NON-ARCHIMEDEAN DIFFERENTIATION

by

W.H. Schikhof

Introduction

This note is a collection of the most important results of a lengthy paper that will appear as a Report, Mathematisch Instituut, Katholieke Universiteit, Nijmegen. There the interested reader may find all the proofs that are missing here, and also a bibliography. Because of the rather complicated formulas and computations that are involved it seems wise to restrict oneself first to $K$-valued functions of one single variable. However, a lot of the results can without any problem be carried over to $E$-valued functions of one variable, where $E$ is a $K$-Banach space. A generalization to functions: $K^n + K^m$ will be less obvious, although it seems clear how to define $C^k$-functions in that case. (For example, in order that $f : K^2 + K$ is $C^1$ one should require (see 3.1) that the difference quotients

$$
\frac{f(x_1, y_1) - f(x_2, y_2)}{x_1 - x_2}, \quad (x, y_1, y_2) \mapsto \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}
$$

can be extended to continuous functions on $K^3$. If we take again difference quotients we get four functions of four variables, required to be continuous in order that $f$ be in $C^2$ (see 6.1). It then follows very easily that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for $f \in C^2$.)

Throughout this note, $K$ will always be a complete non-archimedean valued field, and $X$ a non-empty subset of $K$, without isolated points. We study differentiability properties of functions $f : X \to K$. Besides the analytic functions we have other examples of differentiable
functions such as the locally constant functions (they have derivative zero everywhere). The function $\mathbb{Z}_p \ni p \mapsto p^n$, defined on $\mathbb{Z}_p$ is an example of an injective function with zero derivative and which is in $\text{Lip}_a$ for every $a > 0$. The function $f : \mathbb{Z}_p \ni p \mapsto p^n$ defined via $f(x) = x - p^{2n}$ if $|x - p^n| < p^{-2n}$ and $f(x) = x$ elsewhere has derivative 1 everywhere, but for all $n \in \mathbb{N}$ $f(p^n) = f(p^n - p^{2n}) = p^n - p^{2n}$, hence $f$ is not locally injective at 0.

The above examples show that we do not have a mean value theorem or a local invertibility theorem. We now state the results going from "bad" to "smooth" functions.

1. Nowhere differentiable functions

Let $BC(X)$ be the algebra of the bounded continuous functions $X \to \mathbb{K}$, normed by the sup norm $|| \cdot ||_\infty$. We have, analogous to the classical case:

**THEOREM 1.1.** The collection of those $f \in BC(X)$ that are somewhere differentiable is of first category in $BC(X)$ (in the sense of Baire).

In contrast to the theory of functions on the real line we have

**THEOREM 1.2.** Let $X$ be open in $\mathbb{K}$, and let $f : X \to \mathbb{K}$ be a bounded uniformly continuous function, and let $\varepsilon > 0$. Then there exists a nowhere differentiable $g : X \to \mathbb{K}$ such that $g$ has bounded difference quotients, and such that $||f - g||_\infty < \varepsilon$.

2. Differentiability as such

Contrary to the classical case we have a nice criterion for a
THEOREM 2.1. Let $f : X \to K$. Then $f$ has an antiderivative if and only if $f$ is of Baire class one. (i.e., $f$ is the pointwise limit of a sequence of continuous functions).

Further we have Sard-type theorems. If $K$ is a local field then $Y \subseteq K$ is called a nullset if it has measure zero in the sense of the (real) Haar measure on $K$.

THEOREM 2.2. Let $K$ be a local field and let $f : X \to K$ be differentiable. Then we have:

(1) If $Y \subseteq X$ is a nullset then $f(Y)$ is a nullset ("$f$ has property (N)"")

(2) $\{f(x) : f'(x) = 0\}$ is a nullset.

COROLLARY 2.3. If $f : X \to K$ is differentiable, $f' = 0$ almost everywhere, then $f(X)$ is a nullset.

3. Continuously differentiable functions

If we want the local invertibility theorem to hold for $C^1$-functions we have to take a definition of a $C^1$-function, stronger then just "$f$ is differentiable and $f$ is continuous". For $f : X \to K$, define

$$\phi_1(x,y) = \frac{f(x) - f(y)}{x-y} \quad (x,y \in X, x \neq y).$$

DEFINITION 3.1. $f : X \to K$ is in $C^1(X)$ if $\phi_1 f$ can (uniquely) be extended to a continuous function $\overline{\phi_1} f$ on $X \times X$.

(Notice that for a real valued function $f$ defined on an interval
the continuity of $f'$ already guarantees the existence of a continuous $f'$. 

**Theorem 3.2.** Let $f \in C^1(X)$ and let $a \in X$.

(a) If $f'(a) \neq 0$ then $f$ is locally invertible at $a$. (In fact, $(f'(a))^{-1}f$ is an isometry locally at $a$).

(b) If $X$ is open in $K$ and if $f' \neq 0$ everywhere on $X$ then $f$ is an open mapping.

Let $BC^1(X) = \{f \in C^1(X) : \|f\|_1 := \|f\|_\infty \vee \|f_1\|_\infty\}$. Then $BC^1(X)$ is a Banach space with respect to $\|\|$. We may put a locally convex topology on $C^1(X)$ via the defining seminorms $\|\|_{1,C}$ where $C$ runs through the compact subsets of $X$:

$$\|f\|_{1,C} = \sup_{x \in C} |f(x)| \vee \sup_{x \in C} |f_1(x,y)|$$

Let $N^1(X) = \{f \in C^1(X) : f' = 0\}$ and $BN^1(X) = \{f \in BC^1(X) : f' = 0\}$. Then $N^1(X)$ is closed in $C^1(X)$, $BN^1(X)$ is closed in $BC^1(X)$.

**Theorem 3.3.** The locally linear functions (in $BC^1(X)$) form a dense subset of $C^1(X)$ (of $BC^1(X)$).

The locally constant functions (in $BC^1(X)$) form a dense subset of $N^1(X)$ (of $BN^1(X)$).

**Theorem 3.4.** If either $X$ is compact or $K$ has discrete valuation then $BC^1(X)$ has an orthonormal base (in the sense of the norm).

If $X$ is not compact and $K$ has dense valuation then, for any $a > 0$, $BC^1(X)$ has no $a$-orthogonal base.
Let us choose real numbers $1 > r_1 > r_2 > \ldots$ with $\lim r_n = 0$, and, for each $n$, let $R_n$ be a full set of representatives of the equivalence relation (in $X$): $x \sim y$ if $|x-y| < r_n$. We can arrange that $R_1 \subset R_2 \subset \ldots$. For each $x \in X$, $n \in \mathbb{N}$, let $x_n \in X$ be determined by:

$|x_n - x| < r_n$, $x_n \in R_n$. For a continuous $f : X \to \mathbb{K}$ set

$$
(Pf)(x) = \sum_{n=1}^{\infty} f(x_n)(x_n - x_{n+1})(x \in X).
$$

**Theorem 3.5.** The map $P$ defined above is a continuous linear map:

$$
C(X) \to C^1(X) \text{ and its restriction to } BC(X) \text{ is an isometry: } BC(X) \to BC^1(X). P \text{ is an antiderivation map i.e., } (Pf)' = f \text{ for each } f \in C(X).
$$

**Corollary 3.6.** Every continuous function has a $C^1$-antiderivative.

In fact, by passing through the quotient, differentiation yields a map $p : BC^1(X)/BN^1(X) \to BC(X)$ which is a surjective isometry. Moreover, $BN^1(X)$ has an orthogonal complement $(\text{im } P)$ in $BC^1(X)$.

### 4. $C^1(X)$ for compact $X$

(Throughout section 4, $X$ is compact). The set $\{ |x-y| : x,y \in X \}$ is bounded and has only 0 as an accumulation point, hence it can be written as $\{r_1, r_2, \ldots \} \cup \{0\}$, where $r_1 > r_2 > \ldots$ and $\lim r_n = 0$.

Let $r_0 = \infty$. For each $i$, let $R_i$ be a full set of representatives in $X$ of the equivalence relation "$x \sim y$ if $|x-y| < r_i$" such that $R_0 \subset R_1 \subset \ldots$. Then $R_i$ is finite for each $i$ and $R_0$ consists only of one single point $a_0$. Let $R = \bigcup_{i} R_i$ and define $v : R \to \{0,1,2,\ldots\}$ as follows. For $a \in R$ let $v(a)$ be the nonnegative integer $m$ for which $a \in R_m \setminus R_{m-1}$ ($R_{-1} = \emptyset$ by definition). For each $a \in R$ let
$B_a = \{ x \in X : |x-a| < r_y(a) \},$

and let $e_a$ be the $K$-valued characteristic function of $B_a$. Further, we define

$a \not\in b \iff b \in B_a \quad (a, b \in \mathbb{R})$

Then we have

**Lemma 4.1.** $(\mathbb{R}, \leq)$ is a partially ordered set with a smallest element $a_0$. For each $a \in \mathbb{R}$, the set $\{x \in \mathbb{R} : x \leq a\}$ is finite and linearly ordered by $\leq$.

Define for $a \in \mathbb{R}$, $a \neq a_0$: $a_- = \max \{x \in \mathbb{R} : x \neq a, x \leq a\}$. Then

**Theorem 4.2.** The set $\{e_a : a \in \mathbb{R}\}$ forms an orthonormal base of $C(X)$.

Let $f \in C(X)$ and $f = \sum \lambda_a e_a$ for some $\lambda_a \in K$. Then

$\lambda_{a_0} = f(a_0)$ and for $a \neq a_0$: $\lambda_a = f(a) - f(a_-)$.

The set $\{e_a : a \in \mathbb{R}\} \cup \{p_{e_a} : a \in \mathbb{R}\}$ (as in 3.5) forms an orthogonal base of $C^1(X)$. Let $f \in C^1(X)$, $f = \sum \lambda_a e_a + \sum \mu_b p_{e_b}$ ($\lambda_a, \mu_b \in K$) in the $\| \cdot \|_1$-norm. Then $\lambda_{a_0} = f(a_0)$, $\mu_{a_0} = f'(a_0)$ and for $a \neq a_0$:

$\lambda_a = f(a) - f(a_-) - (a-a_-)f'(a_-)$

$\mu_a = f'(a) - f'(a_-)$.

5. Uniform differentiability

There seem to be two natural notions of "uniform differentiability".

Let $f \in C^1(X)$. $f$ is called **uniformly differentiable** if $\lim_{x \to y} f(x, y) = f'(y)$ uniformly in $y$. $f$ is called **strongly uniformly differentiable** if $f$ is uniformly continuous.

If $X$ is compact both notions are the same and coincide with "continuous differentiable".
THEOREM 5.1. Let \( f : X \to K \) be (strongly) uniformly differentiable. Then
\( f \) has a unique continuous extension \( \tilde{f} : \overline{X} \to K \) (\( \overline{X} \) is the closure of \( X \) in \( K \)).
This \( \tilde{f} \) is (strongly) uniformly differentiable.

THEOREM 5.2. Let \( f : X \to K \) be uniformly differentiable. Then each of the following properties implies strong uniform differentiability of \( f \):
(a) \( \phi_1 f \) is bounded.
(b) Both \( f \) and \( f' \) are bounded
(c) \( X \) is "nice" and \( f \) is bounded.

(\( X \) is called "nice" if for each \( r > 0 \) there is \( s > 0 \) such that for every \( x \in X \) there is \( y \in X \) such that \( s \leq |x-y| \leq r \).)

The theorems 3.3, 3.5., 3.6. each have an analogon for uniformly differentiable functions.

6. \( C^n \)-functions

For \( n \in \mathbb{N} \), let \( V^n(X) = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j \} \). For \( f : X \to K \) we define the \( n \)th difference quotient \( \phi_n f : V^{n+1}X \to K \) inductively as follows \( \phi_0 f = f \) and for \((x_1, \ldots, x_{n+1}) \in V^{n+1}X\):
\[
\phi_n f(x_1, \ldots, x_{n+1}) = (x_1-x_2)^{-1}(\phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})).
\]
Since \( V^n X \) is dense in \( X^n \) for each \( n \) the following definition makes sense.

DEFINITION 6.1. Let \( f : X \to K \), \( n \in \mathbb{N} \cup \{0\} \). We say that \( f \in C^n(X) \) if \( \phi_n f \) can be extended to a continuous function \( \overline{\phi}_n f : X^{n+1} \to K \).
We say that \( f \in B^n(X) \) if \( \phi_0 f, \ldots, \phi_n f \) are bounded functions.
For $f \in B^k_n(X)$ set

$$\|f\|_n = \max_{0 \leq i \leq n} \|\phi_i f\|_\infty.$$ 

Let $BC^k_n(X) = B^k_n(X) \cap C^n(X)$, $C^\infty(X) = \bigcap_{n=1}^{\infty} C^n(X)$, $BC^\infty(X) = \bigcap_{n=1}^{\infty} BC^n(X)$.

**THEOREM 6.2.** $C^1(X) \supset C^2(X) \supset \ldots$

$B^1_n(X) \supset BC^1_n(X) \supset B^2_n(X) \supset BC^2(X) \supset \ldots$

$B^k_n(X)$ is a Banach space with respect to $\|\|_n$ and $BC_k^n(X)$ is closed in $B^k_n(X)$.

For $f \in C^n(X)$ ($n \geq 1$) and $0 \leq j < n$ we define the $j^{th}$ Hasse derivative of $f$ by

$$D_j f(x) = \frac{\partial \phi_j f}{\partial x^j}(x \in X).$$

**THEOREM 6.3.** Let $f \in C^n(X)$. Then for $0 \leq j < n$ we have $D_j f \in C^{n-j}(X)$ and if $i+j \leq n$

$$D_i D_j f = \binom{i+j}{i} D_{i+j} f.$$

$f$ is $n$ times differentiable in the ordinary sense and for $0 \leq i \leq n$ we have

$$f^{(i)} = i! D_i f.$$

$f : X \rightarrow \mathbb{K}$ is called a spline function of degree $\leq n$ if for every $a \in X$ there is a neighborhood $U$ of $a$ such that $f|U \cap X$ is a polynomial function of degree $\leq n$. Spline functions are in $C^\infty(X)$.

**THEOREM 6.4.** Let $f \in C^n(X)$ and $\epsilon > 0$. Then there is a spline function $g$ of degree $\leq n$ such that $f-g \in BC^n(X)$, $\|f-g\|_n < \epsilon$. If $D_i f = D_{i+1} f = \ldots = D_n f = 0$ for some $i \in \{1, \ldots, n\}$ then $g$ can be chosen to be of degree $\leq i-1$. 
THEOREM 6.5. (Local invertibility). Let $f \in C^n(X)$ and $f'(a) \neq 0$ for some $a \in X$. Then there is a neighbourhood $U$ of $a$ ($a \in U \subseteq X$) such that $f : U \to f(U)$ is a bijection, and such that the local inverse $f(U) \to U$ is in $C^n(f(U))$.

THEOREM 6.6. (Taylor formula). Let $f \in C^n(X)$. Then for all $x,y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n^f(x,y).$$

The above result leads to another possible notion of "n-times continuously differentiable".

DEFINITION 6.7. Let $f : X \to K$, $n \in \mathbb{N}$. We say that $f \in C^n(X)$ if there exist functions $D_1f, \ldots, D_{n-1}f : X \to K$ and a continuous $R_n^f : X^2 \to K$ such that for all $x,y \in X$:

$$f(x) = f(y) + (x-y)D_1f(y) + \ldots + (x-y)^{n-1}D_{n-1}f(y) + (x-y)^nR_n^f(x,y).$$

(It follows that the $D_i^f$, $R_n^f$ are uniquely determined and continuous. Further we have $C^1(X) \supseteq C^2(X) \supseteq \ldots$). It is easy to show that $C^i(X) = C^i(X)$ for $i = 1,2$. Also $C^n(X) \subset C^n(X)$ for all $n$, by 6.6. But we have

EXAMPLE 6.8. Let $X = \{0,1,2,\ldots\}$, and let $f : X \to K$ be defined via

$$f(\Sigma a_n p^n) = \Sigma a_n p^{3n!}.$$ 

Then $f \in C^n(X)$ for each $n$, and $D_i^f = 0$ for $i = 1,2,4,5,\ldots$ and $D_3^f = 1$. On the other hand, $f \notin C^3(X)$.

Let $C > 0$ and $\{x_1,\ldots,x_n\}$ a set of $n$ distinct points in $X$. We call $\{x_1,\ldots,x_n\}$ a $C$-polygon if for all $i,j,k,l \in \{1,\ldots,n\}$, $k \neq l$:
DEFINITION 6.9. Let \( n \in \mathbb{N} \). We say that \( X \) has locally property \( B_n \) if for each \( a \in X \) there is \( \delta > 0 \) and \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2, |x_1 - a| < \delta, |x_2 - a| < \delta \) there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (By definition, every \( X \) has locally property \( B_1 \) and \( B_2 \)).

We say that \( X \) has globally property \( B_n \) if there exists \( C > 0 \) such that for all \( x_1, x_2 \in X, x_1 \neq x_2 \), there exist \( x_3, \ldots, x_n \in X \) such that \( \{x_1, x_2, \ldots, x_n\} \) is a \( C \)-polygon. (Every \( X \) has globally property \( B_1 \) and \( B_2 \)).

For example, a ball in \( K \) has globally property \( B_n \) for each \( n \). Every open (non-empty) subset of \( K \) has locally property \( B_n \), for each \( n \).

Let us call \( B^0(X) = \{f \in C^0(X) : ||f||_0 < \infty\} \), where, by definition,

\[
||f||_0 = \max(||f||_\infty, |D_1 f|_\infty, \ldots, |D_{n-1} f|_\infty, |R_n f|_\infty)
\]

(see 6.7). It is very easy to show that \( B^0(X) \) is a Banach space with respect to \( || \cdot ||_0 \). The main theorem:

**THEOREM 6.10.** If \( X \) has locally property \( B_n \), then \( C^n(X) = C^n(X) \).

Let \( X \) have globally property \( B_n \) \( (n > 2) \) in the sense that every two-point set can be extended to a \( C \)-polygon. Then \( B^0(X) = B^0(X) \) and

\[
||f||_0 \leq ||f||_n \leq C^2(n-2) ||f||_0.
\]
(In general we have for \( f \in BC^n(X) : \|f\|_n = \max_{0 \leq i \leq n} \|D_i f\|_{n-i}^i \).

As in 3.5. we want to find an antiderivation map: \( C^{n-1}(X) \to C^n(X) \).

We cannot use the map \( P \) of 3.5. since one can prove: if \( f \in C^1(X) \) then \( Pf \in C^2(X) \) if and only if \( f' = 0 \). Further, if the characteristic of \( K \) equals \( p \neq 0 \) it is easy to see that not every \( C^p \)-function has a \( C^p \)-antiderivative.

**Theorem 6.11.** Let the characteristic of \( K \) be zero and let \( r_1 > r_2 \ldots \)

as in 3.5 but such that \( r_m < \rho r_{m+1} \) for all \( m \), some \( \rho > 0 \). For \( f \in C^{n-1}(X) \), set

\[
P_n f(x) = \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i! (x_{k+1} - x_k)^i} D_{i-1} f(x_k) \quad (x \in X)
\]

Then \( P_n f \in C^n(X) \) and \( (P_n f)' = f \). If \( f \in BC^{n-1}(X) \), then

\[
\|P_n f\|_n \leq c_n \|f\|_{n-1}
\]

where

\[
c_n = \max_{1 \leq i \leq n} \frac{1}{i! \cdot \rho^i}.
\]

It follows that the map \( Q_n = n! P_n P_{n-1} \ldots P_1 \) sends \( C(X) \) into \( C^n(X) \),

\( D_n Q_n \) is the identity on \( C(X) \). A computation yields

\[
Q_n = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} n^{i-i} S_i,
\]

where \( M \) is the multiplication with \( x \) ((\( Mf)(x) = xf(x) \) for \( f \in C(X) \))

and where

\[
S_i f(x) = \sum_{k=1}^{\infty} f(x_k) (x_k^i - x_{k+1}^i) \quad (f \in C(X), \ x \in X)
\]
Note: Theorem 3.4 is also true if we replace $BC^n(X)$ by $BC^n(X)$ ($n \in \mathbb{N}$).

7. $C^\infty$-functions

Spline functions, analytic functions are in $C^\infty(X)$.

Let $f \in C^\infty(X)$ and let $f(a) = 0$ for some $a \in X$. Then $f(x) = (x-a)g(x)$ ($x \in X$) where $g \in C^\infty(X)$.

A derivation on $C^\infty(X)$ is a linear map $\phi : C^\infty(X) \to C^\infty(X)$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) \quad (f,g \in C^\infty(X)).$$

Any derivation $\phi$ has the form $\frac{d}{dx}g$, where $g \in C^\infty(X)$.

THEOREM 7.1. Let $Y \subseteq K$ be a closed subset of $K$. Then there is $f \in C^\infty(K)$ (with $f' = 0$ everywhere) such that $Y$ is the set of zeros of $f$.

THEOREM 7.2. Let $\lambda_0, \lambda_1, \ldots$ be any sequence in $K$. Then there exists an $f \in C^\infty(K)$ such that $D_i f(0) = \lambda_i$ for all $i$.

Open problem: Let the characteristic of $K$ be zero. Does every $f \in C^\infty(X)$ have a $C^\infty$-antiderivative?