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$D_n(A)$ for a Class of Polynomial Automorphisms and Stably Tameness

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In this paper we introduce a set, denoted by $D_n(A)$, for every commutative ring A and every positive integer n. It is shown that the elements of this set can be used to give an explicit description of the class $H_n(A)$ introduced in van den Essen and Hubbers [*J. Algebra* **187** (1997), 214–226]. We deduce that each polynomial map of the form $F = X + H$ with $H \in H_n(A)$ can be written as a finite product of automorphisms of the form $\exp(D)$, where each D is a locally nilpotent derivation satisfying $D^2(X_i) = 0$ for all i. Furthermore we deduce that all such Fs are stably tame.

1. NOTATION, DEFINITIONS, AND AN EXPLICIT DESCRIPTION OF THE CLASS $H_n(A)$

1.1. *Notation*

Throughout this paper A denotes an arbitrary commutative ring and $A[X] := A[X_1, \ldots, X_n]$ denotes the polynomial ring in n variables over A. Furthermore if $G = (G_1, \ldots, G_n) \in A[X]^n$ and $S = (S_{ij}(X)) \in M_{p,q}(A[X])$ then $S(G)$ or $S_{|G}$ denotes the $p \times q$ matrix $(S_{ij}(G_1, \ldots, G_n))_{i,j}$. In particular if $F \in A[X]^n$ $(= M_{n,1}(A[X]))$ then the composition of the polynomial maps F and G, denoted $F \circ G$, is equal to $F(G)$.

Matrix multiplication will be denoted by the symbol '$*$.' So if S, $T \in M_n(A[X])$ then the matrix product of S and T is denoted by $S * T$. By X we denote the column vector $(X_1, \ldots, X_n)^t$. In the sequel we also need another multiplication in $M_n(A[X])$, which we denote by $\triangle$. This multipli-

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cation is defined as follows:

$$S \vartriangle T := S(T * X) * T$$

for all $S, T \in M_n(A[X])$.

One easily verifies that this multiplication is associative, so it makes sense to write

$$S_1 \vartriangle S_2 \vartriangle \cdots \vartriangle S_n$$

for each n-tuple $S_1, \ldots, S_n$ in $M_n(A[X])$. Sometimes we need to extend a vector of length $1 \leq p \leq n-1$ or a $p \times p$ matrix to, respectively, a vector of length n, or an $n \times n$ matrix. This is done as follows: let $1 \leq p \leq n-1$, $c \in A[X]^p$, and $T \in M_p(A[X])$. Then $\tilde{c}^n$ denotes the vector

$$\tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n,$$

obtained by extending c by $n-p$ zeros, and $\tilde{T}^n$ denotes the matrix

$$\tilde{T}^n = \begin{pmatrix} T & 0 \\ 0 & I_{n-p} \end{pmatrix} \in M_n(A[X]),$$

obtained by extending T with the $(n-p) \times (n-p)$ identity matrix. To simplify the notation we drop the superscript n and write $\tilde{c}$ and $\tilde{T}$, even sometimes when it is clear from the context that we mean $\tilde{c}^{n-1}$, respectively, $\tilde{T}^{n-1}$ instead of $\tilde{C}^n$, respectively, $\tilde{T}^n$.

Finally the adjoint of a matrix T is denoted by $\operatorname{Adj}(T)$ and if $a_1, \ldots, a_p$ are elements of a (nonnecessary commutative) ring then $\prod_{i=1}^{p} a_i$ denotes the element $a_1 \cdots a_p$.

1.2. $\mathsf{D}_n(A)$ *and the class* $\mathsf{H}_n(A)$

In [6] we introduced a new class of polynomial maps, denoted by $\mathsf{H}_n(A)$, and showed that for each $H \in \mathsf{H}_n(A)$ the Jacobian matrix JH is nilpotent and that the polynomial map $F = X + H$ is invertible over A with $\det(JF) = 1$.

Let us recall the definition of $\mathsf{H}_n(A)$.

DEFINITION 1.1. First if $n = 1$ we define $\mathsf{H}_1(A) = A$. If $n \geq 2$ we define $\mathsf{H}_n(A)$ inductively as follows: Let $H \in A[X]^n$. Then $H \in \mathsf{H}_n(A)$ if and only if there exist $T \in M_n(A)$, $c \in A^n$, and $H_\star \in \mathsf{H}_{n-1}(A[X_n])$ such that

$$H = \operatorname{Adj}(T) * \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{|T * X} + c. \tag{1}$$

The main aim of this section is to give an explicit description of the elements of $\mathsf{H}_n(A)$. Therefore we introduce some useful objects.

DEFINITION 1.2. Let $n \geq 2$. Then $\mathsf{D}_n(A)$ is the set of $(2n-1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n),$$

where $T_n \in M_n(A)$, $T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_n \in A^n$ $(= M_{n,1}(A))$ and $c_i \in M_{i,1}(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi\colon \mathsf{D}_n(A) \to \mathsf{D}_{n-1}(A[X_n])$ defined by

$$\pi\big((T_2, \ldots, T_n\, c_1, \ldots, c_n)\big) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write (T', c').

DEFINITION 1.3. Let $n \geq 2$ and $0 \leq p \leq n-2$. Then

$$E_{n,p}\colon \mathsf{D}_n(A) \to A[X]^n$$

is given by

1. $E_{n,0}((T, c)) := \mathrm{Adj}(T_n) * \tilde{c}_{n-1 \mid T_n * X}$ for all $(T, c) \in \mathsf{D}_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n-2$, then inductively (with respect to n)

$$E_{n,p}((T, c)) := \mathrm{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}((T', c')) \\ 0 \end{pmatrix}_{\mid T_n * X}$$

Instead of $E_{n,p}((T, c))$ we simply write $E_{n,p}(T, c)$.

Now we are able to give the main result of this section.

PROPOSITION 1.4. *Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in \mathsf{H}_n(A)$ if and only if there exists $(T, c) \in \mathsf{D}_n(A)$ such that*

$$H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.$$

Proof. The proof is by induction on n. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = \mathrm{Adj}(T_n) * \begin{pmatrix} H_\star \\ 0 \end{pmatrix}_{\mid T_n * X} + c_n,$$

where $T_n \in M_n(A)$, $c_n \in A^n$, and $H_\star \in \mathsf{H}_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_\star = \sum_{p=0}^{n-3} E_{n-1,p}(T^*, c^*) + c^*_{n-1}$$

for some $(T^*, c^*) \in \mathsf{D}_{n-1}(A[X_n])$. Put $(T, c) := (T^*, T_n, c^*, c_n)$ and observe that $(T, c) \in \mathsf{D}_n(A)$ and $(T', c') = (T^*, c^*)$. So

$$\begin{aligned}
H &= \sum_{p=0}^{n-3} \operatorname{Adj}(T_n) * E_{n-1,p}(T', c')_{|T_n * X} + \operatorname{Adj}(T_n) * \begin{pmatrix} c^*_{n-1} \\ 0 \end{pmatrix}_{|T_n * X} + c_n \\
&= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n \\
&= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 1.5. *Let $n \geq 2$, $0 \leq p \leq n-2$, and $(T, c) \in \mathsf{D}_n(A)$. Then*

$$E_{n,p}(T, c) = \operatorname{Adj}\left(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\right) * \tilde{c}_{n-p-1\,|\,(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n) * X}.$$

Proof. The proof is by induction on p. The case $p = 0$ is obvious. So let $p \geq 1$. Then

$$\begin{aligned}
E_{n,p}(T, c) &= \operatorname{Adj}(T_n) * \begin{pmatrix} E_{n-1,p-1}(T', c') \\ 0 \end{pmatrix}_{|T_n * X} \\
&= \operatorname{Adj}(T_n) * \Big[\operatorname{Adj}\left(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\right)_{|T_n * X} \\
&\qquad * \left(\tilde{c}_{n-p-1\,|\,(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}) * X}\right)_{|T_n * X}\Big] \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= \operatorname{Adj}\left(\left(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\right)_{|T_n * X} * T_n\right) \\
&\qquad * \tilde{c}_{n-p-1}\left(\left(\left(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1}\right)_{|T_n * X}\right) * T_n * X\right) \\
&= \operatorname{Adj}\left(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\right) * \tilde{c}_{n-p-1\,|\,(\tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n) * X}.
\end{aligned}$$

$\blacksquare$

EXAMPLE 1.6. Consider the polynomial map $F := X + H \colon \mathbb{C}^4 \to \mathbb{C}^4$, where H equals

$$\begin{pmatrix}
-X_2 X_4^2 - e_4 X_3^2 X_4 - 2\dfrac{m_4}{g_4} X_2 X_3 X_4 - g_4 X_1 X_3 X_4 - k_4 X_3^3 - \dfrac{m_4^2}{g_4^2} X_2 X_3^2 - m_4 X_1 X_3^2 \\
-X_3 X_4^2 - e_3 X_3^2 X_4 + g_4 X_2 X_3 X_4 - k_3 X_3^3 + m_4 X_2 X_3^2 + g_4^2 X_1 X_3^2 \\
-\dfrac{1}{3} X_4^3 \\
0
\end{pmatrix}$$

and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This F is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that PFP is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element (T, c) of $D_4(\mathbb{C})$, where

$$T = \left(\begin{pmatrix} 1 & 0 \\ g_4^2 X_3 & g_4 X_4 + m_4 X_3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

and

$$c = \left(\left(\frac{-1}{g_4^2} X_2 \right), \begin{pmatrix} -X_3^2(e_4 X_4 + k_4 X_3) \\ -X_3 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \dfrac{-1}{3} X_4^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$, and $E_{4,2}$ by the method of Proposition 1.5. Note that $c_4 = 0$. Since $T_4 = \tilde{T}_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$E_{4,0} = \operatorname{Adj}(T_4) * \tilde{c}_{3\,|\,T_4 * X} = \tilde{c}_3 = \begin{pmatrix} 0 \\ 0 \\ \dfrac{-1}{3} X_4^3 \\ 0 \end{pmatrix},$$

$$E_{4,1} = \operatorname{Adj}\left(\tilde{T}_3 \triangle T_4\right) * \tilde{c}_{2\,|\,(\tilde{T}_3 \triangle T_4) * X} = \tilde{c}_2 = \begin{pmatrix} -X_3^2(e_4 X_4 + k_4 X_3) \\ -X_3 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \\ 0 \\ 0 \end{pmatrix}.$$

Before we compute $E_{4,2}$ we present the following identities:

$$\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4 = \tilde{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g_4^2 X_3 & g_4 X_4 + m_4 X_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\operatorname{Adj}\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) = \begin{pmatrix} g_4 X_4 + m_4 X_3 & 0 & 0 & 0 \\ -g_4^2 X_3 & 1 & 0 & 0 \\ 0 & 0 & g_4 X_4 + m_4 X_3 & 0 \\ 0 & 0 & 0 & g_4 X_4 + m_4 X_3 \end{pmatrix};$$

$$\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) * X = \begin{pmatrix} X_1 \\ g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3 \\ X_3 \\ X_4 \end{pmatrix};$$

$$\tilde{c}_{1 \mid (\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4) * X} = \begin{pmatrix} -X_1 X_3 - \dfrac{1}{g_4} X_2 X_4 - \dfrac{m_4}{g_4^2} X_2 X_3 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

and finally

$$\begin{aligned} E_{4,2} &= \operatorname{Adj}\left(\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4\right) * \tilde{c}_{1 \mid (\tilde{T}_2 \vartriangle \tilde{T}_3 \vartriangle T_4) * X} \\ &= \begin{pmatrix} -\left(X_4 + \dfrac{m_4}{g_4} X_3\right)\left(g_4 X_1 X_3 + X_2 X_4 + \dfrac{m_4}{g_4} X_2 X_3\right) \\ X_3\left(g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3\right) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2. NICE DERIVATIONS

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated A-algebra and D a subset of $\operatorname{Der}_A(B)$. By B^D we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$.

DEFINITION 2.1. Let $D \subset \mathrm{Der}_A(B)$ be a finite subset and $\tau \in \mathrm{Der}_A(B)$.

1. We say that τ is *derived from D in at most one step* if τ is of the form $\tau = \sum_{d \in D} b_d d$, where $b_d \in B^D$ for all $d \in D$.

2. Let $m \geq 2$. We say that τ is *derived from D in at most m steps* if there exists a sequence of finite subsets

$$D = D_0, D_1, D_2, \dots, D_m$$

of $\mathrm{Der}_A(B)$ such that $\tau \in D_m$ and all elements of D_i are derived from D_{i-1} in at most one step, for all $1 \leq i \leq m$. If furthermore the elements of D satisfy $d_1 d_2(x_i) = 0$ for all $d_1, d_2 \in D$ and all i, then τ is called *nice of order* $\leq m$, with respect to $x_1, \dots, x_n$ and D.

PROPOSITION 2.2. *The notation is as in Definition* 2.1. *If* $d_1 d_2(x_i) = 0$ *for all* $d_1, d_2 \in D$ *and all* i, *then* $d_1 d_2(x_i) = 0$ *for all* $d_1, d_2 \in D_m$ *and all* i. *In particular* $d^2(x_i) = 0$ *for every nice derivation.*

Proof. We use induction on m. The case $m = 0$ is obvious since $D_0 = D$. Now let $m \geq 1$. Then $d_1 = \sum_{d \in D_{m-1}} b_d d, d_2 = \sum_{d' \in D_{m-1}} b'_{d'} d'$ with $b'_{d'}, b_{d'} \in B^{D_{m-1}}$. Then

$$d_1 d_2(x_i) = \sum_{d, d'} bd(b'_{d'}) d(x_i) + \sum_{d, d'} b_d b'_{d'}, dd'(x_i). \qquad (2)$$

Now observe that $d(b'_{d'}) = 0$ since $b'_{d'} \in B^{D_{m-1}}$ and $d \in D_{m-1}$. Finally the induction hypothesis gives $dd'(x_i) = 0$ for all $d, d' \in D_{m-1}$ and all i, so (2) implies $d_1 d_2(x_i) = 0$. ∎

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

EXAMPLE 2.3. Let $\tau = (1 + X_4 X_2 - X_5 X_3)\partial_{X_1} + X_5 \partial_{X_2} + X_4 \partial_{X_3}$, a derivation on $B := A[X_1, X_2, X_3, X_4, X_5]$. Let $D = \{\partial_{X_1}, \partial_{X_2}, \partial_{X_3}\}$. Then τ is nice of order 2 with respect to X_1, X_2, X_3, X_4, X_5, and D. To show that this is true, we present a sequence of finite subsets of $\mathrm{Der}_A(B)$,

$$D = D_0, D_1, D_2.$$

Take $D_1 := \{\partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_3}\}$ and $D_2 := \{\tau\}$. Note that in Definition

2.1 it is not demanded that the set D_i of this sequence is a subset of D_{i+1}. The only demand is that each D_i is a finite subset of $\mathrm{Der}_A(B)$. Since $X_4, X_5 \in B^D$ it follows immediately that ∂_{X_1} and $X_5 \partial_{X_2} + X_4 \partial_{X_3}$ are derived from D in one step. And from $1 + X_4 X_2 - X_5 X_3 \in B^{D_1}$ it follows that τ is derived from D_1 in one step. Obviously we have $d_1 d_2(X_i) = 0$ for all $d_1, d_2 \in D$ and hence with Proposition 2.2 also $\tau^2(X_i) = 0$.

3. DERIVATIONS ASSOCIATED WITH POLYNOMIAL MAPS

The main aim of this section is to show that for each $0 \le p \le n - 2$ the polynomial map $X + E_{n,p}(T,c)$ (where $(T,c) \in \mathcal{D}_n(A)$) is of the form $\exp(d)$, for some nice A-derivation d of $A[X]$. Observe that d is locally nilpotent if d is nice with respect to $X_1, \ldots, X_n$ since $d^2(X_i) = 0$ for all i, by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise some of the notions of Sect. 1 to arbitrary finitely generated A-algebras. So let $B := A[x_1, \ldots, x_n]$ be a finitely generated A-algebra, and let $\varphi\colon A[X_1, \ldots, X_n] \to B$ be the A-ring homomorphism defined by $\varphi(X_i) = x_i$ for all i. For each $p, q \ge 1$ consider the natural extension

$$\varphi\colon M_{p,q}(A[X_1, \ldots, X_n]) \to M_{p,q}(B).$$

Then for each $(T,c) \in \mathcal{D}_n(A)$ we define

$$E_{n,p}(T,c)(x) := \varphi\big(E_{n,p}(T,c)\big) \in B^n.$$

Now let $(\partial_1, \ldots, \partial_n)$ be an n-tuple of A-derivations of B. With each vector $b = (b_1, \ldots, b_n)^t \in B^n$ we associate the following A-derivation of B:

$$D(b; \partial_1, \ldots, \partial_n) := b_1 \partial_1 + \cdots + b_n \partial_n \qquad \left(= b^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \right).$$

To formulate the next lemma we need some more notation: Let $(T,c) \in \mathcal{D}_n(A)$. Put

$$(x'_1, \ldots, x'_n)^t := T_n * (x_1, \ldots, x_n)^t,$$

$$(\partial'_1, \ldots, \partial'_n) := \big(\mathrm{Adj}(T_n)\big)^t * (\partial_1, \ldots, \partial_n)^t,$$

$$x'' := (x'_1, \ldots, x'_{n-1}),$$

$$(T'', c'') := \big(T'(X_n = x'_n), c'(X_n = x'_n)\big) \in \mathcal{D}_{n-1}(A[x'_n]).$$

LEMMA 3.1. *Let $n \geq 3$ and $1 \leq p \leq n-2$. Then*

$$D\big(E_{n,p}(T,c)(x);\, \partial_1,\dots,\partial_n\big) = D\big(E_{n-1,p-1}(T'',c'')(x'');\, \partial'_1,\dots,\partial'_{n-1}\big).$$

Proof.

$$\begin{aligned} &D\big(E_{n,p}(T,c)(x);\, \partial_1,\dots,\partial_n\big) \\ &\quad = \big(E_{n,p}(T,c)(x)\big)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &\quad = \Big(\big(E_{n-1,p-1}(T',c')_{|T_n * x}\big)^t \quad 0\Big) * \big(\mathrm{Adj}(T_n)\big)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\ &\quad = \Big(\big(E_{n-1,p-1}(T'',c'')(x'')\big)^t \quad 0\Big) * \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_n \end{pmatrix} \\ &\quad = D\big(E_{n-1,p-1}(T'',c'')(x'');\, \partial'_1,\dots,\partial'_{n-1}\big). \qquad \blacksquare \end{aligned}$$

LEMMA 3.2. *The notation is as above. Let $a \in A$ and let $\partial_1,\dots,\partial_n$ be A-derivations of B such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j. Then*

$$\partial'_i(x'_j) = a \det(T_n)\, \delta_{ij}$$

for all i, j.

Proof. Denote the ith column of $\mathrm{Adj}(T_n)$ by $(t^*_{1i},\dots,t^*_{ni})^t$ and the jth row of T_n by $(t_{j1},\dots,t_{jn})$. Then

$$\begin{aligned} \partial'_i(x'_j) &= \left(\sum_{s=1}^n t^*_{si}\partial_s\right)\left(\sum_{s=1}^n t_{js}x_s\right) \\ &= \sum_{s=1}^n a t^*_{si} t_{js} \\ &= a\big(T_n * \mathrm{Adj}(T_n)\big)_{ji} \\ &= a \det(T_n)\, \delta_{ij}. \qquad \blacksquare \end{aligned}$$

Now we are able to prove:

THEOREM 3.3. *Let $\partial_1, \dots, \partial_n$ be A-derivations on $A[x_1, \dots, x_n]$ such that there exists an element $a \in A$ such that $\partial_i(x_j) = a\delta_{ij}$ for all i, j. Let $(T, c) \in \mathsf{D}_n(A)$. Then the A-derivation $d := D(E_{n,p}(T, c)(x); \partial_1, \dots, \partial_n)$ is nice with respect to $x_1, \dots, x_n$ and $D_0 := \{\partial_1, \dots, \partial_n\}$, for all $n \geq 2$ and all $0 \leq p \leq n-2$.*

Proof. 1. The hypotheses on the ∂_i imply that $dd'(x_i) = 0$ for all $d, d' \in D_0$ and all i.

2. First we consider the case $p = 0$. Then

$$E_{n,0}(T, c) = \operatorname{Adj}(T_n) * \tilde{c}_{n-1 \mid T_n * X}.$$

So

$$d = \left(\tilde{c}_{n-1 \mid T_n * X}\right)^t * \left(\operatorname{Adj}(T_n)\right)^t * \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix}.$$

Write $\tilde{c}^t_{n-1} = (\gamma_1(X_n), \dots, \gamma_{n-1}(X_n), 0)$. Then the definition of x'_n and the ∂'_j imply that

$$d = \left(\gamma_1(x'_n), \dots, \gamma_{n-1}(x'_n), 0\right) * \left(\partial'_1, \dots, \partial'_n\right)^t = \sum_{i=1}^{n-1} \gamma_i(x'_n)\, \partial'_i. \qquad (3)$$

Put $D_1 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and observe that $D_1 \subset \operatorname{Der}_A(B)$ and that each element of D_1 is derived from D_0 in at most one step. Finally since $\partial'_i(x'_n) = 0$ for all $1 \leq i \leq n-1$ (by Lemma 3.2) we get that $\gamma_i(x'_n) \in B^{D_1}$ for all $1 \leq i \leq n-1$. So (3) implies that d is derived from D_1 in at most one step. Consequently d is derived from D_0 in at most two steps. So d is nice with respect to $x_1, \dots, x_n$ and D_0 by case 1.

3. Now we prove the theorem by induction on n. If $n = 2$, then $p = 0$ and we are in case 2. So let $n \geq 3$. By case 2 we may assume that $p \geq 1$. Then by Lemma 3.1 we have

$$d = D\left(E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \dots, \partial'_{n-1}\right)$$

with $(T'', c'') \in D_{n-1}(A[x'_n])$. By Lemma 3.2 we can apply the induction hypothesis to the ring $A[x'_n]$ and the $(n-1)$-tuple of $A[x'_n]$-derivations $\partial'_1, \dots, \partial'_{n-1}$ on the $A[x'_n]$-algebra $B' := A[x'_n][x'_1, \dots, x'_{n-1}]$. So the $A[x'_n]$-derivation d on B' is nice with respect to $D'_0 := \{\partial'_1, \dots, \partial'_{n-1}\}$ and $x'_1, \dots, x'_{n-1}$. So there exists a sequence

$$D'_0, D'_1, \dots, D'_m$$

of finite subsets of $\mathrm{Der}_{A[x'_n]}(B')$ such that $d \in D'_m$ and D'_i is derived from D'_{i-1} in at most one step for all $1 \le i \le m$. Now observe that $D'_0 \subset \mathrm{Der}_A(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all i. Consequently if d' is an $A[x'_n]$-derivation of B' derived from D'_0 in at most one step, then $d' \in \mathrm{Der}_A(B)$. Hence $D'_1 \subset \mathrm{Der}_A(B)$. Arguing in a similar way we conclude by induction on i that $D'_i \subset \mathrm{Der}_A(B)$ for all $0 \le i \le m$. Since as remarked in case 2 above, all elements of D'_0 $(= D_1$ in case 2) are derived from D_0 in at most one step we deduce that d is derived from D_0 in at most $m + 1$ steps. Just define $D_i := D'_{i-1}$ for all $1 \le i \le m + 1$. Hence d is nice with respect to $x_1, \dots, x_n$ and D_0 by 1. ∎

COROLLARY 3.4. *Let $(T, c) \in \mathsf{D}_n(A)$ and $0 \le p \le n - 2$. Put*

$$D := D\left(E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right).$$

Then D is nice with respect to $X_1, \dots, X_n$ and $\{\partial/\partial X_1, \dots, \partial/\partial X_n\}$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

Proof. The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X_i) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)$. ∎

4. THE MAIN THEOREM

In this section we show that for every $H \in \mathsf{H}_n(A)$ the polynomial map $F = X + H$ is a product of n polynomial automorphisms of the form $\exp(D)$, where each D is a nice derivation on $A[X]$. More precisely

THEOREM 4.1. *Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n$, for some $(T, c) \in \mathsf{D}_n(A)$. Then*

$$F = \exp\left(D\left(c_n; \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)\right) \times \prod_{p=0}^{n-2} \exp\left(D\left(E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)\right).$$

Proof. Observe that

$$\exp\left(-D\left(c_n; \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}\right)\right) \circ F = \sum_{p=0}^{n-2} E_{n,p}(T, c).$$

So the case $n = 2$ follows from Corollary 3.4. Hence we may assume that $n \geq 3$. Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4. ∎

PROPOSITION 4.2. *Let $n \geq 3$, $0 \leq p \leq n - 3$, and $(T, c) \in \mathsf{D}_n(A)$. Then*

$$\exp\big(-D\big(E_{n,p}(T,c)\big)\big) \circ \left(X + \sum_{q=p}^{n-2} E_{n,q}(T,c)\right) = X + \sum_{q=p+1}^{n-2} E_{n,q}(T,c).$$

Proof. Put $G := \exp(-D(E_{n,p}(T,c)))$. So $G = X - E_{n,p}(T,c)$ (by Corollary 3.4). Hence if we put

$$U := \tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n$$

then by Proposition 1.4 we get

$$G = X - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 | U * X}.$$

So if we put

$$f := X + \sum_{q=p}^{n-2} E_{n,q}(T,c)$$

then

$$G \circ f = f - \operatorname{Adj}(U(f)) * \tilde{c}_{n-p-1 | U(f) * f}.$$

Since $U(f) = f$ (by Corollary 4.4 below, with $j = 0$) we get

$$G \circ f = f - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 | U * f}.$$

Now observe that each component of $\tilde{c}_{n-p-1}$ belongs to $A[X_{n-p}, \dots, X_n]$ and that for each $i \geq n - p$ $(U * f)_i = (U * X)_i$ (by Lemma 4.3 below). So $\tilde{c}_{n-p-1|U*f} = \tilde{c}_{n-p-1|U*X}$ and hence

$$\begin{aligned} G \circ f &= f - \operatorname{Adj}(U) * \tilde{c}_{n-p-1 | U * X} \\ &= f - E_{n,p}(T,c) \end{aligned}$$

(by Proposition 1.4) ∎

LEMMA 4.3. *Let $n \geq 3$, $0 \leq p \leq n-2$, $0 \leq j \leq p$, and $(T, c) \in \mathcal{D}_n(A)$. Put $f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c)$. Then*

$$\left[\left(\tilde{T}_{n-p+j} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\right) * f\right]_i = \left[\left(\tilde{T}_{n-p+j} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n\right) * X\right]_i$$

for all $i \geq n-p+j$.

Proof. Put $U := \tilde{T}_{n-p+j} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n$. It suffices to show that for each $q \geq p$

$$\left[U * E_{n,q}(T, c)\right]_i = 0 \tag{4}$$

for all $i \geq n-p+j$. So let $q \geq p$. Then $q \geq p-j$.

1. We first treat the case that $q = p-j$. Then $j = 0$ and $q = p$. Consequently $U = \tilde{T}_{n-p} \vartriangle \cdots \vartriangle \tilde{T}_{n-1} \vartriangle T_n$, $E_{n,q}(T, c) = E_{n,p}(T, c)$, and hence by Proposition 1.4

$$\begin{aligned} U * E_{n,q}(T, c) &= U * \mathrm{Adj}(U) * \tilde{c}_{n-p-1|U*X} \\ &= \det(U) * \tilde{c}_{n-p-1|U*X}. \end{aligned}$$

Since the last $p+1$ coordinates of $\tilde{c}_{n-p-1}$ are zero, we obtain that

$$\left[U * E_{n,q}(T, c)\right]_i = 0$$

for all $i \geq n-p$, which proves the case that $q = p-j$.

2. Now assume that $q \geq p-j+1$. So $n-q \leq n-p+j-1$. Put $V := \tilde{T}_{n-q} \vartriangle \cdots \vartriangle \tilde{T}_{n-p+j-1}$. Then by Proposition 1.4 we can write

$$\begin{aligned} E_{n,q}(T, c) &= \mathrm{Adj}(V \vartriangle U) * \tilde{c}_{n-q-1|(V \vartriangle U)*X} \\ &= \mathrm{Adj}\left(V_{|U*X} * U\right) * \tilde{c}_{n-q-1|(V \vartriangle U)*X} \\ &= \mathrm{Adj}(U) * \mathrm{Adj}\left(V_{|U*X}\right) * \tilde{c}_{n-q-1|(V \vartriangle U)*X}. \end{aligned}$$

Consequently

$$U * E_{n,q}(T, c) = \det(U) * \mathrm{Adj}\left(V_{|U*X}\right) * \tilde{c}_{n-q-1|(V \vartriangle U)*X}. \tag{5}$$

Note that V, and hence $V_{|U*X}$, is of the form $\tilde{B}$ for some $B \in M_{n-p+j-1}(A[X])$. Furthermore $(\tilde{c}_{n-q-1})_i = 0$ if $i \geq n-q$, which implies that $(\tilde{c}_{n-q-1|(V \vartriangle U)*X})_i = 0$ if $i \geq n-p+j$ (since $n-p+j > n-q$). Now the desired result (4) follows from (5). ∎

COROLLARY 4.4. *The notation is as in Lemma* 4.3. *Then*

$$\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) = \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n.$$

Proof. The proof is by induction on $N := p - j$. If $N = 0$ the result is obvious. So let $N \geq 1$. Then

$$\begin{aligned}
&\left(\tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) \\
&\quad = \tilde{T}_{n-p+j\,|\,(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n)(f) * f} * \left(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f) \\
&\quad = \tilde{T}_{n-p+j\,|\,(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n) * f} * \left(\tilde{T}_{n-p+j+1} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n\right)(f)
\end{aligned}$$

by the induction hypothesis. Finally observe that the matrix elements of $\tilde{T}_{n-p+j}$ depend only on $X_{n-p+j+1}, \ldots, X_n$. The result follows immediately from Lemma 4.3 (with $j + 1$ instead of j). ∎

5. STABLY TAMENESS

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class $H_n(A)$, and, we will also show that this result is "sharp": We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in [1–4] and [8]):

CONJECTURE 5.1. *For every invertible polynomial map $F: k^n \to k^n$ over a field k there exist $t_1, \ldots, t_m$ such that*

$$F^{[m]} = (F, t_1, \ldots, t_m): k^{n+m} \to k^{n+m}$$

is tame, i.e., F is stably tame.

THEOREM 5.2. *Let $F = X + H$ with $H \in H_n(A)$. Then F is stably tame.*

To do this we use the following result due to Martha Smith [10]:

PROPOSITION 5.3. *Let D be a locally nilpotent derivation of $A[X]$. Let $a \in \ker(D)$. Extend D to $A[X][t]$ by setting $D(t) = 0$. Note that tD is locally nilpotent. Define $\rho \in \operatorname{Aut}_A A[X][t]$ by $\rho(X_i) = X_i$, $i = 1, \ldots, n$, and $\rho(t) = t + a$. Then*

$$(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).$$

COROLLARY 5.4. *Let D and a be as in Proposition* 5.3 *If D is conjugate by a tame automorphism to a triangular derivation, then* $(\exp(aD, t))$ *is tame.*

LEMMA 5.5. *Let τ be a nice derivation of order m with respect to $X_1, \ldots, X_n$ and $D := \{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$ on $A[X]$. Then* $\exp(a\tau)$ *is stably tame for all $a \in \ker(\tau)$.*

Proof. We use induction on m. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \bigcap_{d \in D} \ker(d) = A$. Hence $\tau(X_i) \in A$ and clearly τ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(a\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \mathrm{Der}_A(A[X])$ of order $m - 1$ with respect to D and $X_1, \ldots, X_n$ and for any commutative ring A we have that $\exp(a\sigma)$ is stably tame for all $a \in \ker(\sigma)$. Let τ be nice of order m. Define ρ and extend τ to $A[X][t]$ as in Proposition 5.3 (in fact we extend all derivations of D_i to $A[X][t]$ in this way). Now from

$$(\exp(a\tau), t) = \rho^{-1} \exp(-t\tau) \rho \exp(t\tau)$$

it follows that it suffices to see that $\exp(t\tau)$ is stably tame. Now we see that $t\tau = \sum_{d \in D_{m-1}} tb_d d$ with $tb_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\begin{aligned} \exp(t\tau) &= \exp\left(\sum_{d \in D_{m-1}} tb_d d \right) \\ &= \prod_{d \in D_{m-1}} \exp(tb_d d). \end{aligned}$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(tb_d d)$ is stably tame to conclude that $\exp(t\tau)$ is stably tame. But d is a nice derivation of order $m - 1$, $tb_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t\tau)$ is stably tame and hence $\exp(a\tau)$ is stably tame. ∎

Proof of Theorem 5.2. Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in \mathcal{H}_n(A)$ can be written as the product of a finite number of $\exp(a_i D_i)$s, where each D_i is a nice derivation with respect to $X_1, \ldots, X_n$ and $\{\partial/\partial X_1, \ldots, \partial/\partial X_n\}$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 n times gives us the desired result: F is stably tame. ∎

Remark 5.6. Note that we do not give an indication of the value of m in Conjecture 5.1. As can be seen from the proof above, this m can be very high. At the highest level we have n $\exp(a_i D_i)$s, but each of these factors can give rise to a great number of extra variables, depending on the "order of niceness" of each D_i.

To conclude this paper we show that in general the automorphisms $F = X + H$ with $H \in \mathsf{H}_2(A)$ need not be tame. Actually, this idea was already presented by Nagata [9].

EXAMPLE 5.7. Let A be a domain, but not a principle ideal domain. Let $a, b \in A$ such that $Aa + Ab$ is not a principal ideal. Let $f(T) \in A[T]$ with $\deg(f) \geq 2$ and let $F = X + H$ with

$$H = \begin{pmatrix} bf(aX_1 + bX_2) \\ -af(aX_1 + bX_2) \end{pmatrix}$$

Since $H \in \mathsf{H}_2(A)$ F is an automorphism of $A[X_1, X_2]$. However, it is shown in [9] that F is not tame.

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