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NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to \( K \)-valued functions of one single variable. However, a lot of the results can without any problem be carried over to \( E \)-valued functions of one variable, where \( E \) is a \( K \)-Banach space. A generalization to functions \( K^n \times K^m \) will be less obvious, although it seems clear how to define \( C^k \)-functions in that case. (For example, in order that \( f : K^2 \to K \) is \( C^1 \) one should require (see 3.1) that the difference quotients

\[
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2}
\]

can be extended to continuous functions on \( K^3 \). If we take again difference quotients we get four functions of four variables, required to be continuous in order that \( f \) be in \( C^2 \) (see 6.1). It then follows very easily that

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]

for \( f \in C^2 \).)

Throughout this note, \( K \) will always be a complete non-archimedean valued field, and \( X \) a non-empty subset of \( K \), without isolated points. We study differentiability properties of functions \( f : X \to K \). Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( \mathbb{Z}_n \rightarrow \mathbb{Z}_n \), defined on \( \mathbb{Z}_n \), is an example of an injective function with zero derivative and which is in Lip \( \alpha \) for every \( \alpha > 0 \). The function \( f : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \) defined via \( f(x) = x - p^n \) if \( |x - p^n| < p^{-2n} \) and \( f(x) = x \) elsewhere has derivative 1 everywhere, but for all \( n \in \mathbb{N} \), \( f(p^n) = f(p^n - p^n) = p^n - p^n \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

   Let \( BC(X) \) be the algebra of the bounded continuous functions: \( X \rightarrow K \), normed by the sup norm \( \| \cdot \|_\infty \). We have, analogous to the classical case:

   **Theorem 1.1.** The collection of those \( f \in BC(X) \) that are somewhere differentiable is of first category in \( BC(X) \) (in the sense of Baire).

   In contrast to the theory of functions on the real line we have

   **Theorem 1.2.** Let \( X \) be open in \( K \), and let \( f : X \rightarrow K \) be a bounded uniformly continuous function, and let \( \varepsilon > 0 \). Then there exists a nowhere differentiable \( g : X \rightarrow K \) such that \( g \) has bounded difference quotients, and such that \( \| f - g \|_\infty < \varepsilon \).

2. **Differentiability as such**

   Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \to K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one. (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subset K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \to K \) be differentiable. Then we have:

1. If \( Y \subset X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for \( C^1 \) functions we have to take a definition of a \( C^1 \)-function, stronger then just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \to K \), define

\[
\phi_1 f(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

**DEFINITION 3.1.** \( f : X \to K \) is in \( C^1(X) \) if \( \phi_1 f \) can (uniquely) be extended to a continuous function \( \overline{\phi}_1 f \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of \( f' \) already guarantees the existence of a continuous \( \bar{f}' \).

**THEOREM 3.2.** Let \( f \in C^1(X) \) and let \( a \in X \).

(a) If \( f'(a) \neq 0 \) then \( f \) is locally invertible at \( a \). (In fact, \( (f'(a))^{-1} \) is an isometry locally at \( a \)).

(b) If \( X \) is open in \( K \) and if \( f' \neq 0 \) everywhere on \( X \) then \( f \) is an open mapping.

Let \( BC^1(X) = \{ f \in C^1(X) : \| f \|_1 := \| f \|_\infty + \| \bar{f}' \|_\infty \} \). Then \( BC^1(X) \) is a Banach space with respect to \( \| \cdot \|_1 \). We may put a locally convex topology on \( C^1(X) \) via the defining seminorms \( \| \cdot \|_{1,C} \) where \( C \) runs through the compact subsets of \( X \):

\[
\| f \|_{1,C} = \sup_{x \in C} |f(x)| v \sup_{x \in C} \bar{f}'(x,y) \quad (f \in C^1(X)).
\]

Let \( N^1(X) = \{ f \in C^1(X) : f' = 0 \} \) and \( BN^1(X) = \{ f \in BC^1(X) : f' = 0 \} \). Then \( N^1(X) \) is closed in \( C^1(X) \), \( BN^1(X) \) is closed in \( BC^1(X) \).

**THEOREM 3.3.** The locally linear functions (in \( BC^1(X) \)) form a dense subset of \( C^1(X) \) (of \( BC^1(X) \)).

The locally constant functions (in \( BC^1(X) \)) form a dense subset of \( N^1(X) \) (of \( BN^1(X) \)).

**THEOREM 3.4.** If either \( X \) is compact or \( K \) has discrete valuation then \( BC^1(X) \) has an orthonormal base (in the sense of the norm).

If \( X \) is not compact and \( K \) has dense valuation then, for any \( \alpha > 0 \), \( BC^1(X) \) has no \( \alpha \)-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subseteq R_2 \subseteq \ldots$ For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

$$|x_n - x| < r_n, \quad x_n \in R_n.$$ For a continuous $f : X \to K$ set

$$(Pf)(x) = \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X)$$

**THEOREM 3.5.** The map $P$ defined above is a continuous linear map: $C(X) \to C^1(X)$ and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map i.e., $(Pf)' = f$ for each $f \in C(X)$.

**COROLLARY 3.6.** Every continuous function has a $C^1$-antiderivative. In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x, y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim r_n = 0$.

Let $x_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subseteq R_1 \subseteq \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_{i=1}^{\infty} R_i$ and define $\nu : R \to \{0, 1, 2, \ldots\}$ as follows. For a $a \in R$ let $\nu(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
\[ B_a = \{x \in X : |x-a| < r_\nu(a)\}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleleft b \text{ iff } b \in B_a \quad (a, b \in R) \]

Then we have

**Lemma 4.1.** \((R, \triangleleft)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in R \), the set \( \{x \in R : x \not\triangleleft a\} \) is finite and linearly ordered by \( \triangleleft \).

Define for \( a \in R \), \( a \not= a_0 \):

\[ a_- = \max \{x \in R : x \not= a, x \not\triangleleft a\} \]. Then

**Theorem 4.2.** The set \( \{e_a : a \in R\} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda_a e_a \) for some \( \lambda_a \in K \). Then

\[ \lambda_0 = f(a_0) \text{ and for } a \not= a_0 : \lambda_a = f(a)-f(a_-). \]

The set \( \{e_a : a \in R\} \cup \{p e_a : a \in R\} \) \((P" as in 3.5)\) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X) \), \( f = \sum \lambda_a e_a + \sum \mu_b p e_b \)

\((\lambda_a, \mu_b \in K)\) in the \( \| \|_1 \)-norm. Then \( \lambda_0 = f(a_0) \), \( \mu_0 = f'(a_0) \) and for \( a \not= a_0 \):

\[ \lambda_a = f(a)-f(a_-)-(a-a_-)f'(a_-) \]

\[ \mu_a = f'(a)-f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability". Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \to y} f(x,y) = f'(y) \)

uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( \Phi f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then

$f$ has a unique continuous extension $\bar{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\bar{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of

the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded.

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for

every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5, 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{IN}$, let $V^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}$. For

$f : X \to K$ we define the $n$th difference quotient $\phi_n f : V^{n+1} X \to K$

inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in V^{n+1} X$:

$$\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1} \phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1}).$$

Since $V^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{IN} \cup \{0\}$. We say that $f \in C^n(X)$ if

$\phi_n f$ can be extended to a continuous function $\bar{\phi}_n f : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For $f \in B^0n(X)$ set

$$
||f||_n = \max_{0 \leq i \leq n} ||f^{(i)}||\infty.
$$

Let $BC^0n(X) = B^0n(X) \cap C^n(X)$, $C^\infty(X) = \cap_{n=1}^\infty C^n(X)$,

$$
BC^\infty(X) = \cap_{n=1}^\infty BC^n(X).
$$

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^1A(X) \supset BC^1(X) \supset B^2A(X) \supset BC^2(X) \supset \ldots$

$B^0n(X)$ is a Banach space with respect to $|| \cdot ||_n$ and

$BC^n(X)$ is closed in $B^n(X)$.

For $f \in C^n(X)$ $(n \geq 1)$ and $0 \leq j \leq n$ we define the $j^{\text{th}}$ Hasse derivative of $f$ by

$$
D_jf(x) = \frac{\phi_j^j}{n!}(x,x,\ldots,x) \quad (x \in X).
$$

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_jf \in C^{n-j}(X)$

and if $i+j \leq n$

$$
D_iD_jf = \binom{i+j}{i} D_{i+j}f
$$

$f$ is $n$ times differentiable in the ordinary sense and

for $0 \leq i \leq n$ we have

$$
f^{(i)} = i! D_i^i f.
$$

$f : X \rightarrow K$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\varepsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \varepsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then

$g$ can be chosen to be of degree $< i-1$. 
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) ( \( a \in U \subset X \) ) such that \( f : U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y),
\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \to k \), \( n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \to k \) and a continuous \( R_nf : X^2 \to k \) such that for all \( x,y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x,y).
\]

(It follows that the \( D_i f, R_n f \) are uniquely determined and continuous. Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1,2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ a \} \in \mathbb{P} : a \in \{ 0,1 \} \) , and let \( f : X \to k \) be defined via

\[
f(\{ a \}) = \{ a \} \Delta_p 3n!.
\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_i f = 0 \) for \( i = 1,2,4,5,\ldots \) and \( D_3 f = 1 \). On the other hand, \( f \notin C^3(X) \).

Let \( C > 0 \) and \( \{ x_1, \ldots, x_n \} \) a set of \( n \) distinct points in \( X \). We call \( \{ x_1, \ldots, x_n \} \) a C-polygon if for all \( i,j,k,l \in \{ 1, \ldots, n \}, k \neq l \):
We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, \ x_1 \neq x_2, \ |x_1 - a| < \delta, \ |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, \ x_1 \neq x_2, \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \), for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC_n(X) = \{ f \in C^n(X) : \|f\|_n^\omega < \infty \} \), where, by definition,

\[
\|f\|_n^\omega = \max(\|f\|_n, |D_1 f|_\omega, \ldots, |D_{n-1} f|_\omega, |R_n f|_\omega)
\]

(see 6.7). It is very easy to show that \( BC_n(X) \) is a Banach space with respect to \( \| \|_n^\omega \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C_n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) (\( n > 2 \)) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC_n(X) = BC^n(X) \) and

\[
\|f\|_n^\omega \leq \|f\|_n \leq C^{2(n-2)} \|f\|_n^\omega.
\]
(In general we have for \( f \in BC^n(X) \) : \( \|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i} \).

As in 3.5. we want to find an antiderivation map: \( C^{n-1}(X) \to C^n(X) \). We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^1(X) \) then \( Pf \in C^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^{p-1} \)-function has a \( C^p \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \)

as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \).

For \( f \in C^{n-1}(X) \), set

\[
P_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X)
\]

Then \( P_n f \in C^n(X) \) and \( (P_n f)' = f \). If \( f \in BC^{n-1}(X) \), then \( Pf \in BC^n(X) \) and

\[
\|P_n f\|_n \leq c_n \|f\|_{n-1}
\]

where

\[
c_n = \max_{1 \leq i \leq n} \frac{1}{i!} \cdot \rho^n.
\]

It follows that the map \( Q_n = n! P_1 P_{n-1} \ldots P_1 \) sends \( C(X) \) into \( C^n(X) \), \( D_n Q_n \) is the identity on \( C(X) \). A computation yields

\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M^{-i} S_i,
\]

where \( M \) is the multiplication with \( x \) \((Mf)(x) = xf(x) \) for \( f \in C(X)\) and where

\[
S_i f(x) = \sum_{k=1}^{i} f(x_k) (x_{k+1} - x_k) \quad (f \in C(X), x \in X)
\]
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{IN}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f,g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $g\frac{d}{dx}$, where $g \in C^\infty(X)$.

**THEOREM 7.1.** Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

**THEOREM 7.2.** Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_i f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?