NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 + K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$(x_1, x_2, y) \mapsto \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad (x, y, y_2) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function \( \mathbb{Z}_n^p \rightarrow \mathbb{Z}_n^p \), defined on \( \mathbb{Z}_p \) is an example of an injective function with zero derivative and which is in \( \text{Lip}_a \) for every \( a > 0 \). The function \( f : \mathbb{Z}_p + \mathbb{Q} \) defined via \( f(x) = x - p^{2n} \) if \( |x - p^n| < p^{-2n} \) and \( f(x) = x \) elsewhere has derivative 1 everywhere, but for all \( n \in \mathbb{N} \) \( f(p^n) = f(p^n - p^n) = p^n - p^n \), hence \( f \) is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. **Nowhere differentiable functions**

   Let \( \text{BC}(X) \) be the algebra of the bounded continuous functions: \( X \rightarrow K \), normed by the sup norm \( \| \cdot \|_\infty \). We have, analogous to the classical case:

   **THEOREM 1.1.** The collection of those \( f \in \text{BC}(X) \) that are somewhere differentiable is of first category in \( \text{BC}(X) \) (in the sense of Baire).

   In contrast to the theory of functions on the real line we have

   **THEOREM 1.2.** Let \( X \) be open in \( K \), and let \( f : X \rightarrow K \) be a bounded uniformly continuous function, and let \( \varepsilon > 0 \). Then there exists a nowhere differentiable \( g : X \rightarrow K \) such that \( g \) has bounded difference quotients, and such that \( \| f - g \|_\infty < \varepsilon \).

2. **Differentiability as such**

   Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \to K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one. (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subset K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \to K \) be differentiable. Then we have:

1. If \( Y \subset X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property \( (N) \))

2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

3. Continuously differentiable functions

   If we want the local invertibility theorem to hold for \( C^1 \)-functions we have to take a definition of a \( C^1 \)-function, stronger then just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \to K \), define

\[
\Phi_1 f(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).
\]

**DEFINITION 3.1.** \( f : X \to K \) is in \( C^1(X) \) if \( \Phi_1 f \) can (uniquely) be extended to a continuous function \( \overline{\Phi_1 f} \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\overline{f}'$. 

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : ||f||_1 := ||f||_\infty \vee ||f'||_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $|| \cdot ||_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $|| \cdot ||_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$||f||_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |\overline{f}'(x,y)|$$

($f \in C^1(X)$).

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $a \geq 0$, $BC^1(X)$ has no $a$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim_{n \to \infty} r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in R_n$ be determined by:

$$|x_n - x| < r_n, \quad x_n \in R_n.$$ 

For a continuous $f: X \to \mathbb{K}$ set

$$Pf(x) = \sum_{n=1}^{\infty} f(x_n) (x_{n+1} - x_n) \quad (x \in X)$$

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:

$$C(X) \to C^1(X)$$

and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map, i.e., $(Pf)' = f$ for each $f \in C(X)$.

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

**4. $C^1(X)$ for compact $X$**

(Throughout section 4, $X$ is compact). The set $\{|x-y| : x,y \in X\}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots\} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim_{n \to \infty} r_n = 0$. Let $x_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_i R_i$ and define $v : R \to \{0,1,2,\ldots\}$ as follows. For a $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
\[ B_a = \{ x \in X : |x-a| < r(v(a)) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleq b \text{ iff } b \in B_a \quad (a, b \in \mathbb{R}) \]

Then we have

**Lemma 4.1.** \((\mathbb{R}, q)\) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in \mathbb{R} \), the set \( \{ x \in \mathbb{R} : x \not< a \} \) is finite and linearly ordered by \( \triangleq \).

Define for \( a \in \mathbb{R} \), \( a \not= a_0 \): \( a_- = \max \{ x \in \mathbb{R} : x \not< a, x \not< a_0 \} \). Then

**Theorem 4.2.** The set \( \{ e_a : a \in \mathbb{R} \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C^1(X) \) and \( f = \sum \lambda a e_a \) for some \( \lambda a \in K \). Then

\[ \lambda a_0 = f(a_0) \text{ and for } a \not= a_0 : \lambda a = f(a) - f(a_-). \]

The set \( \{ e_a : a \in \mathbb{R} \} \cup \{ p e_a : a \in \mathbb{R} \} (p \text{ as in 3.5}) \) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X), f = \sum \lambda a e_a + \sum \mu b p e_b \) (\( \lambda a, \mu b \in K \)) in the \( \| \|_1 \)-norm. Then \( \lambda a_0 = f(a_0), \mu a_0 = f'(a_0) \) and for \( a \not= a_0 \):

\[ \lambda a = f(a) - f(a_-) - (a-a_-) f'(a_-) \quad \mu a = f'(a) - f'(a_-). \]

**5. Uniform differentiability**

There seem to be two natural notions of "uniform differentiability".

Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \to y} f(x,y) = f'(y) \) uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( \Phi f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuously differentiable".
THEOREM 5.1. Let \( f : X \to K \) be (strongly) uniformly differentiable. Then \( f \) has a unique continuous extension \( \overline{f} : \overline{X} \to K \) (\( \overline{X} \) is the closure of \( X \) in \( K \)). This \( \overline{f} \) is (strongly) uniformly differentiable.

THEOREM 5.2. Let \( f : X \to K \) be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of \( f \):

(a) \( \Phi^1 f \) is bounded.

(b) Both \( f \) and \( f' \) are bounded.

(c) \( X \) is "nice" and \( f \) is bounded.

(\( X \) is called "nice" if for each \( r > 0 \) there is \( s > 0 \) such that for every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \)).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. \( C^n \)-functions

For \( n \in \mathbb{N} \), let \( \mathcal{V}^n X = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\} \). For \( f : X \to K \) we define the \( n \)-th difference quotient \( \phi_n f : \mathcal{V}^{n+1} X \to K \) inductively as follows \( \phi_0 f = f \) and for \( (x_1, \ldots, x_{n+1}) \in \mathcal{V}^{n+1} X \):

\[
\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).
\]

Since \( \mathcal{V}^n X \) is dense in \( X^n \) for each \( n \) the following definition makes sense.

DEFINITION 6.1. Let \( f : X \to K \), \( n \in \mathbb{N} \cup \{0\} \). We say that \( f \in C^n(X) \) if \( \phi_n f \) can be extended to a continuous function \( \overline{\phi}_n f : X^{n+1} \to K \). We say that \( f \in B^n(X) \) if \( \phi_0 f, \ldots, \phi_n f \) are bounded functions.
For $f \in B\Delta^n(X)$ set
\[ ||f||_n = \max_{0 \leq i \leq n} ||\phi_i f||_\infty. \]
Let $BC^n(X) = B\Delta^n(X) \cap C^n(X)$, $C^\omega(X) = \bigcap_{n=1}^\infty C^n(X)$, $BC^\omega(X) = \bigcap_{n=1}^\infty BC^n(X)$.

**Theorem 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$ $B\Delta^1(X) \supset BC^1(X) \supset B\Delta^2(X) \supset BC^2(X) \supset \ldots$ $B\Delta^n(X)$ is a Banach space with respect to $|| \cdot ||_n$ and $BC^n(X)$ is closed in $B\Delta^n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative of $f$ by
\[ D^j f(x) = \delta_j f(x, x, \ldots, x) \quad (x \in X). \]

**Theorem 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D^j f \in C^{n-j}(X)$ and if $i+j \leq n$
\[ D_i D^j f = \binom{i+j}{i} D^i+j f \]
$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have
\[ f^{(i)} = i! D^i f. \]

$f : X \to K$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\omega(X)$.

**Theorem 6.4.** Let $f \in C^n(X)$ and $\epsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \epsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let \( f \in C^n(X) \) and \( f'(a) \neq 0 \) for some \( a \in X \). Then there is a neighbourhood \( U \) of \( a \) (\( a \in U \subset X \)) such that \( f : U \to f(U) \) is a bijection, and such that the local inverse: \( f(U) \to U \) is in \( C^n(f(U)) \).

THEOREM 6.6. (Taylor formula). Let \( f \in C^n(X) \). Then for all \( x, y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n f(x, y, \ldots, y).
\]

The above result leads to another possible notion of "n-times continuously differentiable":

DEFINITION 6.7. Let \( f : X \to K, n \in \mathbb{N} \). We say that \( f \in C^n(X) \) if there exist functions \( D_1f, \ldots, D_{n-1}f : X \to K \) and a continuous \( R_n f : X^2 \to K \) such that for all \( x, y \in X \):

\[
f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n f(x, y).
\]

(It follows that the \( D_i f, R_n f \) are uniquely determined and continuous.

Further we have \( C^1(X) \supset C^2(X) \supset \ldots \). It is easy to show that \( C^i(X) = C^i(X) \) for \( i = 1, 2 \). Also \( C^n(X) \subset C^n(X) \) for all \( n \), by 6.6. But we have

EXAMPLE 6.8. Let \( X = \{ \Sigma a_n p^n : a_n \in \{0,1\} \} \), and let \( f : X \to K \) be defined via

\[
f(\Sigma a_n p^n) = \Sigma a_n p^{3n}.\]

Then \( f \in C^n(X) \) for each \( n \), and \( D_i f = 0 \) for \( i = 1, 2, 4, 5, \ldots \) and \( D_3 f = 1 \). On the other hand, \( f \notin C^3(X) \).

Let \( C > 0 \) and \( \{x_1, \ldots, x_n\} \) a set of \( n \) distinct points in \( X \). We call \( \{x_1, \ldots, x_n\} \) a C-polygon if for all \( i, j, k, l \in \{1, \ldots, n\}, k \neq l \):
\[ \frac{x_i - x_j}{x_k - x_1} \leq C. \]

**Definition 6.9.** Let \( n \in \mathbb{N} \). We say that \( X \) has **locally property** \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \).)

We say that \( X \) has **globally property** \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \).)

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : \|f\|_n^\omega < \omega \} \), where, by definition,

\[ \|f\|_n^\omega = \max(\|f\|_\omega, \|D_1 f\|_\omega, \ldots, \|D_{n-1} f\|_\omega, \|R_n f\|_\omega) \]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( \| \| \)\( ^\omega \)\( _n \). The main theorem:

**Theorem 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = \mathcal{C}^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n \geq 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[ \| \|_n^\omega \leq \| \|_n \leq C^2(n-2) \| \|_n^\omega. \]
(In general we have for \( f \in BC^n(X) \) : \(|f|_n = \max_{0 \leq i \leq n} |D_i f|_{n-i}^\nu \).

As in 3.5, we want to find an antiderivation map: \( C^{n-1}(X) \to C^n(X) \). We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^1(X) \) then \( Pf \in C^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^{n-1} \)-function has a \( C^n \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \) as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \).

For \( f \in C^{n-1}(X) \), set
\[
P_n f(x) = \sum_{k=1}^{n} \sum_{i=1}^{n} \frac{1}{i} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X)
\]

Then \( P_n f \in C^n(X) \) and \( (P_n f)' = f \). If \( f \in BC^{n-1}(X) \), then
\[
\| P_n f \|_n \leq c_n \| f \|_{n-1}
\]

where
\[
c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot \rho^n}.
\]

It follows that the map \( Q_n = n! P_n P_{n-1} \ldots P_1 \) sends \( C(X) \) into \( C^n(X) \), \( D_n Q_n \) is the identity on \( C(X) \). A computation yields
\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} n^{n-i} M^{-1} S_i,
\]
where \( M \) is the multiplication with \( x \) \((M f)(x) = xf(x) \) for \( f \in C(X) \)
and where
\[
S_i f(x) = \sum_{k=1}^{n} f(x_k) (x_{k+1} - x_k) \quad (f \in C(X), \ x \in X)
\]
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}'$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subseteq K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_1 f(0) = \lambda_1$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?