A commuting derivations theorem on UFD’s

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Abstract

Let $A$ be the polynomial ring over $k$ (a field of characteristic zero) in $n + 1$ variables. The commuting derivations conjecture states that $n$ commuting locally nilpotent derivations on $A$, linearly independent over $A$, must satisfy $AD_1, \ldots, AD_n = k[f]$ where $f$ is a coordinate. The conjecture can be formulated as stating that a $(G_m)^n$-action on $k^{n+1}$ must have invariant ring $k[f]$ where $f$ is a coordinate. In this paper we prove a statement (theorem 2.1) where we assume less on $A$ ($A$ is a UFD over $k$ of transcendence degree $n + 1$ satisfying $A^* = k$) and prove less ($A/(f - \alpha)$ is a polynomial ring for all but finitely many $\alpha$). Under certain additional conditions (the $D_i$ are linearly independent modulo $(f - \alpha)$ for each $\alpha \in k$) we prove that $A$ is a polynomial ring itself and $f$ is a coordinate. This statement is proven even more generally by replacing “free unipotent action of dimension $n$” for “$G_a^n$-action”.

We make links with the (Abhyankar-)Sataye conjecture and give a new equivalent formulation of the Sataye conjecture.

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1 Preliminaries and introduction

Notations: $k$ will denote a field of characteristic zero. For a $k$-algebra $A$ we define $\text{LND}(A)$ as the set of all locally nilpotent derivations, and $\text{DER}(A)$ as the set of derivations. We will denote by $A^{D_1, \ldots, D_m} := \{a \in A; D_1(a) = \ldots = D_m(a) = 0\}$.

In the paper [7], the following conjecture is posed:

**Commuting Derivations Conjecture**: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $D_1, \ldots, D_n \in \text{LND}(A)$ be commuting, linearly independent over $A$, locally nilpotent derivations. Then $A^{D_1, \ldots, D_n} = k[f]$ and $f$ is a coordinate.

**Geometric version**: Suppose we have a $G := (G_a)^n$-action on $k^{n+1}$. Then $k[X_1, \ldots, X_{n+1}]^G = k[f]$ and $f$ is a coordinate.

In the elegant paper [1], it is shown that this conjecture is equivalent to the following:

**Weak Abhyankar-Sataye Conjecture**: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1, \ldots, X_n] \cong k(f)[Y_1, \ldots, Y_{n-1}]$. Then $f$ is a coordinate in $A$.

For completeness sake, let us state

**Abhyankar-Sataye Conjecture**: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f) \cong k[Y_1, \ldots, Y_n]$. Then $f$ is a coordinate.

**Sataye Conjecture**: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k[Y_1, \ldots, Y_n]$ for all $\alpha \in \mathbb{C}$. Then $f$ is a coordinate.

In [7], the Commuting Derivations Conjecture is proven for $n = 3$. But there is no indication that it might be true in higher dimensions. Even more, the Vénéréau polynomials (see[8]) (or similar objects), which are candidate counterexamples to the Abhyankar-Sataye conjecture, could very well spoil things for the Commuting Derivations Conjecture in higher dimensions. In any case, it seems like a proof is far away.

Therefore, it seems a good idea to be a little less ambitious. In this paper, we consider the weaker statement that $A$ is a UFD (instead of a polynomial ring). It turns out that the situation can be quite different and interesting. Let us consider a famous example:

**Example 1.1.** Let $A := \mathbb{C}[x, y, z, t] = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3)$ and let $D_1 := 2z \frac{\partial}{\partial y} - x^2 \frac{\partial}{\partial z}$ and $D_2 := 3t^2 \frac{\partial}{\partial y} - x^2 \frac{\partial}{\partial t}$. $A$ is a UFD of transcendence degree 3 which is not a polynomial ring (see [6], or use the fact that the commuting derivations conjecture in dimension 3 holds). $D_1$ and $D_2$ commute, and $A^{D_1, D_2} = \mathbb{C}[x]$. Now $A/(x - \alpha) \cong \mathbb{C}[Y_1, Y_2]$ except in the case that $\alpha = 0$. 
Also, \( D_1 \text{ mod } (x - \alpha), D_2 \text{ mod } (x - \alpha) \) are independent over \( A/(x - \alpha) \) if and only if \( \alpha \neq 0 \).

## 2 The UFD Commuting derivations theorem

The following theorem is the main result of this paper.

**Theorem 2.1.** Let \( A \) be a UFD over \( k \) with \( \text{trdeg}_k Q(A) = n + 1 (\geq 1) \), \( A^* = k^* \), and let \( D_1, \ldots, D_n \) be commuting locally nilpotent derivations (linearly independent over \( A \)). Now \( AD_1, \ldots, D_n = k[f] \) for some \( f \in A \setminus k \), and

1. If \( D_1 \text{ mod } (f - \alpha), \ldots, D_n \text{ mod } (f - \alpha) \) are independent over \( A/(f - \alpha) \), then \( A/(f - \alpha) \cong \mathbb{C}[n] \). There are only finitely many \( \alpha \in \mathbb{C} \) for which \( D_1 \text{ mod } (f - \alpha), \ldots, D_n \text{ mod } (f - \alpha) \) are dependent over \( A/(f - \alpha) \).

2. In the case that \( D_1 \text{ mod } (f - \alpha), \ldots, D_n \text{ mod } (f - \alpha) \) are independent over \( A/(f - \alpha) \) for each \( \alpha \in k \), then \( A = k[s_1, \ldots, s_n, f] \), a polynomial ring in \( n + 1 \) variables.

**Geometric Version:** Let \( V \) be a factorial affine surface over \( k \) of dimension \( n + 1 \) such that \( \mathcal{O}(V)^* = k^* \). Suppose there exists a \( G := (G_a)^n \)-action on \( V \). Then \( \mathcal{O}(V)^G = k[f] \) and

1. Suppose that the fiber \( f = \alpha \) has a point with trivial stabilizer. Then the fiber \( f = \alpha \) is isomorphic to \( \mathbb{C}^n \). There are only finitely many \( \alpha \) for which \( f = \alpha \) has no point with trivial stabilizer.

2. Suppose that all fibers \( f = \alpha \) have a point with trivial stabilizer. (Then, all points have trivial stabilizers.) Then \( V \cong \mathbb{C}^{n+1} \) and the action \( G \times V \rightarrow V \) is a translation on the first \( n \) coordinates.

In the last section we will prove a more general geometric statement of part 2 for unipotent groups in stead of \( G_a^n \)-actions, but we will stick with this description for the moment, as this is the most interesting case for us, and has a simpler, direct, algebraic proof.

Before we give a proof of the above theorem, let us meditate on this a bit. The example 1.1 is a typical case of part 1 of the above theorem. But there is a connection with the Sataye Conjecture. Let us consider the following conjecture:

**Modified Sataye Conjecture:** Let \( A := k[X_1, \ldots, X_{n+1}] \), and let \( f \in A \) be such that \( A/(f - \alpha) \cong k[Y_1, \ldots, Y_n] \) for all \( \alpha \in \mathbb{C} \). Then there exist \( n \) commuting locally nilpotent derivations \( D_1, \ldots, D_n \) on \( A \) such that \( AD_1, \ldots, D_n = \mathbb{C}[f] \) and the \( D_i \) are linearly independent modulo \( (f - \alpha) \) for each \( \alpha \in \mathbb{C} \).

**Proposition 2.2.** The Modified Sataye Conjecture is equivalent to the Sataye Conjecture.
Proof. Let us abbreviate the conjectures by SC and MSC. Suppose we have proven the MSC. Then for any \( f \) satisfying \( A/(f - \alpha) \cong k[Y_1, \ldots, Y_n] \) for all \( \alpha \in \mathbb{C} \) we can find commuting derivations as stated in the MSC. But using theorem 2.1 part 2 we get that \( f \) is a coordinate in \( A \). So the SC is true in that case.

Now suppose we have proven the SC. Let \( f \) satisfy the requirements of the MSC, that is, \( A/(f - \alpha) \cong k[Y_1, \ldots, Y_n] \) for all \( \alpha \in \mathbb{C} \). Since \( f \) satisfies the requirements of the SC, \( f \) then must be a coordinate. So it has \( n \) so-called mates: \( \mathbb{C}[f, f_1, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_{n+1}] \). But then each of these \( n + 1 \) polynomials \( f, f_1, \ldots, f_n \) defines a locally nilpotent derivation, all of them commute, and the intersection of the last \( n \) derivations is \( \mathbb{C}[f] \); so the MSC holds.

But now it is time to stop daydreaming about big conjectures, and start doing some hard-core proofs. Since the following proof uses the tools of the next section, the reader is encouraged to read section 3 before reading the following proof in detail.

Proof. (of theorem 2.1) Using lemma 3.4 we have \( p_i \in A \) such that \( D_j(p_i) = 0 \) if \( i \neq j \), and \( D_i(p_i) = q_i(f) \in \mathbb{C}[f] \) of lowest possible degree.

Part 1: \( D_1, \ldots, D_n \) are independent over \( A \), but they may become dependent modulo \( (f - \alpha) \). Let us first consider the case where they are independent modulo \( (f - \alpha) \): then \( D_1, \ldots, D_n \) are linearly independent over \( A/(f - \alpha) \). Then, by proposition 3.1 we have that \( A/(f - \alpha) \cong k^{[n]} \).

So, left to prove is that \( D_1, \ldots, D_n \) can only be linearly dependent modulo finitely many \( (f - \alpha) \). But this follows directly from lemma 3.5, as there are only finitely many zeroes in \( q_1q_2 \cdots q_n \).

Part 2: Lemma 3.5 tells us directly that for each \( 1 \leq i \leq n \) and \( \alpha \in k \), we have \( q_i(\alpha) \neq 0 \). But this means that the \( q_i \in k^* \), so the \( p_i \) are in fact slices, and using 3.3 we are done. \( \square \)

3 Tools

The tools proven in this section focus on the situation of theorem 2.1 part 1, and are interesting in their own respect.

In this section, \( A \) is a \( k \)-domain, and \( \text{trdeg}(A) = n + 1 (\geq 1) \).

The following two propositions are proposition 3.2 and 3.4 in [7].

**Proposition 3.1.** Let \( D_1, \ldots, D_{n+1} \) be commuting locally nilpotent \( k \)-derivations on \( A \) which are linearly independent over \( A \). Then

(i). There exist \( s_i \) in \( A \) such that \( D_i s_j = \delta_{ij} \) for all \( i, j \) and

(ii). \( A = k[s_1, \ldots, s_{n+1}] \) a polynomial ring in \( n + 1 \) variables over \( k \).

**Proposition 3.2.** Let \( A \) be a \( \text{VFDO} \) and let \( A^* = k^* \). Let \( D_1, \ldots, D_n \) be commuting locally nilpotent derivations, linearly independent over \( A \). Then \( A^{D_1, \ldots, D_n} = k[f] \) for some \( f \in A \backslash k \), and \( f - \alpha \) is irreducible for each \( \alpha \in \mathbb{C} \).
Proposition 3.3. Let A, D_i, f as in proposition 3.2. Suppose there exist s_1, \ldots, s_n such that D_i(s_i) = 1. Then A = \kappa[s_1, \ldots, s_n, f], a polynomial ring in n + 1 variables.

Proof. This is an easy consequence of the fact that, if D \in LND(A) having an s \in A such that D(s) = 1, then A^D[s] = A.

Define the following abbreviation:

(S1:) Let A be a UFD and let A^* = \kappa^*. Let D_1, \ldots, D_n be commuting locally nilpotent derivations, linearly independent over A.

Lemma 3.4. Assume (S1).
(1) Then there exist p_i \in A such that D_j(p_i) = 0 if j \neq i, and D_i(p_i) \in k[f]\{0\}. Furthermore, k[p_1, \ldots, p_n, f] \subseteq A is algebraic.
(2) Define P_i := \{p_i \in A | D_j(p_i) = 0 if i \neq j and D_i(p_i) \in k[f]\}. then D_i(P_i) = q_i(f)k[f] for some nonzero polynomial q_i. Taking p_i such that D_i(p_i) is of lowest possible degree yields D_i(p_i) \in kq_i(f).

Proof. (1) We assume that all n derivations commute, so D_i(A) \subseteq A^D_i, and therefore D_i sends A := A^{D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n} to itself. Taking some a \in A \setminus \mathbb{C}[f] nonzero, we use the fact that D_i is locally nilpotent to find the lowest m \in \mathbb{N} such that D^m(a) = 0. Now define p_i := D^{m-1}(a) (indeed m \geq 2). The rest is easy.
(2) Take p_i such that D_i(p_i) = q_i(f) \neq 0 has lowest possible degree. Let \tilde{p}_i \in P_i. then D_i(\tilde{p}_i) = h_i(f)q_i(f) + r_i(f) where deg(r_i) < deg(q_i). Now D_i(\tilde{p}_i - h_i(f)p_i) = r_i(f) so r_i = 0. So D_i(\tilde{p}_i) \in q_i(f)\mathbb{C}[f].

Lemma 3.5. Assume (S1). Choose p_i such that D_i(p_i) = q_i(f) as in lemma 3.4, where q_i is of lowest possible degree. The D_i are linearly dependent modulo f - \alpha if and only if q_i(\alpha) = 0 for some i.

Proof. (\Rightarrow): Write "bars" for "modulo f - \alpha". Suppose that 0 \neq D := g_1D_1 + \ldots + g_nD_n satisfies \overline{D} = 0 where \overline{g}_i \in A, and not all \overline{g}_i = 0. Now \overline{g}_iD_i(\overline{p}_i) = \overline{D}(\overline{p}_i) = 0 for each i, so for each i, either \overline{g}_i = 0 or \overline{q}_i(f) = 0 (as f - \alpha is irreducible by proposition 3.2). Since not all \overline{g}_i = 0, at least one \overline{q}_i(f) = 0. Since f - \alpha is irreducible for each \alpha, we not only have \overline{(f - \alpha)q_i}(f), but even \overline{(X - \alpha)q_i}(X), so q_i(\alpha) = 0.

(\Leftarrow): Assume f - \alpha divides q_i(f). We need to show that the D_i mod (f - \alpha) are linearly dependent over A/(f - \alpha). Suppose the \overline{D}_i are linearly independent over \overline{A}. Then we have n commuting, linearly independent LNDs on a domain of transcendence degree n, so we can use proposition 3.1 and conclude that \overline{A}^{\overline{D}_1, \ldots, \overline{D}_n} = k. This means, since \overline{q_i}(f) = 0, that \overline{p}_i \in k. So, p_i = (f - \alpha)a + \lambda where a \in A, \lambda \in k. Now taking a \in A we still have D_j(a) = 0 for all j \neq i, and D_i(a) = q_i(f - \alpha)^{-1} \in \mathbb{C}[f]. This contradicts the assumption that q_i was minimal, so our assumption that the D_i are linearly independent was incorrect.
Now we want to point out the following phenomenon:

**Example 3.6.** Let $D_1 = Z\partial_X + \partial_Y, D_2 = \partial_Y$ on $A = \mathbb{C}[X, Y, Z]$. Now $A^{D_1, D_2} = \mathbb{C}[Z]$. The $D_1, D_2$ are linearly independent modulo $Z - \alpha$ as long as $\alpha \neq 0$. But it is clear that a different set of derivations, namely $E_1 = \partial_X, E_2 = \partial_Y$ commute, their $\mathbb{C}[Z]$-span contains $D_1, D_2$ and the $E_i$ are linearly independent for more fibers $f - \alpha$.

The $E_i$ of the example are an improvement over the $D_i$: all the same properties, but they are linearly independent for more $f - \alpha$. Perhaps for your given space $A$ and derivations $D_i$ it is impossible to find $E_i$ such that the $E_i$ are independent modulo every $f - \alpha$, giving more information on your ring $A$. Before we elaborate on this, let us give a lemma that enables construction of the $E_i$:

**Lemma 3.7.** Assume (S1). Define $M := k(f)D_1 + \ldots + k(f)D_n \cap \text{DER}(A)$. Then $M = k[f]E_1 \oplus \ldots \oplus k[f]E_n$ for some $E_i \in M$, and the $E_i$ have all the properties that the $D_i$ have (i.e. commuting locally nilpotent, linearly independent over $A$). Furthermore, if the $D_i$ are linearly independent modulo $(f - \alpha)$, then the $E_i$ are too (but not necessary the other way around).

**Proof.** Use lemma 3.4 we find preslices $p_i$ and $D(p_i) = q_i(f)$ as stated there.

If $D \in M$ then $D = g_1(f)D_1 + \ldots + g_n(f)D_n$ where $g_i(f) \in k(f)$. Now since $D \in \text{DER}(A)$ we have $D(p_i) \in A$. Also $D(p_i) = g_i(f)D_i(p_i) = g_i(f)q_i(f) \in k(f)$ thus $D(p_i) \in A \cap k(f)$, which equals $k[f]$ since $A^* = k^*$.

Therefore the map $\varphi : M \rightarrow k[f]^n$ sending $D \rightarrow (D(p_1), \ldots, D(p_n))$ is well-defined. If $0 = \varphi(g_1(f)D_1 + \ldots + g_n(f)D_n)$ then $g_i(f)D_i(p_i) = 0$ and therefore $g_i(f) = 0$; thus $\varphi$ is injective.

Since $\varphi$ is an injective map, $M$ must be a free $k[f]$-module. Note that $M$ can only have dimension $n$. Therefore we can find $E_1, \ldots, E_n$ as required.

Any derivation in $M$ is locally nilpotent. Even more, any two derivations of $M$ commute! Next to that, the $E_i$ are clearly independent over $A$. \qed

Note that the $E_i$ can be constructively made, given the injective map $\varphi$ in the above proof. This actually gives an interesting concept. Given the situation (S1), one can improve the derivations $D_i$ (by replacing them by the $E_i$) and then they are linearly independent modulo as much as possible $f - \alpha$. For every such $\alpha$ we have that $A/(f - \alpha)$ is a polynomial ring. The question is if the converse holds:

**Question:** Assume (S1). Additionally, assume $k[f]D_1 + \ldots + k[f]D_n = (k(f)D_1 + \ldots + k(f)D_n) \cap \text{DER}(A)$. Is the set $\{\alpha \in \mathbb{C} | D_1, \ldots, D_n \text{ linearly dependent modulo } (f - \alpha)\}$ equal to the set $\{\alpha \in \mathbb{C} | A/(f - \alpha) \text{ is not a polynomial ring}\}$? (One always has $\geq$.) Or, if this equality does not hold, what type of rings $A$ do have equality?

Note that the requirement “$A$ UFD” is absolutely necessary, as for a simple Danielewski surface $\mathbb{C}[X, Y, Z]/(X^2Y - Z^2)$ we find a LND $2Z\partial_Y + X^2\partial_Z$ which
is nonzero modulo each $X - \alpha$. (But $A/(f - \alpha)$ is not always a domain in this case, even.)

4 Unipotent actions

The authors would like to thank prof. Kraft for pointing out the generalization of theorem 2.1 part 2, which has become the below theorem 4.2.

Proposition 4.1. If $U \times V \rightarrow V$ is an action of a unipotent group $U$ on an affine variety $V$, then for each $u \in U$, the map $u^* : O(V) \rightarrow O(V)$ is an exponent of a locally nilpotent derivation.

For the proof we can refer to proposition 2.1.3 in [2], or ask the reader to verify that $u^* - Id$ is a locally nilpotent endomorphism, and that thus “log($u^*$)” can be defined, and is a derivation.

This proposition has some immediate consequences, like that the invariants of a unipotent group action are the intersection of kernels of locally nilpotent derivations. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, so the invariants of a unipotent group is factorially closed.

In the below theorem, $\mathbb{C}$ is a field of characteristic zero, which is algebraically closed.

Theorem 4.2. Let $U$ be a unipotent algebraic group of dimension $n$, acting freely on $X$, a factorial variety of dimension $n + 1$ satisfying $O(X)^* = \mathbb{C}^*$. Then $X$ is $U$-isomorphic to $U \times \mathbb{C}$. In particular, $X \simeq \mathbb{C}^{n+1}$.

Proof. The fact that $U$ acts free means that each $x \in X$ has trivial stabilizer: $U_x = \{u \in U; ux = x\} = \{id\}$. So, each orbit $Ux$ is of dimension $n$. This means that $X//U$ is of dimension 1. Also, as remarked above, $X^U$ is factorial. But then it is also normal, and smooth. So $X//U$ is a smooth, rational, affine curve, in other words, an open subvariety of $\mathbb{C}$. Now suppose that $X//U \not\simeq \mathbb{C}$, so $X//U = \mathbb{C} - \{p_1, \ldots, p_n\}$, then $O(X)^U = O(\mathbb{C} - \{p_1, \ldots, p_n\}) = \mathbb{C}[t, (t - p_1)^{-1}, \ldots, (t - p_n)^{-1}]$. This means that $O(X)$ contains invertible elements $(t - p_1)^{-1}$, giving a contradiction with the assumption $O(X)^* = \mathbb{C}^*$. Hence, $X//U \simeq \mathbb{C}$, so $O(X)^U = O(X//U) = O(\mathbb{C}) \cong \mathbb{C}[f]$ for some $f$. Now every $f - \lambda$ ($\lambda \in \mathbb{C}$) is irreducible, as otherwise any irreducible factor of $f - \lambda$ would be in $O(X)^U$ too.

Now consider the map $f : X \rightarrow \mathbb{C}$. This is in fact the map $X \rightarrow X//U$ (as it corresponds to the map $O(X) \leftarrow O(X)^U = \mathbb{C}[f]$) and thus surjective. Also note that the fibers $f^{-1}(\lambda)$ are invariant under $U$: they correspond to the function space $O(X)/(f - \lambda)$. By assumption, $U$ acts free on each fiber of $X \rightarrow X//U$, which means exactly that $U$ acts free on $f^{-1}(\lambda)$ for each $\lambda$. Let $x \in f^{-1}(\lambda)$. Then $Ux$ is of dimension $n$ (it is just a copy of $U$). Also, each orbit of a unipotent group is closed (see Satz 4 from [3]), and therefore the inclusion $Ux \subseteq f^{-1}(\lambda)$ is an equality. So orbits of $U$ are the same as fibers of $f$, i.e. we have an orbit fibration (or $U$-fibration).
X_{\text{sing}} is closed and $U$-stable, hence a union of $U$-orbits, and so $\text{codim } X_{\text{sing}} = 1$ or $X_{\text{sing}}$ is empty. But $X$ is factorial, so in particular normal, which implies $\text{codim}(X_{\text{sing}}) \geq 2$. So $X_{\text{sing}}$ is empty, in other words: $X$ is smooth.

Now we claim that $f : X \to \mathbb{C}$ is smooth. To see this, first note that $\mathcal{O}(f^{-1}(\lambda)) = \mathcal{O}(X)/(f - \lambda)$ is reduced as $f - \lambda$ is irreducible, as seen before. And, as we already implied, the set of functions vanishing on $f^{-1}(\lambda)$ is the ideal $(f - \lambda)$. Now consider the tangent map $df_x : T_x X \to T_0 \mathbb{C} = \mathbb{C}$ where $x \in f^{-1}(\lambda)$. Using “Satz 2”, page 269 in [3] we see that, $\ker df \supseteq T_x f^{-1}(\lambda)$, but since $f^{-1}(\lambda)$ is reduced, we even have equality $\ker df = T_x f^{-1}(\lambda)$. Now remember that the fiber $f^{-1}(\lambda)$ is an orbit, hence smooth (as any orbit is smooth!). This implies $\dim T_x f^{-1}(\lambda) = n$ and thus $\dim \ker df = n$. Since $\dim T_x X = n + 1$ we have $\dim \text{Im}(df_x) = 1$, hence $df_x$ is surjective. A morphism between smooth varieties is smooth if and only if the differential is surjective. So we have shown that $f$ is smooth.

So: $f : X \to \mathbb{C}$ is surjective, and smooth. Let $K := \ker df|_x \subset T_x X$. Take some linear subspace $C$ such that $K \oplus C = T_x X$. Note that $C$ has dimension 1. Seeing $X$ as a subset of some $\mathbb{C}^N$, we can find hyperplanes $H$ that contains $C$. We even want $H \cap T_x X = C$, so this means that $H \oplus T_x \subseteq \mathbb{C}^N$, so let us take a hyperplane $H$ of codimension $n$ such that $H \cap T_x X = C$. Now let $Z$ be an irreducible component of $H \cap X$ which contains $x$. Also, $\dim_x H \cap X \geq 1$, thus $\dim_x Z = 1$ and $Z$ is smooth at $x$. Now $Z$ and $C$ are smooth, and the differential of $f|_Z : Z \to \mathbb{C}$ is an isomorphism at $x$ (implying surjective), thus we have that $f|_Z$ is smooth at $x$. Replacing $Z$, if necessary, by a (special) open subset $Z' \subset Z$, we have $f|_Z$ is étale.

Now look at the following diagram

$$
\begin{array}{ccc}
Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow f & & \downarrow f \\
Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
$$

where $Z \times_{\mathbb{C}} X = \{(x,z) \in X \times Z \mid f(x) = f|_Z(z)\}$ is the (schematic) fiber product. Since $f$ is smooth, the same holds for $f$ and so $Z \times_{\mathbb{C}} X$ is smooth. Moreover, $U$ acts on $Z \times_{\mathbb{C}} X$ by $u(z,x) = (z,ux)$ and $p(u(x,z)) = ux$ ($p$ is $U$-equivariant) and $f(u(x,z)) = z = f(x,z)$ ($f$ is $U$-invariant). The fibers of $\tilde{f}$ are $\tilde{f}^{-1}(\alpha) = \{(x,z) \mid \tilde{f}(x) = \tilde{f}|_Z(z)\} = \{x \mid f(x) = \alpha\} = f^{-1}(\alpha)$ where $\alpha = f_Z(z)$. Now $f$ has a section $\sigma : Z \to Z \times_{\mathbb{C}} X$ given by $z \mapsto (z,z)$, i.e. $\tilde{f} \circ \sigma = \text{id}_Z$. Therefore, we can extend the diagram above

$$
\begin{array}{ccc}
U \times Z & \xrightarrow{q} & Z \times_{\mathbb{C}} X & \xrightarrow{p} & X \\
\downarrow \text{pr}_Z & & \downarrow \tilde{f} & & \downarrow f \\
Z & \xrightarrow{f|_Z} & \mathbb{C}
\end{array}
$$

where $q : U \times Z \to Z \times_{\mathbb{C}} X$ is given by $(u,z) \mapsto (z,uz)$. By construction, $q$ is bijective, hence an isomorphism, since the second variety is normal (see [4].
proposition 5.7). Note that the role of $x$ was arbitrary: for each $x$ we find a neighborhood $Z$ where $Z \times_{\mathbb{C}} X = Z \times_{\mathbb{C}} U$. This last statement exactly means that the map $f: X \to \mathbb{C}$ is a locally trivial principal $U$-bundle with respect to the étale topology: for every point $\lambda \in \mathbb{C}$ there is an étale map $Z \to \mathbb{C}$ such that $\lambda$ is in the image and the fiber product $Z \times_{\mathbb{C}} X$ is a trivial $U$-bundle, i.e. isomorphic to $U \times Z \xrightarrow{pr_2} Z$.

In the paper [5] we now find a result that tells us that a principal $G$-bundle where $G$ is a unipotent group is trivial over any affine variety, and then we are done.

\[ \square \]

References


[6] L. Makar-Limanov, On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$ -like threefold which is not $\mathbb{C}^3$, Israel J. Math., 96(1996), 419-429.
