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A SIMPLE SOLUTION OF HILBERT’S FOURTEENTH PROBLEM
IN DIMENSION FIVE

BY

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Abstract. We give a short proof of a counterexample (due to Daigle and Freudenburg) to Hilbert’s fourteenth problem in dimension five.

Introduction. In 1900 at the International Congress of Mathematicians in Paris David Hilbert presented a list of 23 problems, intended to challenge the mathematicians of the new century. The fourteenth problem of this list can be stated as follows: let \( k \) be a field, \( k[x] := k[x_1, \ldots, x_n] \) the polynomial ring, \( k(x) \) its quotient field and \( L \) a subfield containing \( k \).

Is \( L \cap k[x] \) a finitely generated \( k \)-algebra?

A positive answer was given by Zariski ([7]) in case trdeg\(_k L \leq 2 \). However in 1958 Nagata ([5]) constructed a counterexample in dimension 32. Then in 1988 Roberts ([6]) found a new counterexample in dimension 7. Recently, in 1998 Freudenburg ([2]), studying Roberts’s example, found a 6-dimensional counterexample, from which a 5-dimensional example was obtained in 1999 by Daigle and Freudenburg in [1]: they consider on \( B := k[X, S, T, U, V] \) the derivation \( D := X^3 \partial_S + S \partial_T + T \partial_U + X^2 \partial_V \) and show that \( B^D := \ker D : B \to B \) is not finitely generated over \( k \) (then the quotient field \( L \) of \( B^D \) is a counterexample to Hilbert fourteen, since \( L \cap B = B^D \)).

The main aim of this note is to give a short proof of this result, by substantially simplifying the arguments given in [1] and [2].

Finally, I would like to mention that recently S. Kuroda has constructed new counterexamples to Hilbert fourteen in the missing dimensions 4 and 3 ([3], [4]).

1. The main result. Throughout this paper we use the following notations: \( k \) is a field of characteristic zero,

\[ B := k[X, S, T, U, V], \quad D_0 := X^3 \partial_S + S \partial_T + T \partial_U, \quad D := D_0 + X^2 \partial_V. \]

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Furthermore,\[ A := k[S, T, U], \quad D_1 := \partial_S + S \partial_T + T \partial_U. \]

Finally, for any \(0 \neq f \in B\), \(\deg f\) denotes the usual degree of \(f\). We also use another grading on \(A\) given by a vector \(w \in \mathbb{N}^3\) and we write \(w\)-deg to denote the degree with respect to this grading. The main aim of this note is to give a short proof of

**Theorem 1.1** (Daigle–Freudenburg). \(B^D\) is not a finitely generated \(k\)-algebra.

The proof is based on the following result which will be proved in the next section.

**Proposition 1.2.** Let \(e : \mathbb{N} \to \mathbb{N}\) be defined by 
\[ e(3l) = 2l, \quad e(3l + 1) = e(3l + 2) = 2l + 1 \quad \text{for all} \; l \geq 0. \]
There exist \(c_0 = 1, c_1, c_2, \ldots \) in \(A\) with 
\[ D_1c_i = c_{i-1} \quad \text{and} \quad \deg c_i \leq e(i) \quad \text{for all} \; i \geq 1. \]

**Proof of Theorem 1.1.** (i) Define
\[ a_i := X^{2i+1}c_i\left(\frac{S}{X^3}, \frac{T}{X^3}, \frac{U}{X^3}\right) \quad \text{for} \; i \geq 0. \]

Then one easily verifies that 
\[ D_0a_i = X^2a_{i-1} \quad \text{for all} \; i \geq 1 \quad \text{and} \quad \text{that} \]
\[ F_n := \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} a_i V^{n-i} \in B^D \quad \text{for all} \; n \geq 1. \]

Suppose now that \(B^D\) is finitely generated by \(g_1, \ldots, g_s\) over \(k\). We may assume that \(g_i(0) = 0\) for all \(i\). Write \(g_i = \sum g_{ij}V^j\) with \(g_{ij} \in k[X, S, T, U]\).

By (ii) below we find that \(g_{ij} \in (X, S, T, U)\) for all \(i, j\). Let \(d\) denote the maximum of the \(V\)-degrees of all \(g_i\). Consider \(F_{d+1} = XV^{d+1}+\text{lower degree} V\)-terms as above. So \(F_{d+1} \in B^D = k[g_1, \ldots, g_s]\). Looking at the coefficient of \(V^{d+1}\), we deduce that 
\(X \in (X, S, T, U)^2\), a contradiction.

(ii) To prove that \(g_{ij} \in (X, S, T, U)\) for all \(i, j\) it suffices to show that if \(g = \sum g_{ij}V^j \in B^D\) satifies \(g(0) = 0\) then each \(g_j \in (X, S, T, U)\). First, clearly \(g_0 \in (X, S, T, U)\). So let \(j \geq 1\). From \(Dg = 0\) we get \(jg_jX^2 = D_0(-g_{j-1}) \in D_0([X, S, T, U] \subset (X^3, S, T)\) for all \(j \geq 1\). If \(g_j(0) \in k^*\), then \(X^2 \in (X^3, S, T, UX^2)\), contradiction. So \(g_j(0) = 0\), i.e. \(g_j \in (X, S, T, U)\).

2. The proof of Proposition 1.2. Put
\[ T_1 := T - \frac{1}{2} S^2, \quad U_1 := U - ST + \frac{1}{3} S^3. \]

Then \(A = k[T_1, U_1][S]\). Since \(D_1T_1 = D_1U_1 = 0\) and \(D_1S = 1\) we get \(A_1^D = k[T_1, U_1]\). Consider on \(A\) the grading defined by \(w(S) = 1, w(T) = 2\) and \(w(U) = 3\). Then \(D_1(A_n) \subset A_{n-1}\) for all \(n \geq 1\), where \(A_n\) is the \(k\)-span of all monomials of \(A\) of \(w\)-degree \(n\). By induction on \(n\) we construct \(c_n \in A\).
So assume that \( c_n \) is already constructed. Write \( c_n = \sum_{i=0}^{n} H_{n-i}S^i \) with \( H_{n-i} \in A_{n-i} \cap A^{D_1} \) (this is possible since \( A = A^{D_1} | S \) and \( c_n \in A_n \)). Then

\[
\tilde{c}_{n+1} := \sum_{i=0}^{n} \frac{1}{i+1} H_{n-i}S^{i+1} \in A_{n+1}
\]

and \( D_1(\tilde{c}_{n+1}) = c_n \). Finally, by Lemma 2.1 below, there exists \( h \in A_{n+1} \cap A^{D_1} \) such that \( \tilde{c}_{n+1} := c_{n+1} - h \) satisfies \( \deg \tilde{c}_{n+1} \leq e(n+1) \).

**Lemma 2.1.** If \( f \in A_{n+1} \) is such that \( \deg D_1 f \leq e(n) \), then there exists \( h \in A_{n+1} \cap A^{D_1} \) such that \( \deg(f - h) \leq e(n+1) \).

**Proof.** (i) Let \( n = 3l \) (the cases \( n = 3l+1 \) and \( n = 3l+2 \) are treated similarly) and let \( M \) be the \( k \)-span of all \( f \in A_{n+1} \) such that \( \deg D_1 f \leq 2l \) (= \( e(3l) \)). Write \( f = \sum \alpha_{ijk} S^i T^j U^k \) with \( i + 2j + 3k = 3l + 1 \) and \( \alpha_{ijk} \in k \). Then

\[
D_1 f = \sum_{i+2j+3k=3l+1} (i\alpha_{ijk} + (j+1)\alpha_{i-2,j+1,k} + (k+1)\alpha_{i-1,j-1,k+1}) S^{i-1} T^j U^k.
\]

So

\( (*) \quad \deg D_1 f \leq 2l \quad \text{iff} \quad i\alpha_{ijk} + (j+1)\alpha_{i-1,j-1,k+1} + (k+1)\alpha_{i-2,j+1,k} = 0 \)

for all \( i, j, k \) satisfying \( i + 2j + 3k = 3l + 1 \) and \( (i-1) + j + k \geq 2l + 1 \), i.e. \( i + j + k \geq 2l + 2 \). For such a triple we have \( i > 0 \). Hence by \( (*) \) each \( \alpha_{ijk} \) is a linear combination of certain \( \alpha_{pqr} \)'s with \( p + q + r < i + j + k \). Consequently, each \( \alpha_{ijk} \) is a linear combination of the \( \alpha_{pqr} \)'s satisfying \( p + q + r = 2l + 2 \). Since there are \( [(l-1)/2] + 1 \) of them (just solve the equations \( p+2q+3r = 0 \) and \( p+q+r = 2l+2 \) it follows that \( \dim \pi(M) \leq [(l-1)/2] + 1 \), where for any \( g \in A \), \( \pi(g) \) denotes the sum of all monomials of \( g \) of degree \( \geq 2l + 2 \).

(ii) Put \( N := A^{D_1} \cap A_{n+1} \). Then \( N \) is the \( k \)-span of all “monomials”

\[
n_p := T_1^{3p+2} U_1^{l-(2p+1)}, \quad \text{where} \ 0 \leq p \leq [(l-1)/2].
\]

CLAIM. The \( \pi(n_p) \) are linearly independent over \( k \).

It then follows from (i) and the inclusion \( \pi(N) \subset \pi(M) \) that \( \pi(N) = \pi(M) \), which proves the lemma.

(iii) To see the claim put

\[
w_p := (-2)^{3p+2} S^{l-(2p+1)} \pi(n_p)|_{T=0, U=\frac{1}{2} S}, \quad \pi((S^2)^{3p+2}(S + S^3)^{l-(2p+1)}).
\]

Observe that

\[
(S^2)^{3p+2}(S + S^3)^{l-(2p+1)} = \sum_{j=0}^{l-(2p+1)} \binom{l-(2p+1)}{j} S^{3l+1-2j}.
\]
Since $3l + 1 - 2j \geq 2l + 2$ iff $0 \leq j \leq [(l - 1)/2]$ we get

$$w_p = \sum_{j=0}^{[(l-1)/2]} \binom{l - (2p + 1)}{j} S^{3l+1-2j}.$$ 

Then the linear independence of the $w_p$ (and hence of the $\pi(n_p)$) follows since

$$\det \left( \binom{l - (2p + 1)}{j} \right)_{0 \leq p,j \leq [(l-1)/2]} \neq 0.$$ 

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