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Abstract of the lecture

NON - ARCHIMEDEAN DIFFERENTIATION

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by

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§ 1. Introduction.

The subject is part of the so-called non-archimedean (or ultrametric) analysis. Roughly speaking, one may say that this is the analysis that one obtains when replacing in the "classical" analysis \( \mathbb{R} \) or \( \mathbb{C} \) by a non-archimedean valued field \( K \).

A non-archimedean valued field is a (commutative) field \( K \), together with a map \( | \cdot | : K \rightarrow \mathbb{R} \) (the valuation) satisfying

\[
|a| \geq 0 \quad , \quad |a| = 0 \text{ iff } a = 0
\]

\[
|ab| = |a| |b|
\]

\[
|a+b| \leq \max(|a|,|b|) \quad \text{(the strong triangle inequality)}
\]

for all \( a,b \in K \).

We have the following remarks.

(1) Apart from \( \mathbb{R} \) or \( \mathbb{C} \), every complete valued field is non-archimedean.

(2) If \( K \) is a non-archimedean valued field and if \( L \supset K \) is an overfield of \( K \) then the valuation on \( K \) can be extended to a non-archimedean valuation on \( L \).

(3) If \( K \) is a (non-archimedean) valued field then its completion \( \sim K \) (with respect to the metric \( (x,y) \mapsto |x-y| \)) can, in a natural

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way, be given the structure of a non-archimedean valued field.
In the sequel we exclude the so-called trivial valuation given by

\[ |x|' = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0.
\end{cases} \]

The non-archimedean analysis has several branches, similar to the classical analysis. Thus we have non-archimedean functional analysis, harmonic analysis, theory of analytic functions in one or several variables, etc.

In this talk we consider a more elementary subject, namely infinitesimal calculus in K. More specifically, we want to see what remains of the so-called Fundamental Theorem of Calculus (in \( \mathbb{R} \)) that states that the operations of differentiation and integration are in some sense each other's inverses.

§ 2. Differentiation in K. Let \( X \subset K \) be a subset without isolated points. A function \( f : X \to K \) is called differentiable if for all \( a \in X \)

\[ f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

exists. The proof of the well-known rules (sum-, product-, chain-rule) can formally be taken over from the classical theory. Thus, a rational function is differentiable if it has no poles on \( X \). An analytic function \( x + \sum_{n=1}^{\infty} a_n x^n \) is differentiable on \( \{ x : |x| < \lim \sqrt[n]{|a_n|}^{-1} \} \).

Deviations from the classical theory appear when we look at the functions whose derivative vanishes everywhere. For example,
let $\epsilon > 0$, $a \in K$. Then $B(a, \epsilon) := \{x \in K : |x-a| < \epsilon\}$ is an open-and-closed subset of $K$, hence $\xi_{B(a, \epsilon)}$, defined by

$$\xi_{B(a, \epsilon)}(x) := \begin{cases} 1 & \text{if } x \in B(a, \epsilon) \\ 0 & \text{elsewhere} \end{cases}$$

is differentiable and $\xi'_{B(a, \epsilon)} = 0$.

Locally constant functions all have derivative zero. On the other hand they form a uniformly dense subset of $C(X)$, the space of all continuous functions: $X \to K$.

Even worse: let $\mathbb{Q}_p$ the field of the $p$-adic numbers and let $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$. Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f(\ell a_p^n) = \ell a_p^{2n} \quad (\ell a_p^n \in \mathbb{Z}_p)$$

satisfies $|f(x) - f(y)| = |x-y|^2$ for all $x, y \in \mathbb{Z}_p$. So $f' = 0$ but $f$ is injective, hence not locally constant.

The above example shows also that a Mean Value Theorem is necessarily absent in our theory.

Notice that the difficulties encountered above also appear when we study differentiability of functions $f : D \to \mathbb{R}$, where $D \subset [0,1]$ is the Cantor set. So it is the domain of $f$ that is responsible for the troubles rather than its range.

§ 3. Continuously differentiable functions.

If we follow naively the path of the classical analysis and define

$$C^1(X) := \{f : X \to K, f \text{ is differentiable, } f' \text{ is continuous}\}$$

then we run up against difficulties.

First of all, one can prove that $C^1(\mathbb{Z}_p)$ (with the norm
max(\|f\|_\infty, \|f'\|_\infty) is not a Banach space. In fact one shows that for every pair of continuous functions \(f, g : \mathbb{R}_+ \to \mathbb{R}_+\) there exists a sequence \(f_1, f_2, \ldots\) in \(C^1(\mathbb{R}_+)\) for which both \(f_n \to f\) and \(f'_n \to g\) uniformly.

What is worse, we have no local invertibility theorem for such \(C^1\)-functions.

In fact, let \(f : \mathbb{R}_+ \to \mathbb{R}_+\) be defined by
\[
f(x) = \begin{cases} 
  x^{-2n} & \text{if } |x^{-n}| < p^{-2n} \\
  x & \text{elsewhere}
\end{cases} 
\]

Then \(f'(x) = 1\) for all \(x \in \mathbb{R}_+\). But \(f(p^n) = f(p^{-n}2^n)\) for all \(n \in \mathbb{N}\), so \(f\) is not even locally injective at 0.

Therefore we are led to define:

Let \(f : X \to K\). Put
\[
\Phi f(x,y) := \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

We say that \(f \in C^1(X)\) if \(\Phi f\) can continuously be extended to a function \(\Phi f : X \times X \to K\).

Then \(BC^1(X) := \{ f \in C^1(X) : f \text{ and } \Phi f \text{ are bounded} \}\) is a Banach space under \(f \mapsto \|f\|_1 := \max(\|f\|_\infty, \|\Phi f\|_\infty)\).

Further, if \(f \in C^1(X)\), \(f'(a) \neq 0\) for some \(a \in X\), then \(f\) has a \(C^1\)-inverse, locally at \(a\).

**Theorem.** Differentiation is a continuous surjection \(BC^1(X) \overset{D}{\to} BC(X)\).

(here \(BC(X)\) is the space of all bounded continuous functions with the supremum norm)

§ 4. "Integration".

Next, we want to define an "indefinite integral" \(P : BC(X) \to BC^1(X)\)
(an analogue of \((Pf)(x) := \int_0^x f(t)dt\) for real functions) such that \(DP\) is the identity on \(BC(X)\).

A natural try is first to find an analogue of the Lebesgue measure in \(K\). But this turns out to be a dead end road. For example if \(K = \mathbb{Q}_p\) there does not exist a nonzero translation invariant bounded additive \(\mathbb{Q}_p\)-valued function \(m\) defined on the compact open subsets of \(\mathcal{Z}\). (By translation invariance \(|m(p^n\mathcal{Z}_p)| = p^n|m(\mathcal{Z}_p)| \to \infty\) if \(m(\mathcal{Z}_p) \neq 0\). For similar reasons it goes wrong for every local field \(K\).

Following the ideas of Dieudonné, Treiber, we define for \(f \in BC(X)\)

\[
(Pf)(x) := \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

Here the \(x_n\) are defined as follows. For each \(n \in \mathbb{N}\) the equivalence relation defined by \(x \sim y\ if \ |x-y| < \frac{1}{n}\) yields a partition of \(X\) into balls. Choose a center in each ball and let \(R_n\) be the set of these centers.

For each \(x \in X\) and \(n \in \mathbb{N}\), \(x_n\) is defined by \(x_n \in R_n, \ |x_n - x| < \frac{1}{n}\).

**Theorem.** (A NON-ARCHIMEDEAN FORM OF THE FUNDAMENTAL THEOREM).

\(P\) is a linear isometry of \(BC(X)\) into \(BC^1(X)\). \(DP\) is the identity on \(BC(X)\), whereas \(PD\) is a projection of \(BC^1(X)\) onto a complement of \(\{f \in BC^1(X) : f' = 0\}\).

§ 5. Generalizations of the Fundamental Theorem.

We may ask whether there exists some form of the Fundamental Theorem for functions belonging to spaces, larger than \(BC(X)\), \(BC^1(X)\)
respectively. (For example, compare the classical theorem on \(L^1\)-functions versus absolutely continuous functions).

We have the following striking fact that has no counterpart in classical analysis. We say that \(g: X \rightarrow K\) is of the first class of Baire if there exists a sequence \(g_1, g_2, \ldots\) of continuous functions \(X \rightarrow K\) such that \(\lim g_n = g\) pointwise.

**THEOREM.**

(a) Let \(f: X \rightarrow K\) be differentiable. Then \(f'\) is of the first class of Baire.

(b) Let \(g: X \rightarrow K\) be of the first class of Baire. Then \(g\) has an antiderivative.

Let \(B^1(X)\) be the Banach space consisting of all bounded functions \(X \rightarrow K\) of the first class of Baire with respect to the supremum norm. Let \(BD(X)\) be the Banach space of all differentiable \(f: X \rightarrow K\) for which both \(f\) and \(\Phi f\) are bounded, with respect to the norm \(f + |f|_\infty + |\Phi f|_\infty\). Then we have

**THEOREM.** Differentiation is a quotient map \(BD(X) \xrightarrow{\Phi} B^1(X)\).

If \(K\) has discrete valuation then there exists a continuous linear \(P: B^1(X) \rightarrow BD(X)\) for which \(DP\) is the identity on \(B^1(X)\).

**Notes.**

1. The construction of the above \(P\) is awful and, contrary to § 4, \(P\) does not resemble an indefinite integral in any way.

2. If the valuation of \(K\) is dense the existence of such a \(P\) is still an open question.
5.6. Restriction of the Fundamental Theorem.

In classical analysis, we have that if $f \in C^n$ then $x \mapsto \int_0^x f(t) \, dt$ is in $C^{n+1}$. In our situation we define for $f : X \to \mathbb{K}$:

if $f \in C^2(X)$ if the function $\Phi_2 f$, defined by

$$\Phi_2 f(x,y,z) = \frac{\Phi_1 f(x,z) - \Phi_1 f(y,z)}{x-y} \quad (x,y,z \in X, \ x \neq y, \ y \neq z, \ x \neq z)$$

can continuously be extended to $\Phi_2 f : X^3 \to \mathbb{K}$. Similarly, we define $C^3(X), C^4(X), \ldots$. Let $C^\infty(X) := \bigcap_{n=1}^\infty C^n(X)$.

The map $P$, defined in § 4, does not always map $C^1$-functions into $C^2$-functions. But we have (notations as in § 4)

**THEOREM.** Let the characteristic of $\mathbb{K}$ be unequal to 2. Then the map

$$P_2 \text{ defined via}$$

$$(P_2 f)(x) := \sum_{n=1} f(x_n) (x_{n+1} - x_n) + \frac{1}{2} \sum_{n=1} f'(x_n) (x_{n+1} - x_n)^2 \quad (x \in X)$$

maps $C^1(X)$ into $C^2(X)$ and $(Pf)' = f$ for all $f \in C^1(X)$.

Similarly, one can define antiderivation maps $P_n : C^{n-1}(X) \to C^n(X)$ (in case the characteristic of $\mathbb{K}$ is unequal to $2, 3, \ldots, n$).

**OPEN QUESTION.** Let $\mathbb{K}$ have characteristic 0. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?

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Reference