(x) = f(a) \text{ for some } x \in X \text{ implies } x = a.

Now suppose that we have } a \in f(X), \text{ not isolated, for which we are in alternative I. Then for a sequence } x_1, x_2, \ldots \text{ with } \lim f(x_n) = a, \text{ we have } \lim x_n = a \in X \text{ so } (a, a) = \lim (x_n, f(x_n)), \text{ so by Cor.4.5 we have } a = f(a). \text{ Thus, } a \in f(X). \text{ f is injective at } a; \text{ if } f(b) = f(a) \text{ then since } \lim f(b) = a \text{ we must have } \lim b = a \text{ i.e. } b = a. \text{ Further, } f \text{ is continuous at } a \text{ (see 2.13 (2)(a)).}

If each bounded subset of } X \text{ is precompact we never can be in case II. This is also true if } f \in M_b(X) \text{ and } |X| \text{ is discrete i.e. if } x_1, y_1 \in X, |x_1 - y_1| > |x_2 - y_2| > \ldots \text{ then } \lim |x_n - y_n| = 0. \text{ Proof: let } a \in f(X) \text{ and let } \lim f(x_n) = a, f(x_n) \neq a \text{ for all } n. \text{ Without loss of generality we may assume } |a - f(x_1)| > |a - f(x_2)| > \ldots \text{ hence } |f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots \text{ and, since } f \in M_b(X) \text{ } |x_1 - x_2| > |x_2 - x_3| > \ldots \text{ Since } |X| \text{ is discrete, the sequence } x_1, x_2, \ldots \text{ is convergent. So we have case I. We find}
THEOREM 4.9 Let either each closed and bounded subset of $X$ be compact and $f \in M_w(X)$, or let $|X|$ be discrete and $f \in M_b(X)$. Then

(i) $f(X)$ is closed (in $K$).

(ii) If $f(a) \in f(X)$ is not isolated then $f$ is injective at $a$, $f$ is continuous at $a$. In particular if $f(X)$ has no isolated points then $f$ is a homeomorphism $X \cong f(X)$.

Proof. If $a \in f(X) \setminus f(X)$ then $a$ is not isolated. Since we are in alternative I of Lemma 4.8, $a \in f(X)$. Contradiction. Thus, $f(X)$ is closed and we have (i). The first part of (ii) follows from the observation above. The second part from 2.13 (2)(a).

It follows that an $f$ satisfying the conditions of 4.9 maps closed subsets of $X$ (without isolated points) into closed sets. But we can say more.

THEOREM 4.10 Let $f : X \to K$.

(i) If $f \in M_w(X)$ and if $Y \subseteq X$ is closed and the closed and bounded subsets of $Y$ are compact then $f(Y)$ is spherically complete.

(ii) If $f \in M_b(X)$ and if $Y \subseteq X$ is spherically complete then so is $f(Y)$.

(iii) If $f \in M_s(X)$ and if $A \subseteq f(X)$ is spherically complete then so is $f^{-1}(A)$.

Proof. (i) Let $B_1 \supsetneq B_2 \supsetneq \ldots$ be balls in $f(Y)$. Choose $y_1, y_2, \ldots \in Y$ such that $f(y_1) \in B_1 \setminus B_2$, $f(y_2) \in B_2 \setminus B_3$, $\ldots$. Then we have

$$|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots$$

and, by the weak monotony of $f$
\(|y_1 - y_2| \preceq |y_2 - y_3| \preceq \ldots\)

Suppose first that \(\lim |y_n - y_{n+1}| = 0\). Then \(y := \lim y_n\) and there are infinitely many \(k\) for which

\(|y_k - y_{k-1}| \preceq |y_{k+1} - y_k|\).

Now \(|y - y_k| \leq \max(|y_k - y_{k+1}|, |y_{k+1} - y_{k+2}|, \ldots) \leq |y_k - y_{k+1}|\). So we get for infinitely many \(k\)

\(|y - y_k| < |y_k - y_{k-1}|\)

whence

\(|f(y) - f(y_k)| \leq |f(y_k) - f(y_{k-1})|\)

so \(f(y) \in B_{k-1}\) for infinitely many \(k\), i.e., \(f(y) \in \bigcap_k B_k\).

Next, suppose that \(|y_{k+1} - y_k| \geq \varepsilon > 0\) for all \(k\). Then since \(y_1, y_2, \ldots\)

is bounded it has a convergent subsequence \(y_{n_1}, y_{n_2}, \ldots\). Let \(y := \lim y_{n_i}\). Then we have for infinitely many \(i\)

\(|y - y_{n_i}| < \varepsilon \leq |y_{n_i} - y_{n_{i+1}}|\)

whence

\(|f(y) - f(y_{n_i})| \leq |f(y_{n_i}) - f(y_{n_{i+1}})|\),

so \(f(y) \in B_{n_i}\) for infinitely many \(i\) i.e., \(f(y) \in \bigcap_k B_k\).

(ii) Let \(B_1 \supsetneq B_2 \supsetneq \ldots\) be balls in \(f(Y)\) and let \(y_1, y_2, \ldots \in Y\) such that \(f(y_1) \in B_1 \setminus B_2, f(y_2) \in B_2 \setminus B_3, \ldots\). Then we have

\(|f(y_1) - f(y_2)| > |f(y_2) - f(y_3)| > \ldots\)

and since \(f \in M_b(X)\):

\(|y_1 - y_2| > |y_2 - y_3| > \ldots\)

Since \(Y\) is spherically complete, there is \(y \in Y\) such that

\(|y - y_n| \leq |y_n - y_{n+1}|\) for all \(n\), hence \(|f(y) - f(y_n)| \leq |f(y_n) - f(y_{n+1})|\) for all \(n\). It follows that \(f(y) \in B_n\) for all \(n\).

(iii) Let \(B_1 \supsetneq B_2 \supsetneq \ldots\) be balls in \(f^{-1}(A)\). Choose \(x_1 \in B_1 \setminus B_2, x_2 \in B_2 \setminus B_3, \ldots\). Then \(|x_1 - x_2| > |x_2 - x_3| > \ldots\) whence \(|f(x_1) - f(x_2)| > |f(x_2) - f(x_3)| > \ldots\)

There is \(x \in f^{-1}(A)\) such that \(|f(x) - f(x_n)| \leq |f(x_n) - f(x_{n+1})|\) for all \(n\).

Hence \(|x - x_n| \leq |x_n - x_{n+1}|\) for all \(n\) i.e., \(x \in \bigcap_n B_n\).
DEFINITION 4.11 Let \( f : X \to K \). The oscillation function \( \omega_f : X \to [0, \infty] \) is defined by

\[
\omega_f(a) := \lim_{n \to \infty} \sup \{|f(x) - f(y)| : |x - a| \leq \frac{1}{n}, |y - a| \leq \frac{1}{n}, x, y \in X\} \quad (a \in X).
\]

\[
= \lim_{n \to \infty} \sup \{|f(x) - f(a)| : |x - a| \leq \frac{1}{n}, x \in X\}.
\]

THEOREM 4.12 Let \( f \in M_w(X) \). Then

\[
\omega_f(a) = \inf_{z \neq a} |f(z) - f(a)| \quad (a \in X).
\]

Proof. For \( x \neq a \) we have \( |f(x) - f(a)| \geq \inf_{z \neq a} |f(z) - f(a)| \) and (since \( a \) is not isolated) consequently

\[
\omega_f(a) \geq \inf_{z \neq a} |f(z) - f(a)|.
\]

Conversely, let \( z \neq a \). Then for all \( x \) such that \( |x - a| < |z - a| \) we have

\[
|f(x) - f(a)| < |f(z) - f(a)|
\]

so

\[
\omega_f(a) \leq |f(z) - f(a)|
\]

whence

\[
\omega_f(a) \leq \inf_{z \neq a} |f(z) - f(a)|.
\]

THEOREM 4.13 Let \( f \in M_w(X) \), \( a \in X \). If \( x_1, x_2, \ldots \in X \), \( \lim_{n \to \infty} x_n = a \) (\( x_n \neq a \) for all \( n \)) then \( \lim_{n \to \infty} |f(x_n) - f(a)| = \omega_f(a) \).

Proof. By 4.12 we have \( \lim_{n \to \infty} |f(x_n) - f(a)| \geq \omega_f(a) \). Conversely, \( \lim_{n \to \infty} |f(x_n) - f(a)| \leq \omega_f(a) \) is clear from the definition of \( \omega_f \).
5. MONOTONE FUNCTIONS FOR SPECIAL K

We further develop the theory of monotone functions, but now we consider the special cases: \( K \) is local, \( k \) is finite, \( K \) has discrete valuation. Also we can sometimes say a little more if we assume \( X \) to be convex. For the time being, let \( X \) be as in Section 4 (closed, no isolated points). We first collect the results from Section 4 in case \( K \) is a local field.

**THEOREM 5.1** Let \( K \) be a local field, and let \( f \in M_{w}(X) \). Then

(i) \( f \) is continuous.

(ii) If \( Y \subseteq X \) is closed then \( f(Y) \) is closed.

(iii) If \( f(X) \) is bounded and \( f \) is not constant then \( X \) is bounded.

(iv) Let \( a \in X \). Then the following are equivalent

(a) \( f \) is not injective at \( a \)

(b) \( f \) is locally constant at \( a \)

(c) \( f(a) \) is isolated in \( f(X) \).

(v) The following conditions are equivalent

(a) \( f \) is injective

(b) \( f(X) \) has no isolated points

(c) \( f \) is a homeomorphism of \( X \) onto \( f(X) \).

**Proof.** Obvious corollaries of 4.4, 4.10(i), 4.7, 4.9(ii).

We now want to derive results for \( M_{b} \) - and \( M_{s} \) -functions in case \( X \) is convex and \( K \) is a local field. First, a lemma that is valid in a more general situation.
LEMMA 5.2 Let the residue class field $k$ of $K$ be finite. Let $X$ be convex and let $f \in M_b(X)$. Then

(i) If $a, b, c \in X$, $|a-b| < |a-c|$, $f(a) \neq f(c)$ then

$$|f(a)-f(b)| < |f(a)-f(c)|.$$ 

(ii) If $C \subset X$ is convex then $f(C)$ is convex in $f(X)$ (f is weakly Darboux continuous, see 2.5).

(iii) If $f$ is injective, then $f \in M_s(X)$.

Proof. (i) Let $B := \{x \in K : |x-a| < |a-c| \}$. Then $B \subset X$ and $f(B) \subset [f(a), f(c)]$. Define an equivalence relation on $B$ by: $x \sim y$ if $|f(x)-f(y)| < |f(a)-f(c)|$.

Since $k$ is finite we get finitely many equivalence classes $B_1, B_2, \ldots, B_n$. Since $a \neq c$ we have $n \geq 2$. The diameter of $f(B)$ equals $|f(a)-f(c)|$, the distance between $f(B_i)$ and $f(B_j)$ equals $|f(a)-f(c)|$ ($i \neq j$). Since $[f(a), f(c)]$ can contain at most $q := \chi(k)$ sets having distances $|f(a)-f(c)|$ to one another we have $n \leq q$. Hence $2 \leq n \leq q$. By 2.2 (β), each $B_i$ is convex. If $x, y \in B_i$ and $|x-y| = |a-c|$ then $B_i = B$, contradicting $n \geq 2$. Thus $B$ is a disjoint union of $n$ balls $B_1, \ldots, B_n$, where $2 \leq n \leq q$ and $|x-y| < |a-c|$ whenever $x, y \in B_i$ ($i = 1, \ldots, n$). It follows that $n = q$ and that each $B_i$ has the form $\{x \in K : |x-b_i| < |a-c| \}$ ($b_i \in B$). Hence, if $|a-b| < |a-c|$ then there is $i$ for which $a, b \in B_i$.

So $|f(a)-f(b)| < |f(a)-f(c)|$.

(ii) Let $a, b \in C$ and let $a \in f(X)$ with $a \in [f(a), f(b)]$. We show that $a \in f(C)$. If $f(a) = f(b)$ this is clear. If $f(a) \neq f(b)$, set $a = f(x)$ where $x \in X$. Then $|f(x)-f(a)| \leq |f(b)-f(a)|$. If $|x-a|$ were $> |b-a|$ then $f(x) \neq f(a)$ (since $f \in M_b(X)$) and by (i) we then had $|f(b)-f(a)| < |f(x)-f(a)|$, a contradiction. Hence $|x-a| \leq |b-a|$ i.e., $x \in [a, b] \subset C$, so $a = f(x) \in f(C)$. 

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(iii) This is clear from (i).

Remark. 5.2 is false if we drop the condition on k, see 2.10.

COROLLARY 5.3 Let K be a local field and let \( f \in M_b(X) \) and X convex.

Then the following conditions are equivalent.

(a) \( f \in M_s(X) \).

(b) \( f \) is injective.

(c) \( f \in M_{bs}(X) \).

(d) \( f(X) \) has no isolated points.

Proof. 5.2 and 5.1.

THEOREM 5.4 Let K be a local field and let X be the unit ball of K
(or any bounded convex set, for that matter). If either

\( f \in M_s(X) \) or \( f \in M_b(X) \) then \( f \) has bounded difference quotients.

Proof. \( f \) is bounded, let \( M := \sup \{ |f(x) - f(y)| : x, y \in X \} \).

It suffices to prove that \( |f(x) - f(0)| \leq M|x| \) for all \( x \).

Let \( \pi \in K \), \( |
\pi| < 1 \), be a generator of the value group.

By induction on \( n \) we prove:

if \( |x| = |
\pi|^n \) then \( |f(x) - f(0)| \leq |\pi|^nM \).

The statement is clear for \( n = 0 \). Now suppose the statement is true for \( 0, 1, \ldots, n-1 \).

Let \( x \in X \), \( |a| = |
\pi|^n \). Then \( |x - 0| < |\pi^{n-1}| \).

If \( f(\pi^{n-1}) \neq f(0) \) we have either since \( f \in M_s(X) \) or by 5.2(i)

\[ |f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1}M \]

hence

\[ |f(x) - f(0)| \leq |\pi|^nM \]

If \( f(\pi^{n-1}) = f(0) \) then \( |f(x) - f(0)| \leq |f(\pi^{n-1}) - f(0)| = 0 \), so certainly

\[ |f(x) - f(0)| \leq |\pi|^nM. \]
Notes.

(a) 5.4 cannot be extended to the case $X = K$. In fact, let
\[ f : \mathbb{Q}_p \to \mathbb{Q}_p \] be the map $\mathbb{Z}_n p^n \mapsto \mathbb{Z}_n p^{2n}$. (\( \mathbb{Z}_n p^n \in \mathbb{Q}_p \)). Then
\[ f \in M_{bs}(\mathbb{Q}_p) \text{ but } |p^n f(p^{-n})| = p^n \to \infty. \]

(b) If we lose the condition on $K$, for example by requiring that the valuation is discrete then 3.22 and 2.4(5) show that the conclusion of 5.4 is false both for $M_b$-functions and $M_s$-functions. On the other hand, it is clear from the proof of 5.4 that a bounded $M_s$-function on $X$ has bounded difference quotients.

(c) One may wonder how difference quotients of $M_w$-functions behave. See the example below.

EXAMPLE 5.5 Let $p \neq 2$. Then there is an $f \in M_w(\mathbb{Z}_p \to \mathbb{Q}_p)$ that has unbounded difference quotients.

**Proof.** Let $a_0, a_1, \ldots$ be defined via $a_{2n} := p^n (n = 0, 1, 2, \ldots)$ and $a_{2n+1} := 2p^n (n = 0, 1, 2, \ldots)$. Thus $(a_0, a_1, a_2, \ldots) = (1, 2, p, p^2, 2p^2, \ldots)$. Then $|a_0| \geq |a_1| \geq |a_2| \geq \ldots$, $\lim a_n = 0$, $|a_n - a_m| = |a_m| (n > m)$.

Set
\[ f(x) = \begin{cases} a_n & \text{if } |x| = p^{-n} (n = 0, 1, 2, \ldots) \\ 0 & \text{if } x = 0 \end{cases} \quad (x \in \mathbb{Z}_p) \]

Then the difference quotients of $f$ are not bounded (for $n \in \mathbb{N}$: $f(p^{2n}) = p^n$, so $|p^{-2n} f(p^{2n})| = p^n \to \infty$ if $n \to \infty$). We show that $f \in M_w(\mathbb{Z}_p)$. Since $f$ is continuous it suffices to show that if $x, y, z$ are $\neq 0$, $|x - y| < |x - z|$ then $|f(x) - f(y)| < |f(x) - f(z)|$. This is clear if $|x| = |y|$. If $|x| < |y|$, then $|x| < |y| < |z|$. If $|x| > |y|$, then $|y| < |x| < |z|$. Let $f(x) = a_n$, $f(y) = a_m$, $f(z) = a_t$. Then in both cases $n \neq m, t < \min(n, m)$: $|f(x) - f(y)| = |a_n - a_m| \leq |a_t|$ and $|f(x) - f(z)| = |a_n - a_t| = |a_t|$ and we are done.
On the other hand (how surprising is life!)

THEOREM 5.6 Let \( k \) be the field of two elements. Then \( M_w(X) = M_b(X) \).

Proof. We prove that \( |x-y| = |y-z| \) implies \( |f(x)-f(y)| \leq |f(y)-f(z)| \) 
\((x \neq y, y \neq z, x, y, z \in X)\). There is \( a \in K^* \) such that \( |a(x-y)| = |a(y-z)| = 1 \). So since \( k = \mathbb{F}_2 \), \( a(x-y) = a(y-z) = 1 \), whence \( a(x-z) = 0 \) or \( |a(x-z)| < 1 \). Thus, \( |x-z| < |x-y| = |y-z| \). Since \( f \in M_w(X), |f(x)-f(z)| \leq \min(|f(x)-f(y)|, |f(y)-f(z)|) \). Consequently, 
\[ |f(x)-f(y)| \leq \max(|f(x)-f(z)|, |f(z)-f(y)|) \leq |f(y)-f(z)|. \]

Of particular interest may be monotone functions mapping convex sets onto convex sets.

THEOREM 5.7 Let \( K \) be a local field, let \( X \) be a bounded open convex set, and let \( f : X \to X \) be surjective. Then the following are equivalent.

(a) \( f \in M_b(X) \)

(b) \( f \in M_s(X) \)

(γ) \( f \in M_{bs}(X) \)

(δ) \( f \) is an isometry.

Proof. (a) \( \Rightarrow \) (β). Since \( f(X) \) has no isolated points, \( f \) is a homeomorphism, by 5.1(v). Then \( f \in M_s(X), \) by 5.3. (β) \( \Rightarrow \) (γ). \( f^{-1} \in M_b(X) \).

We just have shown (a) \( \Rightarrow \) (β), so \( f^{-1} \in M_s(X) \) i.e., \( f \in M_b(X) \).

(γ) \( \Rightarrow \) (δ). From the proof of 5.4 we have seen that \( |f(x)-f(y)| \leq M|x-y| \), where \( M = \sup|f(x)-f(y)| = 1 \). Hence \( |f(x)-f(y)| \leq |x-y| \) for all \( x, y \in X \), but by the same token this also holds for \( f^{-1} \). Then \( f \) is an isometry. (δ) \( \Rightarrow \) (a) is obvious.
COROLLARY 5.8 Let \( X \) be an open convex subset of \( K \) and \( f : X \to K \), not constant, \( f(X) \) convex. Then the following conditions are equivalent.

(a) \( f \in M_b(X) \)

(b) \( f \in M_s(X) \)

(c) \( f \in M_{bs}(X) \)

(d) \( f \) is a scalar multiple of an isometry.

Proof. If \( X \) is bounded then \( f(X) \) is bounded and by a linear transformation we can arrange that \( X = f(X) \). The equivalence follows easily. If \( X \) is unbounded, then \( X = K \), and from 5.1 (iii) we get \( f(X) \) is unbounded, so \( f(X) = K \). The equivalence of (a) (b) (c) is now easy. To prove (d) \( \Rightarrow \) (a) we may assume \( f(0) = 0 \), \( f(1) = 1 \). Let

\[ X_n := \{ x \in K : |x| \leq n \}. \]

Then \( f(X_n) \) is convex for each \( n \in \mathbb{N} \), so there is \( c_n \) such that \( |f(x)-f(y)| = c_n |x-y| \) \( (x, y \in X_n) \). By substituting \( x = 1 \), \( y = 0 \) we see that \( c_n = 1 \).

Similar to what we did in example 3.3, (3) we try to express the condition \( f \in M_{\text{ubs}}(Z_p) \) into conditions on the coefficients of \( f \) with respect to the orthonormal base \( e_0, e_1, \ldots \) of \( C(Z_p) \). So let the notations be as in 3.3(3), and suppose first \( f \in M_{\text{ubs}}(Z_p) \) i.e.

\[ |x-y| = |s-t| \iff |f(x)-f(y)| = |f(s)-f(t)|. \]

Let \( n, m \in \mathbb{N} \). If \( |n-n_\_| = |m-m_\_| \) then \( |f(n)-f(n_\_)| = |f(m)-f(m_\_)| \), so if we write \( f = \Sigma_n a_n \) we find \( |\lambda_n| = |\lambda_m| \). Let \( n = a_0 + a_1 p + \ldots + a_k p^k \) \( (a_k \neq 0) \) then \( |n-n_\_| = p^{-k} \) where \( k = \lfloor \log n / \log p \rfloor \). We find

\[
\text{if } \left[ \frac{\log n}{\log p} \right] < \left[ \frac{\log m}{\log p} \right] \text{ then } |\lambda_n| > |\lambda_m|.
\]

\[
\text{if } \left[ \frac{\log n}{\log p} \right] = \left[ \frac{\log m}{\log p} \right] \text{ then } |\lambda_n| = |\lambda_m|.
\]
Moreover, if \( \left\lfloor \frac{\log n}{\log p} \right\rfloor = \left\lfloor \frac{\log m}{\log p} \right\rfloor = k \) and \( n - m \) is divisible by \( p^k \), i.e., \( n - m = p^k \cdot i \log p \), then \( |f(n) - f(m)| = |\lambda_n - \lambda_m| \). If \( n > m \) then \( |f(n) - f(0)| = |\lambda_n - \lambda_m| = |\lambda_n| \).

We have found the first half of

**Theorem 5.9** Let \( f = \sum \lambda_n e_n \in C(\mathbb{Z}_p) \). In order that \( f \in M_{\text{ubs}}(\mathbb{Z}_p) \) it is necessary and sufficient that condition (*) below holds

\[
(*) \quad \begin{cases} \frac{\log n}{\log p} \quad \text{is a strictly decreasing function of} \quad \left( \frac{\log n}{\log p} \right) \quad (n \in \mathbb{N}) \\
\frac{\log n}{\log p} = \frac{\log m}{\log p}, \quad n \neq m, \quad n - m \quad \text{implies} \\
|\lambda_n - \lambda_m| = |\lambda_n| = |\lambda_m| \quad (n, m \in \mathbb{N}).
\end{cases}
\]

We have shown \( f \in M_{\text{ubs}}(\mathbb{Z}_p) \rightarrow (*) \). Now suppose (*) and let \( |x - y| = p^{-k} \). We show that \( |f(x) - f(y)| = |\lambda_p^k| \). Now \( e_n(x) = e_n(y) \) for \( n < p^k \), so

\[
f(x) - f(y) = \sum_{n \geq p^k} \lambda_n (e_n(x) - e_n(y)).
\]

Set \( x := a_0 + a_1 p + \ldots + a_k p^k + a_{k+1} p^{k+1} + \ldots \); \( a_k \neq b_k \)

\[
y := a_0 + a_1 p + \ldots + b_k p^k + b_{k+1} p^{k+1} + \ldots
\]

Then

\[
\left| \sum_{n \geq p^k} \lambda_n e_n(x) \right| = \left| \lambda_{p^k}^k \cdot \lambda_n^k a_k p^k + \lambda_{p^k}^{k+1} a_k p^{k+1} + \ldots \right|
\]

\[
\left| \sum_{n \geq p^k} \lambda_n e_n(y) \right| = \left| \lambda_{p^k}^k \cdot \lambda_n^k b_k p^k + \lambda_{p^k}^{k+1} b_k p^{k+1} + \ldots \right|
\]

Now either \( a_k \) or \( b_k \) is \( \neq 0 \), say, \( a_k \neq 0 \). If \( b_k = 0 \) then by (*)

\[
\left| \sum_{n \geq p^k} \lambda_n e_n(y) \right| < \left| \lambda_{p^k}^k \right| = \left| \sum_{n \geq p^k} \lambda_n e_n(x) \right|, \quad \text{so} \quad |f(x) - f(y)| = |\lambda_{p^k}^k| = |\lambda_p^k|.
\]

If \( b_k \neq 0 \) then by (*) \( |\lambda_p^k| = |\lambda_p^{k-1} - \lambda_p^k| = |f(x) - f(y)| \).

Note. Using similar methods, we can prove: \( f = \sum \lambda_n e_n \) is in \( M_{\text{ubs}}(\mathbb{Z}_p) \) if and only if we have (**) for all \( n, m \in \mathbb{N} \):
\[
\left\lfloor \frac{\log n}{\log p} \right\rfloor > \left\lfloor \frac{\log m}{\log p} \right\rfloor = k \quad \text{n-m divisible by } p^k
\]

\[n = m_n, n \neq m_m\]

\[|\lambda_n| < |\lambda_m| \quad \Rightarrow |\lambda_n| = |\lambda_m| = |\lambda_n - \lambda_m|.
\]

If we assume only that \(K\) has a discrete valuation then example 2.10 shows that theorem 5.7 becomes a falsity. But we do have

**THEOREM 5.10** Let \(X\) be the unit ball of a discretely valued field. Let \(f : X \rightarrow X\) be surjective, \(f \in M_{bs}(X)\). Then \(f\) is an isometry.

**Proof.** It is clear from previous theory that \(f\) is a homeomorphism of the unit ball. It suffices to show that \(|f(x) - f(y)| \leq |x - y|\) for all \(x, y \in X\). (Apply this result also for \(f^{-1}\). Then \(f\) is an isometry.)

Let \(\pi \in K, |\pi| < 1\), be a generator of \(|K^*|\). We prove by induction

\[|x| = |\pi|^n \text{ then } |f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)|.
\]

For \(n = 0\) this is clear. (\(|x - 0| \leq |1 - 0|\), so \(|f(x) - f(0)| \leq |f(1) - f(0)|\)).

Suppose the statement is true for \(n = 1\). Let \(|x| = |\pi|^n\). Then

\[|x - 0| < |\pi^{n-1} - 0|, \text{so } |f(x) - f(0)| < |f(\pi^{n-1}) - f(0)| \leq |\pi|^{n-1} |f(1) - f(0)|,
\]

so \(|f(x) - f(0)| \leq |\pi|^n |f(1) - f(0)|\) and we are done. (In fact, we have shown that a bounded \(M\) function has bounded difference quotients.)

Further, from 4.9 we infer

**THEOREM 5.11** Let \(K\) have discrete valuation and let \(f \in M_b(X)\). Then the following conditions are equivalent.

(a) \(f(X)\) has no isolated points.

(b) \(f\) is injective and continuous.

(c) \(f\) is a homeomorphism \(X \sim f(X)\).
Proof. (a) \( (\gamma) \) is 4.9(ii). \( (\gamma) \) \( (\beta) \) is clear. \( (\beta) \) \( (\gamma) \): if \( f(a) \) were an isolated point of \( f(X) \), then \( \{ x : f(x) = f(a) \} \) is open in \( X \). Since \( f \) is injective \( \{ a \} \) is open. But \( X \) has no isolated points. Contradiction.

To show that 5.11 may not be true if \( K \) has a dense valuation we construct

**EXAMPLE 5.12** Let \( |K| = [0,\infty) \). Then we construct an \( \mathbb{M}_s \)-homeomorphism sending

\[ \{ x \in K : \frac{1}{4} < |x| \leq 1 \} \text{ onto } \{ x \in K : 0 < |x| \leq 1 \} \]

**Proof.** Let \( \phi : [\frac{1}{4},1] \to [0,1] \) be the map \( x \mapsto 2(x-\frac{1}{4}) \) \( (x \in (\frac{1}{4},1]) \). For each \( v \in (\frac{1}{4},1] \), choose \( \beta_v \in K \) such that \( |\beta_v| = \frac{\phi(v)}{v} \). Define \( f : \{ x \in K : \frac{1}{4} < |x| \leq 1 \} \to \{ x \in K : 0 < |x| \leq 1 \} \) as follows

\[ f(x) = \beta_{|x|} x \quad (\frac{1}{4} < |x| \leq 1) \]

Clearly, \( |f(x)| = |\beta_{|x|}| \cdot |x| = \phi(|x|) \in (0,1] \). The inverse of \( f \) is given by \( y \mapsto \beta_{|y|}^{-1}(|y|) \), so \( f \) is a bijection. Since \( f^{-1} \) can be defined in the same way as \( f \) (only with \( \phi^{-1} \) instead of \( \phi \)) it suffices to show that \( f \in \mathbb{M}_s \). Let \( |x-y| < |x-z| \).

Suppose \( |x| > |z| \). Then \( |x-z| = |x| \) and \( |y| = \max(|x-y|,|x|) = |x| \).

Then \( \beta_{|x|} = \beta_{|y|} \), so \( |f(x)-f(y)| = |\beta_{|x|}| |x-y| \) and \( |f(x)-f(z)| = |f(x)| = \beta_{|x|} |x-z| \), so we are done in this case. Suppose \( |x| < |z| \).

Then \( |x-z| = |z| \) and \( |y| = \max(|x-y|,|x|) < |z| \). Then \( |f(x)-f(y)| \leq \max(|f(x)|,|f(y)|) < |f(z)| = |f(z)-f(x)| \).

Suppose \( |x| = |z| \). Then \( |y| \leq \max(|x-y|,|x|) \leq |x| \); if \( |y| \) were < \( |x| \) then \( |x-y| = |x| = |z| < |x-z| \), a contradiction, so \( |y| = |x| = |z| \), and \( |f(x)-f(y)| = \beta_{|x|} |x-y|, |f(x)-f(z)| = \beta_{|x|} |x-z| \) whence \( |f(x)-f(y)| < |f(x)-f(z)| \).
EXAMPLE 5.13 Extend $f$ to a surjection $g$ of $\{x \in K : |x| \leq 1\}$ onto itself by defining $g(x) = 0$ if $|x| \leq \frac{1}{2}$. We claim that $g \in M_b$. Let $|x-y| \leq |x-z|$. To check whether $|g(x)-g(y)| \leq |g(x)-g(z)|$ we only have to consider the cases $|x| \leq \frac{1}{2}$ and $|y| > \frac{1}{2}$ and $|y| \leq \frac{1}{2}$. In the first case, $|x-y| = |y| \leq |x-z|$, so $|z| = \max(|z-x|, |x|) = |z-x| \geq |y|$. Then $|g(x)-g(y)| = |f(y)| \leq |f(z)| = |g(z)-g(x)|$. In the second case $|g(x)-g(y)| = |f(x)|$. If $|x| < |z|$ then $|f(x)| < |f(z)| = |f(x)-f(x)| = |g(z)-g(x)|$. If $|x| > |z|$ then $|f(x)| = |g(x)-g(z)|$.

If $|x| = |z|$ then $|f(x)-f(z)| = \beta|_{|x|} |x-z| \geq \beta|_{|x|} |x-y| = \beta|_{|x|} |x| = |f(x)|$.

Thus we have found a continuous surjection $g : \{x \in K : |x| \leq 1\} \to \{x \in K : |x| \leq 1\}$, $g \in M_b$, such that $g = 0$ on $\{x : |x| \leq \frac{1}{2}\}$. (Compare 5.11). 

EXAMPLE 5.14 Let $h : \{x \in K : |x| \leq 1\} \to K$ be defined as

$$h(x) = \begin{cases} f^{-1}(x) & \text{if } x \neq 0 \quad (f \text{ as in 5.12)} \\ 0 & \text{if } x = 0. \end{cases}$$

Then $h$ is a non-continuous $M_{bs}$-function.

**Proof.** That $h$ is not continuous at $0$ is clear. Further, $h$, restricted to $\{x : 0 < |x| \leq 1\}$ is in $M_{bs}$ (see 5.12). Further, since $g \circ h$ is the identity ($g$ as in 5.12), we see that $h \in M_b$. It suffices to show that $|x-y| = |x-z|$ implies $|h(x)-h(y)| = |h(x)-h(z)|$ in case $0 \in \{x,y,z\}$.

We may suppose $x \neq y, y \neq z, x \neq z$. Let $x = 0$. Then $|y| = |z|$, so $|f^{-1}(y)| = |f^{-1}(z)|$ i.e., $|h(x)-h(y)| = |h(x)-h(z)|$. Now let $y = 0$.

Then $|x| = |x-z|$. Choose $0 < |t| \leq 1$ such that $|t| < |x|$. Then $|x-t| = |x-z|$ so $|f^{-1}(x) - f^{-1}(t)| = |f^{-1}(x)-f^{-1}(z)|$ i.e., $|h(x)| = |h(x)-h(z)|$, and we are done.
6. FUNCTIONS OF BOUNDED VARIATION

In this section \( X \) is the unit ball of \( K \), and \( \mathcal{B}(X) := \{ f : X \to K : \sup_{x \neq y} \frac{|f(x) - f(y)|}{x-y} < \infty \} \). Let us define

\[
\|f\|_\Delta := \sup \left\{ \frac{|f(x) - f(y)|}{x-y} : x, y \in X, x \neq y \right\} (f \in \mathcal{B}(X)).
\]

It will turn out that, in a natural way, \( \mathcal{B}(X) \) can be regarded as the space of functions of bounded variation, and that \( \| \|_\Delta \) plays the role of the total variation.

**Theorem 6.1** Let \( f : X \to K \). Then the following are equivalent

(a) \( f \in \mathcal{B}(X) \).

(b) \( f \) is a linear combination of two increasing functions.

If \( |K| \) is discrete (a), (b) are equivalent to

(y) \( f \) is the difference of two bounded monotone functions of some type \( c \).

(δ) \( f \in [M_{\Delta S}(X)] \).

If \( K \) is a local field then (a)-(δ) are equivalent to

(e) \( f \in [M_{\Delta D}(X)] \).

(η) \( f \in [M_{\Delta S}(X)] \).

**Proof.** We only prove (a) \( \Rightarrow \) (b). The rest follows from (5.10), (5.4).

So let \( f \in \mathcal{B}(X) \) and choose \( \lambda \in X \) such that \( |f(x) - f(y)| < |\lambda| |x-y| \) for all \( x, y \in X, x \neq y \). Then \( \lambda^{-1} f \) is a pseudocontraction, \( f(x) = \lambda x + \lambda (\lambda^{-1} f(x) - x) \) (\( x \in X \)), where \( x \to x \) and \( x \to \lambda^{-1} f(x) - x \) are increasing.

In the real case, we can define for a function \([0,1] \to \mathbb{R} \), of bounded variation
\[ V(f) := \inf \{ \text{Var } g + \text{Var } h : f = g + h, \text{ } g, h \text{ monotone} \}. \]

It is an easy exercise to show that \( f \mapsto V(f) \) is a seminorm on the space of all functions of bounded variation and that \( V \) is equivalent to the total variation \( \text{Var} \), defined via

\[ \text{Var } f = \sup \{ \sum |f(x_i) - f(x_{i-1})| : x_0 < x_1 < \ldots < x_n \text{ is a partition of } [0,1] \}. \]

So in the non-archimedean situation we define for \( f : X \rightarrow K \)

\[ J(f) = \sup \{ |f(x) - f(y)| : x, y \in X \}. \]

(If \( f \) is considered to be "monotone" then \( J(f) \) can be interpreted as the "total variation" of \( f \).) We are led to the following definitions for \( f \in BA(X) \):

\[ \text{Var } f := \inf \{ \max(J(g),J(h)) : f = g + h, \text{ } g, h \text{ are scalar multiples of increasing functions} \}. \]

(If \( |K| \) is discrete) \( \text{Var}^f := \inf \{ \max(J(g),J(h)) : f = g + h \text{ } g, h \text{ in } M(X) \}. \)

(If \( K \) is local) \( \text{Var}_f := \inf \{ \max(J(g),J(h)) : f = g + h \text{ } g, h \text{ in } M(X) \}. \)

Let us first compare \( \text{Var } f \) and \( \|f\|_\Delta \). If \( f = g + h \) and \( g, h \) are scalar multiples of increasing functions we have for \( x, y \in X, \ x \neq y \)

\[ \frac{|f(x) - f(y)|}{x-y} \leq \max\left( \frac{|g(x) - g(y)|}{x-y}, \frac{|h(x) - h(y)|}{x-y} \right) \leq \max(J(g),J(h)) \]

so \( \|f\|_\Delta \leq \text{Var } f \). Conversely, if \( |\lambda| > \sup \frac{|f(x) - f(y)|}{x-y} \) then

\[ f(x) = \lambda x + \lambda^{-1} f(x) - x \quad (x \in X) \]

whence

\[ \text{Var } f \leq |\lambda| \]
So, if \( |K| \) is dense we have \( \text{Var} f = \| f \|_\Delta (f \cdot B\Delta(X)) \). Otherwise we have at least
\[
\| f \|_\Delta \leq \text{Var} f \leq c \| f \|_\Delta \quad (f \in B\Delta(X))
\]
(\( c \) is the smallest value \( > 1 \)).

If \( |K| \) is discrete we clearly have \( \text{Var}_1 f \leq \text{Var} f \). Conversely, let \( f = g+h \), where \( g,h \in M_{bs}(X) \). It follows from the proof of 5.10 that
\[
|g(x) - g(y)| \leq M|x-y| \quad (x,y \in X)
\]
\[
|h(x) - h(y)| \leq N|x-y|
\]
where \( M = \sup |g(x) - g(y)| = J(g) \) and \( N = J(h) \).

So
\[
\frac{f(x) - f(y)}{x-y} \leq \max(J(g), J(h)), \quad \text{whence}
\]
\[
\| f \|_\Delta \leq \text{Var}_1 f.
\]

Similar proofs work for \( \text{Var}_2 f, \text{Var}_3 f \). We have

THEOREM 6.2 The seminorms \( \text{Var}, \text{Var}_1, \text{Var}_2, \text{Var}_3 \), on \( B\Delta(X) \) (whenever defined) are all equivalent to \( \| \|_\Delta \).
REFERENCES


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