Factorial moments, cumulants and correlation integrals in $\pi^+p$ and $K^+p$ interactions at 250 GeV/c

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Abstract. A selected sample of 59200 $\pi^+p$ and $K^+p$ non-single-diffractive interactions at $\sqrt{s} = 22$ GeV is used to investigate one, two- and three-dimensional factorial moments, factorial cumulant moments, as well as correlation integrals. The rise of factorial moments and cumulants with decreasing phase-space volume is stronger when evaluated in three than in lower dimensions. Ratios of slopes are easier to obtain than the slopes themselves. Contrary to earlier findings, they turn out to depend on the dimension. The order dependence of the averaged ratios is better described by a Lévy stable law solution with $\alpha = 1.6$ than by Gaussian approximation of the $\alpha$-model ($\alpha = 2$) or a second order phase transition ($\alpha = 0$), but values $\alpha > 2$ inconsistent with Lévy-type fluctuations are reached in a three-dimensional analysis. The multi-particle contributions to the factorial moments are calculated by means of factorial cumulant moments. A particular improvement of the method is that of correlation (or density) integrals. It leads to the conclusion that Bose-Einstein interference plays an important role in the intermittency effect, but indication is found for an interpretation alternative to the conventional view of Bose-Einstein correlations.

1 Introduction

Recent experimental effort has established the existence of the empirical phenomenon of "intermittency" in multi-particle production. Basically, this phenomenon is defined [1] as a power-law rise for small intervals $\delta$ of phase space,

$$F_q(\delta) \sim \delta^{-\alpha_q},$$

of bin-averaged normalized factorial moments $F_q$ of order $q$, defined in its "vertical" form as

$$F_q^V(\delta) = \frac{1}{M^d} \sum_{m=1}^{M^d} \left\langle n_m^q \right\rangle,$$

and its "horizontal" form as

$$F_q^H(\delta) = \left\langle \left( \frac{n_m^q}{M^d} \right) \right\rangle.$$

For the computation of these moments, a binning of an original $d$-dimensional region $A$ into $M^d$ intervals of size $\delta$ is introduced and the number $n_m$ of particles in bin $m$ is counted. The region $A$ and its binning can be chosen in any phase-space variable, such as rapidity $y$, azimuthal angle $\phi$, transverse momentum $p_T$, or in a two- or three-dimensional combination of these. The superscript $[q]$ identifies the factorial of order $q$, i.e. $n_m^q = n_m(n_m - 1)(n_m - q + 1)$, and the bracket $\langle \rangle$ indicates the average over all events in the sample. The powers $\phi_q$ measure the strength of the effect and are related to the "anomalous" dimensions $d_q$ [2] via the relation

$$d_q = \frac{1}{q - 1} \phi_q.$$

The power law (1), if extrapolated to $\delta \to 0$, leads to a singularity in the multi-particle density at small sepa-
racion in phase space and, therefore, goes beyond the
conventional study of short range correlations in particle
production.

Data are now available for $e^+e^-$ [3–8], lepton-
hadron [9, 10], hadron-hadron [11–14], hadron-nucleus
[15, 16, 17] and nucleus-nucleus [15, 18–21] collisions and
the recent evidence for a power law is mainly based on
the analysis in two and three dimensions. At present, it
cannot be excluded that the basic origin of intermittency
is the same in all types of collision. Recent reviews are
given in [22].

Experimentally established, the effect is still far from
understood theoretically. QCD inspired parton-shower
models can explain the general behavior of factorial mo-
mments in the simplest case of $e^+e^-$ collisions, but pre-
ently used models fail quite dramatically for all other
types of collision.

To help in improving existing models, or to construct
new ones, more experimental insight is needed into the
details and the possible origin of the phenomenon. Fac-
torial moments $F_n$ are integrals of $q$-th order inclusive
densities $\rho_q$ over a particular $q$-dimensional phase-space
region. Besides these, other related quantities should be
studied, together with observables derived from the (con-
ected) correlation functions into which $\rho_q$ may be ex-
panded [23].

The plan of the paper is as follows. In Sect. 2 we de-
scribe the data used in the analysis, discuss the experi-
mental biases that might affect our results and define the
variables. Section 3 contains results on factorial moments
in one, two and three dimensions. These data definitively
exclude a second order phase transition as a possible
explanation of “intermittency” in this experiment. Fac-
torial cumulant moments are discussed in Sect. 4. These
provide clear evidence for high-order correlations in small
phase-space domains. Section 5 is devoted to an impor-
tant recent modification of the factorial moment method
which significantly improves the sensitivity of the analysis
and holds promise of much more refined studies in the
future. Our conclusions are summarized in Sect. 6.

2 Experimental procedure

2.1 Event selection

In this CERN experiment, the European Hybrid Spec-
trometer (EHS) is equipped with the Rapid Cycling Bub-
ble Chamber (RCBC) as an active vertex detector and
exposed to a 250 GeV/c tagged positive, meson enriched
beam. In data taking, a minimum bias interaction trigger is
used. The details of the spectrometer and the trigger
can be found in previous publications [24, 25].

Charged particle tracks are reconstructed from hits in the
wire- and drift-chambers of the two lever-arm mag-
netic spectrometer and from measurements in the bubble
chamber. The average momentum resolution $\langle \Delta p/p \rangle$
varies from a maximum of 2.5% at 30 GeV/c to around
1.5% above 100 GeV/c.

Events are accepted for the analysis when measured
and reconstructed charge multiplicity is the same, charge
balance is satisfied, no electron is detected among the
secondary tracks and the number of badly reconstructed
(and therefore rejected) tracks is 0. The loss of events
during measurement and reconstruction is corrected for
by means of the topological cross section data [24]. Elas-
tic events are excluded. Furthermore, an event is called
single-diffractive and excluded from the sample if the
total charge multiplicity is smaller than 8 and at least one
of the positive tracks has $|x_p| > 0.88$. After these cuts,
the inelastic non-single-diffractive sample consists of
59 200 $\pi^+p$ and $K^+p$ events. The sample averages in (2)
include events without tracks in the c.m. rapidity interval
$-2.0 < y < 2.0$.

For momenta $p_{LAB} < 0.7$ GeV/c, the range in the
bubble chamber and/or the change of track curvature is
used for proton identification. In addition, a visual ioni-
zation scan has been used for $p_{LAB} < 1.2$ GeV/c on the
full $K^+p$ and 62% of the $\pi^+p$ sample. Positive particles
with $p_{LAB} > 150$ GeV/c are given the identity of the beam
particle. Other particles with momenta $p_{LAB} > 1.2$
GeV/c are not identified in the present analysis and are
treated as pions.

2.2 Resolution and biases

The analysis described in this paper is performed in the
three phase-space variables rapidity $y$, azimuthal angle $\phi$
and transverse momentum variable $p_T$. The experi-
mental resolution in these variables is shown in Fig. 1a
for single particles. The number of bins $M=50$ corre-
sponds to the largest number of divisions used in the
intermittency analysis. Each bin contains the same num-
ber of particles and the local bin sizes are indicated by
the horizontal bars in Fig. 1a. In the rapidity region $\Delta Y$
under consideration ($-2.0 < y < 2.0$), the averaged ex-
perimental resolution in rapidity varies between 0.007 and
0.023 units, that in $\phi$ between 0.015 and 0.022 rad and
that in $p_T$ is between 0.05 and 0.27. The experimental
single particle resolution thus lies far below the corre-
spanding bin size.

Of particular importance for our type of analysis is
the resolution in the phase-space distance between par-
ticles. In Fig. 1b we, therefore, give the error on the dis-
tance as a function of this distance for the three variables
used (assuming no correlation between errors on single
tracks). In all cases, the error stays (considerably) below the
distance itself.

A number of checks have been applied to the data
(see [26] for details):

1. The use of event weights to correct for multiplicity
dependent event losses increases the intermittency signal,
but it has been checked that a significant signal remains
for unweighted events.

2. Exclusion of all events with local density $\delta n/\delta y \geq 100$
reduces the signal, but a significant signal remains even
then. It should be stressed, however, that the method of
factorial moments is in fact designed to study the influ-
ence of just those high density events by reducing the
non-spiky background.

3. Random track losses lead to a reduced signal, but
4. Limited two-track resolution can cause a serious bias, even if the limit is on the level of only a few percent of the bin size [27]. In this experiment, tracks are resolved visually in RCBC and matched to spectrometer tracks over a 40 m lever arm with very strong criteria. For illustration, in Fig. 2 the rapidity gap distribution (number of particle pairs with a certain rapidity difference $|y_i - y_{i+1}|$) is plotted. From Fig. 2a, measuring a smallest gap of 0.01 units in rapidity, it can be seen that the gap distribution increases exponentially with decreasing gap size. Fig. 2b measures gap sizes down to 0.001 units, i.e. 1% of the smallest bin size $\delta y$ used for the factorial moments in this paper. In Fig. 2, a limited two-track resolution would show up as a dip at low $|y_i - y_{i+1}|$ values. Of course, these distributions are integrated over the azimuthal angle $\varphi$. However, limiting the analysis to tracks within azimuthal intervals of $\Delta \varphi = 2 \pi / 10$ (lower distributions in Fig. 2) does not give any indication that the experiment suffers from a limited two-track resolution.

5. A possible bias leading to a dramatic increase of the intermittency signal is double counting of tracks, e.g. due to track match failures. This would be visible in Fig. 2 as a sharp increase at the smallest gap sizes. In our experiment it is excluded from the track-following and hybridization procedure and the track selection criteria.

6. It has been verified by Monte Carlo simulation that $K^0$ or $A^0$ decay have little effect on the intermittency signal. If present at all, the effect is a small decrease rather than an increase of the signal. This is expected, since the correlation length of these decays is of the order of one rapidity unit, considerably larger than the smallest bin sizes used for the analysis.

7. Dalitz decay and $\gamma$-conversion near the primary vertex will be studied in detail in Sect. 5. Recently, it has been shown [28] that the FRITIOF model [29] overestimates $\eta$ production in our data by roughly a factor 2. Since even this high $\eta$ rate the model does not show a considerable intermittency signal, the signal observed in the data cannot be due to resonance decay.

8. The factorial moments depend strongly on the high-multiplicity tail of the multiplicity distribution in the phase-space cell considered. Because of the finite number of events in an analysis, the multiplicity distribution is truncated. This so-called "empty bin effect" has been studied in detail in [30]. At small enough bin sizes, this, on the average, leads to an underestimate of the factorial moments. For the conditions encountered in the one-dimensional analyses, little distortion is found except possibly in the 5th order. It has been checked that this also remains true for higher-dimensional analysis, at least for the bin sizes used in this paper.

Fig. 1. a Experimental single particle resolution in the phase-space variables used, b experimental error on the distance between two particles as a function of this distance, for the three phase-space variables used.
2.3 Definition of variables

In order to perform a meaningful analysis in terms of the bin-averaged moments (2), the underlying inclusive distribution has to be translationally invariant in the phase-space region considered. This is certainly not the case for the distribution in transverse momentum \( p_T \) or even \( \ln p_T^2 \). For this reason, Ochs [31] as well as Bialas and Gazdzicki [32] have proposed normalized cumulative variables

\[
X(y) = \frac{\int_{y_{\min}}^{y_{\max}} \rho(y')dy'}{\int_{y_{\min}}^{y_{\max}} \rho(y')dy'} ,
\]

where \( y \) stands for any phase-space variable under consideration. A strong correlation is observed between the magnitude of the factorial moments and the number of bins chosen to define the grid of the cumulative distribution. However, in general, a stable situation is reached when 100,000 bins are used in \( y \). Unless stated otherwise, the analysis done in this paper will be in terms of the normalized cumulative variables. By definition, the difference between “vertical” and “horizontal” averaging vanishes for these variables. In Figs. 3-10 the transformed variables (5) are referred to by their parent variable names.

To perform the analysis in higher dimensions, Ochs assumes that the three-dimensional density function factorizes according to

\[
\rho(y, \varphi, p_T) = \rho(y)\rho(\varphi)\rho(p_T).
\]

Using this rather strong assumption, it is sufficient to calculate the cumulative distribution for all three variables, independently. On the other hand, this assumption is not necessary in [32]. Since in our data the two methods lead to rather similar results [26], we present here results obtained with (6). Because of correlation between the variables, the distribution (5) is not completely flat, even when using transformed variables (5). This leads to a small difference between the results obtained with vertical and horizontal averaging. In the following, the results from the vertical analysis will be given.

3 Factorial moments

In Fig. 3*, a compilation is shown of the factorial moments \( F_2 \) to \( F_3 \) as a function of \( \ln M \), where \( M^d \) is the total number of boxes in the \( d \)-dimensional analysis using the variable transform (5)-(6). For a given order \( q \), the first point (\( \ln M = 0 \)) has the same value for the 1-, 2- and 3-dimensional distributions, since the same initial interval \((-2 < y < 2, \ 0 \leq \varphi \leq 2\pi, \ -18 < \ln p_T^2 < 6\) has been used for all distributions.

The rise of the moments is clearly strongest in the 3-dimensional case (rightmost column in Fig. 3). Since it is faster than a power law, no slopes have been fitted (see Sect. 4.2, however). While the rise is still faster than a power law for the two-dimensional analysis in \( y \) and \( \varphi \) (middle column), the \( F_2 \) show the usual flattening in the one-dimensional projections onto \( y \) and \( \ln p_T^2 \) (leftmost column). Due to anti-correlations at large difference in azimuthal angle \( \varphi \) from transverse momentum conservation, the moments in \( \varphi \) first decrease up to \( \ln M \approx 2 \), but increase above that value.

The influence of biases (Dalitz decay and \( y \)-conversion) has been studied using the FRITIOF Monte Carlo generator. Several parameter settings were used, as well as different versions of FRITIOF (2.0, 3.0 and 3.1). A correction factor obtained from a comparison of the ‘plain’ and the ‘biased’ (i.e. contaminated by Dalitz decay and \( y \)-conversion) FRITIOF data has been applied to the NA22 data (not shown here). The corrected data are lower than the original data, in particular for 3 dimensions, but the upward bending of the factorial moments remains.

* Data shown in this and the following figures are available in numerical form on request from U63207 at HNYKUN11.bitnet
To investigate the modified power-law assumption of Ochs and Wosiek [31],

\[ F_q(\delta) = b_q (g(\delta))^e, \]  

\[ \ln F_q(\delta) = \ln b_q + \frac{\alpha_q}{\alpha_2} (\ln F_2(\delta) - \ln b_2) \]

\[ = \delta_q + \frac{\alpha_q}{\alpha_2} \ln F_2(\delta) = \delta_q + r_q \ln F_2(\delta) \]  

the Ochs-Wosiek plot, \( \ln F_q(\delta) \) versus \( \ln F_2(\delta) \), is given in Fig. 4 for the indicated variables. While in [31] data at small scales, where \( F_2 \) does not vary significantly, are rejected, here all points are presented. In rapidity, third- and higher-order moments vary stronger than second-order ones. This leads to the accumulation of the data at \( \ln F_2 = 0.3 \). Data for \( \phi \) are not given, but it has been checked that they follow a straight line, even for the larger intervals. This is remarkable, since from Fig. 3 it can be seen that the \( F_q \) themselves decrease for these interval sizes. Thus, even the anti-correlations follow a modified power law.

To be able to compare the NA22 results with those of other experiments, the modified power law (7) has been fitted to the data. To obtain meaningful results, only factorial moments in the larger intervals (down to \( M = 10 \)) are used, where \( F_2 \) still grows significantly. Ochs-Wosiek plots for various combinations of variables are given in Fig. 5. The lines are individual fits to these data according to (8). Contrary to the observation in [31], the slope \( r_q = \alpha_q / \alpha_2 \) is larger for 3 dimensions than for lower ones. Slopes would only be similar if the fits were constrained to have \( \delta_q = 0 \).

A comparison of the slopes \( r_q \) for the different variables as well as for the combined sample (all five variable combinations taken together) and for the weighted average is given in Table 1. The data of other experiments are from [31]. From this table it can be concluded that the weighted average slopes are similar to those found from 1-dimensional DELPHI data (\( \sqrt{s} = 91 \) GeV) [6] and from 1- and 2-dimensional TASSO data (\( \sqrt{s} = 35 \) GeV) [4]. The fit of the combined NA22 sample corresponds more or less to the 1- and 2-dimensional EMC data (\( \sqrt{s} = 4-20 \) GeV) [9]. None of the samples agrees with the \( hh \) data quoted in [31]. This might be due to the fact that these early results were obtained from rapidity, and not from the transformed rapidity given by (4).

The slopes \( r_q \) become of theoretical importance when they are related to the anomalous dimensions \( d_q \),

\[ r_q = \frac{d_q}{\alpha_2 (q - 1)} = \frac{\alpha_q}{\alpha_2 (q - 1)}. \]
Brax and Peschanski have shown in [33] that the so-called Lévy stable law can be used as an approximation to the density distribution in the case of a self-similar multiplicative cascade process, as for example the α-model. The order dependence of the ratio (9) is then given by

$$\frac{d_x}{d_2} = q^{\mu - q} \frac{1}{2^\mu - 2^{q - 1}}.$$  \hspace{1cm} (10)

The Lévy index $\mu$, in principle, determines the tail of the density distribution. An important case is the Gaussian distribution corresponding to $\mu = 2$. If the cascade processes were sufficiently long, such as to render the central limit theorem applicable (however shown not to be realized in practice [34]), one would obtain

$$\frac{d_x}{d_2} = q \frac{1}{2}.$$  \hspace{1cm} (11)

If the self-similarity is not due to a cascade process, but due to a second-order phase transition (e.g. quark-gluon plasma to hadrons) at the critical temperature, a monofractal behavior is expected [35], resulting in a constant ratio. In the Brax-Peschanski picture this would correspond to $\mu = 0$, a value not in the allowed range for the Lévy index ($0 < \mu \leq 2$).

In Fig. 6a the ratio $d_x/d_2$ is plotted for the 3-dimensional data and for the combined and average fits from Table 1. The lines represent (10) with the $\mu$ values indicated. The 3-dimensional data have $\mu > 2$ which is not allowed in the sense of Lévy stable laws. Even larger values of $\mu$ ranging from 3.2 to 3.5 have been found for muon-proton deep inelastic scattering in [33]. According to [36, 37], this is evidence that the procedure to obtain the Lévy-index is used outside its domain of validity. A more general method of double trace moments leads to $\mu$-values within the mathematically allowed boundaries. However, the latter method may be criticized on other grounds. The true significance of the Lévy-law approach clearly needs further clarification.

The combined sample and the weighted average are close to the fit of [31]. The NA22 data presented here are higher than the $hh$ data quoted in [31] and shown in Fig. 6b. The latter were based on eye-ball fits to results from non-transformed variables. All results exclude a second-order phase transition as a possible mechanism of multiparticle production. This also applies to AA interactions [31].

The conclusion of this section is that in three-dimensions the factorial moments increase faster than in two or one. To a certain extent, the data follow a modified power law (7). However, the slope $r_q$ derived from the

Table 1. Fits to the modified power-law equation (8). The values labeled “average” are the weighted averages of the results from the separate fits. “Combined fit” stands for all NA22 data fitted together. The lower four rows are from [31].

<table>
<thead>
<tr>
<th></th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA22</td>
<td>$y$</td>
<td>2.63 ± 0.09</td>
<td>4.9 ± 0.2</td>
</tr>
<tr>
<td>NA22</td>
<td>$p_T$</td>
<td>3.0 ± 0.3</td>
<td>6.2 ± 0.5</td>
</tr>
<tr>
<td>NA22</td>
<td>$y - p_T$</td>
<td>3.4 ± 0.2</td>
<td>6.1 ± 0.6</td>
</tr>
<tr>
<td>NA22</td>
<td>$y - p_T$</td>
<td>2.75 ± 0.07</td>
<td>5.2 ± 0.2</td>
</tr>
<tr>
<td>NA22</td>
<td>$\mu$</td>
<td>3.4 ± 0.2</td>
<td>7.8 ± 0.5</td>
</tr>
<tr>
<td>NA22</td>
<td>average</td>
<td>3.0 ± 0.3</td>
<td>6.2 ± 0.5</td>
</tr>
<tr>
<td>NA22</td>
<td>combined fit</td>
<td>2.64 ± 0.04</td>
<td>4.87 ± 0.09</td>
</tr>
<tr>
<td>$e^+ e^-$</td>
<td>TASSO, DELPHI</td>
<td>2.83 ± 0.11</td>
<td>5.32 ± 0.20</td>
</tr>
<tr>
<td>$\mu$</td>
<td>EMC</td>
<td>2.60 ± 0.10</td>
<td>4.76 ± 0.22</td>
</tr>
<tr>
<td>$hh$</td>
<td>UA1, UA5, NA22</td>
<td>2.43 ± 0.10</td>
<td>4.3 ± 0.2</td>
</tr>
<tr>
<td>$pA, AA$</td>
<td>KLM, EMU01</td>
<td>2.86 ± 0.07</td>
<td>5.23 ± 0.22</td>
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</table>
Fig. 6. a The ratio $d_q/d_2$ for the 3-dimensional data, the combined sample, and fit average. b Data from experiments indicated. The dashed lines correspond to $\langle \text{10} \rangle$, the dotted lines to mono-fractal behavior. The dashed-dotted line is a fit from [31] with $\mu = 1.6$

Ochs-Wosiek relation is not independent of the dimension in which the factorial moment analysis is performed. From the 3-dimensional analysis, we obtain a value for the Lévy index beyond the mathematically allowed region.

4 Factorial cumulants

4.1 Definitions

Although intermittency as originally considered by Bialas and Peschanski, was formulated in terms of particle densities, it is known from many branches of physics, that inclusive densities $\rho_q$ are often ill-suited to reveal dynamical effects.

Besides genuine $q$-particle correlations, the $\rho_q$ contain contributions from “random associations” of lower-order correlated and uncorrelated $q'$-particle groups ($q' \leq q$). For most applications, it is more convenient to eliminate the latter and to concentrate the analysis on “connected” correlation functions.

This is easily accomplished via a “cluster-expansion” familiar from statistical mechanics [38]. For particles of the same species, the cluster expansion leads, after integration over a suitable $q$-dimensional phase-space region, to a set of equations relating the factorial moments to their connected counterparts, the factorial cumulant moments [39, 23]. The relation between the non-normalized factorial moments $\langle n_m^{(q)} \rangle$ and the non-normalized factorial cumulants $k^{(m)}_q$ is

$$k^{(m)}_q = q! \sum_{i_q} \left( -1 \right)^{\left( q-1 \right)} \prod_{p=1}^{q} \left( \frac{1}{(i_p-2)} \right)!$$

(12)

where the first sum runs over all $i_p \geq i_p+1$, $i_1 \geq 1$ and $p = 1, \ldots, q$ and $i_{q+1} = 0$. For the orders $q = 1,\ldots, 4$ one finds:

$$k^{(m)}_1 = \langle n_m \rangle$$

(13)

$$k^{(m)}_2 = \langle n_m^{(2)} \rangle - \langle n_m \rangle^2$$

(14)

$$k^{(m)}_3 = \langle n_m^{(3)} \rangle - 3 \langle n_m^{(2)} \rangle \langle n_m \rangle + 2 \langle n_m \rangle^3$$

(15)

$$k^{(m)}_4 = \langle n_m^{(4)} \rangle - 4 \langle n_m^{(3)} \rangle \langle n_m \rangle - 3 \langle n_m^{(2)} \rangle^2$$

(16)

$$+ 12 \langle n_m^{(2)} \rangle \langle n_m \rangle^2 - 6 \langle n_m \rangle^4$$

Bin-averaged normalized factorial cumulants are defined as

$$K_q(\delta) = \frac{1}{M^d} \sum_{n=1}^{M^d} \frac{k^{(m)}_q \delta}{\langle n_m \rangle^{q-1}} = \frac{1}{M^d} \sum_{n=1}^{M^d} K^{(m)}_q(\delta).$$

(17)

As noted above, since combinations of lower order correlations are removed, the cumulant of order $q$ is a measure of the genuine $q$-particle correlation.

As the $k^{(m)}_q$ are combinations of $\langle n_m^{(q)} \rangle$, $K_q$ can become negative and the errors on $\langle n_m^{(q)} \rangle$ can accumulate to large errors on $k^{(m)}_q$ and $K_q$.

Fig. 7. Comparison of $K_q$ and $K_q$ for different variables and dimensions, as indicated in the uppermost subfigures. Full lines correspond to a fit by (18), dashed lines to a fit by (19)
4.2 Results of multidimensional analysis

The ln(K_q) are plotted as a function of ln(M) in Fig. 7, for the analysis in the various variables and dimensions (of course, only non-zero positive values can be shown on the logarithmic plot). The K_q are considered for bin sizes where F_{g+1} is non-zero and the relative error on F_q is smaller than 50%.

In one-dimensional rapidity space, the data (open squares in Fig. 7a and 7d) show the presence of genuine higher order multiparticle correlations in rapidity. K_4 still has the same trend of an increase with decreasing bin size, but is not shown here because of large errors. K_3 is consistent with zero (not shown). The same analysis applied to the variable y (open circles in Fig. 7a) does not show evidence for correlations of order higher than 2. In p_T (triangles in Fig. 7a) an increase is observed for K_2 and K_3, but results are not shown for K_4 due to too large errors. Also here, K_3 and K_4 are consistent with zero.

For the three possible two-dimensional combinations of y, \phi, and p_T (Fig. 7b) an increase is observed for K_3 and K_2, but results are not shown for K_4 due to too large errors. Again, K_3 and K_4 are consistent with zero.

In three dimensions (full squares in Fig. 7c and e), the existence of K_2 > 0 up to order 3 suggests the presence of genuine higher order multiparticle correlations in 3 dimensions. K_4 still has the same trend of an increase with decreasing bin size, but is not shown here because of too large errors. K_3 is consistent with zero.

In Fig. 7 one can further see that the ln(K_q) show an approximately linear rise with ln(M). In fact, it has been pointed out [40,41] that it may be the cumulants that have the feature of scaling, rather than the factorial moments. In order to test this hypothesis, the power law

\[ k_q(\delta) = a\delta^{-b}, \]

has been fitted to the data (solid lines in Fig. 7). The fit parameters are collected in Table 2, together with \chi^2/NDF and the range of the fit in M. Except for K_3 in the third dimensions, the fit quality is good. K_2, however, shows a clear upward bending and, compared to the behavior of F_2 in Fig. 3, there is very little improvement in the direction of linearity.

![Fig. 8. a \ln(K_2 - c) versus \ln M for 3 dimensions, where c is obtained by fitting (19). The straight line represents the fit, b as in a, but for horizontal averaging, c as in a, but for horizontal averaging and using the proper covariance matrix](image)

To allow for a possible deviation from a simple power law, Fialkowski [42,43] suggested to add a constant c, i.e. to fit

\[ K_2(\delta) = a\delta^{-b} + c, \]

(19)

where c takes into account possible non-singular long-range correlations (dashed lines in Fig. 7). Then, \ln(K_2 - c) should show a linear dependence on \ln M. This is shown for K_2 in the case of 3 dimensions in Fig. 8a. The parameters are given in Table 3. For the 3-dimensional factorial cumulant the addition of the constant in (19) improves the fit quality. The value c = 0.16 ± 0.02 found here is compatible with an old model [44] and a recent estimate [45]. Since for q > 2 the K_q show good linear behavior already on ln(M), an analogous fit (19) to higher orders gives a value of c compatible with zero.

To see the influence of bin-size correlations on the fit results, fits are repeated for horizontal 3-dimensional moments without (Fig. 8b) and with (Fig. 8c) the use of the covariance matrix. Horizontal and vertical averaging gives the same result for the fit, but the constant term is reduced to c = 0.13 ± 0.01 if bin-size correlations are included.

The factorial cumulants can be used to study the contributions of genuine multiparticle correlations to the factorial moments [40,41,46]. Inverting formula (12) gives [23]

### Table 2. Fit results according to (18)

<table>
<thead>
<tr>
<th>Order</th>
<th>y</th>
<th>\phi</th>
<th>p_T</th>
<th>y - \phi</th>
<th>y - p_T</th>
<th>\phi - p_T</th>
<th>y - \phi - p_T</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>a</td>
<td>0.299 ± 0.002</td>
<td>0.086 ± 0.001</td>
<td>0.205 ± 0.002</td>
<td>0.162 ± 0.002</td>
<td>0.305 ± 0.003</td>
<td>0.065 ± 0.001</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>0.036 ± 0.002</td>
<td>0.081 ± 0.004</td>
<td>0.042 ± 0.002</td>
<td>0.118 ± 0.002</td>
<td>0.073 ± 0.002</td>
<td>0.195 ± 0.005</td>
</tr>
<tr>
<td></td>
<td>\chi^2/NDF</td>
<td>8.1/35</td>
<td>10.6/35</td>
<td>4.4/35</td>
<td>5.4/12</td>
<td>2.6/12</td>
<td>6.1/12</td>
</tr>
<tr>
<td></td>
<td>range M</td>
<td>4-40</td>
<td>4-40</td>
<td>4-40</td>
<td>5-18</td>
<td>5-18</td>
<td>5-18</td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>0.049 ± 0.003</td>
<td>0.036 ± 0.002</td>
<td>0.009 ± 0.002</td>
<td>0.18 ± 0.02</td>
<td>0.04 ± 0.01</td>
<td>0.026 ± 0.005</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>0.47 ± 0.02</td>
<td>0.30 ± 0.03</td>
<td>0.74 ± 0.04</td>
<td>0.15 ± 0.03</td>
<td>0.24 ± 0.08</td>
<td>0.60 ± 0.05</td>
</tr>
<tr>
<td></td>
<td>\chi^2/NDF</td>
<td>5.9/35</td>
<td>3.7/35</td>
<td>2.7/7</td>
<td>2.6/7</td>
<td>0.7/5</td>
<td>0.8/4</td>
</tr>
<tr>
<td></td>
<td>range M</td>
<td>4-40</td>
<td>4-40</td>
<td>5-13</td>
<td>5-13</td>
<td>5-11</td>
<td>1-6</td>
</tr>
<tr>
<td>4</td>
<td>a</td>
<td>0.014 ± 0.003</td>
<td>1.46 ± 0.07</td>
<td>8.92/28</td>
<td>0.006 ± 0.005</td>
<td>1.28 ± 0.24</td>
<td>0.26/2</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td></td>
<td></td>
<td>4-33</td>
<td>1-4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3. Fit results according to (19), range as in Table 2

<table>
<thead>
<tr>
<th>Order</th>
<th>$a'$</th>
<th>$b'$</th>
<th>$c$</th>
<th>$\chi^2$/NDF</th>
<th>$y - \phi$</th>
<th>$y - p_T$</th>
<th>$\phi - p_T$</th>
<th>$y - \phi - p_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.58 ± 0.02</td>
<td>0.70 ± 0.02</td>
<td>-1.30 ± 0.02</td>
<td>8.1/34</td>
<td>0.02 ± 0.08</td>
<td>0.04 ± 0.24</td>
<td>-1.30 ± 0.02</td>
<td>0.16 ± 0.02</td>
</tr>
<tr>
<td>3</td>
<td>0.07 ± 0.01</td>
<td>1.8 ± 1.3</td>
<td>0.101 ± 0.004</td>
<td>4.4/34</td>
<td>0.24 ± 0.63</td>
<td>1.1 ± 0.7</td>
<td>0.13 ± 0.02</td>
<td>4.3/34</td>
</tr>
<tr>
<td>4</td>
<td>-1.30 ± 0.02</td>
<td>0.101 ± 0.004</td>
<td>-0.49 ± 0.002</td>
<td>8.1/34</td>
<td>0.29 ± 0.33</td>
<td>0.13 ± 0.02</td>
<td>4.4/34</td>
<td></td>
</tr>
</tbody>
</table>

The substitution $k_q = 0$ for $q > 2$ in the above relations then give the contribution from two-particle correlations. After normalization and bin averaging, one obtains

$$F_q^{(2)}(\delta) = 1 + 3 K_2(\delta)$$

$$F_q^{(3)}(\delta) = 1 + 6 K_2(\delta) + \frac{3}{M} \sum_{m=1}^M (K_2^{(m)}(\delta))^2$$

The combination of two and three-particle contributions is obtained by setting $k_2 = 0$ for $q > 3$, which leads to

$$F_q^{(3)}(\delta) = 1 + 6 K_2(\delta) + \frac{3}{M} \sum_{m=1}^M (K_2^{(m)}(\delta))^2 + 4 K_3(\delta)$$

Higher order contributions can be obtained via analogous procedures.

The $\ln F_q$ (open squares) and $\ln F_q^{(p)}$ (full circles) are compared as a function of $\ln M$ in Fig. 9 for $p = 2$ and 3 and $q = 3$ and 4, in one-, two- and three-dimensional phase space ($y$, $y - \phi$ and $y - \phi - p_T$). In general, the difference increases with increasing $\ln M$ (decreasing bin size), proving that the contribution of higher order multiparticle correlations to the factorial moments increases with decreasing bin size. An exception is the variable $\phi$ for which only two-particle correlations are seen (not shown).

4.3 Test of the LPA

In order to understand the nature of the higher order multiparticle correlations, a number of attempts have been
made to express the higher order normalized correlation functions in terms of linked second order normalized correlation functions $K_q (y_1, y_2)$ [40, 41, 46–50].

We here test the linking procedure proposed by Car ruthers et al. [41, 46, 47], known as the linked pair approximation (LPA):

$$K_q (y_1, y_2, \ldots, y_q) = \frac{A_q}{q!} \sum_{\text{perm.}} K_2 (y_1, y_2) K_2 (y_2, y_3) \cdots K_2 (y_{q-1}, y_q),$$  \hspace{1cm} (27)

where the parameters $A_q$ of the model are a set of constants. If one assumes that the single particle distribution in $y$ changes slowly within each bin, one can write:

$$K_q^{(m)} (\delta) \approx \frac{1}{(\delta y)^q} \int_{(m-1) \delta}^{m \delta} dy_1 \cdots dy_q K_q (y_1, \ldots, y_q).$$  \hspace{1cm} (28)

After a second assumption,

$$K_2 (y_1, y_2) \approx K_2 (|y_1 - y_2|),$$ \hspace{1cm} (29)

within each bin separately, and applying the so-called strip approximation* [41, 46, 47], one finds

$$K_q^{(m)} (\delta) = A_q (K_2^{(m)} (\delta))^{q-1}.$$ \hspace{1cm} (30)

So, after bin averaging one obtains

$$A_q = \frac{1}{M} \sum_{m=1}^{M} (K_2^{(m)} (\delta))^{q-1},$$ \hspace{1cm} (31)

with $A_q$ independent of $\delta$.

If, furthermore, the underlying multiplicity distribution is a negative binomial (NBD), $K_2^{(m)} (\delta)$ is related to the NBD parameter $1/k$ by

$$K_2^{(m)} (\delta) = \frac{1}{k^{(m)}}$$ \hspace{1cm} (32)

and the linking parameters are fixed [48]:

$$A_q^{\text{NBD}} = (q - 1)!. \hspace{1cm} (33)$$

Experimentally, relation (26) with coefficients close to (33) was found to be valid at UA1 energies [46].

From the above derivation it is clear that the approximation only holds if $1/k$ is constant over all bins. This is not the case for our data. In [51] reasonable agreement has been found with the negative binomial, but a variation of $1/k$ is observed from 0.4 at $y < 1$ to 0 at $y \approx 2$. Furthermore, the tail of the distribution on which this analysis is based is not well described.

In Fig. 10 we show $A_q$ vs. $\ln M$ for the cumulative variable obtained from one-dimensional rapidity space. Contrary to (30), $A_q$ is observed to increase with decreasing bin size (increasing $\ln M$). The dotted line in Fig. 10 represents the value of $A_q$ for the case of a NBD. As may be expected from a variation of $1/k$ in our data, the experimental values are not equal to the predicted value (Fig. 10a, d). A similar trend is observed for $A_3$ in two- and three-dimensional analysis (not shown). An improvement can be expected when decreasing the initial interval from $[-2, 2]$ to $[-0.75, 0.75]$ and when excluding the "spike event" [52]. In both cases the variation of the parameter $1/k$ is reduced. An improvement is indeed seen in Fig. 10b, d and Fig. 10c, f.

We conclude from this section that non-zero genuine multiparticle correlations exist. They increase in magnitude with decreasing phase-space volume. Our data do not support the linked pair approximation, but that may be due to a deviation from translational invariance (variation of $1/k$) over the rapidity interval at our energy and the presence of the exceptionally dense spike event.

5 Correlation (or density) strip integral

5.1 The method

The most promising recent development in the study of density fluctuations is the correlation (or better density) strip integral method [53]. By means of integrals of the
inclusive density over a strip domain, rather than a sum of box domains, one not only avoids splitting up density spikes, but also drastically increases the integration volume at given resolution.

In terms of density integrals, the (vertical) $F_q$ can be written (for an analysis in one dimension)

$$F_q^V(\delta y) = \frac{1}{M} \sum_{m=1}^{M} \left\{ \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \right\}^M = \frac{1}{M} \sum_{m=1}^{M} \left\{ \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \right\}^M \int \prod_{i} d y_i \rho_q(y_1, \ldots, y_q)$$

The integration domain $\Omega_\beta = \sum_{m=1}^{M} \Omega_m$ thus consists of $M$ $q$-dimensional boxes $\Omega_m$ of edge length $\delta y$. For the case $q=2$, $\Omega_\beta$ is the domain in Fig. 11a. A point in the $m$-th box corresponds to a pair $(y_1, y_2)$ of distance $|y_1 - y_2| < \delta y$ and both particles in the same bin. Points with $|y_1 - y_2| < \delta y$ which happen not to lie in the same bin but in adjacent bins (e.g. the point $x$ in Fig. 11a) are left out. The statistics can be approximately doubled when replacing $\Omega_\beta$ by the strip domain of Fig. 11b. For higher orders $q$, the enhancement of integration volume (and reduction of squared statistical error) is in fact roughly proportional to the order of the correlation. The gain is even larger when working in two or three phase-space variables.

In terms of the strips (or hyper-tubes for $q > 2$) the (vertical) density integrals become

$$F_q^{VS}(\delta y) = \int \prod_{i} d y_i \rho_q(y_1, \ldots, y_q)$$

The strip equivalent of the horizontally normalized factorial moments

$$F_q^H(\delta y) = M^{q-1} \sum_{m=1}^{M} \left\{ \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \right\}^M = \int \prod_{i} d y_i \rho_q(y_1, \ldots, y_q)$$

$$= M^{q-1} \sum_{m=1}^{M} \left\{ \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \right\}^M \int \prod_{i} d y_i \rho_q(y_1, \ldots, y_q)$$

This can be written as

$$F_q^{VS}(\delta y) = \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

$$F_q^{H}(\delta y) = M^{q-1} \sum_{m=1}^{M} \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

$$= \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

$$= M^{q-1} \sum_{m=1}^{M} \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

$$= \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

These integrals can be evaluated directly from the data, after selection of a proper distance measure ($|y_1 - y_2|$, $[(y_1 - y_2)^2 + (\phi_1 - \phi_2)^2]^{1/2}$, box volume $= \max\{|y_1 - y_2|, |\phi_1 - \phi_2|\}$ in the transformed variables (35)-(36), or better the four-momentum difference $Q_\beta^2 = - (p_1 - p_2)^2$, and after definition of a proper multiparticle topology (GHP [54] or snake integral [53]). The advantage of $Q_\beta^2$ is that it combines the features of a three-dimensional analysis with the large statistics of the one-dimensional projection. For the case of two-particles $Q_\beta^2$ is related to the invariant mass of a particle pair,

$$Q_\beta^2 = (p_1 + p_2)^2 = Q_{12}^2 + 2m^2 + 2m^2$$

To actually compute the numerator of $F_q^H(\delta y)$ it is sufficient to count the number of $q$-tuples that have a distance smaller than $\delta y$. Mathematically, this can be expressed as

$$\int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

$$= \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

In case of vertical normalization, the denominator is obtained from "mixed" events by using a track pool. The multiplicity of a mixed event is taken to be a Poissonian random variable, thereby ensuring that no extra correlations are introduced. Such a mixed event undergoes the same procedure as a real event. Because the tracks of a mixed event are not correlated, $\rho_q(y_1, \ldots, y_q)$ factorizes to $\rho_1(y_1) \ldots \rho_1(y_q)$ and the denominator is obtained. Furthermore, a correlation factor has to be applied for the difference in average multiplicity of the Poissonian and the experimental distribution.

For many cases, an analytical expression exists for the strip volume, so the denominator of the horizontally normalized $F_q^{H}$ can be calculated very precisely, even for very small phase-space domains.

5.2 Application to 3 dimensions (box volume)

To be able to compare the results of the correlation integral with the conventional $F_q$, it is convenient to calculate $F_q^H$ in three dimensions, where the transformed variable are used and the distance is defined by the smallest box volume that encloses the $q$-tuple. Since for this definition the strip volume can be expressed explicitly

$$\text{strip volume} = (q - 1)! \frac{\langle n_q \rangle^y}{\langle n \rangle^y} \int \prod d y_i \rho_q(y_1, \ldots, y_q)$$

the horizontal normalization is preferred.

* For $n$ particles, there are $\binom{n}{q}$ ordered $q$-tuples. The factor $q!$ takes into account the number of permutations within a $q$-tuple.
In Fig. 12 the conventional $F^H$ and $F^H_2$ are compared to the strip versions $F^H_{2\text{IS}}$ and $F^H_{3\text{IS}}$. For the conventional normalized factorial moments the box volume equals $M^{-3}$. As anticipated, we indeed observe that statistical errors are strongly reduced in the $F^H_{2\text{IS}}$. This, in principle, allows the analysis to be carried down to much smaller box volumes for $F^H_{2\text{IS}}$. It, furthermore, makes it possible to compare different charge combinations, for which we distinguish the sample of all charged tracks (“all charged”) used in the previous sections, from the sample of negative tracks (“negatives only”) and that of positive tracks (“positives only”). For the case of $q=2$ we also consider pairs of unlike charge (“unlike charged”). The solid lines on the plots for $q=2$ correspond to the Fialkowski fit (19). Results are given in Table 4 for the range given in the first line and indicated in the figures. The fits in Fig. 12a start with the second point, since the first point is not compatible with the form (19), in particular for the unlike-charged sample. The fits in Fig. 12c correspond to the corresponding box volume. Results for box volumes smaller than 0.001 are not used in the fit because of possible bias (see below). Whenever, in the process of fitting, the constant $c$ was found to be compatible with zero, it was fixed to zero in order to obtain more precise values for the other parameters. Results are given for fits in which bin-size correlations were neglected (and consequently the $\chi^2/\text{NDF}$ values were very low). It has been verified on the $F^H_2$, however, that parameter values are the same within errors if bin-size correlations are taken into account.

The parameter values given in Table 4 show a striking difference for unlike- and like-charged pairs. While (+−) pairs are dominated by long range correlations (large $c$), this is small or absent in the case of (−−) and (++). Correspondingly, the parameter $a'$ is compatible with zero for (+−), but relatively large for (−−) and (++). Consequently, $b'$ is determined well only for the latter.

As in the case of the conventional $F^H$, the modified power-law assumptions (8) can be fitted to the data. The experimental slopes serve as input for the determination of the Lévy index $\mu$ by means of (10). Results are collected in Table 5. Again, $\mu$ is found to be larger than 2, but now the fit quality of (10) is very bad.

Since transformed variables with a flat distribution are used, the horizontal and the vertical normalization should lead to approximately the same results. It was found that the vertical $F^H_q$ differ from the horizontal ones by an almost constant factor close to 1. This is due to the fact that in generating “mixed” events, an upper limit is put on the multiplicity. This leads to a deviation from a true poissonian. This factor has no influence on the intermit-
tency $b'$, but reduces slightly the long-range constant $c$. Despite this small disadvantage of the mixed-event technique, it turns out a good method to calculate the denominator of (35). This is important, because one is forced to use the vertical normalization for variables which do not have a flat inclusive distribution.

### Table 4. Results of fits to the data presented in Fig. 12 according to (1) and (19), respectively

<table>
<thead>
<tr>
<th>Charge</th>
<th>$F^H_2$</th>
<th>$F^H_{2\text{IS}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All charged</td>
<td>$a'$</td>
<td>$0.01 \pm 0.01$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.47 \pm 0.06$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0.20 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>$\chi^2/\text{NDF}$</td>
<td>$2.3/6$</td>
</tr>
<tr>
<td>Unlike charged</td>
<td>$a'$</td>
<td>$0.003 \pm 0.0005$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.8 \pm 0.2$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0.36 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>$\chi^2/\text{NDF}$</td>
<td>$1.3/30$</td>
</tr>
<tr>
<td>Negatives only</td>
<td>$a'$</td>
<td>$0.070 \pm 0.003$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.39 \pm 0.05$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0.05 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>$\chi^2/\text{NDF}$</td>
<td>$2.0/30$</td>
</tr>
<tr>
<td>Positives only</td>
<td>$a'$</td>
<td>$0.028 \pm 0.001$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.45 \pm 0.05$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0.03 \pm 0.01$</td>
</tr>
<tr>
<td></td>
<td>$\chi^2/\text{NDF}$</td>
<td>$9.2/30$</td>
</tr>
</tbody>
</table>

5.3 Application to four-momenta

Because of non-flat distribution in the four-momenta $p_t$, vertical normalization has to be used in the case of the
Table 5. Fits to the modified power-law equation (8) and the Lévy index \( \mu \) obtained from (10), by means of \( F^2 \) (box vol) (Range: largest value = 1, smallest value as given in table)

<table>
<thead>
<tr>
<th>Order</th>
<th>Range</th>
<th>All charged</th>
<th>Negatives only</th>
<th>Positives only</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( b_3 )</td>
<td>( -0.23 \pm 0.02 )</td>
<td>( 0.17 \pm 0.02 )</td>
<td>( -0.15 \pm 0.01 )</td>
</tr>
<tr>
<td></td>
<td>( r_3 )</td>
<td>( 4.0 \pm 0.1 )</td>
<td>( 4.5 \pm 0.2 )</td>
<td>( 5.4 \pm 0.2 )</td>
</tr>
<tr>
<td></td>
<td>( \chi^2/NDF )</td>
<td>( 38/46 )</td>
<td>( 7.5/38 )</td>
<td>( 84/42 )</td>
</tr>
<tr>
<td>4</td>
<td>( b_4 )</td>
<td>( 0.68 \pm 0.06 )</td>
<td>( -0.53 \pm 0.06 )</td>
<td>( -0.78 \pm 0.05 )</td>
</tr>
<tr>
<td></td>
<td>( r_4 )</td>
<td>( 8.7 \pm 0.3 )</td>
<td>( 10.7 \pm 0.6 )</td>
<td>( 18.4 \pm 0.7 )</td>
</tr>
<tr>
<td></td>
<td>( \chi^2/NDF )</td>
<td>( 68/39 )</td>
<td>( 12/32 )</td>
<td>( 69/32 )</td>
</tr>
<tr>
<td>5</td>
<td>( b_5 )</td>
<td>( -1.0 \pm 0.1 )</td>
<td>( -0.9 \pm 0.2 )</td>
<td>( -1.47 \pm 0.08 )</td>
</tr>
<tr>
<td></td>
<td>( r_5 )</td>
<td>( 13.4 \pm 0.7 )</td>
<td>( 17.0 \pm 2.0 )</td>
<td>( 33.0 \pm 1.0 )</td>
</tr>
<tr>
<td></td>
<td>( \chi^2/NDF )</td>
<td>( 39/26 )</td>
<td>( 18/21 )</td>
<td>( 63/25 )</td>
</tr>
</tbody>
</table>

distance measure \( Q^2 \). A distance measure generalized to more than 2 particles is

\[
\text{distance}(p_{i_1}^{eq}, \ldots, p_{i_q}^{eq}) = \max_{\text{all pairs } k_1, k_2} \{ - (p_{i_1}^{eq} - p_{i_2}^{eq})^2 \}. \quad (41)
\]

This gives for \( F^2_q(Q^2) \)

\[
F^2_q(Q^2) = \frac{1}{\text{Norm}} \left\langle 
q! \sum_{i_1 < \ldots < i_q} \prod_{k_1, k_2} \Theta(Q^2 - Q^2_{i_1, i_2}^{(ev)}) \right\rangle. \quad (42)
\]

In Fig. 13 the NA22 data are plotted as a function of \(-\ln Q^2\). On this figure, the following observations can be made.

i) The errors and fluctuations are largely reduced, as compared to Fig. 3.

ii) The (one-dimensional) distance measure \( Q^2 \) essentially shows a similarly steep rise as the three-dimensional analysis*.

iii) Contrary to the results in \( y \) [11], the positives-only and negatives-only samples behave similarly here, but are now much steeper than the all-charged sample.

iv) \( F^2 \) is flatter for unlike-charged pairs than for the all-charged or negatives-only samples.

The first two observations demonstrate the strength of the method and the proper variable. The last two observations directly demonstrate the large influence of like-charged particle combinations on the rise of the factorial moments. These results agree very well with (preliminary) results from the UA1 collaboration [56].

The solid lines correspond to fits according to (1), the dashed ones (for \( q = 2 \), only) according to (19). The fit parameters are given in Table 6a. It can be seen that the negatives-only sample now is a factor 1.2 (\( \phi_3 \)) to 1.6 (\( \phi_2 \)) steeper than all-charged sample. A factor 2 has been predicted [57] on the basis of Bose-Einstein correlations, but

* Note that for \( F \), the 3-D-cell size is related to the range in invariant mass of the pair [42, 55] for small bins and \( \rho_T \) not too far from average.

has not been observed so far in the analysis in \( y \), \( \varphi \) and \( \rho_T \).

In Fig. 14 we show \( \ln F_q^2 \) in \( \ln F^2_q \) as a function of \(-\ln Q^2\) for event samples where each particle in the \( q \)-tuple has transverse momentum \( \rho_T < 0.15 \) (upper plots)
Table 6. Results of fits to the data presented in Figs. 13 and 14 according to (1) and (19), respectively (Range: $Q^2$ in (GeV/c)$^2$, largest value = 1, smallest value as given in table)

<table>
<thead>
<tr>
<th>Order</th>
<th>All charged</th>
<th>Negatives only</th>
<th>Positives only</th>
<th>Unlike charged</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Data from Fig. 13</td>
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<td></td>
<td></td>
</tr>
<tr>
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<td>range</td>
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<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2$</td>
<td>1.219 ± 0.003</td>
<td>1.131 ± 0.002</td>
<td>1.026 ± 0.002</td>
</tr>
<tr>
<td></td>
<td>$\phi_3$</td>
<td>0.051 ± 0.001</td>
<td>0.081 ± 0.001</td>
<td>0.067 ± 0.001</td>
</tr>
<tr>
<td></td>
<td>$x^2$/NDF</td>
<td>2.4/35</td>
<td>17/35</td>
<td>36/35</td>
</tr>
<tr>
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<td>range</td>
<td>0.02674</td>
<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2'$</td>
<td>0.228 ± 0.003</td>
<td>0.153 ± 0.001</td>
<td>0.11 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>$b_2'$</td>
<td>0.207 ± 0.006</td>
<td>0.363 ± 0.004</td>
<td>0.37 ± 0.05</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>-0.7 ± 0.3</td>
</tr>
<tr>
<td></td>
<td>$x^2$/NDF</td>
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<td>4.6/35</td>
<td>1.5/34</td>
</tr>
<tr>
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<td>0.02674</td>
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<td>$x^2$/NDF</td>
<td>13/35</td>
<td>31/35</td>
<td>65/35</td>
</tr>
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<td>$a_4$</td>
<td>2.90 ± 0.02</td>
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<td>$\phi_3$</td>
<td>0.358 ± 0.002</td>
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</tr>
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<td>$x^2$/NDF</td>
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<td>52/35</td>
</tr>
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<td>0.02674</td>
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<tr>
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<td>$a_5$</td>
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<td>0.66 ± 0.02</td>
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</tr>
<tr>
<td></td>
<td>$x^2$/NDF</td>
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<td>13/31</td>
<td>23/31</td>
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</table>

Data from Fig. 14

$p_T < 0.15$ GeV/c

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<th>Negatives only</th>
<th>Positives only</th>
<th>Unlike charged</th>
</tr>
</thead>
<tbody>
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<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2$</td>
<td>0.792 ± 0.002</td>
<td>0.756 ± 0.004</td>
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<tr>
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<td>$\phi_3$</td>
<td>0.046 ± 0.002</td>
<td>0.053 ± 0.002</td>
<td>0.046 ± 0.002</td>
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<td>$x^2$/NDF</td>
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<td>3.6/35</td>
<td>8/35</td>
</tr>
<tr>
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<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2'$</td>
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<td>0.772 ± 0.064</td>
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<tr>
<td></td>
<td>$b_2'$</td>
<td>0.18 ± 0.13</td>
<td>0.33 ± 0.12</td>
<td>0.56 ± 0.19</td>
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<tr>
<td></td>
<td>$c$</td>
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<td>-0.35 ± 0.02</td>
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<tr>
<td></td>
<td>$x^2$/NDF</td>
<td>1.1/34</td>
<td>1.3/34</td>
<td>48/34</td>
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<td>$a_3$</td>
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<td>$\phi_3$</td>
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<tr>
<td></td>
<td>$x^2$/NDF</td>
<td>2.9/35</td>
<td>3.3/35</td>
<td>3.8/35</td>
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</table>

$p_T > 0.15$ GeV/c

<table>
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<tr>
<th>Order</th>
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<th>Negatives only</th>
<th>Positives only</th>
<th>Unlike charged</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>range</td>
<td>0.02674</td>
<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2$</td>
<td>1.196 ± 0.002</td>
<td>1.085 ± 0.002</td>
<td>0.999 ± 0.002</td>
</tr>
<tr>
<td></td>
<td>$\phi_3$</td>
<td>0.032 ± 0.001</td>
<td>0.081 ± 0.001</td>
<td>0.061 ± 0.001</td>
</tr>
<tr>
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<td>$x^2$/NDF</td>
<td>11.6/35</td>
<td>18/35</td>
<td>30/35</td>
</tr>
<tr>
<td>2</td>
<td>range</td>
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<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_2'$</td>
<td>0.200 ± 0.002</td>
<td>0.19 ± 0.06</td>
<td>0.09 ± 0.02</td>
</tr>
<tr>
<td></td>
<td>$b_2'$</td>
<td>0.154 ± 0.004</td>
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<td>0.39 ± 0.06</td>
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<tr>
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<td>$c$</td>
<td>0</td>
<td>-0.09 ± 0.06</td>
<td>-0.07 ± 0.03</td>
</tr>
<tr>
<td></td>
<td>$x^2$/NDF</td>
<td>9.9/35</td>
<td>1.7/34</td>
<td>1.1/34</td>
</tr>
<tr>
<td>3</td>
<td>range</td>
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<td>0.02674</td>
<td>0.02674</td>
</tr>
<tr>
<td></td>
<td>$a_3$</td>
<td>0.691 ± 0.007</td>
<td>1.26 ± 0.01</td>
<td>1.085 ± 0.006</td>
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<td></td>
<td>$\phi_3$</td>
<td>0.107 ± 0.003</td>
<td>0.269 ± 0.006</td>
<td>0.202 ± 0.004</td>
</tr>
<tr>
<td></td>
<td>$x^2$/NDF</td>
<td>27/35</td>
<td>21/35</td>
<td>22/35</td>
</tr>
</tbody>
</table>

and $p_T > 0.15$ GeV/c (lower plots), respectively. The results of the fits with formulae (1) and (19) are summarized in Table 6.b.

Figs. 13 and 14 reveal several interesting features:

i) The difference in $-\ln Q^2$ dependence of $F_2^S(- -)$ and $F_2^S(- -)$ seen in Fig. 13 is essentially due to the particles with $p_T > 0.15$ GeV/c, where $F_2^S(- -)$ even decreases.

ii) The slope $\phi_2(- -)$ is larger for $p_T > 0.15$ GeV/c than for $p_T < 0.15$ GeV/c. This effect is much less pronounced with a $p_T$-cut of 0.3 GeV/c (not shown here).
iii) $\phi_2(-,-)$ is smaller for $p_T < 0.15$ GeV/c than for the uncut data sample.

From the last two observations (for negative) one would conclude that in $Q^2$ the intermittency effect is weaker at low $p_T$ than at higher $p_T$. However, in a one-dimensional (rapidity) analysis of the all-charged sample in this experiment, it was concluded that, on the contrary, intermittency is strongest when small-$p_T$ particles are selected [11]. This is still visible when comparing $\phi_2$ and $\phi_3$ for all-charged sample for $p_T < 0.15$ GeV/c and $p_T > 0.15$ GeV/c in Table 6b.

The reason for this apparent contradiction lies in the fact that factorial moments such as $F_2$ have very different $-\ln Q^2$ and $p_T$-dependence for like-charged ($\pm$) and unlike-charged ($\mp$) pairs. This hampers easy interpretation of the all-charged sample. It is therefore dangerous to base conclusions about dynamical properties on all-charged data only, without proper analysis of the different charge combinations.

A further warning is needed concerning the interpretation of correlation-integral results for data samples in restricted $p_T$ intervals. In [55] it is shown that the stronger rise of $F_2(\delta y)$ with decreasing $\delta y$, and the slower rise of $F_2^S(Q^2)$ with decreasing $Q^2$ for small-$p_T$ particles, as observed in our data, is to a large extent a consequence of a kinematical cut on the two-particle invariant-mass distribution. This result is obtained under the assumption that the two-particle correlation function is a rapidly decreasing function of the invariant-mass in restricted $p_T$-intervals (or $Q^2$) without an explicit dependence on the other kinematical variables of the pair. Integration of such a correlation function over the appropriate variables
in a $p_T$-restricted phase space leads to the observed results.

For the uncut sample, the modified power law (8) and the Lévy index $\mu$ have been examined. The Ochs-Wosiek plot is given in Fig. 15, the fit results are given in Table 7. Although the values of $\mu$ are smaller than those obtained for $F_2^S$ (box vol), they are still larger than 2, thus confirming that the density fluctuations are not of the Lévy type.

5.4 Influence of possible biases

At very small $Q^2$, the increase of $F_2^S(+-)$ with decreasing $Q^2$ is at least partially due to Dalitz misidentification and undetected $\gamma$-conversions. To estimate this effect, a procedure borrowed from [58] is followed. First, the differential form of the density integral, $D_2^S$ (for definition analogous to (43), below), is determined with distance measure $M_{inv} - 2m_e$ near threshold, where this variable is very sensitive to biases like $e^\pm$ misidentification and double counting of tracks.

In Fig. 16, $D_2^S(M_{inv} - 2m_e)$ is shown for the different charge combinations. In all cases the data indeed exhibit a sharp peak near zero, pointing to a possible bias of double counting (Fig. 16c and d) and Dalitz decay or $\gamma$-conversion (Fig. 16b). Events contributing to the first bin

in Figs. 16c and d have been investigated visually on the scanning table for double counting of single tracks. The corresponding tracks could be positively identified as double tracks by charge conservation, double minimum ionization and/or visible separation at the end of the sensitive volume of the bubble chamber. Furthermore, it has been verified that the small mass peaking can be qualitatively reproduced by the FRITIOF Monte Carlo when including Bose-Einstein correlations (see further Sect. 5.5 below). The peak for like-charged pairs is therefore considered to be real. The peak at small mass for unlike-charged pairs can be reproduced by FRITIOF only when $\gamma$ conversion is introduced as a bias.

To see the influence of the peaks on $F_2^S(Q^2)$, every pair that contributes to $(M_{inv} - 2m_e) < 0.002$ GeV is given a weight such that $D_2^S(M_{inv} - 2m_e) = 2$ in our case (of course this cut can be applied separately and with different cut values for the various charge combinations). The influence of the peaks on $F_2^S(Q^2)$ is shown in Fig. 17. Removing the $(+-)$-peak gives the most dramatic effect.

The influence on the fit parameters of (1) and (19) can be judged from Table 8. The intermittency indices $\phi_2$ and $b^\prime$ decrease, but do not become compatible with zero. Also the constant $c$ of (19), which is supposed to take into account possible non-singular long-range correla-

![Fig. 16. The second-order differential density function in $M_{inv} - 2m_e$ for a $cc$ pairs, b $++$ pairs, c $--$ pairs and d $++$ pairs. This plot serves as a test for possible biases, such as: Dalitz misidentification, undetected $\gamma$-conversions and track double counting](image)

![Fig. 17. Influence of suppression of small $M_{inv} - 2m_e$ values on a $F_2^S(+-)$, b $F_2^S(-+)$, c $F_2^S(++)$ and d $F_2^S(cc)$ where $Q^2$ is used as distance measure](image)
Table 8. Results of fits to the data presented in Fig. 17 according to (1) and (19), respectively (Range in $Q^2$: $1.0 - 0.9419 \times 10^{-3} (\text{GeV})^2$)

<table>
<thead>
<tr>
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<th>No cut</th>
<th>Cut on unlike charged pairs</th>
<th>Cut of like charged pairs</th>
<th>Cut on both</th>
</tr>
</thead>
<tbody>
<tr>
<td>All charged</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>$a'$</td>
<td>$0.09 \pm 0.01$</td>
<td>$0.15 \pm 0.02$</td>
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</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.36 \pm 0.02$</td>
<td>$0.27 \pm 0.02$</td>
<td>$0.31 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
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<td>$0.11 \pm 0.02$</td>
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<tr>
<td></td>
<td>$\chi^2$/NDF</td>
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<td>$15/67$</td>
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<td>Unlike charged</td>
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<td>$\chi^2$/NDF</td>
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<tr>
<td>Negatives only</td>
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</tr>
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<td>$\phi_2$</td>
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<td></td>
<td>$0.084 \pm 0.001$</td>
</tr>
<tr>
<td></td>
<td>$\chi^2$/NDF</td>
<td>$97/68$</td>
<td>$32/68$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a'$</td>
<td>$0.25 \pm 0.03$</td>
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<td>$0.52 \pm 0.09$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.26 \pm 0.02$</td>
<td></td>
<td>$0.15 \pm 0.02$</td>
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<tr>
<td></td>
<td>$c$</td>
<td>$-0.11 \pm 0.03$</td>
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<td>$-0.38 \pm 0.09$</td>
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<td>$\chi^2$/NDF</td>
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<td></td>
<td>$c$</td>
<td>$-0.13 \pm 0.02$</td>
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<td>$18/67$</td>
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Table 9. Results of fits to the data presented in Fig. 18 according to (1) and (19), respectively (Fit range in box volume: $0.125 - 0.000125$)

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<th>Cut of like charged pairs</th>
<th>Cut on both</th>
</tr>
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<td>$0.017 \pm 0.004$</td>
<td>$0.025 \pm 0.007$</td>
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<td>$b'$</td>
<td>$0.37 \pm 0.03$</td>
<td>$0.32 \pm 0.03$</td>
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</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0.19 \pm 0.01$</td>
<td>$0.17 \pm 0.01$</td>
<td>$0.18 \pm 0.01$</td>
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<td></td>
<td>$\chi^2$/NDF</td>
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<td>$8.9/43$</td>
<td>$7.5/43$</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.58 \pm 0.07$</td>
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<td></td>
</tr>
<tr>
<td>Negatives only</td>
<td>$a'$</td>
<td>$0.057 \pm 0.0007$</td>
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<td>$0.062 \pm 0.001$</td>
</tr>
<tr>
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<td>$b'$</td>
<td>$0.303 \pm 0.002$</td>
<td></td>
<td>$0.282 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\chi^2$/NDF</td>
<td>$8.5/44$</td>
<td></td>
<td>$11/44$</td>
</tr>
<tr>
<td>Positives only</td>
<td>$a'$</td>
<td>$0.0293 \pm 0.0004$</td>
<td></td>
<td>$0.0304 \pm 0.0004$</td>
</tr>
<tr>
<td></td>
<td>$b'$</td>
<td>$0.344 \pm 0.002$</td>
<td></td>
<td>$0.334 \pm 0.002$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\chi^2$/NDF</td>
<td>$17/44$</td>
<td></td>
<td>$19/44$</td>
</tr>
</tbody>
</table>

The same analysis has been repeated for $F_2^S$ (box vol) and the ordinary three-dimensional factorial moments. It leads to the same conclusions. Because of smaller errors only $F_2^S$ (box vol) will be discussed. As in the case of $Q^2$, only the second order is sensitive to the peaks in Fig. 16. The $F_2^S(+ +)$ and $F_2^S(- -)$ change within errors (see Fig. 18). However, $F_2^S(+-)$ becomes almost completely flat and the fit parameters $\phi_2$ and $b'$ diminish dramatically (Table 9). The effect of the different cuts on the all-charged sample gives results that are very close to each other.
One should be aware that in the analysis done in this section, one assumes that the peaks in $D_1^{S}(M_{\gamma \gamma} - 2m_{\pi})$ near zero are completely caused by experimental biases (already disproved for the case of like-charged pairs). The results therefore reflect the worst possible case. In addition, note that in Figs. 13–15 the data are only given for $-\ln Q^2 < 3.64$ ($Q^2 > 0.0264 \text{GeV}^2$), which is far away from the region where possible biases have influence on the data.

5.5 Monte-Carlo models

In previous analysis [11] it has been shown that the increase of $F_\gamma(\delta \gamma)$ with decreasing $\delta \gamma$ cannot be reproduced by presently used models for hadron-hadron collisions. We here repeat the analysis with FRITIOF, but shall add Bose-Einstein correlation as well as a bias from Dalitz decay and $\gamma$-conversion. For consistency with earlier NA22 model comparisons, FRITIOF 2.0 [29] is used here, since this version is best tuned to our data. In particular, in JETSET, $\rho$, $\eta$ and $\eta'$ production is reduced with respect to the standard version to reproduce our [59, 28] and other [60] data on these resonances. We refer to this version as "plain".

Dalitz decay is treated according to the procedure used in the Monte Carlo. Contamination from undetected $\gamma$-conversions has been studied in [26] and is introduced into the model using the rate (0.25% of all $\gamma$'s), $\gamma$ effective mass distribution and electron-energy ratio as estimated from the detected $\gamma$'s.

Bose-Einstein correlations have been studied in NA22 data [61] and are introduced into FRITIOF by means of the routine LUBOEI of JETSET 7.3 using an exponential $(1 + A \exp(-r Q^2))$, with $A$ and $r$ taken from a fit to the NA22 data (see Table 10, below). This form is steeper than the conventionally used Gaussian form $(1 + \lambda \exp(-r^2 Q^2))$, but flatter than the power-law behavior of intermittency.

The Monte Carlo results are given in Fig. 19a for the case of $F_2^{S}$ and Fig. 19b for the case of $F_1^{S}$. The 'plain' version is not able to describe the data for any charge combination of any order. The predictions show no increase at all; they even decrease for small $Q^2$. After including Bose-Einstein correlations, the model results for $F_2^{S}(--)$ differ from the data almost only by a shift. However, for $F_1^{S}$ the Bose-Einstein effect used in the model is too strong.

After adding a bias in order to take into account Dalitz decay and (0.25%) undetected $\gamma$-conversions, the FRITIOF results for $F_1^{S}$ show the same increase as the data, but the values stay too low.

5.6 Bose-Einstein correlation: Gaussian or power law?

Instead of counting the number of $q$-tuples with distance smaller than $Q^2$, it is possible to count the number of $q$-tuples with a distance in the interval $[Q^2 - \delta, Q^2]$. This leads to a differential form of $F_1^{S}$, the correlation (density) function $D_1^{S}$. Equation (42) becomes

$$D_1^{S}(Q^2) = \frac{1}{\text{Norm}} \left[ \sum_{l_1 < \ldots < l_q} \prod_{k_1, k_2}^{\text{all pairs}} \{\Theta(Q^2 - Q_{l_1, l_2}^{(\text{ev})}) \times \Theta(Q_{l_1, l_2}^{(\text{ev})} - Q^2 + \delta)\} \right],$$

where the normalization factor is determined analogously to the one in (42).

For $q = 2$, this function, plotted in Fig. 20 for different charge combinations, is closely related to the function $R(Q^2)$ used in pion-interferometry (see [61] for application to our data). In order to increase statistics, the negative-only sample and the positive-only sample are combined into the like-charged sample. As in the case of the integral form, the strongest increase is found for the like-charged sample. A cut at $(M_{\gamma \gamma} - 2m_{\pi}) < 0.002 \text{GeV}$ does not influence the data shown in Fig. 20, since this corresponds to $Q^2 \approx 0.001 \text{GeV}^2$.

In an attempt to understand the role of BE correlations in the steep rise of $D_1^{S}$ (like charged), the following functions are fitted to the data (see [56] for a similar analysis of UA1 data):

* Note that on Fig. 19, as in Fig. 17, the data are shown for $Q^2$ down to $10^{-3} \text{GeV}^2$, while in Figs. 13 and 14, the smallest $Q^2$ is 0.0026
Fig. 19. a $\ln F_2^f$ in the data (full circles) compared to FRITIOF2.0, FRITIOF2.0 with BE and FRITIOF2.0 with BE, Dalitz decay and $\gamma$-conversion (open symbols), for (cc), (---), and (---) combinations, as indicated; b same as Fig. a for $F_2^f$.

- power law
  
  $D_2^f(Q^2) = a + b(Q^2)^{-\alpha}$

- exponential
  
  $D_2^f(Q^2) = a \exp(-rQ)$

- double exponential
  
  $D_2^f(Q^2) = a(1 + 2\lambda(1-\lambda)\exp(-rQ) + \lambda^2\exp(-2rQ))$

Fig. 20. The second-order differential density function in $Q^2$ for a like-charged particles, b unlike-charged particles and c all charged particles. In a the following fits are shown: power law (full line), exponential (dashed line), double exponential (dotted line) and Gaussian (dash-dot).
Table 10. Results of fits to the data presented in Fig. 20a

\[
a + b(Q^2)^{-\gamma}
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (with errors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0.56 \pm 0.09)</td>
</tr>
<tr>
<td>(b)</td>
<td>(0.40 \pm 0.09)</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>(0.17 \pm 0.03)</td>
</tr>
<tr>
<td>(\chi^2/\text{NDF})</td>
<td>(22.0/36)</td>
</tr>
</tbody>
</table>

\[
a(1 + b \exp(-rQ))
\]

<table>
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<tr>
<th>Parameter</th>
<th>Value (with errors)</th>
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</thead>
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<tr>
<td>(a)</td>
<td>(0.97 \pm 0.01)</td>
</tr>
<tr>
<td>(b)</td>
<td>(0.72 \pm 0.03)</td>
</tr>
<tr>
<td>(r(\text{GeV}^{-1}))</td>
<td>(4.0 \pm 0.3)</td>
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<tr>
<td>(r(\text{fm}))</td>
<td>(0.79 \pm 0.05)</td>
</tr>
<tr>
<td>(\chi^2/\text{NDF})</td>
<td>(40.4/36)</td>
</tr>
</tbody>
</table>

\[
a(1 + 2 \lambda (1 - \lambda) \exp(-rQ) + \lambda^2 \exp(-2rQ))
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (with errors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0.96 \pm 0.01)</td>
</tr>
<tr>
<td>(b)</td>
<td>(0.53 \pm 0.03)</td>
</tr>
<tr>
<td>(r(\text{GeV}^{-1}))</td>
<td>(3.3 \pm 0.2)</td>
</tr>
<tr>
<td>(r(\text{fm}))</td>
<td>(0.65 \pm 0.04)</td>
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<tr>
<td>(\chi^2/\text{NDF})</td>
<td>(35.7/36)</td>
</tr>
</tbody>
</table>

\[
a(1 + b \exp(-Q^2/(2 \sigma^2)))
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (with errors)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(0.996 \pm 0.005)</td>
</tr>
<tr>
<td>(b)</td>
<td>(0.41 \pm 0.01)</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>(0.25 \pm 0.01)</td>
</tr>
<tr>
<td>(\chi^2/\text{NDF})</td>
<td>(123/36)</td>
</tr>
</tbody>
</table>

- **Gaussian**

\[
D_S^S(Q^2) = a(1 + b \exp(-Q^2/(2 \sigma^2))). \quad (47)
\]

While the power law (44) would lead to intermittency, the exponential and mainly the Gaussian forms correspond to conventional parametrization of Bose-Einstein correlation.

All fits are superposed on Fig. 20a, the values of the fit parameters are collected in Table 10. The power-law fit gives the best result, but in the region where the distinction can be made the statistical errors are large.

For consistency check, the parameters (Table 10) of the exponential fit have been used to parametrize Bose-Einstein correlations in FRITIOF, as already done in Fig. 19. It has been verified that the FRITIOF results indeed follow the exponential form in Fig. 20a (dashed). Since a reflection from \(\eta'\) decay is expected at low \(Q^2\), the FRITIOF run has been repeated without \(\eta'\) decay. An effect of \(\eta'\) is present around \(Q^2 = 10^{-2} \text{ GeV}^2\), but not at smaller \(Q^2\).

If the power law is confirmed in the small \(Q^2\) region by future experiments, this is in contradiction to the conventional Gaussian or Bessel-type parametrization of Bose-Einstein correlations. Furthermore, it is important to note that even in the larger \(Q^2\) region (0.006 < \(Q^2\) < 1 GeV\(^2\)) conventional Bose-Einstein parametrization and power law are indistinguishable. So, self-similarity of the correlation function is in fact even there an interpretation alternative to the conventional view of Bose-Einstein correlations (the latter relating the low \(Q^2\) enhancement to the static size of an interaction region).

Historically speaking, it should be noted that another differential form of the correlation-integral technique was introduced and used a long time ago by Berger et al. [62, 63] in a study of the invariant-mass \((M_{\text{inv}})\) dependence of the two-pion inclusive correlation function. Using data from a 205 GeV/c \(pp\) experiment at FNAL, these authors have shown that the \((+ -)\) and \((- -)\) correlation function is significantly different from zero only for invariant masses below 1.5 and 0.6 GeV/c\(^2\), respectively. Moreover, at small invariant mass, the second-order cumulants \(K_2^+(M_{\text{inv}})\) and \(K_2^-(M_{\text{inv}})\) show power-law behavior, with the ratio

\[
\frac{K_2^+(M_{\text{inv}})}{K_2^-(M_{\text{inv}})}
\]

behaving as \((\frac{1}{M_{\text{inv}}^2})\). The results were interpreted in the Mueller-Regge picture as well as an “exclusive” picture, where most of the correlation in the threshold region is explained from resonance decay into three or more pions [63].

These early results agree qualitatively and quantitatively* with the data presented here and confirm that the strong rise of factorial moments with decreasing \(Q^2\) must be attributed to like-charged pion effects, an obvious candidate being a low-mass enhancement caused by B.E. symmetrization. The latter conclusion may support the view recently developed in [65]. There, intermittency is explained from Bose-Einstein correlation between (like-charged) pions with the power-law behavior obtained from fluctuations in the size and/or the shape of the source. This latter effect can be explained e.g. from the self-organized criticality of parton-avalanches. Though at this moment only speculative, it is an interesting new view. This explanation does, for example, not need local parton hadron duality (LPHD) in fragmentation, since intermittency is explained as a final state interaction.

If intermittency is indeed caused by final-state effects (including B.E. symmetrization), one should expect the correlations to exhibit “universal” properties at small invariant mass, irrespective of the process producing the multi-hadron final state. The observation recently made by Fialkowski [42] for hadronic collisions is consistent with such an interpretation. On the contrary, if intermittency is somehow related to parton-shower evolution (an approximately scale-invariant branching process) and LPHD, one expects to find significant differences in the low-mass correlation function for \(e^+e^-\), on the one hand, and \(hh, hA\) and \(AA\) collisions, on the other hand. We, therefore, encourage a systematic reanalysis of the large volume of data now available, using differential analysis techniques, in terms of physically sensible and Lorentz-invariant kinematical variables.

* A detailed analysis of the NA22 data using the differential method of [62] will be published elsewhere [64]. Preliminary results are given in [55].
6 Conclusions

1. The rise of the factorial moments with decreasing bin size is stronger when evaluated in three than in lower dimensions. The ratios of slopes are easier to obtain than the slopes themselves. Contrary to earlier observation, they turn out to depend on the dimension of the analysis and, in 3 dimensions, become too large to be interpreted as due to fluctuations of the Lévy type. The $\phi$-dependence of these ratios excludes a phase transition as possible source of intermittency.

2. Factorial cumulants show evidence for non-zero genuine multi-particle correlations, increasing in magnitude with decreasing phase-space volume. In the last section a promising method of correlation study is applied. The results show that a large part of the observed intermittency effect originates from correlations among like-charged particles. Indication is, however, found for an interpretation alternative to the conventional view of Bose-Einstein correlations.

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