NON-ARCHIMEDEAN DIFFERENTIATION

by

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Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to K-valued functions of one single variable. However, a lot of the results can without any problem be carried over to E-valued functions of one variable, where E is a K-Banach space. A generalization to functions: $K^n \times K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 \rightarrow K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \rightarrow K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{I}_{a} p^n \rightarrow \mathbb{I}_{a} p^n !$, defined on $\mathbb{Z}$ is an example of an injective function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{Z} + \mathbb{Q}$ defined via $f(x) = x - p^n$ if $|x-p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^n) = p^n - p^n$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $\mathcal{BC}(X)$ be the algebra of the bounded continuous functions: $X \rightarrow \mathbb{K}$, normed by the sup norm $\| \|_{\infty}$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in \mathcal{BC}(X)$ that are somewhere differentiable is of first category in $\mathcal{BC}(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $\mathbb{K}$, and let $f : X \rightarrow \mathbb{K}$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \rightarrow \mathbb{K}$ such that $g$ has bounded difference quotients, and such that $\|f-g\|_{\infty} < \varepsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
function to possess an antiderivative:

**THEOREM 2.1.** Let \( f : X \to K \). Then \( f \) has an antiderivative if and only if \( f \) is of Baire class one. (i.e., \( f \) is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If \( K \) is a local field then \( Y \subseteq K \) is called a nullset if it has measure zero in the sense of the (real) Haar measure on \( K \).

**THEOREM 2.2.** Let \( K \) be a local field and let \( f : X \to K \) be differentiable. Then we have:

1. If \( Y \subseteq X \) is a nullset then \( f(Y) \) is a nullset ("\( f \) has property (N)"")
2. \( \{ f(x) : f'(x) = 0 \} \) is a nullset.

**COROLLARY 2.3.** If \( f : X \to K \) is differentiable, \( f' = 0 \) almost everywhere, then \( f(X) \) is a nullset.

### 3. Continuously differentiable functions

If we want the local invertibility theorem to hold for \( C^1 \) functions we have to take a definition of a \( C^1 \)-function, stronger than just "\( f \) is differentiable and \( f \) is continuous". For \( f : X \to K \), define

\[
\phi^1 f(x,y) = \frac{f(x) - f(y)}{x - y} \quad (x,y \in X, x \neq y).
\]

**DEFINITION 3.1.** \( f : X \to K \) is in \( C^1(X) \) if \( \phi^1 f \) can (uniquely) be extended to a continuous function \( \bar{\phi}^1 f \) on \( X \times X \).

(Notice that for a real valued function \( f \) defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $\overline{\psi}_1f$.

**THEOREM 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : ||f||_1 := ||f||_\infty \vee ||\overline{\psi}_1f||_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $||\ |\ |_1$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $||\ |\ |_1,C$ where $C$ runs through the compact subsets of $X$:

$$||f||_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C, y \in C} |\overline{\psi}_1f(x,y)|$$

$f \in C^1(X))$.

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**THEOREM 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**THEOREM 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm). If $X$ is not compact and $K$ has dense valuation then, for any $a > 0$, $BC^1(X)$ has no $a$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subseteq R_2 \subseteq \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

\[ |x_n - x| < r_n, \quad x_n \in R_n. \]

For a continuous $f : X \to K$ set

\[ (Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_{n+1} - x_n) \quad (x \in X). \]

**THEOREM 3.5.** The map $P$ defined above is a continuous linear map:

- $C(X) \to C^1(X)$ and its restriction to $BC(X)$ is an isometry: $BC(X) \to BC^1(X)$. $P$ is an antiderivation map, i.e., $(Pf)' = f$ for each $f \in C(X)$.

**COROLLARY 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{ |x-y| : x, y \in X \}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{ r_1, r_2, \ldots \} \cup \{ 0 \}$, where $r_1 > r_2 > \ldots$ and $\lim r_n = 0$. Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subseteq R_1 \subseteq \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_i R_i$ and define $v : R \to \{ 0, 1, 2, \ldots \}$ as follows. For a $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
\[ B_a = \{ x \in X : |x-a| < r_x(a) \}, \]

and let \( e_a \) be the \( K \)-valued characteristic function of \( B_a \). Further, we define

\[ a \triangleleft b \text{ iff } b \in B_a \quad (a,b \in R) \]

Then we have

**Lemma 4.1.** \( (R,\triangleleft) \) is a partially ordered set with a smallest element \( a_0 \). For each \( a \in R \), the set \( \{ x \in R : x \triangleleft a \} \) is finite and linearly ordered by \( \triangleleft \).

Define for \( a \in R \), \( a \neq a_0 \):

\[ a_- = \max \{ x \in R : x \neq a, x \triangleleft a \}. \]

**Theorem 4.2.** The set \( \{ e_a : a \in R \} \) forms an orthonormal base of \( C(X) \).

Let \( f \in C(X) \) and \( f = \sum \lambda e_a \) for some \( \lambda \in K \). Then

\[ \lambda_a = f(a_0) \quad \text{and for } a \neq a_0: \quad \lambda_a = f(a)-f(a_-). \]

The set \( \{ e_a : a \in R \} \cup \{ p_e : e \in R \} \) (\( P \) as in 3.5) forms an orthogonal base of \( C^1(X) \). Let \( f \in C^1(X) \), \( f = \sum \lambda e_a \sum \mu_e p_e \) \((\lambda, \mu \in K)\) in the \( \| f \|_1 \)-norm. Then \( \lambda_{a_0} = f(a_0), \mu_{a_0} = f'(a_0) \) and for \( a \neq a_0:

\[ \lambda_a = f(a)-f(a_-)-(a-a_-)f'(a_-), \]

\[ \mu_a = f'(a)-f'(a_-). \]

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let \( f \in C^1(X) \). \( f \) is called uniformly differentiable if \( \lim_{x \rightarrow y} f(x,y) = f'(y) \) uniformly in \( y \). \( f \) is called strongly uniformly differentiable if \( f \) is uniformly continuous.

If \( X \) is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let $f : X \to K$ be (strongly) uniformly differentiable. Then $f$ has a unique continuous extension $\overline{f} : \overline{X} \to K$ ($\overline{X}$ is the closure of $X$ in $K$).

This $\overline{f}$ is (strongly) uniformly differentiable.

THEOREM 5.2. Let $f : X \to K$ be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of $f$:

(a) $\phi_1 f$ is bounded.

(b) Both $f$ and $f'$ are bounded.

(c) $X$ is "nice" and $f$ is bounded.

($X$ is called "nice" if for each $r > 0$ there is $s > 0$ such that for every $x \in X$ there is $y \in X$ such that $s \leq |x-y| \leq r$).

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. $C^n$-functions

For $n \in \mathbb{N}$, let $V^n X = \{ (x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j \}$. For $f : X \to K$ we define the $n$th difference quotient $\phi_n f : V^{n+1} X \to K$ inductively as follows $\phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in V^{n+1} X$:

$$\phi_n f(x_1, \ldots, x_{n+1}) = (x_1 - x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1}))$$

Since $V^n X$ is dense in $X^n$ for each $n$ the following definition makes sense.

DEFINITION 6.1. Let $f : X \to K$, $n \in \mathbb{N} \cup \{0\}$. We say that $f \in C^n(X)$ if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f} : X^{n+1} \to K$.

We say that $f \in B^n(X)$ if $\phi_0 f, \ldots, \phi_n f$ are bounded functions.
For $f \in B^\infty(X)$ set

$$||f||_n = \max_{0 \leq i \leq n} ||f_i||_\infty.$$  

Let $BC^1(X) = B^\infty(X) \cap C^1(X)$, $C^\infty(X) = \bigcap_{n=1}^\infty C^n(X)$, $BC^\infty(X) = \bigcap_{n=1}^\infty BC^n(X)$.

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$  

$B^1(X) \supset BC^1(X) \supset B^2(X) \supset BC^2(X) \supset \ldots$  

$B^n(X)$ is a Banach space with respect to $||\cdot||_n$ and $BC^n(X)$ is closed in $B^n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j \leq n$ we define the $j^{th}$ Hasse derivative of $f$ by

$$D_j f(x) = \bar{\phi}_j f(x, x, \ldots, x) \quad (x \in X).$$

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j \leq n$ we have $D_j f \in C^{n-j}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f$$

$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$

$f : X \to K$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighbourhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\epsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $||f-g||_n < \epsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ (a $\in U \subset X$) such that $f : U \to f(U)$ is a bijection, and such that the local inverse: $f(U) \to U$ is in $C^n(f(U))$.

**THEOREM 6.6. (Taylor formula).** Let $f \in C^n(X)$. Then for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x, y, \ldots, y).$$

The above result leads to another possible notion of "n-times continuously differentiable":

**DEFINITION 6.7.** Let $f : X \to \mathbb{R}$, $n \in \mathbb{IN}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to \mathbb{R}$ and a continuous $R_nf : X^2 \to \mathbb{R}$ such that for all $x, y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_nf(x, y).$$

(It follows that the $D_i f$, $R_n f$ are uniquely determined and continuous.

Further we have $C^1(X) \supset C^2(X) \supset \ldots$). It is easy to show that $C^i(X) = C^i(X)$ for $i = 1, 2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

**EXAMPLE 6.8.** Let $X = \{ \Sigma a_n p^n : a_n \in \{0, 1\} \}$, and let $f : X \to \mathbb{K}$ be defined via

$$f(\Sigma a_n p^n) = \Sigma a_n p^{3n!}.$$ Then $f \in C^n(X)$ for each $n$, and $D_i f = 0$ for $i = 1, 2, 4, 5, \ldots$ and $D_3 f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1, \ldots, x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1, \ldots, x_n\}$ a $C$-polygon if for all $i, j, k, l \in \{1, \ldots, n\}$, $k \neq l$: 
DEFINITION 6.9. Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), \( |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon.

(Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( BC^n(X) = \{ f \in C^n(X) : ||f||^\omega_n < \infty \} \), where, by definition,

\[
||f||^\omega_n = \max(||f||^\omega, ||D_1 f||^\omega, \ldots, ||D_{n-1} f||^\omega, ||R_n f||^\omega)
\]

(see 6.7). It is very easy to show that \( BC^n(X) \) is a Banach space with respect to \( || \cdot ||^\omega_n \). The main theorem:

THEOREM 6.10. If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) (\( n > 2 \)) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( BC^n(X) = BC^n(X) \) and

\[
||f||^\omega_n \leq ||f||_n \leq C^2(n-2)||f||^\omega_n.
\]
(In general we have for $f \in \text{BC}^n(X) : \|f\|_n = \max_{0<i\leq n} \|D_i f\|_{n-i}^\nu$.)

As in 3.5, we want to find an antiderivation map: $\text{C}^{n-1}(X) \to \text{C}^n(X)$. We cannot use the map $P$ of 3.5. since one can prove: if $f \in \text{C}^1(X)$ then $Pf \in \text{C}^2(X)$ if and only if $f' = 0$. Further, if the characteristic of $K$ equals $p \neq 0$ it is easy to see that not every $\text{C}^{p-1}$-function has a $\text{C}^p$-antiderivative.

**THEOREM 6.11.** Let the characteristic of $K$ be zero and let $r_1 > r_2 \ldots$ as in 3.5 but such that $r_m < \rho r_{m+1}$ for all $m$, some $\rho > 0$.

For $f \in \text{C}^{n-1}(X)$, set

$$p_n f(x) = \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{i!} (x_{k+1} - x_k)^i D_{i-1} f(x_k) \quad (x \in X)$$

Then $p_n f \in \text{C}^n(X)$ and $(p_n f)' = f$. If $f \in \text{BC}^{n-1}(X)$, then $Pf \in \text{BC}^n(X)$ and

$$\|p_n f\|_n \leq c_n \|f\|_{n-1}$$

where

$$c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot \rho^n}.$$

It follows that the map $Q_n = n! P_{n-1} \ldots P_1$ sends $\text{C}(X)$ into $\text{C}^n(X)$, $D_n Q_n$ is the identity on $\text{C}(X)$. A computation yields

$$Q_n = \sum_{i=1}^{n} (-1)^{i+1} (\frac{n}{i}) M^{n-i} S_i,$$

where $M$ is the multiplication with $x$ $(Mf)(x) = xf(x)$ for $f \in \text{C}(X)$ and where

$$S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_{k+1}^i - x_k^i) \quad (f \in \text{C}(X), x \in X)$$
Note: Theorem 3.4 is also true if we replace $BC^1(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f, g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subset K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(X)$ such that $D_i f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?