NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to \( K \)-valued functions of one single variable. However, a lot of the results can without any problem be carried over to \( E \)-valued functions of one variable, where \( E \) is a \( K \)-Banach space. A generalization to functions \( K^n + K^m \) will be less obvious, although it seems clear how to define \( C^k \)-functions in that case. (For example, in order that \( f : K^2 \rightarrow K \) is \( C^1 \) one should require (see 3.1) that the difference quotients

\[
\begin{align*}
(x_1, x_2, y) \mapsto \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, & \quad (x, y_1, y_2) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
\end{align*}
\]

can be extended to continuous functions on \( K^3 \). If we take again difference quotients we get four functions of four variables, required to be continuous in order that \( f \) be in \( C^2 \) (see 6.1). It then follows very easily that \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \) for \( f \in C^2 \).

Throughout this note, \( K \) will always be a complete non-archimedean valued field, and \( X \) a non-empty subset of \( K \), without isolated points. We study differentiability properties of functions \( f : X \rightarrow K \). Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $E_a: H^p \to H^p$, defined on $H^p$ is an example of an injective function with zero derivative and which is in $Lip_a$ for every $a > 0$. The function $f: H^p \to H^p$ defined via $f(x) = x - p^{2n}$ if $|x - p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^n) = p^n - p^{2n}$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions $X \to K$, normed by the sup norm $||\cdot||_\infty$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $K$, and let $f: X \to K$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g: X \to K$ such that $g$ has bounded difference quotients, and such that $||f-g||_\infty < \varepsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
THEOREM 2.1. Let $f : X \rightarrow K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one, (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subset K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

THEOREM 2.2. Let $K$ be a local field and let $f : X \rightarrow K$ be differentiable. Then we have:

1. If $Y \subset X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"")

2. $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X \rightarrow K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger then just "$f$ is differentiable and $f$ is continuous". For $f : X \rightarrow K$, define

$$\phi_1 f(x,y) = \frac{f(x) - f(y)}{x - y} \quad (x, y \in X, x \neq y).$$

DEFINITION 3.1. $f : X \rightarrow K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\overline{\phi_1 f}$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of \( f' \) already guarantees the existence of a continuous \( \overline{f}' \).

**THEOREM 3.2.** Let \( f \in C^1(X) \) and let \( a \in X \).

(a) If \( f'(a) \neq 0 \) then \( f \) is locally invertible at \( a \). (In fact, \( (f'(a))^{-1}f \) is an isometry locally at \( a \)).

(b) If \( X \) is open in \( K \) and if \( f' \neq 0 \) everywhere on \( X \) then \( f \) is an open mapping.

Let \( BC^1(X) = \{ f \in C^1(X) : \| f \|_1 := \| f \|_\infty \vee \| \overline{f}' \|_\infty \} \). Then \( BC^1(X) \) is a Banach space with respect to \( \| \cdot \|_1 \). We may put a locally convex topology on \( C^1(X) \) via the defining seminorms \( \| \cdot \|_{1,C} \) where \( C \) runs through the compact subsets of \( X \):

\[
\| f \|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} \left| \overline{f}'(x,y) \right| \quad (f \in C^1(X)).
\]

Let \( N^1(X) = \{ f \in C^1(X) : f' = 0 \} \) and \( BN^1(X) = \{ f \in BC^1(X) : f' = 0 \} \). Then \( N^1(X) \) is closed in \( C^1(X) \), \( BN^1(X) \) is closed in \( BC^1(X) \).

**THEOREM 3.3.** The locally linear functions (in \( BC^1(X) \)) form a dense subset of \( C^1(X) \) (of \( BC^1(X) \)).

The locally constant functions (in \( BC^1(X) \)) form a dense subset of \( N^1(X) \) (of \( BN^1(X) \)).

**THEOREM 3.4.** If either \( X \) is compact or \( K \) has discrete valuation then \( BC^1(X) \) has an orthonormal base (in the sense of the norm).

If \( X \) is not compact and \( K \) has dense valuation then, for any \( a > 0 \), \( BC^1(X) \) has no \( a \)-orthogonal base.
Let us choose real numbers \(1 > r_1 > r_2 > \ldots\) with \(\lim_{n \to \infty} r_n = 0\), and, for each \(n\), let \(R_n\) be a full set of representatives of the equivalence relation (in \(X\)): \(x \sim y\) if \(|x - y| < r_n\). We can arrange that 
\[R_1 \subset R_2 \subset \ldots\] For each \(x \in X\), \(n \in \mathbb{N}\), let \(x_n \in R_n\) be determined by: \(|x_n - x| < r_n\). For a continuous \(f : X \to \mathbb{K}\) set 
\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

**Theorem 3.5.** The map \(P\) defined above is a continuous linear map: 
\(C(X) \to C^1(X)\) and its restriction to \(BC(X)\) is an isometry: \(BC(X) \to BC^1(X)\). \(P\) is an antiderivation map i.e., \((Pf)' = f\) for each \(f \in C(X)\).

**Corollary 3.6.** Every continuous function has a \(C^1\)-antiderivative.

In fact, by passing through the quotient, differentiation yields a map \(p : BC^1(X)/BN^1(X) \to BC(X)\) which is a surjective isometry. Moreover, \(BN^1(X)\) has an orthogonal complement \((\text{im } P)\) in \(BC^1(X)\).

### 4. \(C^1(X)\) for compact \(X\)

(Throughout section 4, \(X\) is compact). The set \(|x-y| : x,y \in X\) is bounded and has only 0 as an accumulation point, hence it can be written as \(\{r_1, r_2, \ldots\} \cup \{0\}\), where \(r_1 > r_2 > \ldots\) and \(\lim_{n \to \infty} r_n = 0\). Let \(r_0 = \infty\). For each \(i\), let \(R_i\) be a full set of representatives in \(X\) of the equivalence relation "\(x \sim y\) if \(|x - y| < r_i\)" such that 
\[R_0 \subset R_1 \subset \ldots\] Then \(R_i\) is finite for each \(i\) and \(R_0\) consists only of one single point \(a_0\). Let \(R = \bigcup_{i} R_i\) and define \(\nu : R \to \{0,1,2,\ldots\}\) as follows. For a \(a \in R\) let \(\nu(a)\) be the nonnegative integer \(m\) for which 
\[a \in R_m \setminus R_{m-1} \quad (R_{-1} = \emptyset \text{ by definition})\] For each a \(a \in R\) let
Let $B_a = \{ x \in X : |x-a| < r_v(a) \}$, and let $e_a$ be the $\mathbb{K}$-valued characteristic function of $B_a$. Further, we define

$$a \preceq b \quad \text{iff} \quad b \in B_a \quad (a,b \in \mathbb{R})$$

Then we have

**Lemma 4.1.** $(\mathbb{R}, \preceq)$ is a partially ordered set with a smallest element $a_0$. For each $a \in \mathbb{R}$, the set $\{ x \in \mathbb{R} : x \geq a \}$ is finite and linearly ordered by $\preceq$.

Define for $a \in \mathbb{R}$, $a \neq a_0$: $a_- = \max \{ x \in \mathbb{R} : x \leq a, x \not\equiv a \}$. Then

**Theorem 4.2.** The set $\{ e_a : a \in \mathbb{R} \}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda^a e_a$ for some $\lambda^a \in \mathbb{K}$. Then

$$\lambda^a_0 = f(a_0) \quad \text{and} \quad \lambda^a \neq a_0: \lambda^a = f(a)-f(a_-).$$

The set $\{ e_a : a \in \mathbb{R} \} \cup \{ P_a e_a : a \in \mathbb{R} \}$ (as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda^a e_a + \sum \mu^b P_b e_b$ ($\lambda^a, \mu^b \in \mathbb{K}$) in the $||f||_1$-norm. Then $\lambda^a_0 = f(a_0)$, $\mu^a_0 = f'(a_0)$ and for $a \neq a_0$:

$$\lambda^a = f(a)-f(a_-)-(a-a_-)f'(a_-),$$

$$\mu^a = f'(a)-f'(a_-).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called uniformly differentiable if $\lim_{x \to y} f(x,y) = f'(y)$ uniformly in $y$. $f$ is called strongly uniformly differentiable if $f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let \( f : X \to K \) be (strongly) uniformly differentiable. Then

\( f \) has a unique continuous extension \( \overline{f} : \overline{X} \to K \) (\( \overline{X} \) is the closure of \( X \) in \( K \)).

This \( \overline{f} \) is (strongly) uniformly differentiable.

THEOREM 5.2. Let \( f : X \to K \) be uniformly differentiable. Then each of

the following properties implies strong uniform differentiability of \( f \):

(a) \( \phi_1 f \) is bounded.

(b) Both \( f \) and \( f' \) are bounded

(c) \( X \) is "nice" and \( f \) is bounded.

(\( X \) is called "nice" if for each \( r > 0 \) there is \( s > 0 \) such that for every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \)).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. \( C^n \)-functions

For \( n \in \mathbb{IN} \), let \( V^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \rightarrow x_i \neq x_j \} \). For \( f : X \to K \) we define the \( n \)th difference quotient \( \phi_n f : V^{n+1} X \to K \)

inductively as follows \( \phi_0 f = f \) and for \( (x_1, \ldots, x_{n+1}) \in V^{n+1} X \):

\[
\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).
\]

Since \( V^n X \) is dense in \( X^n \) for each \( n \) the following definition makes sense.

DEFINITION 6.1. Let \( f : X \to K \), \( n \in \mathbb{IN} \cup \{0\} \). We say that \( f \in C^n(X) \) if

\( \phi_n f \) can be extended to a continuous function \( \overline{\phi_n f} : X^{n+1} \to K \).

We say that \( f \in \mathbb{B}^n(X) \) if \( \phi_0 f, \ldots, \phi_n f \) are bounded functions.
For \( f \in \mathcal{B}^n(X) \) set
\[
\| f \|_n = \max_{0 \leq i \leq n} \| \phi_i f \|_{\infty}.
\]

Let \( \mathcal{B}^n(X) = \mathcal{B}^n(X) \cap \mathcal{C}^n(X) \), \( \mathcal{C}^\infty(X) = \bigcap_{n=1}^{\infty} \mathcal{C}^n(X) \),
\( \mathcal{B}^\infty(X) = \bigcap_{n=1}^{\infty} \mathcal{B}^n(X) \).

**Theorem 6.2.** \( \mathcal{C}^1(X) \supset \mathcal{C}^2(X) \supset \ldots \)
\( \mathcal{B}^1(X) \supset \mathcal{B}^2(X) \supset \mathcal{B}^3(X) \supset \ldots \)
\( \mathcal{B}^n(X) \) is a Banach space with respect to \( \| \|_n \) and \( \mathcal{B}^n(X) \) is closed in \( \mathcal{B}^n(X) \).

For \( f \in \mathcal{C}^n(X) \) \( (n \geq 1) \) and \( 0 \leq j \leq n \) we define the \( j \)th Hasse derivative of \( f \) by
\[
D_j f(x) = \phi_j f(x, \ldots, x) \quad (x \in X).
\]

**Theorem 6.3.** Let \( f \in \mathcal{C}^n(X) \). Then for \( 0 \leq j \leq n \) we have \( D_j f \in \mathcal{C}^{n-j}(X) \)
and if \( i+j \leq n \)
\[
D_i D_j f = \binom{i+j}{i} D_{i+j} f
\]

\( f \) is \( n \) times differentiable in the ordinary sense and
for \( 0 \leq i \leq n \) we have
\[
f^{(i)} = i! D_i f.
\]

\( f : X \to K \) is called a spline function of degree \( \leq n \) if for every \( a \in X \) there is a neighbourhood \( U \) of \( a \) such that \( f|U \cap X \) is a polynomial function of degree \( \leq n \). Spline functions are in \( \mathcal{C}^\infty(X) \).

**Theorem 6.4.** Let \( f \in \mathcal{C}^n(X) \) and \( \varepsilon > 0 \). Then there is a spline function 
\( g \) of degree \( \leq n \) such that \( f-g \in \mathcal{B}^n(X) \), \( \| f-g \|_n < \varepsilon \). If
\[
D_i f = D_{i+1} f = \ldots = D_n f = 0 \text{ for some } i \in \{1, \ldots, n\} \text{ then } g \text{ can be chosen to be of degree } \leq i-1.
\]
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subseteq X \)) such that \( f : U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nD_nf(x,y,\ldots,y).
\]

The above result leads to another possible notion of "\( n \)-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \to K \), \( n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \to K \) and a continuous \( D_nf : X^2 \to K \) such that for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nD_nf(x,y,\ldots,y).
\]

(It follows that the \( D_if, D_nf \) are uniquely determined and continuous. Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1,2 \). Also \( C^n(X) \subseteq C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ \Sigma a_n p^n : a_n \in \{0,1\} \} \), and let \( f : X \to K \) be defined via

\[
f(\Sigma a_n p^n) = \Sigma a_n p^{3n}.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_if = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3f = 1 \). On the other hand, \( f \not\in C^3(X) \).

Let \( C > 0 \) and \( \{x_1,\ldots,x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1,\ldots,x_n\} \) a C-polygon if for all \( i,j,k,l \in \{1,\ldots,n\}, k \neq l \):
\[ \frac{|x_i - x_j|}{|x_k - x_1|} \leq C. \]

**Definition 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{f \in C^n(X) : \|f\|^n_n < \infty\} \), where, by definition,

\[ \|f\|^n_n = \max(\|f\|_m^m, |D_1 f|_o, \ldots, |D_{n-1} f|_o, |R_n f|_o) \]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( \| \|_n^n \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n > 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[ \|\|_n^n \leq \|\|_n \leq C^2(n-2) \|\|_n^n. \]
(In general we have for \( f \in \mathcal{B}C^1(X) : ||f||_n = \max_{0 \leq i \leq n} ||D_i f||_{n-i}.\)

As in 3.5, we want to find an antiderivation map: \( C^{n-1}(X) \rightarrow C^n(X).\)

We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in \mathcal{C}^1(X) \) then
\[ Pf \in \mathcal{C}^2(X) \text{ if and only if } f' = 0. \]
Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^{p-1} \)-function has a \( C^p \)-antiderivative.

**THEOREM 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \)

as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m, \) some \( \rho > 0. \)

For \( f \in C^{n-1}(X), \) set

\[ P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X) \]

Then \( P_n f \in C^n(X) \text{ and } (P_n f)' = f. \) If \( f \in \mathcal{B}C^{n-1}(X), \) then

\[ Pf \in \mathcal{B}C^n(X) \text{ and } \]
\[ ||P_n f||_n \leq c_n ||f||_{n-1} \]

where

\[ c_n = \max_{1 \leq i \leq n} \frac{1}{i \cdot \rho^n}. \]

It follows that the map \( Q_n = n! P_1 P_{n-1} \ldots P_1 \) sends \( C(X) \) into \( C^n(X), \)

\( D_n Q_n \) is the identity on \( C(X). \) A computation yields

\[ Q_n = \sum_{i=1}^{n} (-1)^{i+1} \frac{n!}{i!} n^{-i} S_i, \]

where \( M \) is the multiplication with \( x \) ((\( Mf)(x) = xf(x) \text{ for } f \in C(X) \))

and where

\[ S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (f \in C(X), \ x \in X) \]
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

**THEOREM 7.1.** Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

**THEOREM 7.2.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_i f(0) = \lambda_i$ for all $i$.

**Open problem:** Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?