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Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis IR or C by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map $|\cdot| : K \rightarrow \mathbb{R}$ (the valuation) satisfying

\[ |a| \geq 0 \quad \text{and} \quad |a| = 0 \quad \text{iff} \quad a = 0 \]

\[ |ab| = |a| |b| \]

\[ |a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)} \]

for all $a,b \in K$.

We have the following remarks.

(1) Apart from IR or C, every complete valued field is non-archimedean.

(2) If $K$ is a non-archimedean valued field and if $L \supset K$ is an overfield of $K$ then the valuation on $K$ can be extended to a non-archimedean valuation on $L$.

(3) If $K$ is a (non-archimedean) valued field then its completion $\overline{K}$ (with respect to the metric $(x,y) \mapsto |x-y|$) can, in a natural way, be constructed.

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way, be given the structure of a non-archimedean valued field.
In the sequel we exclude the so-called \textit{trivial} valuation given by
\[
|x|^\prime = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases}
\]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean \textit{functional analysis}, \textit{harmonic analysis}, \textit{theory of analytic functions in one or several variables}, etc.

In this talk we consider a more elementary subject, namely \textit{infinitesimal calculus} in $K$. More specifically, we want to see what remains of the so-called \textit{Fundamental Theorem of Calculus} (in $\mathbb{R}$) that states that the operations of differentiation and integration are in some sense each other's inverses.

\section*{§ 2. Differentiation in $K$.} Let $X \subset K$ be a subset without isolated points. A function $f : X \to K$ is called \textit{differentiable} if for all $a \in X$
\[
f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]
exists. The proof of the well known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on $X$. An analytic function $x + \sum a_n x^n$ is differentiable on $\{x : |x| < (\lim \sqrt[n]{|a_n|})^{-1}\}$.

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\varepsilon > 0$, $a \in K$. Then $B(a, \varepsilon) := \{x \in K : |x-a| < \varepsilon\}$ is an open- and-closed subset of $K$, hence $\xi_{B(a, \varepsilon)}$, defined by

$$
\xi_{B(a, \varepsilon)}(x) := \begin{cases} 
1 & \text{if } x \in B(a, \varepsilon) \\
0 & \text{elsewhere}
\end{cases}
$$

is differentiable and $\xi_{B(a, \varepsilon)}' = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \rightarrow K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ defined by

$$f(\sum_{n} a_n p^n) = \sum_{n} a_n p^{2n} \quad (\sum_{n} a_n p^n \in \mathbb{Z}_p)$$

satisfies $|f(x) - f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : D \rightarrow \mathbb{R}$, where $D \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \rightarrow K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
f * max( |f|₂, |f'|₂ ) is not a Banach space. In fact one shows that for every pair of continuous functions f, g : Z → K, there exists a sequence f₁, f₂, ... in C¹(Z) for which both f_n → f and f'_n → g uniformly.

What is worse, we have no local invertibility theorem for such C¹-functions.

In fact, let f : Z → K be defined by

\[ f(x) = \begin{cases} x-p^n & \text{if } |x-p^n| < p^{-2n} \\ x & \text{elsewhere} \end{cases} \quad (n \in \mathbb{N}) \]

Then f'(x) = 1 for all x ∈ Z. But f(p^n) = f(p^n - p^{-2n}) for all n ∈ Z, so f is not even locally injective at 0.

Therefore we are led to define:

Let f : X → K. Put

\[ \Phi f(x,y) := \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y) .\]

We say that f ∈ C¹(X) if \( \Phi f \) can continuously be extended to a function \( \overline{\Phi f} : X × X \rightarrow K \).

Then BC¹(X) := \{ f ∈ C¹(X) : f and \( \Phi f \) are bounded \} is a Banach space under f * max( |f|₁, |\( \Phi f \)|₁ ) := max( |f|₁, |\( \Phi f \)|₁ ) .

Further, if f ∈ C¹(X), f'(a) ≠ 0 for some a ∈ X, then f has a C¹-inverse, locally at a.

**Theorem.** Differentiation is a continuous surjection BC¹(X) \( \overset{D}{\rightarrow} \) BC(X).

(here BC(X) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" P : BC(X) → BC¹(X)
(an analogue of \((Pf)(x) := \int_0^X f(t)dt\) for real functions) such that DP is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathcal{Z}\). (By translation invariance \(|m(p^n\mathcal{Z}_p)| = p^n|m(\mathcal{Z}_p)| \to \infty\) if \(m(\mathcal{Z}_p) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim_n\) defined by \(x \sim_n y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n, \ |x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\(P\) is a linear isometry of \(BC(X)\) into \(BC^1(X)\). DP is the identity on \(BC(X)\), whereas PD is a projection of \(BC^1(X)\) onto a complement of \(\{f \in BC^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X)\), \(BC^1(X)\).
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm.

Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\Phi f$ are bounded, with respect to the norm $f + ||f||_\infty + ||\Phi f||_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \xrightarrow{D} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \to BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5.6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if \( f \in C^n \) then 
\( x \mapsto \int_0^x f(t) \, dt \) is in \( C^{n+1} \). In our situation we define for \( f : X \rightarrow K \):

\[ f \in C^2(X) \text{ if the function } \phi_2 f, \text{ defined by} \]

\[
\phi_2 f(x, y, z) = \frac{\phi_1 f(x, z) - \phi_1 f(y, z)}{x - y} \quad (x, y, z \in X, x \neq y, y \neq z, x \neq z)
\]

can continuously be extended to \( \Phi_2 f : X^3 \rightarrow K \). Similarly, we define 
\( C^3(X), C^4(X), \ldots \). Let \( C^\infty(X) := \bigcap_{n=1}^\infty C^n(X) \).

The map \( P \), defined in §4, does not always map \( C^1 \)-functions into \( C^2 \)-functions. But we have (notations as in §4)

**THEOREM.** Let the characteristic of \( K \) be unequal to 2. Then the map 
\( P_2 \) defined via

\[
(P_2 f)(x) := \sum_{n=1}^\infty \left( n+1 \frac{f(x)}{n} (x_{n+1} - x_n) + \frac{n}{2} \sum_{n=1}^\infty \left( x_{n+1} - x_n \right)^2 \right) \quad (x \in X)
\]

maps \( C^1(X) \) into \( C^2(X) \) and \((Pf)' = f \) for all \( f \in C^1(X) \).

Similarly, one can define antiderivation maps \( P_n : C^{n-1}(X) \rightarrow C^n(X) \) (in case the characteristic of \( K \) is unequal to \( 2, 3, \ldots, n \)).

**OPEN QUESTION.** Let \( K \) have characteristic 0. Does every \( f \in C^\infty(X) \) have a \( C^\infty \)-antiderivative?

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**Reference**