A commuting derivations theorem on UFD’s

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Abstract

Let A be the polynomial ring over k (a field of characteristic zero) in n + 1 variables. The commuting derivations conjecture states that n commuting locally nilpotent derivations on A, linearly independent over A, must satisfy AD1,...,Dn = k[f] where f is a coordinate. The conjecture can be formulated as stating that a (Gm)n-action on kn+1 must have invariant ring k[f] where f is a coordinate. In this paper we prove a statement (theorem 2.1) where we assume less on A (A is a UFD over k of transcendence degree n + 1 satisfying A* = k) and prove less (A/(f − α) is a polynomial ring for all but finitely many α). Under certain additional conditions (the Di are linearly independent modulo (f − α) for each α ∈ k) we prove that A is a polynomial ring itself and f is a coordinate. This statement is proven even more generally by replacing “free unipotent action of dimension n” for “Gn-action”.

We make links with the (Abhyankar-)Sataye conjecture and give a new equivalent formulation of the Sataye conjecture.

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1 Preliminaries and introduction

Notations: $k$ will denote a field of characteristic zero. For a $k$-algebra $A$ we define $\text{LND}(A)$ as the set of all locally nilpotent derivations, and $\text{DER}(A)$ as the set of derivations. We will denote by $A^{D_1,\ldots,D_m} := \{a \in A; D_1(a) = \ldots = D_m(a) = 0\}$.

In the paper [7], the following conjecture is posed:

**Commuting Derivations Conjecture:** Let $A := k[X_1,\ldots,X_{n+1}]$, and let $D_1,\ldots,D_n \in \text{LND}(A)$ be commuting, linearly independent over $A$, locally nilpotent derivations. Then $A^{D_1,\ldots,D_n} = k[f]$ and $f$ is a coordinate.

**Geometric version:** Suppose we have a $\mathcal{G} := (\mathcal{G}_a)^n$-action on $k^{n+1}$. Then $k[X_1,\ldots,X_{n+1}]^\mathcal{G} = k[f]$ and $f$ is a coordinate.

In the elegant paper [1], it is shown that this conjecture is equivalent to the following:

**Weak Abhyankar-Sataye Conjecture:** Let $A := k[X_1,\ldots,X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1,\ldots,X_n] \cong k(f)[Y_1,\ldots,Y_{n-1}]$. Then $f$ is a coordinate in $A$.

For completeness sake, let us state

**Abhyankar-Sataye Conjecture:** Let $A := k[X_1,\ldots,X_{n+1}]$, and let $f \in A$ be such that $A/(f) \cong k[Y_1,\ldots,Y_n]$. Then $f$ is a coordinate.

**Sataye Conjecture:** Let $A := k[X_1,\ldots,X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1,\ldots,Y_n]$ for all $\alpha \in \mathbb{C}$. Then $f$ is a coordinate.

In [7], the Commuting Derivations Conjecture is proven for $n = 3$. But there is no indication that it might be true in higher dimensions. Even more, the Vénéreau polynomials (see[8]) (or similar objects), which are candidate counterexamples to the Abhyankar-Sataye conjecture, could very well spoil things for the Commuting Derivations Conjecture in higher dimensions. In any case, it seems like a proof is far away.

Therefore, it seems a good idea to be a little less ambitious. In this paper, we consider the weaker statement that $A$ is a UFD (instead of a polynomial ring). It turns out that the situation can be quite different and interesting. Let us consider a famous example:

**Example 1.1.** Let $A := \mathbb{C}[x,y,z,t] = \mathbb{C}[X,Y,Z,T]/(X^2Y+X+Z^2+T^3)$ and let $D_1 := 2z \frac{\partial}{\partial y} - x^2 \frac{\partial}{\partial z}$ and $D_2 := 3t^2 \frac{\partial}{\partial y} - x^2 \frac{\partial}{\partial t}$. $A$ is a UFD of transcendence degree 3 which is not a polynomial ring (see [6], or use the fact that the commuting derivations conjecture in dimension 3 holds). $D_1$ and $D_2$ commute, and $A^{D_1,D_2} = \mathbb{C}[x]$. Now $A/(x - \alpha) \cong \mathbb{C}[Y_1, Y_2]$ except in the case that $\alpha = 0$. 

\[ 2 \]
Also, $D_1 \mod (x - \alpha), D_2 \mod (x - \alpha)$ are independent over $A/(x - \alpha)$ if and only if $\alpha \neq 0$.

2 The UFD Commuting derivations theorem

The following theorem is the main result of this paper.

**Theorem 2.1.** Let $A$ be a UFD over $k$ with $\text{trdeg}_k Q(A) = n + 1 (\geq 1)$, $A^* = k^*$, and let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations (linearly independent over $A$). Now $A^{D_1, \ldots, D_n} = k[f]$ for some $f \in A \setminus k$, and

1. If $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are independent over $A/(f - \alpha)$, then $A/(f - \alpha) \cong \mathbb{C}^{[\alpha]}$. There are only finitely many $\alpha \in \mathbb{C}$ for which $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are dependent over $A/(f - \alpha)$.

2. In the case that $D_1 \mod (f - \alpha), \ldots, D_n \mod (f - \alpha)$ are independent over $A/(f - \alpha)$ for each $\alpha \in k$, then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n + 1$ variables.

**Geometric Version:** Let $V$ be a factorial affine surface over $k$ of dimension $n + 1$ such that $\mathcal{O}(V)^* = k^*$. Suppose there exists a $G := (G_\alpha)^n$-action on $V$. Then $\mathcal{O}(V)^G = k[f]$ and

1. Suppose that the fiber $f = \alpha$ has a point with trivial stabilizer. Then the fiber $f = \alpha$ is isomorphic to $\mathbb{C}^n$. There are only finitely many $\alpha$ for which $f = \alpha$ has no point with trivial stabilizer.

2. Suppose that all fibers $f = \alpha$ have a point with trivial stabilizer. (Then, all points have trivial stabilizers.) Then $V \cong \mathbb{C}^{n+1}$ and the action $G \times V \longrightarrow V$ is a translation on the first $n$ coordinates.

In the last section we will prove a more general geometric statement of part 2 for unipotent groups in stead of $G_\alpha^n$-actions, but we will stick with this description for the moment, as this is the most interesting case for us, and has a simpler, direct, algebraic proof.

Before we give a proof of the above theorem, let us meditate on this a bit. The example 1.1 is a typical case of part 1 of the above theorem. But there is a connection with the Sataye Conjecture. Let us consider the following conjecture:

**Modified Sataye Conjecture:** Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then there exist $n$ commuting locally nilpotent derivations $D_1, \ldots, D_n$ on $A$ such that $A^{D_1, \ldots, D_n} = \mathbb{C}[f]$ and the $D_i$ are linearly independent modulo $(f - \alpha)$ for each $\alpha \in \mathbb{C}$.

**Proposition 2.2.** The Modified Sataye Conjecture is equivalent to the Sataye Conjecture.
Proof. Let us abbreviate the conjectures by SC and MSC. Suppose we have proven the MSC. Then for any \( f \) satisfying \( A/(f - \alpha) \cong k[Y_1, \ldots, Y_n] \) for all \( \alpha \in \mathbb{C} \) we can find commuting derivations as stated in the MSC. But using theorem 2.1 part 2 we get that \( f \) is a coordinate in \( A \). So the SC is true in that case.

Now suppose we have proven the SC. Let \( f \) satisfy the requirements of the MSC, that is, \( A/(f - \alpha) \cong k[Y_1, \ldots, Y_n] \) for all \( \alpha \in \mathbb{C} \). Since \( f \) satisfies the requirements of the SC, \( f \) then must be a coordinate. So it has \( n \) so-called mates: \( \mathbb{C}[f, f_1, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_{n+1}] \). But then each of these \( n + 1 \) polynomials \( f, f_1, \ldots, f_n \) defines a locally nilpotent derivation, all of them commute, and the intersection of the last \( n \) derivations is \( \mathbb{C}[f] \); so the MSC holds. \( \square \)

But now it is time to stop daydreaming about big conjectures, and start doing some hard-core proofs. Since the following proof uses the tools of the next section, the reader is encouraged to read section 3 before reading the following proof in detail.

Proof. (of theorem 2.1) Using lemma 3.4 we have \( p_i \in A \) such that \( D_j(p_i) = 0 \) if \( i \neq j \), and \( D_i(p_i) = q_i(f) \in \mathbb{C}[f] \) of lowest possible degree.

Part 1: \( D_1, \ldots, D_n \) are independent over \( A \), but they may become dependent modulo \( (f - \alpha) \). Let us first consider the case where they are independent modulo \( (f - \alpha) \): then \( D_1, \ldots, D_n \) are linearly independent over \( A/(f - \alpha) \). Then, by proposition 3.1 we have that \( A/(f - \alpha) \cong k^{[n]} \).

So, left to prove is that \( D_1, \ldots, D_n \) can only be linearly dependent modulo finitely many \( (f - \alpha) \). But this follows directly from lemma 3.5, as there are only finitely many zeroes in \( q_1 q_2 \cdots q_n \).

Part 2: Lemma 3.5 tells us directly that for each \( 1 \leq i \leq n \) and \( \alpha \in k \), we have \( q_i(\alpha) \neq 0 \). But this means that the \( q_i \in k^* \), so the \( p_i \) are in fact slices, and using 3.3 we are done. \( \square \)

3 Tools

The tools proven in this section focus on the situation of theorem 2.1 part 1, and are interesting in their own respect.

In this section, \( A \) is a \( k \)-domain, and \( \text{trdeg}(A) = n + 1 (\geq 1) \).

The following two propositions are proposition 3.2 and 3.4 in [7].

**Proposition 3.1.** Let \( D_1, \ldots, D_{n+1} \) be commuting locally nilpotent \( k \)-derivations on \( A \) which are linearly independent over \( A \). Then

(i). There exist \( s_i \) in \( A \) such that \( D_is_i = \delta_{ij} \) for all \( i, j \) and

(ii). \( A = k[s_1, \ldots, s_{n+1}] \) a polynomial ring in \( n + 1 \) variables over \( k \).

**Proposition 3.2.** Let \( A \) be a UFD and let \( A^* = k^* \). Let \( D_1, \ldots, D_n \) be commuting locally nilpotent derivations, linearly independent over \( A \). Then

\( A^{D_1, \ldots, D_n} = k[f] \) for some \( f \in A \setminus k \), and \( f - \alpha \) is irreducible for each \( \alpha \in \mathbb{C} \).
Proposition 3.3. Let $A$, $D_i$, $f$ as in proposition 3.2. Suppose there exist $s_1, \ldots, s_n$ such that $D_i(s_i) = 1$. Then $A = k[s_1, \ldots, s_n, f]$, a polynomial ring in $n + 1$ variables.

Proof. This is an easy consequence of the fact, if $D \in LND(A)$ having an $s \in A$ such that $D(s) = 1$, then $A^D[s] = A$. \hfill $\square$

Define the following abbreviation:

(S1:) Let $A$ be a UFD and let $A^* = k^*$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent derivations, linearly independent over $A$.

Lemma 3.4. Assume (S1).

(1) Then there exist $p_i \in A$ such that $D_j(p_i) = 0$ if $j \neq i$, and $D_i(p_i) \in k[f]\{0\}$. Furthermore, $k[p_1, \ldots, p_n, f] \subseteq A$ is algebraic.

(2) Define $P_i := \{p_i \in A \mid D_j(p_i) = 0$ if $i \neq j$ and $D_i(p_i) \in k[f]\}$. Then $D_i(P_i) = q_i(f)k[f]$ for some nonzero polynomial $q_i$. Taking $p_i$ such that $D_i(p_i)$ is of lowest possible degree yields $D_i(p_i) \in k[q_i(f)]$.

Proof. (1) We assume that all $n$ derivations commute, so $D_i(A^P) \subseteq A^P$, and therefore $D_i$ sends $A_i := A^{D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n}$ to itself. Taking some $a \in A_i \setminus k[f]$ nonzero, we use the fact that $D_j$ is locally nilpotent to find the lowest $m \in \mathbb{N}$ such that $D^m(a) = 0$. Now define $p_i := D^{m-2}(a)$ (indeed $m \geq 2$). The rest is easy.

(2) Take $p_i$ such that $D_i(p_i) = q_i(f) \neq 0$ has lowest possible degree. Let $\bar{p}_i \in P_i$. Then $D_i(\bar{p}_i) = h_i(f)q_i(f) + r_i(f)$ where $\deg(r_i) < \deg(q_i)$. Now $D_i(\bar{p}_i - h_i(f)p_i) = r_i(f)$ so $r_i = 0$. So $D_i(\bar{p}_i) \in q_i(f)k[f]$. \hfill $\square$

Lemma 3.5. Assume (S1). Choose $p_i$ such that $D_i(p_i) = q_i(f)$ as in lemma 3.4, where $q_i$ is of lowest possible degree. The $D_i$ are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some $i$.

Proof. $(\Rightarrow)$: Write “bars” for “modulo $f - \alpha$”. Suppose that $0 \neq D := q_1D_1 + \ldots + q_nD_n$ satisfies $\bar{D} = 0$ where $q_i \in A$, and not all $\bar{g}_i = 0$. Now $\bar{g}_i\bar{D}_i(\bar{p}_i) = \bar{D}(\bar{p}_i) = 0$ for each $i$, so for each $i$, either $\bar{g}_i = 0$ or $\bar{q}_i(f) = 0$ (as $f - \alpha$ is irreducible by proposition 3.2). Since not all $\bar{g}_i = 0$, at least one $\bar{q}_i(f) = 0$. Since $f - \alpha$ is irreducible for each $\alpha$, we not only have $(f - \alpha)|q_i(f)$, but even $(X - \alpha)|\bar{q}_i(X)$, so $q_i(\alpha) = 0$.

$(\Leftarrow)$: Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $D_i$ mod $(f - \alpha)$ are linearly dependent over $A/(f - \alpha)$. Suppose the $D_i$ are linearly independent over $A$. Then we have $n$ commuting, linearly independent LNDs on a domain of transcendence degree $n$, so we can use proposition 3.1 and conclude that $A^{D_1, \ldots, D_n} = k$. This means, since $\bar{q}_i(f) = 0$, that $\bar{p}_i \in k$. So, $p_i = (f - \alpha)a + \lambda$ where $a \in A, \lambda \in k$. Now taking $a \in A$ we still have $D_j(a) = 0$ for all $j \neq i$, and $D_i(a) = q_i(f)(f - \alpha)^{-1} \in k[f]$. This contradicts the assumption that $q_i$ was minimal, so our assumption that the $D_i$ are linearly independent was incorrect. \hfill $\square$
Now we want to point out the following phenomenon:

**Example 3.6.** Let $D_1 = Z \partial_X + \partial_Y, D_2 = \partial_Y$ on $A = \mathbb{C}[X, Y, Z]$. Now $A^{D_1, D_2} = \mathbb{C}[Z]$. The $D_1, D_2$ are linearly independent modulo $Z - \alpha$ as long as $\alpha \neq 0$. But it is clear that a different set of derivations, namely $E_1 = \partial_X, E_2 = \partial_Y$ commute, their $\mathbb{C}[Z]$-span contains $D_1, D_2$ and the $E_i$ are linearly independent for more fibers $f - \alpha$.

The $E_i$ of the example are an improvement over the $D_i$: all the same properties, but they are linearly independent for more $f - \alpha$. Perhaps for your given space $A$ and derivations $D_i$ it is impossible to find $E_i$ such that the $E_i$ are independent modulo every $f - \alpha$, giving more information on your ring $A$. Before we elaborate on this, let us give a lemma that enables construction of the $E_i$:

**Lemma 3.7.** Assume (S1). Define $M := k(f)D_1 + ... + k(f)D_n \cap \text{DER}(A)$. Then $M = k[f]E_1 \oplus \ldots \oplus k[f]E_n$ for some $E_i \in M$, and the $E_i$ have all the properties that the $D_i$ have (i.e. commuting locally nilpotent, linearly independent over $A$). Furthermore, if the $D_i$ are linearly independent modulo $(f - \alpha)$, then the $E_i$ are too (but not necessary the other way around).

**Proof.** Use lemma 3.4 we find preslices $p_i$ and $D(p_i) = q_i(f)$ as stated there.

If $D \in M$ then $D = g_1(f)D_1 + \ldots + g_n(f)D_n$ where $g_i(f) \in k(f)$. Now since $D \in \text{DER}(A)$ we have $D(p_i) \in A$. Also $D(p_i) = g_1(f)D_1(p_i) = g_2(f)q_1(f) \in k(f)$ thus $D(p_i) \in A \cap k(f)$, which equals $k[f]$ since $A^* = k^*$.

Therefore the map $\varphi : M \longrightarrow k[f]^n$ sending $D \longrightarrow (D(p_1), \ldots, D(p_n))$ is well-defined. If $0 = \varphi(g_1(f)D_1 + \ldots + g_n(f)D_n)$ then $g_i(f)D_i(p_i) = 0$ and therefore $g_i(f) = 0$; thus $\varphi$ is injective.

Since $\varphi$ is an injective map, $M$ must be a free $k[f]$-module. Note that $M$ can only have dimension $n$. Therefore we can find $E_1, \ldots, E_n$ as required.

Any derivation in $M$ is locally nilpotent. Even more, any two derivations of $M$ commute! Next to that, the $E_i$ are clearly independent over $A$. □

Note that the $E_i$ can be constructively made, given the injective map $\varphi$ in the above proof. This actually gives an interesting concept. Given the situation (S1), one can improve the derivations $D_i$ (by replacing them by the $E_i$) and then they are linearly independent modulo as much as possible $f - \alpha$.

For every such $\alpha$ we have that $A/(f - \alpha)$ is a polynomial ring. The question is if the converse holds:

**Question:** Assume (S1). Additionally, assume $k[f]D_1 + \ldots + k[f]D_n = (k(f)D_1 + \ldots + k(f)D_n) \cap \text{DER}(A)$. Is the set $\{\alpha \in \mathbb{C} \mid D_1, \ldots, D_n$ linearly dependent modulo $(f - \alpha)$\} equal to the set $\{\alpha \in \mathbb{C} \mid A/(f - \alpha)$ is not a polynomial ring\}? (One always has $\supset$.) Or, if this equality does not hold, what type of rings $A$ do have equality?

Note that the requirement “$A$ UFD” is absolutely necessary, as for a simple Danielewski surface $\mathbb{C}[X, Y, Z]/(X^2Y - Z^2)$ we find a LND $2Z\partial_Z + X^2\partial_Z$ which
is nonzero modulo each $X - \alpha$. (But $A/(f - \alpha)$ is not always a domain in this case, even.)

## 4 Unipotent actions

The authors would like to thank prof. Kraft for pointing out the generalization of theorem 2.1 part 2, which has become the below theorem 4.2.

**Proposition 4.1.** If $U \times V \rightarrow V$ is an action of a unipotent group $U$ on an affine variety $V$, then for each $u \in U$, the map $u^*: O(V) \rightarrow O(V)$ is an exponent of a locally nilpotent derivation.

For the proof we can refer to proposition 2.1.3 in [2], or ask the reader to verify that $u^* - Id$ is a locally nilpotent endomorphism, and that thus “log($u^*$)” can be defined, and is a derivation.

This proposition has some immediate consequences, like that the invariants of a unipotent group action are the intersection of kernels of locally nilpotent derivations. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, so the invariants of a unipotent group is factorially closed.

In the below theorem, $C$ is a field of characteristic zero, which is algebraically closed.

**Theorem 4.2.** Let $U$ be a unipotent algebraic group of dimension $n$, acting freely on $X$, a factorial variety of dimension $n + 1$ satisfying $O(X)^* = C^*$. Then $X$ is $U$-isomorphic to $U \times C$. In particular, $X \simeq C^{n+1}$.

**Proof.** The fact that $U$ acts free means that each $x \in X$ has trivial stabilizer: $U_x = \{u \in U; ux = x\} = \{id\}$. So, each orbit $Ux$ is of dimension $n$. This means that $X/U$ is of dimension 1. Also, as remarked above, $X^U$ is factorial. But then it is also normal, and smooth. So $X/U$ is a smooth, rational, affine curve, in other words, an open subvariety of $C$. Now suppose that $X/U \not\subseteq C$, so $X/U = C - \{p_1, \ldots, p_n\}$, then $O(X)^U = O(C - \{p_1, \ldots, p_n\}) = C[t, (t - p_1)^{-1}, \ldots, (t - p_n)^{-1}]$. This means that $O(X)$ contains invertible elements $(t - p_1)^{-1}$, giving a contradiction with the assumption $O(X)^* = C^*$. Hence, $X/U \simeq C$, so $O(X)^U = O(X/U) = O(C) \cong C[f]$ for some $f$. Now every $f - \lambda (\lambda \in C)$ is irreducible, as otherwise any irreducible factor of $f - \lambda$ would be in $O(X)^U$ too.

Now consider the map $f: X \rightarrow C$. This is in fact the map $X \rightarrow X/U$ (as it corresponds to the map $O(X) \leftarrow O(X)^U = C[f]$) and thus surjective. Also note that the fibers $f^{-1}(\lambda)$ are invariant under $U$: they correspond to the function space $O(X)/(f - \lambda)$. By assumption, $U$ acts free on each fiber of $X \rightarrow X/U$, which means exactly that $U$ acts free on $f^{-1}(\lambda)$ for each $\lambda$. Let $x \in f^{-1}(\lambda)$. Then $Ux$ is of dimension $n$ (it is just a copy of $U$). Also, each orbit of a unipotent group is closed (see Satz 4 from [3]), and therefore the inclusion $Ux \subseteq f^{-1}(\lambda)$ is an equality. So orbits of $U$ are the same as fibers of $f$, i.e. we have an orbit fibration (or $U$-fibration).
\(X_{\text{sing}}\) is closed and \(U\)-stable, hence a union of \(U\)-orbits, and so \(\text{codim} \ X_{\text{sing}} = 1\) or \(X_{\text{sing}}\) is empty. But \(X\) is factorial, so in particular normal, which implies \(\text{codim}(X_{\text{sing}}) \geq 2\). So \(X_{\text{sing}}\) is empty, in other words: \(X\) is smooth.

Now we claim that \(f : X \to \mathbb{C}\) is smooth. To see this, first note that \(\mathcal{O}(f^{-1}(\lambda)) = \mathcal{O}(X)/(f - \lambda)\) is reduced as \(f - \lambda\) is irreducible, as seen before. And, as we already implied, the set of functions vanishing on \(f^{-1}(\lambda)\) is the ideal \((f - \lambda)\). Now consider the tangent map \(df_x : T_x X \to T_x \mathbb{C} = \mathbb{C}\) where \(x \in f^{-1}(\lambda)\). Using "Satz 2", page 269 in [3] we see that, \(\ker df \supseteq T_x f^{-1}(\lambda)\), but since \(f^{-1}(\lambda)\) is reduced, we even have equality \(\ker df = T_x f^{-1}(\lambda)\). Now remember that the fiber \(f^{-1}(\lambda)\) is an orbit, hence smooth (as any orbit is smooth!). This implies \(\dim T_x f^{-1}(\lambda) = n\) and thus \(\dim \ker df = n\). Since \(\dim T_x \mathbb{C} = n + 1\) we have \(\dim \text{Im}(df_x) = 1\), hence \(df_x\) is surjective. A morphism between smooth varieties is smooth if and only if the differential is surjective. So we have shown that \(f\) is smooth.

So: \(f : X \to \mathbb{C}\) is surjective, and smooth. Let \(K := \ker df|_X \subseteq T_x X\). Take some linear subspace \(C\) such that \(K \oplus C = T_x X\). Note that \(C\) has dimension 1. Seeing \(X\) as a subset of some \(\mathbb{C}^N\), we can find hyperplanes \(H\) that contains \(C\). We even want \(H \cap T_x X = C\), so this means that \(H \oplus T_x X \subseteq \mathbb{C}^N\), so let us take a hyperplane \(H\) of codimension \(n\) such that \(H \cap T_x X = C\). Now let \(Z\) be an irreducible component of \(H \cap X\) which contains \(x\). Also, \(\dim_x H \cap X \geq 1\), thus \(\dim_x Z = 1\) and \(Z\) is smooth at \(x\). Now \(Z\) and \(C\) are smooth, and the differential of \(f|_Z : Z \to \mathbb{C}\) is an isomorphism at \(x\) (implying surjective), thus we have that \(f|_Z\) is smooth at \(x\). Replacing \(Z\), if necessary, by a (special) open subset \(Z' \subset Z\), we have \(f|_Z\) is étale.

Now look at the following diagram

\[
\begin{array}{ccc}
Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
\]

where \(Z \times_{\mathbb{C}} X = \{(x,z) \in X \times Z \mid f(x) = f|_Z(z)\}\) is the (schematic) fiber product. Since \(f\) is smooth, the same holds for \(f|_Z\) so \(Z \times_{\mathbb{C}} X\) is smooth. Moreover, \(U\) acts on \(Z \times_{\mathbb{C}} X\) by \(u(z,x) = (z, ux)\) and \(p(u(z,x)) = ux\) (\(p\) is \(U\)-equivariant) and \(f(u(z,x)) = z = f(x,z)\) (\(f\) is \(U\)-invariant). The fibers of \(\tilde{f}\) are \(\tilde{f}^{-1}(\alpha) = \{(x,z) \mid f(x) = f|_Z(z)\} = \{x \mid f(x) = \alpha\} = f^{-1}(\alpha)\) where \(\alpha = f|_Z(z)\). Now \(\tilde{f}\) has a section \(\sigma : Z \to Z \times_{\mathbb{C}} X\) given by \(z \mapsto (z,z)\), i.e. \(\tilde{f} \circ \sigma = \text{id}_Z\). Therefore, we can extend the diagram above

\[
\begin{array}{ccc}
U \times Z & \xrightarrow{q} & Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow \text{pr} \times Z & & \downarrow \tilde{f} & & \downarrow f \\
Z & = & Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
\]

where \(q : U \times Z \to Z \times_{\mathbb{C}} X\) is given by \((u, z) \mapsto (z, uz)\). By construction, \(q\) is bijective, hence an isomorphism, since the second variety is normal (see [4]
proposition 5.7). Note that the role of \( x \) was arbitrary: for each \( x \) we find a neighborhood \( Z \) where \( Z \times_{\mathbb{C}} X = Z \times_{\mathbb{C}} U \). This last statement exactly means that the map \( f : X \rightarrow \mathbb{C} \) is a locally trivial principal \( U \)-bundle with respect to the étale topology: for every point \( \lambda \in \mathbb{C} \) there is an étale map \( Z \rightarrow \mathbb{C} \) such that \( \lambda \) is in the image and the fiber product \( Z \times_{\mathbb{C}} X \) is a trivial \( U \)-bundle, i.e. isomorphic to \( U \times_{\mathbb{C}} Z \rightarrow Z \).

In the paper [5] we now find a result that tells us that a principal \( G \)-bundle where \( G \) is a unipotent group is trivial over any affine variety, and then we are done.

\[ \square \]

References


[6] L. Makar-Limanov, On the hypersurface \( x + x^2y + z^2 + t^3 = 0 \) in \( \mathbb{C}^4 \) or a \( \mathbb{C}^3 \)-like threefold which is not \( \mathbb{C}^3 \), Israel J. Math., 96(1996), 419-429
