Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis $\mathbb{R}$ or $\mathbb{C}$ by a non-archimedean valued field $K$.

A non-archimedean valued field is a (commutative) field $K$, together with a map $| | : K \to \mathbb{R}$ (the valuation) satisfying

$$
|a| \geq 0 \quad , \quad |a| = 0 \text{ iff } a = 0
$$
$$
|ab| = |a| \cdot |b|
$$
$$
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
$$

for all $a, b \in K$.

We have the following remarks.

(1) Apart from $\mathbb{R}$ or $\mathbb{C}$, every complete valued field is non-archimedean.

(2) If $K$ is a non-archimedean valued field and if $L \supset K$ is an overfield of $K$ then the valuation on $K$ can be extended to a non-archimedean valuation on $L$.

(3) If $K$ is a (non-archimedean) valued field then its completion

$\hat{K}$ (with respect to the metric $(x,y) \mapsto |x-y|$) can, in a natural
way, be given the structure of a non-archimedean valued field. In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in \( K \). More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in \( \mathbb{R} \)) that states that the operations of differentiation and integration are in some sense each other's inverses.

§ 2. Differentiation in \( K \). Let \( X \subset K \) be a subset without isolated points. A function \( f : X \to K \) is called differentiable if for all \( a \in X \)

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well-known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on \( X \). An analytic function \( x + \sum a_n x^n \) is differentiable on \( \{ x : |x| < (\lim \sqrt[n]{|a_n|})^{-1} \} \).

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
and-closed subset of $K$, hence $\xi_{B(a,\varepsilon)}$, defined by

$$
\xi_{B(a,\varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a,\varepsilon) \\
0 & \text{elsewhere}
\end{cases}
$$

is differentiable and $\xi_{B(a,\varepsilon)}' = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \rightarrow K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ defined by

$$
f(\sum_{n=0}^{\infty} a_n p^n) = \sum_{n=0}^{\infty} a_n p^{2n} \quad (\sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p)
$$

satisfies $|f(x) - f(y)| = |x - y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : \mathbb{D} \rightarrow \mathbb{R}$, where $\mathbb{D} \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$
C^1(X) := \{f : X \rightarrow K, f \text{ is differentiable, } f' \text{ is continuous}\}
$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
f \mapsto \max(|f|_\infty, |f'|_\infty) \) is not a Banach space. In fact one shows that for every pair of continuous functions \( f, g : \mathbb{Z}_p \to \mathbb{Q}_p \) there exists a sequence \( f_1, f_2, \ldots \) in \( C^1(\mathbb{Z}_p) \) for which both \( f_n \to f \) and \( f'_n \to g \) uniformly.

What is worse, we have no local invertibility theorem for such \( C^1 \)-functions.

In fact, let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be defined by
\[
f(x) = \begin{cases} x-p^{-2n} & \text{if } |x-p^n| < p^{-2n} \\ x & \text{elsewhere} \end{cases} \quad (n \in \mathbb{N})
\]

Then \( f'(x) = 1 \) for all \( x \in \mathbb{Z}_p \). But \( f(p^n) = f(p^n-p^{2n}) \) for all \( n \in \mathbb{N} \), so \( f \) is not even locally injective at 0.

Therefore we are led to define:

Let \( f : X \to K \). Put
\[
\Phi f(x,y) := \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \( f \in C^1(X) \) if \( \Phi f \) can continuously be extended to a function \( \bar{\Phi} f : X \times X \to K \).

Then \( BC^1(X) := \{ f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded} \} \) is a Banach space under \( f \mapsto ||f||_1 := \max(||f||_\infty, ||\Phi f||_\infty) \).

Further, if \( f \in C^1(X) \), \( f'(a) \neq 0 \) for some \( a \in X \), then \( f \) has a \( C^1 \)-inverse, locally at \( a \).

**Theorem.** Differentiation is a continuous surjection \( BC^1(X) \overset{D}{\to} BC(X) \).

(here \( BC(X) \) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" \( P : BC(X) \to BC^1(X) \)
(an analogue of $(Pf)(x) := \int_{-\infty}^{x} f(t) dt$ for real functions) such that
$DP$ is the identity on $BC(X)$.

A natural try is first to find an analogue of the Lebesgue
measure in $K$. But this turns out to be a dead end road. For example
if $K = \mathbb{Q}_p$ there does not exist a nonzero translation invariant
bounded additive $\mathbb{Q}_p$-valued function $m$ defined on the compact open
subsets of $\mathbb{Z}_p$. (By translation invariance
$|m(p^n\mathbb{Z}_p)| = p^n|m(\mathbb{Z}_p)| \to \infty$ if $m(\mathbb{Z}_p) \neq 0$). For similar reasons it
goes wrong for every local field $K$.

Following the ideas of Dieudonné, Treiber, we define for
$f \in BC(X)$

$$(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)$$

Here the $x_n$ are defined as follows. For each $n \in \mathbb{N}$ the equivalence
relation $\sim_n$ defined by $x \sim_n y$ if $|x - y| < \frac{1}{n}$ yields a partition of $X$
into balls. Choose a center in each ball and let $R_n$ be the set of
these centers.

For each $x \in X$ and $n \in \mathbb{N}$, $x_n$ is defined by $x_n \in R_n$, $|x_n - x| < \frac{1}{n}$.

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).
$P$ is a linear isometry of $BC(X)$ into $BC^1(X)$. $DP$ is the identity on
$BC(X)$, whereas $PD$ is a projection of $BC^1(X)$ onto a complement of
$\{f \in BC^1(X) : f' = 0\}$.

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental
Theorem for functions belonging to spaces, larger than $BC(X), BC^1(X)$

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respectively. (For example, compare the classical theorem on \( L^1 \)-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that \( g : X \to K \) is of the first class of Baire if there exists a sequence \( g_1, g_2, \ldots \) of continuous functions \( X \to K \) such that \( \lim g_n = g \) pointwise.

**THEOREM.** (a) Let \( f : X \to K \) be differentiable. Then \( f' \) is of the first class of Baire.

(b) Let \( g : X \to K \) be of the first class of Baire. Then \( g \) has an antiderivative.

Let \( B^1(X) \) be the Banach space consisting of all bounded functions \( X \to K \) of the first class of Baire with respect to the supremum norm.

Let \( BD(X) \) be the Banach space of all differentiable \( f : X \to K \) for which both \( f \) and \( \psi f \) are bounded, with respect to the norm \( f \to ||f||_\infty \vee ||\psi f||_\infty \). Then we have

**THEOREM.** Differentiation is a quotient map \( BD(X) \xrightarrow{D} B^1(X) \).

If \( K \) has discrete valuation then there exists a continuous linear \( P : B^1(X) \to BD(X) \) for which \( DP \) is the identity on \( B^1(X) \).

**Notes.**

1. The construction of the above \( P \) is awful and, contrary to § 4, \( P \) does not resemble an indefinite integral in any way.

2. If the valuation of \( K \) is dense the existence of such a \( P \) is still an open question.
5 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to K$: $f \in C^2(X)$ if the function $\Phi_2^f$, defined by

$$
\Phi_2^f(x,y,z) = \frac{\Phi_1^f(x,y,z) - \Phi_1^f(y,z)}{x-y} \quad (x,y,z \in X, \, x \neq y, \, y \neq z, \, x \neq z)
$$

can continuously be extended to $\Phi_2^f : X^3 \to K$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$(P_2 f)(x) := \sum_{n=0}^2 f(x_n) (x_{n+1} - x_n) + \frac{1}{2} \sum_{n=1}^2 f'(x_n) (x_{n+1} - x_n)^2 \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to 2,3,\ldots,n).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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**Reference**