Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

held on Tuesday, June 5, 1979 at the "VI Jornadas de Matemáticas Hispano-Lusas" organized by the University of SANTANDER,

by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis IR or C by a non-archimedean valued field K.

A non-archimedean valued field is a (commutative) field K, together with a map \(||: K \rightarrow IR\) (the valuation) satisfying

\[|a| \geq 0, \quad |a| = 0 \text{ iff } a = 0\]
\[|ab| = |a| \cdot |b|\]
\[|a+b| \leq \max(|a|,|b|)\] (the strong triangle inequality)

for all \(a,b \in K\).

We have the following remarks.

(1) Apart from IR or C, every complete valued field is non-archimedean.

(2) If K is a non-archimedean valued field and if \(L \supset K\) is an overfield of K then the valuation on K can be extended to a non-archimedean valuation on L.

(3) If K is a (non-archimedean) valued field then its completion \(\hat{K}\) (with respect to the metric \((x,y) \mapsto |x-y|\) can, in a natural
way, be given the structure of a non-archimedean valued field.

In the sequel we exclude the so-called trivial valuation given by

$$|x|' = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$$

The non-archimedean analysis has several branches, similar to

the classical analysis. Thus we have non-archimedean functional

analysis, harmonic analysis, theory of analytic functions in one or

several variables, etc.

In this talk we consider a more elementary subject, namely

infinitesimal calculus in $K$. More specifically, we want to see what

remains of the so-called Fundamental Theorem of Calculus (in $\mathbb{R}$)

that states that the operations of differentiation and integration

are in some sense each others inverses.

§ 2. Differentiation in $K$. Let $X \subset K$ be a subset without isolated

points. A function $f : X \rightarrow K$ is called differentiable if for all

$a \in X$

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The proof of the well known rules (sum-, product-, chain-

rule) can formally be taken over from the classical theory. Thus, a

rational function is differentiable if it has no poles on $X$. An

analytic function $x + \sum a_n x^n$ is differentiable on

$$\{x : |x| < \left(\lim \sqrt[|a_n|]{}\right)^{-1}\}.$$
let $\epsilon > 0$, $a \in K$. Then $B(a, \epsilon) := \{x \in K : |x-a| < \epsilon\}$ is an open-and-closed subset of $K$, hence $\xi_{B(a, \epsilon)}$, defined by

$$\xi_{B(a, \epsilon)}(x) := \begin{cases} 1 & \text{if } x \in B(a, \epsilon) \\ 0 & \text{elsewhere} \end{cases}$$

is differentiable and $\xi'_{B(a, \epsilon)} = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let 

$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\sum a_n^p n) = \sum a_n^2 p^{2n} \quad (\sum a_n^p n \in \mathbb{Z}_p)$$

satisfies $|f(x)-f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
f\mapsto \max(\|f\|_\infty, \|f'\|_\infty) is not a Banach space. In fact one shows that for every pair of continuous functions \(f, g : \mathcal{P} \to \mathcal{Q}\) there exists a sequence \(f_0, f_1, f_2, \ldots\) in \(C^1(\mathcal{P})\) for which both \(f_n \to f\) and \(f_n' \to g\) uniformly.

What is worse, we have no local invertibility theorem for such \(C^1\)-functions.

In fact, let \(f : \mathcal{P} \to \mathcal{Q}\) be defined by
\[
f(x) = \begin{cases} 
 x-p^{-2n} & \text{if} \quad |x-p^n| < p^{-2n} \\
 x & \text{elsewhere}
\end{cases} \quad (n \in \mathbb{N})
\]

Then \(f'(x) = 1\) for all \(x \in \mathcal{P}\). But \(f(p^n) = f(p^n-p^{-2n})\) for all \(n \in \mathbb{N}\), so \(f\) is not even locally injective at 0.

Therefore we are led to define:

\[
\text{Let} \quad f : X \to K. \quad \text{Put} \\
\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \(f \in C^1(X)\) if \(\Phi f\) can continuously be extended to a function \(\bar{\Phi} f : X \times X \to K\).

Then \(BC^1(X) := \{f \in C^1(X) : f\text{ and }\Phi f\text{ are bounded}\}\) is a Banach space under \(f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi(f)\|_\infty)\).

Further, if \(f \in C^1(X)\), \(f'(a) \neq 0\) for some \(a \in X\), then \(f\) has a \(C^1\)-inverse, locally at \(a\).

**Theorem.** Differentiation is a continuous surjection \(BC^1(X) \overset{D}{\to} BC(X)\).

(here \(BC(X)\) is the space of all bounded continuous functions with the supremum norm)

§ 4. **"Integration".**

Next, we want to define an "indefinite integral" \(P : BC(X) \to BC^1(X)\)
(an analogue of \((Pf)(x) := \int_0^x f(t)dt\) for real functions) such that \(DP\) is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathbb{R}_p\). (By translation invariance \(|m(p^n\mathbb{Z}_p)| = p^n|m(\mathbb{Z}_p)| \rightarrow \infty\) if \(m(\mathbb{Z}_p) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_1)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation \(\sim_n\) defined by \(x \sim_n y\) if \(|x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n\), \(|x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\(P\) is a linear isometry of \(BC(X)\) into \(BC^1(X)\). \(DP\) is the identity on \(BC(X)\), whereas \(PD\) is a projection of \(BC^1(X)\) onto a complement of \(\{f \in BC^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X)\), \(BC^1(X)\)
respectively. (For example, compare the classical theorem on $L^1$-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that $g : X \to K$ is of the first class of Baire if there exists a sequence $g_1, g_2, \ldots$ of continuous functions $X \to K$ such that $\lim g_n = g$ pointwise.

**THEOREM.** (a) Let $f : X \to K$ be differentiable. Then $f'$ is of the first class of Baire.

(b) Let $g : X \to K$ be of the first class of Baire. Then $g$ has an antiderivative.

Let $B^1(X)$ be the Banach space consisting of all bounded functions $X \to K$ of the first class of Baire with respect to the supremum norm. Let $BD(X)$ be the Banach space of all differentiable $f : X \to K$ for which both $f$ and $\phi f$ are bounded, with respect to the norm $\|f\|_\infty \vee \|\phi f\|_\infty$. Then we have

**THEOREM.** Differentiation is a quotient map $BD(X) \xrightarrow{D} B^1(X)$.

If $K$ has discrete valuation then there exists a continuous linear $P : B^1(X) \to BD(X)$ for which $DP$ is the identity on $B^1(X)$.

**Notes.**

1. The construction of the above $P$ is awful and, contrary to § 4, $P$ does not resemble an indefinite integral in any way.

2. If the valuation of $K$ is dense the existence of such a $P$ is still an open question.
5 6. **Restriction of the Fundamental Theorem.**

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to \mathbb{K}$:

If $f \in C^2(X)$ if the function $\phi_2^f$, defined by

$$
\phi_2^f(x,y,z) = \frac{\phi_1^f(x,z) - \phi_1^f(y,z)}{x-y} \quad (x, y, z \in X, x \neq y, y \neq z, x \neq z)
$$

can continuously be extended to $\phi_2^f : X^3 \to \mathbb{K}$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^{\infty} C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $K$ be unequal to 2. Then the map $P_2$ defined via

$$(P_2 f)(x) := \sum_{n} f_n(x_n - x_n) + \frac{1}{2} \sum_{n} f_n(x_{n+1} - x_n)^2 \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $K$ is unequal to 2, 3, \ldots, $n$).

**OPEN QUESTION.** Let $K$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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**Reference**