Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis \( \mathbb{R} \) or \( \mathbb{C} \) by a non-archimedean valued field \( K \).

A non-archimedean valued field is a (commutative) field \( K \), together with a map \( | | : K \to \mathbb{R} \) (the valuation) satisfying

\[
|a| \geq 0 \quad , \quad |a| = 0 \text{ iff } a = 0 \\
|ab| = |a| \cdot |b| \\
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
\]

for all \( a,b \in K \).

We have the following remarks.

(1) Apart from \( \mathbb{R} \) or \( \mathbb{C} \), every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \sim K \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural...
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in $K$. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called differentiable if for all $a \in X$

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum a_n x^n$ is differentiable on

\[ \{ x : |x| < (\lim \sqrt[n]{|a_n|})^{-1} \} . \]

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let \( \varepsilon > 0, a \in K \). Then \( B(a, \varepsilon) := \{ x \in K : |x-a| < \varepsilon \} \) is an open-
and-closed subset of \( K \), hence \( \xi_{B(a, \varepsilon)} \), defined by

\[
\xi_{B(a, \varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \varepsilon) \\
0 & \text{elsewhere}
\end{cases}
\]
is differentiable and \( \xi'_{B(a, \varepsilon)} = 0 \).

Locally constant functions all have derivative zero. On the other
hand they form a uniformly dense subset of \( C(X) \), the space of all
continuous functions: \( X \to K \).

Even worse: let \( \mathbb{Q}_p \) the field of the \( p \)-adic numbers and let
\( Z_p := \{ x \in \mathbb{Q}_p : |x| \leq 1 \} \). Then the function \( f : Z_p \to \mathbb{Q}_p \) defined by

\[
f(\Sigma a_n p^n) = \Sigma a_n p^{2n} \quad (\Sigma a_n p^n \in Z_p)
\]
satisfies \( |f(x)-f(y)| = |x-y|^2 \) for all \( x,y \in Z_p \). So \( f' = 0 \) but \( f \)
is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessa-
arily absent in our theory.

Notice that the difficulties encountered above also appear when we
study differentiability of functions \( f : \mathcal{D} \to \mathbb{R} \), where \( \mathcal{D} \subset [0,1] \)
is the Cantor set. So it is the domain of \( f \) that is responsible
for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

\[
C^1(X) := \{ f : X \to K, f \text{ is differentiable, } f' \text{ is continuous} \}
\]
then we run up against difficulties.

First of all, one can prove that \( C^1(Z_p) \) (with the norm
$f \mapsto \max(\|f\|_\infty, \|f'\|_\infty)$ is not a Banach space. In fact one shows that for every pair of continuous functions $f, g : \mathbb{Z}_p \to \mathbb{Q}_p$ there exists a sequence $f_1, f_2, \ldots$ in $C^1(\mathbb{Z}_p)$ for which both $f_n \to f$ and $f'_n \to g$ uniformly.

What is worse, we have no local invertibility theorem for such $C^1$-functions.

In fact, let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be defined by

$$
f(x) = \begin{cases} 
  x-p^{2n} & \text{if } |x-p^n| < p^{-2n} \quad (n \in \mathbb{N}) \\
  x & \text{elsewhere}
\end{cases}
$$

Then $f'(x) = 1$ for all $x \in \mathbb{Z}_p$. But $f(p^n) = f(p^n - 2^n)$ for all $n \in \mathbb{N}$, so $f$ is not even locally injective at 0.

Therefore we are led to define:

Let $f : X \to K$. Put

$$
\Phi f(x, y) := \frac{f(x) - f(y)}{x-y} \quad (x, y \in X, x \neq y).
$$

We say that $f \in C^1(X)$ if $\Phi f$ can continuously be extended to a function $\overline{\Phi}f : X \times X \to K$.

Then $BC^1(X) := \{f \in C^1(X) : f$ and $\Phi f$ are bounded$\}$ is a Banach space under $f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi f\|_\infty)$.

Further, if $f \in C^1(X), f'(a) \neq 0$ for some $a \in X$, then $f$ has a $C^1$-inverse, locally at $a$.

Theorem. Differentiation is a continuous surjection $BC^1(X) \overset{D}{\to} BC(X)$.

(here $BC(X)$ is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" $P : BC(X) \to BC^1(X)$.
(an analogue of $(Pf)(x) := \int_0^x f(t)\,dt$ for real functions) such that $DP$ is the identity on $BC(X)$.

A natural try is first to find an analogue of the Lebesgue measure in $K$. But this turns out to be a dead end road. For example if $K = \mathbb{Q}_p$ there does not exist a nonzero translation invariant bounded additive $\mathbb{Q}_p$-valued function $m$ defined on the compact open subsets of $\mathbb{Z}_p$. (By translation invariance $|m(p^n\mathbb{Z}_p)| = p^n|m(\mathbb{Z}_p)| \to \infty$ if $m(\mathbb{Z}_p) \neq 0$). For similar reasons it goes wrong for every local field $K$.

Following the ideas of Dieudonné, Treiber, we define for $f \in BC(X)$

$$(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1}-x_n) \quad (x \in X)$$

Here the $x_n$ are defined as follows. For each $n \in \mathbb{N}$ the equivalence relation $\sim_n$ defined by $x \sim_n y$ if $|x-y| < \frac{1}{n}$ yields a partition of $X$ into balls. Choose a center in each ball and let $R_n$ be the set of these centers.

For each $x \in X$ and $n \in \mathbb{N}$, $x_n$ is defined by $x_n \in R_n$, $|x_n-x| < \frac{1}{n}$.

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

$P$ is a linear isometry of $BC(X)$ into $BC^1(X)$. $DP$ is the identity on $BC(X)$, whereas $PD$ is a projection of $BC^1(X)$ onto a complement of $\{f \in BC^1(X) : f' = 0\}$.

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than $BC(X), BC^1(X)$.
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \rightarrow K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \rightarrow K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \rightarrow K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \rightarrow K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \rightarrow K$ of the first class of Baire with respect to the supremum norm.

Let $BD(X)$ be the Banach space of all differentiable $f : X \rightarrow K$ for which both $f$ and $\phi f$ are bounded, with respect to the norm $f + \|f\|_\infty + \|\phi f\|_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \overset{D}{\rightarrow} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \rightarrow BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$:

If $f \in C^2(X)$ if the function $\phi_2 f$, defined by

$$\phi_2 f(x,y,z) = \frac{\phi_1 f(x,z) - \phi_1 f(y,z)}{x - y}$$

$(x,y,z \in X, x \neq y, y \neq z, x \neq z)$

can continuously be extended to $\phi_2 f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map

$P_2$ defined via

$$P_2 f(x) := \sum_{n=1}^\infty (x_n - x_n - x_{n+1}) + \frac{1}{2} \sum_{n=1}^\infty (x_n - x_{n+1})^2$$

maps $C^1(X)$ into $C^2(X)$ and $(P_2 f)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to $2, 3, \ldots, n$).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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**Reference**

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