A REDUCTION OF THE JACOBIAN CONJECTURE
TO THE SYMMETRIC CASE

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(Communicated by Bernd Ulrich)

Abstract. The main result of this paper asserts that it suffices to prove the
Jacobian Conjecture for all polynomial maps of the form $x + H$, where $H$ is
homogeneous (of degree 3) and $JH$ is nilpotent and symmetric. Also a 6-
dimensional counterexample is given to a dependence problem posed by de
Bondt and van den Essen (2003).

Introduction

Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e. each $F_i$ is a polyno-
mial in $n$ variables over $\mathbb{C}$, and denote by $JF := (\frac{\partial F_i}{\partial x_j})_{1 \leq i, j \leq n}$ the Jacobian matrix
of $F$. Then the Jacobian Conjecture asserts that if $\det JF \in \mathbb{C}^*$, then $F$ is in-
vertible. It was shown in the classical papers [1] and [13] by Bass-Connell-Wright
and Yagzhev, respectively, that it suffices to prove the Jacobian Conjecture for all
$n \geq 2$ and all polynomial maps of the form $F = x + H$, where $H$ is homogeneous
(of degree 3) and $JH$ is nilpotent.

In [12] and [7] the cubic homogeneous cases in dimension 3 (resp. 4) were treated
by Wright (resp. Hubbers).

Recently, in [6] Washburn and the second author treated one more special case,
namely they showed that if $n \leq 4$, then the Jacobian Conjecture holds for all
polynomial maps of the form $F = x + H$, where $JH$ is homogeneous, nilpotent and
symmetric.

At first glance the condition that $JH$ is symmetric seems rather special. However
the main result of this paper, Theorem 1.1, asserts that it suffices to prove the
Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$,
where $JH$ is homogeneous, nilpotent and symmetric!

The technique to obtain this result is used in section 2 to give a negative answer
in dimension 6 to a dependence problem posed in [2] (which, if true, would have
implied the Jacobian Conjecture). We refer to section 2 for more details. Finally
we would like to mention that in [3] the authors have obtained the following exten-
sions of the results from [6]: the Jacobian Conjecture holds for all $F$ of the form
$x + H$, where $JH$ is nilpotent and symmetric in the case $n \leq 4$ ($H$ need not be
homogeneous) and in the case $n = 5$ when $H$ is homogeneous.

Received by the editors June 30, 2003.
2000 Mathematics Subject Classification. Primary 14R15, 14R10.
Key words and phrases. Jacobian Conjecture, Hessian Conjecture, dependence problems.

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1. Reduction to symmetric matrices

Throughout this paper we use the following notation:
\( \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n] \) is the polynomial ring in \( n \) variables over \( \mathbb{C} \) and \( H := (H_1, \ldots, H_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is a polynomial map. Its Jacobian matrix is denoted by \( JH \). It follows from the Poincaré lemma (see for example [5], 1.3.53) that \( JH \) is symmetric iff there exists \( f \in \mathbb{C}[x] \) such that \( H = (f_{x_1}, \ldots, f_{x_n}) \) or equivalently such that \( JH = (\frac{\partial^2 f}{\partial x_i \partial x_j}) \), the Hessian matrix of \( f \). We denote this matrix by \( \partial^2 f \).

Observe that
\[
\partial^2 f = JH.
\]

For \( A \in M_n(\mathbb{C}) \) we put \( f \circ A := f(Ax) \). It is well known that
\[
\partial^2 (f \circ A) = A \partial^2 f |_{Ax} A.
\]

Now we introduce \( n \) new variables \( y_1, \ldots, y_n \) and to \( H \) as above we associate the polynomial \( f_H \in \mathbb{C}[x,y] \) defined by
\[
f_H := (-i)H_1(x_1 + iy_1, \ldots, x_n + iy_n)y_1 + \cdots + (-i)H_n(x_1 + iy_1, \ldots, x_n + iy_n)y_n.
\]

So if \( S \) is the (invertible) linear map given by
\[
S := (x_1 - iy_1, \ldots, x_n - iy_n, y_1, \ldots, y_n),
\]
then \( g_H := f_H \circ S = (-i)H_1(x_1 + iy_1, \ldots, x_n + iy_n)y_1 + \cdots + (-i)H_n(x_1 + iy_1, \ldots, x_n + iy_n)y_n \).

One readily verifies that \( h(g_H) \) is of the form
\[
h(g_H) = \begin{pmatrix} * & (-i)(JH)^t \\ (-i)JH & 0 \end{pmatrix}.
\]

In order to formulate the main result of this paper we introduce

**Hessian Conjecture HC(n).** Let \( f \in \mathbb{C}[x] \). If \( h(f) \) is nilpotent, then \( F := (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}) \) is invertible.

It follows from (1) that if the \( n \)-dimensional Jacobian Conjecture is true, then \( HC(n) \) is true as well. The surprising point is now

**Theorem 1.1.** The Jacobian Conjecture is equivalent to the Hessian Conjecture. More precisely, if \( HC(2n) \) holds, then \( x + H \) is invertible for every \( H : \mathbb{C}^n \rightarrow \mathbb{C}^n \) with \( JH \) nilpotent.

The proof of this result is based on the following lemma.

**Lemma 1.2.** Let \( H = (H_1, \ldots, H_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n \) and let \( f_H \in \mathbb{C}[x,y] \) be as defined in (3). Then \( JH \) is nilpotent iff \( h(f_H) \) is nilpotent.

**Proof.** Introduce an extra variable \( z \) and write \( f \) (resp. \( g \)) instead of \( f_H \) (resp. \( g_H \)). Then \( h(f) \) is nilpotent iff \( \det(zI_{2n} - h(f)) = z^{2n} \). Put \( q := (1/2) \sum_{j=1}^n (x_j^2 + y_j^2) \). Then \( h(qz) = zI_{2n} \), so
\[
h(qz - f) = zI_{2n} - h(f).
\]
Since \( \det S = 1 \), it follows from (2) and (5) that
\[
\det h(qz \circ S - g) = \det h(qz - f)_{S(x,y)}.
\]
Since \( q \circ S = \frac{1}{2} \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} i x_j y_j \), it follows from (4) that
\[
h(zq \circ S - g) = \begin{pmatrix}
-izI_n + i(JH)^t \\
0
\end{pmatrix},
\]
Consequently
\[
(7) \quad \det h(zq \circ S - g) = \det(zI_n - JH) \det(zI_n - (JH)^t).
\]
So by (6) and (7) we obtain
\[
\det(zI_{2n} - h(f))|_{S(x,y)} = \det(zI_n - JH) \det(zI_n - (JH)^t).
\]
Hence \( h(f) \) is nilpotent iff \( \det(zI_{2n} - h(f)) = z^2n \) iff \( \det(zI_n - JH) = z^n \) iff \( JH \) is nilpotent.

Proof of Theorem 1.1. Let \( H = (H_1, \ldots, H_n) \) be such that \( JH \) is nilpotent and let \( f_H \) be as in (3). Then by Lemma 1.2 \( h(f) \) is nilpotent. So the assumption \( HC(2n) \) implies that \( F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}, y_1 + f_{y_1}, \ldots, y_n + f_{y_n}) \) is invertible. Consequently \( F \circ S \) is invertible. An easy calculation shows that
\[
F \circ S = \begin{pmatrix}
x_1 - iy_1 - i \sum_j H_{jx_1}(x)y_j, \ldots, x_n - iy_n - i \sum_j H_{jx_n}(x)y_j, \\
y_1 + \sum_j H_{jx_1}(x)y_j - iH_1, \ldots, y_n + \sum_j H_{jx_n}(x)y_j - iH_n
\end{pmatrix}.
\]
Hence \( S^{-1} \circ F \circ S = (x_1 + H_1(x), \ldots, x_n + H_n(x), *, \ldots, *) \) is invertible, which in turn implies that \( x + H \) is invertible.

Corollary 1.3. It suffices to prove the Jacobian Conjecture for all \( n \geq 2 \) and all \( F \) of the form \( F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}) \), where \( h(f) \) is nilpotent and \( f \) is homogeneous of degree 4 (or equivalently for all \( n \geq 2 \) and all \( F \) of the form \( F = x + H \) with \( JH \) nilpotent and symmetric and \( H \) homogeneous of degree 3).

Proof. Follows immediately from Theorem 1.1 and Corollary 2.2 of \[1\].

2. Dependence problems

In the search for the Jacobian Conjecture the following problems were formulated by several authors (see \[8\], Conjecture 1, p. 80, \[10\], Conjecture B, p. 135, \[11\], Conjecture 11.3, \[4\] and \[5\], 7.1.7)

(Homogeneous) Dependence Problem (H)DP(n). Let \( H := (H_1, \ldots, H_n) \) with \( H(0) = 0 \) be (homogeneous of degree \( d \geq 1 \)) such that \( JH \) is nilpotent. Are the \( H_i \) linearly dependent over \( \mathbb{C} \)?

One easily verifies that the linear dependence of the \( H_i \) is equivalent to the linear dependence of the rows of \( JH \) over \( \mathbb{C} \). It is shown in \[11\], Theorem 7.1.7, that DP(2) has an affirmative answer and that for each \( n \geq 3 \) there are counterexamples. The easiest such example is the following:
\[
(8) \quad H_1 = x_2 - x_1^2, \quad H_2 = x_3 + 2x_1(x_2 - x_1^2), \quad H_3 = -(x_2 - x_1^2)^2.
\]
The homogeneous dependence problem is still open; but in the cases \( n = 3, d = 3 \) and \( n = 4, d = 3 \), affirmative answers were obtained by Wright in \[12\] and Hubbers...
in [7]. Recently in [2] the corresponding dependence problems were formulated for
Hessian matrices, i.e.

(Homogeneous) Symmetric Dependence Problem (H)SDP(n). Let $H$ with
$H(0) = 0$ be (homogeneous of degree $d \geq 1$) such that $JH$ is nilpotent and sym-
metric. Are the $H_i$ linearly dependent over $\mathbb{C}$?

The importance of these problems becomes clear if one combines Theorem 1.1
with the following result of [2].

Theorem 2.1 ([2 Theorem 2.1]). i) If SDP(p) has an affirmative answer for all
$p \leq n$, then HC(n) holds.

ii) If SDP(p) has an affirmative answer for all $p \leq n - 2$ and HSDP(p) for
$p = n - 1$ and $p = n$, then HC(n) holds for all homogeneous $f \in \mathbb{C}[x]$.

The aim of this section is to relate the dependence problems stated before with
the symmetric dependence problems. As a consequence we obtain a negative answer
to SDP(6). More precisely

Example. Let $H = (H_1, H_2, H_3)$ be as in (8). Then $JH$ is nilpotent and $H_1, H_2, H_3$
are linearly independent over $\mathbb{C}$. Now let $f_H$ be as in (3). Then it follows from
the next result and the fact that DP(2) holds, that $h(f_H)$ is a counterexample to
SDP(6).

Proposition 2.2. If $n$ is minimal such that (H)DP(n) does not hold, then
(H)SDP(2n) does not hold either.

Proof. i) Suppose (H)DP(n) does not hold and $n$ is minimal with this property.
Then there exists $H : \mathbb{C}^n \to \mathbb{C}^n$ with $H(0) = 0$ such that $JH$ is nilpotent and
the rows of $JH$ are independent over $\mathbb{C}$.

Claim. The columns of $JH$ are also independent over $\mathbb{C}$.

Namely, if the columns of $JH$ are dependent over $\mathbb{C}$, then there exists $0 \neq v \in \mathbb{C}^n$
with $JH \cdot v = 0$. Let $T \in Gl_n(\mathbb{C})$ be such that its last column equals $v$. Then
the last column of $JH \cdot T$ equals zero. So if we put $\tilde{H} := T^{-1} \circ H \circ T$, then
$\tilde{JH} = T^{-1} JH(Tx)T$ is nilpotent and also its last column equals zero. In particular
$H_1, \ldots , H_{n-1} \in \mathbb{C}[x_1, \ldots , x_{n-1}]$. Finally put $H_* := (H_1, \ldots , H_{n-1})$. Since the last
column of $JH$ is zero, it follows readily that $JH_*$ is nilpotent and that the rows
of $JH_*$ are linearly independent over $\mathbb{C}$ (since the rows of $JH$ are those of
$JH_*$ are by hypothesis). So $H_*$ contradicts the minimality of $n$.

ii) Therefore, the columns of $JH$ are independent over $\mathbb{C}$. Let $g_H$ and $f_H$ be as
above. Then $h(g_H)$ has the form (4).

Claim. The rows $R_j$ of $h(g_H)$ are independent over $\mathbb{C}$: namely suppose that
$\sum_{j=1}^{2n} c_j R_j = 0$ for some $c_j \in \mathbb{C}$. Since the rows of $(-i)(JH)^t$ are independent over $\mathbb{C}$
(since the columns of $JH$ are by i), the zero matrix in the right corner of $h(g_H)$
in (4) implies that $c_1 = \ldots = c_n = 0$. So $\sum_{j=n+1}^{2n} c_j R_j = 0$. However the rows of
$(-i)JH$ are also independent over $\mathbb{C}$ (by hypothesis), so also $c_j = 0$ if $j > n$, which
proves the claim.

iii) Finally, since $f_H = g_H \circ T$ ($T := S^{-1}$) it follows from (2) that $h(f_H) =
T^t h(g_H) T (x,y) T$. Therefore, the rows of $h(f_H)$ are also independent over $\mathbb{C}$, which
concludes the proof. \[\square\]
3. Final remarks

Almost three months after this paper was submitted, the authors were notified by David Wright that the paper [3] by Guowu Meng had appeared on the internet, in which he obtained a result similar to ours. He also formulates a Hessian Conjecture and shows that the Jacobian Conjecture is equivalent to his Hessian Conjecture. Meng’s Hessian Conjecture states that the Jacobian Conjecture holds for all gradient maps $\nabla f := (f_{x_1}, \ldots, f_{x_n})$. The difference between our Hessian Conjecture and the one formulated by Meng is that he considers all polynomial maps of the form $\nabla f$ with $\det h(f) \in \mathbb{C}^*$, whereas we only need to consider all polynomial maps of the form $x + \nabla f$, with $h(f)$ nilpotent. So our reduction is more refined in the sense that it preserves the nilpotency as formulated in the classical reduction theorems of [1] and [3].

Added in proof

In a recent paper the authors gave an affirmative answer to HDP(3). Also, the first author found counterexamples to HDP(n) for all $n \geq 5$.

References


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