A REDUCTION OF THE JACOBIAN CONJECTURE TO THE SYMMETRIC CASE

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Abstract. The main result of this paper asserts that it suffices to prove the Jacobian Conjecture for all polynomial maps of the form $x + H$, where $H$ is homogeneous (of degree 3) and $JH$ is nilpotent and symmetric. Also a 6-dimensional counterexample is given to a dependence problem posed by de Bondt and van den Essen (2003).

Introduction

Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e. each $F_i$ is a polynomial in $n$ variables over $\mathbb{C}$, and denote by $JF := \left( \frac{\partial F_i}{\partial x_j} \right)_{1 \leq i, j \leq n}$ the Jacobian matrix of $F$. Then the Jacobian Conjecture asserts that if $\det JF \in \mathbb{C}^*$, then $F$ is invertible. It was shown in the classical papers [1] and [13] by Bass-Connell-Wright and Yagzhev, respectively, that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where $H$ is homogeneous (of degree 3) and $JH$ is nilpotent.

In [12] and [7] the cubic homogeneous cases in dimension 3 (resp. 4) were treated by Wright (resp. Hubbers). Recently, in [6] Washburn and the second author treated one more special case, namely they showed that if $n \leq 4$, then the Jacobian Conjecture holds for all polynomial maps of the form $F = x + H$, where $JH$ is homogeneous, nilpotent and symmetric.

At first glance the condition that $JH$ is symmetric seems rather special. However the main result of this paper, Theorem 1.1, asserts that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where $JH$ is homogeneous, nilpotent and symmetric!

The technique to obtain this result is used in section 2 to give a negative answer in dimension 6 to a dependence problem posed in [2] (which, if true, would have implied the Jacobian Conjecture). We refer to section 2 for more details. Finally we would like to mention that in [3] the authors have obtained the following extensions of the results from [6]: the Jacobian Conjecture holds for all $F$ of the form $x + H$, where $JH$ is nilpotent and symmetric in the case $n \leq 4$ ($H$ need not be homogeneous) and in the case $n = 5$ when $H$ is homogeneous.

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1. Reduction to Symmetric Matrices

Throughout this paper we use the following notation:
\[ \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n] \] is the polynomial ring in \( n \) variables over \( \mathbb{C} \) and \( H := (H_1, \ldots, H_n) : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map. Its Jacobian matrix is denoted by \( JH \). It follows from the Poincaré lemma (see for example [5], 1.3.53) that \( JH \) is symmetric iff there exists \( f \in \mathbb{C}[x] \) such that \( H = (f_{x_1}, \ldots, f_{x_n}) \) or equivalently such that \( JH = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \), the Hessian matrix of \( f \). We denote this matrix by \( h(f) \).

Observe that
\[ h(f) = J(f_{x_1}, \ldots, f_{x_n}). \]

For \( A \in M_n(\mathbb{C}) \) we put \( f \circ A := f(Ax) \). It is well known that
\[ h(f \circ A) = A^t h(f) A. \]

Now we introduce \( n \) new variables \( y_1, \ldots, y_n \) and to \( H \) as above we associate the polynomial \( f_H \in \mathbb{C}[x, y] \) defined by
\[ f_H := (-i)H_1(x_1 + iy_1, \ldots, x_n + iy_n)y_1 + \ldots + (-i)H_n(x_1 + iy_1, \ldots, x_n + iy_n)y_n. \]

So if \( S \) is the (invertible) linear map given by
\[ S := (x_1 - iy_1, \ldots, x_n - iy_n, y_1, \ldots, y_n), \]
then \( g_H := f_H \circ S = (-i)H_1(x_1 + y_1) + \ldots + (-i)H_n(x_n) y_n. \)

One readily verifies that \( h(g_H) \) is of the form
\[ h(g_H) = \begin{pmatrix} * & (-i)(JH)^t \\ (-i)JH & 0 \end{pmatrix}. \]

In order to formulate the main result of this paper we introduce

**Hessian Conjecture HC(n).** Let \( f \in \mathbb{C}[x] \). If \( h(f) \) is nilpotent, then \( F := (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}) \) is invertible.

It follows from (1) that if the \( n \)-dimensional Jacobian Conjecture is true, then HC(n) is true as well. The surprising point is now

**Theorem 1.1.** The Jacobian Conjecture is equivalent to the Hessian Conjecture. More precisely, if HC(\( 2n \)) holds, then \( x + H \) is invertible for every \( H : \mathbb{C}^n \to \mathbb{C}^n \) with \( JH \) nilpotent.

The proof of this result is based on the following lemma.

**Lemma 1.2.** Let \( H = (H_1, \ldots, H_n) : \mathbb{C}^n \to \mathbb{C}^n \) and let \( f_H \in \mathbb{C}[x, y] \) be as defined in (3). Then \( JH \) is nilpotent iff \( h(f_H) \) is nilpotent.

**Proof.** Introduce an extra variable \( z \) and write \( f \) (resp. \( g \)) instead of \( f_H \) (resp. \( g_H \)). Then \( h(f) \) is nilpotent iff \( \det(zI_{2n} - h(f)) = z^{2n} \). Put \( q := (1/2) \sum_{j=1}^n (x_j^2 + y_j^2). \) Then \( h(qz) = zI_{2n} \), so
\[ h(qz - f) = zI_{2n} - h(f). \]

Since \( \det S = 1 \), it follows from (2) and (5) that
\[ \det h(qz \circ S - g) = \det h(qz - f)|_{S(x,y)}. \]
Since \( q \circ S = \frac{1}{2} \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} ix_j y_j \), it follows from (4) that
\[
h(zq \circ S - g) = \begin{pmatrix} * & -izI_n + iJH & 0 \\
-izI_n & 0 & 
\end{pmatrix},
\]
Consequently
\[
(7) \quad \det h(zq \circ S - g) = \det(zI_n - JH) \det(zI_n - (JH)^t).
\]
So by (6) and (7) we obtain
\[
\det(zI_n - h(f))_{|S(x,y)} = \det(zI_n - JH) \det(zI_n - (JH)^t).
\]
Hence \( h(f) \) is nilpotent iff \( \det(zI_n - h(f)) = z^n \) iff \( \det(zI_n - JH) = z^n \) iff \( JH \) is nilpotent.

**Proof of Theorem 1.1.** Let \( H = (H_1, \ldots, H_n) \) be such that \( JH \) is nilpotent and let \( f_H \) be as in (3). Then by Lemma 1.2 \( h(f) \) is nilpotent. So the assumption \( HC(2n) \implies JH \) is nilpotent and symmetric and \( H \) homogeneous of degree 4 (or equivalently for all \( n \geq 2 \) and all \( F \) of the form \( F = x + H \) with \( JH \) nilpotent and symmetric and \( H \) homogeneous of degree 3).

**Corollary 1.3.** It suffices to prove the Jacobian Conjecture for all \( n \geq 2 \) and all \( F \) of the form \( F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}) \), where \( h(f) \) is nilpotent and \( f \) is homogeneous of degree 4 (or equivalently for all \( n \geq 2 \) and all \( F \) of the form \( F = x + H \) with \( JH \) nilpotent and symmetric and \( H \) homogeneous of degree 3).

**Proof.** Follows immediately from Theorem 1.1 and Corollary 2.2 of [1].

## 2. Dependence problems

In the search for the Jacobian Conjecture the following problems were formulated by several authors (see [8], Conjecture 1, p. 80, [10], Conjecture B, p. 135, [11], Conjecture 11.3, [3] and [5], 7.1.7)

**Homogeneous) Dependence Problem (H)DP(n).** Let \( H := (H_1, \ldots, H_n) \) with \( H(0) = 0 \) be (homogeneous of degree \( d \geq 1 \) such that \( JH \) is nilpotent. Are the \( H_i \) linearly dependent over \( \mathbb{C} \)?

One easily verifies that the linear dependence of the \( H_i \) is equivalent to the linear dependence of the rows of \( JH \) over \( \mathbb{C} \). It is shown in [5], Theorem 7.1.7, that \( DP(2) \) has an affirmative answer and that for each \( n \geq 3 \) there are counterexamples. The easiest such example is the following:

\[
(8) \quad H_1 = x_2 - x_1^2, \quad H_2 = x_3 + 2x_1(x_2 - x_1^2), \quad H_3 = -(x_2 - x_1^2)^2.
\]

The homogeneous dependence problem is still open; but in the cases \( n = 3, d = 3 \) and \( n = 4, d = 3 \), affirmative answers were obtained by Wright in [12] and Hubbers.
in [7]. Recently in [2] the corresponding dependence problems were formulated for Hessian matrices, i.e.

(Homogeneous) Symmetric Dependence Problem (H)SDP(n). Let $H$ with $H(0) = 0$ be (homogeneous of degree $d \geq 1$) such that $JH$ is nilpotent and symmetric. Are the $H_i$ linearly dependent over $\mathbb{C}$?

The importance of these problems becomes clear if one combines Theorem 1.1 with the following result of [2].

**Theorem 2.1** ([2] Theorem 2.1). i) If SDP$(p)$ has an affirmative answer for all $p \leq n$, then $HC(n)$ holds.

ii) If SDP$(p)$ has an affirmative answer for all $p \leq n - 2$ and $HSDP(p)$ for $p = n - 1$ and $p = n$, then $HC(n)$ holds for all homogeneous $f \in \mathbb{C}[x]$.

The aim of this section is to relate the dependence problems stated before with the symmetric dependence problems. As a consequence we obtain a negative answer to SDP(6). More precisely

**Example.** Let $H = (H_1, H_2, H_3)$ be as in (8). Then $JH$ is nilpotent and $H_1, H_2, H_3$ are linearly independent over $\mathbb{C}$. Now let $f_H$ be as in (3). Then it follows from the next result and the fact that DP(2) holds, that $h(f_H)$ is a counterexample to SDP(6).

**Proposition 2.2.** If $n$ is minimal such that (H)DP$(n)$ does not hold, then (H)SDP$(2n)$ does not hold either.

**Proof.** i) Suppose (H)DP$(n)$ does not hold and $n$ is minimal with this property. Then there exists $H: \mathbb{C}^n \to \mathbb{C}^n$ with $H(0) = 0$ such that $JH$ is nilpotent and the rows of $JH$ are independent over $\mathbb{C}$.

**Claim.** The columns of $JH$ are also independent over $\mathbb{C}$.

Namely, if the columns of $JH$ are dependent over $\mathbb{C}$, then there exists $0 \neq v \in \mathbb{C}^n$ with $JH \cdot v = 0$. Let $T \in GL_n(\mathbb{C})$ be such that its last column equals $v$. Then the last column of $JH \cdot T$ equals zero. So if we put $H := T^{-1} \circ H \circ T$, then $\tilde{J}H = T^{-1}JH(Tx)T$ is nilpotent and also its last column equals zero. In particular $H_1, \ldots, H_{n-1} \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. Finally put $H_* := (H_1, \ldots, H_{n-1})$. Since the last column of $\tilde{J}H$ is zero, it follows readily that $JH_*$ is nilpotent and that the rows of $JH_*$ are linearly independent over $\mathbb{C}$ (since the rows of $\tilde{J}H$ are because those of $JH$ are by hypothesis). So $H_*$ contradicts the minimality of $n$.

ii) Therefore, the columns of $JH$ are independent over $\mathbb{C}$. Let $g_H$ and $f_H$ be as above. Then $h(g_H)$ has the form (4).

**Claim.** The rows $R_j$ of $h(g_H)$ are independent over $\mathbb{C}$: namely suppose that $\sum_{j=1}^{2n} c_j R_j = 0$ for some $c_j \in \mathbb{C}$. Since the rows of $(-i)(JH)^t$ are independent over $\mathbb{C}$ (since the columns of $JH$ are by i)), the zero matrix in the right corner of $h(g_H)$ in (4) implies that $c_1 = \ldots = c_n = 0$. So $\sum_{j=n+1}^{2n} c_j R_j = 0$. However the rows of $(-i)JH$ are also independent over $\mathbb{C}$ (by hypothesis), so also $c_j = 0$ if $j > n$, which proves the claim.

iii) Finally, since $f_H = g_H \circ T$ ($T := S^{-1}$) it follows from (2) that $h(f_H) = T^t h(g_H)T(x,y)T$. Therefore, the rows of $h(f_H)$ are also independent over $\mathbb{C}$, which concludes the proof. $\square$
Almost three months after this paper was submitted, the authors were notified by David Wright that the paper [9] by Guowu Meng had appeared on the internet, in which he obtained a result similar to ours. He also formulates a Hessian Conjecture and shows that the Jacobian Conjecture is equivalent to his Hessian Conjecture. Meng’s Hessian Conjecture states that the Jacobian Conjecture holds for all gradient maps \( \nabla f := (f_{x_1}, \ldots, f_{x_n}) \). The difference between our Hessian Conjecture and the one formulated by Meng is that he considers all polynomial maps of the form \( \nabla f \) with \( \det(h(f)) \in \mathbb{C}^* \), where we only need to consider all polynomial maps of the form \( x + \nabla f \), with \( h(f) \) nilpotent. So our reduction is more refined in the sense that it preserves the nilpotency as formulated in the classical reduction theorems of [1] and [3].

**Added in proof**

In a recent paper the authors gave an affirmative answer to HDP(3). Also, the first author found counterexamples to HDP(n) for all \( n \geq 5 \).

**References**