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NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 \to K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x_1, y) - f(x, y_2)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that

$$
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
$$

for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{Z}_n \mathbb{P}^n \to \mathbb{Z}_n \mathbb{P}^n!$, defined on $\mathbb{Z}_n$ is an example of an injective function with zero derivative and which is in Lip for every $a > 0$. The function $f : \mathbb{Z}_n \mathbb{P}^n \to \mathbb{Z}_n \mathbb{P}^n!$ defined via $f(x) = x - p^n$ if $|x - p^n| < p^{2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence $f$ is not locally injective at $0$.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

   Let $BC(X)$ be the algebra of the bounded continuous functions: $X \to K$, normed by the sup norm $|| \cdot ||_\infty$. We have, analogous to the classical case:

   **Theorem 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

   In contrast to the theory of functions on the real line we have

   **Theorem 1.2.** Let $X$ be open in $K$, and let $f : X \to K$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to K$ such that $g$ has bounded difference quotients, and such that $||f - g||_\infty < \varepsilon$.

2. **Differentiability as such**

   Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let $f : X \rightarrow K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one, (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subseteq K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

**THEOREM 2.2.** Let $K$ be a local field and let $f : X \rightarrow K$ be differentiable. Then we have:

1. If $Y \subseteq X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"")
2. $\{f(x) : f'(x) = 0\}$ is a nullset.

**COROLLARY 2.3.** If $f : X \rightarrow K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger then just "$f$ is differentiable and $f$ is continuous". For $f : X \rightarrow K$, define

$$\phi_1f(x,y) = \frac{f(x)-f(y)}{x-y} \quad (x,y \in X, x \neq y).$$

**DEFINITION 3.1.** $f : X \rightarrow K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\bar{\phi}_1 f$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\overline{\phi}_1(f)$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : ||f||_1 := ||f||_\infty \vee ||\phi_1 f||_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $|| \cdot ||_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $|| \cdot ||_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$||f||_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C, y \in C} |\overline{\phi}_1 f(x,y)| \quad (f \in C^1(X)).$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $\alpha > 0$, $BC^1(X)$ has no $\alpha$-orthogonal base.
Let us choose real numbers \( 1 > r_1 > r_2 > \ldots \) with \( \lim_{n \to \infty} r_n = 0 \), and, for each \( n \), let \( R_n \) be a full set of representatives of the equivalence relation (in \( X \)): \( x \sim y \) if \( |x-y| < r_n \). We can arrange that \( R_1 \subset R_2 \subset \ldots \). For each \( x \in X \), \( n \in \mathbb{N} \), let \( x_n \in X \) be determined by:

\[
|x_n - x| < r_n, \quad x_n \in R_n.
\]

For a continuous \( f : X \to K \) set

\[
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X)
\]

**THEOREM 3.5.** The map \( P \) defined above is a continuous linear map:

\[
C(X) \to C^1(X) \quad \text{and its restriction to } BC(X) \text{ is an isometry: } BC(X) \to BC^1(X).
\]

\( P \) is an antiderivation map, i.e., \( (Pf)' = f \) for each \( f \in C(X) \).

**COROLLARY 3.6.** Every continuous function has a \( C^1 \)-antiderivative.

In fact, by passing through the quotient, differentiation yields a map \( p : BC^1(X)/BN^1(X) \to BC(X) \) which is a surjective isometry. Moreover, \( BN^1(X) \) has an orthogonal complement \( \text{im } P \) in \( BC^1(X) \).

4. \( C^1(X) \) for compact \( X \)

(Throughout section 4, \( X \) is compact). The set \( \{|x-y| : x, y \in X\} \) is bounded and has only 0 as an accumulation point, hence it can be written as \( \{r_1, r_2, \ldots\} \cup \{0\} \), where \( r_1 > r_2 > \ldots \) and \( \lim_{n \to \infty} r_n = 0 \). Let \( r_0 = \infty \). For each \( i \), let \( R_i \) be a full set of representatives in \( X \) of the equivalence relation "\( x \sim y \) if \( |x-y| < r_i \)" such that

\[
R_0 \subset R_1 \subset \ldots .
\]

Then \( R_i \) is finite for each \( i \) and \( R_0 \) consists only of one single point \( a_0 \). Let \( R = \bigcup_{i} R_i \) and define \( v : R \to \{0,1,2,\ldots\} \) as follows. For a \( a \in R \) let \( v(a) \) be the nonnegative integer \( m \) for which \( a \in R_m \setminus R_{m-1} \) (\( R_{-1} = \emptyset \) by definition). For each \( a \in R \) let
$B_a = \{x \in X : |x-a| < r_v(a)\}$,

and let $e_a$ be the $K$-valued characteristic function of $B_a$. Further, we define

$$a \triangleq b \text{ iff } b \in B_a \quad (a,b \in R)$$

Then we have

**Lemma 4.1.** $(R,\triangleq)$ is a partially ordered set with a smallest element $a_0$. For each $a \in R$, the set $\{x \in R : x \triangleq a\}$ is finite and linearly ordered by $\triangleq$.

Define for $a \in R$, $a \neq a_0$: $a_0 = \max \{x \in R : x \neq a, x \triangleq a\}$. Then

**Theorem 4.2.** The set $\{e_a : a \in R\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then $\lambda_{a_0} = f(a_0)$ and for $a \neq a_0$: $\lambda_a = f(a) - f(a_0)$.

The set $\{e_a : a \in R\} \cup \{Pe_a : a \in R\}$ $(P$ as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_a e_a + \sum \mu_b Pe_b$ $(\lambda_a, \mu_b \in K)$ in the $\| \|_1$-norm. Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ and for $a \neq a_0$:

$$\lambda_a = f(a) - f(a_0) - (a-a_0)f'(a)$$

$$\mu_a = f'(a) - f'(a_0).$$

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called uniformly differentiable if $\lim \limits_{x \to y} f(x,y) = f'(y)$ uniformly in $y$. $f$ is called strongly uniformly differentiable if $\Phi f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\overline{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\overline{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_{1} f$ is bounded.

(b) Both $f$ and $f'$ are bounded

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5, 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $\nu^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For $f : X \to K$ we define the $n^{th}$ difference quotient $\phi_n f : \nu^{n+1} X \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in \nu^{n+1} X$:

$$
\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).
$$

Since $\nu^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f} : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For \( f \in B^\infty(X) \) set
\[
||f||_n = \max_{0 \leq i \leq n} ||f_i||_{\infty}.
\]

Let \( BC^1(X) = B^1(X) \cap C^1(X) \), \( C^\infty(X) = \bigcap_{n=1}^{\infty} C^n(X) \),
\( BC^\infty(X) = \bigcap_{n=1}^{\infty} BC^n(X) \).

**THEOREM 6.2.** \( C^1(X) \supset C^2(X) \supset \ldots \)
\( B^1(X) \supset BC^1(X) \supset B^2(X) \supset BC^2(X) \supset \ldots \)
\( B^n(X) \) is a Banach space with respect to \( || \ | \ |_n \) and
\( BC^n(X) \) is closed in \( B^n(X) \).

For \( f \in C^n(X) \) \((n \geq 1)\) and \( 0 \leq j \leq n \) we define the \( j^{\text{th}} \) Hasse derivative
of \( f \) by
\[
D_j f(x) = \frac{\partial f}{\partial x_j}(x, x, \ldots, x) \quad (x \in X).
\]

**THEOREM 6.3.** Let \( f \in C^n(X) \). Then for \( 0 \leq j \leq n \) we have \( D_j f \in C^{n-j}(X) \)
and if \( i+j \leq n \)
\[
D_i D_j f = {i+j \choose i} D_{i+j} f
\]
\( f \) is \( n \) times differentiable in the ordinary sense and
for \( 0 \leq i \leq n \) we have
\[
f^{(i)} = i! \ D_i f.
\]

\( f : X \to K \) is called a spline function of degree \( \leq n \) if for every
\( a \in X \) there is a neighbourhood \( U \) of \( a \) such that \( f|U \cap X \) is a polynomial
function of degree \( \leq n \). Spline functions are in \( C^\infty(X) \).

**THEOREM 6.4.** Let \( f \in C^n(X) \) and \( \varepsilon > 0 \). Then there is a spline function
\( g \) of degree \( \leq n \) such that \( f-g \in BC^n(X) \), \( ||f-g||_n < \varepsilon \). If
\( D_i f = D_{i+1} f = \ldots = D_n f = 0 \) for some \( i \in \{1, \ldots, n\} \) then
\( g \) can be chosen to be of degree \( \leq i-1 \).
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subset X \)) such that \( f: U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y)\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f: X \to \mathbb{K} \), \( n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \to \mathbb{K} \) and a continuous \( R_n : X^2 \to \mathbb{K} \) such that for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y).
\]

(It follows that the \( D_if, R_nf \) are uniquely determined and continuous. Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1,2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ \Sigma a_n p^n : a_n \in \{0,1\} \} \), and let \( f: X \to \mathbb{K} \) be defined via

\[
f(\Sigma a_n p^n) = \Sigma a_n p^{3n!}.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_if = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3f = 1 \). On the other hand, \( f \not\in C^3(X) \).

Let \( C > 0 \) and \( \{x_1,\ldots,x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1,\ldots,x_n\} \) a \( C \)-polygon if for all \( i,j,k,l \in \{1,\ldots,n\} \), \( k \neq l \):
\[ \left| \frac{x_i - x_j}{x_k - x_l} \right| \leq C. \]

**DEFINITION 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has **locally property** \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \).)

We say that \( X \) has **globally property** \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||_{n} < \omega \} \), where, by definition,

\[ ||f||_{n} = \max( ||f||_{\omega}, ||D_{1}f||_{\omega}, \ldots, ||D_{n-1}f||_{\omega}, ||R_{n}f||_{\omega}) \]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||_{n}^{\omega} \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = BC^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n > 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[ ||f||_{n}^{\omega} \leq ||f||_{n} \leq c^{2(n-2)} ||f||_{n}^{\omega}. \]
(In general we have for \( f \in BC^n(X) \): \( \|f\|_n = \max_{0 \leq i \leq n} \|D_if\|_{n-i} \).

As in 3.5, we want to find an antiderivation map: \( C^{n-1}(X) \to C^n(X) \).

We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^1(X) \) then \( Pf \in C^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^p \)-function has a C\( ^p \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \)

as in 3.5 but such that \( r_m \leq \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \).

For \( f \in C^{n-1}(X) \), set

\[
P_{n}f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1}f(x_k) \quad (x \in X)
\]

Then \( P_{n}f \in C^n(X) \) and \( (P_{n}f)' = f \). If \( f \in BC^{n-1}(X) \), then \( Pf \in BC^n(X) \) and

\[
\|P_{n}f\|_n \leq c_n \|f\|_{n-1}
\]

where

\[
c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot \rho^n}.
\]

It follows that the map \( Q_n = n! P_{n} \ldots P_1 \) sends \( C(X) \) into \( C^n(X) \),

\( D_nQ_n \) is the identity on \( C(X) \). A computation yields

\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{-i} S_i,
\]

where \( M \) is the multiplication with \( x \) ((\( Mf \))(x) = xf(x) for \( f \in C(X) \))

and where

\[
S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (f \in C(X), x \in X)
\]
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $g\frac{d}{dx}$, where $g \in C^\infty(X)$.

**THEOREM 7.1.** Let $Y \subseteq K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

**THEOREM 7.2.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_i f(0) = \lambda_i$ for all $i$.

**Open problem:** Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?