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D_n(A) for a Class of Polynomial Automorphisms and Stably Tameness

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In this paper we introduce a set, denoted by D_n(A), for every commutative ring A and every positive integer n. It is shown that the elements of this set can be used to give an explicit description of the class H_n(A) introduced in van den Essen and Hubbers [J. Algebra 187 (1997), 214–226]. We deduce that each polynomial map of the form F = X + H with H ∈ H_n(A) can be written as a finite product of automorphisms of the form exp(D), where each D is a locally nilpotent derivation satisfying D^i(X) = 0 for all i. Furthermore we deduce that all such F's are stably tame.

1. NOTATION, DEFINITIONS, AND AN EXPLICIT DESCRIPTION OF THE CLASS H_n(A)

1.1. Notation

Throughout this paper A denotes an arbitrary commutative ring and A[X] := A[X_1, ..., X_n] denotes the polynomial ring in n variables over A. Furthermore if G = (G_1, ..., G_n) ∈ A[X]^n and S = (S_i(X)) ∈ M_p,q(A[X]) then S(G) or S^G denotes the p × q matrix (S_i(G_1, ..., G_n)). In particular if F ∈ A[X]^n (= M_n,1(A[X])) then the composition of the polynomial maps F and G, denoted F = G, is equal to F(G).

Matrix multiplication will be denoted by the symbol ‘*’. So if S, T ∈ M_n(A[X]) then the matrix product of S and T is denoted by S * T. By X we denote the column vector (X_1, ..., X_n). In the sequel we also need another multiplication in M_n(A[X]), which we denote by ∆. This multipli-
cation is defined as follows:

\[ S \triangle T := S(T \ast X) \ast T \]

for all \( S, T \in \mathcal{M}_n(A[X]) \).

One easily verifies that this multiplication is associative, so it makes sense to write

\[ S_1 \triangle S_2 \triangle \cdots \triangle S_n \]

for each \( n \)-tuple \( S_1, \ldots, S_n \) in \( \mathcal{M}_n(A[X]) \). Sometimes we need to extend a vector of length \( 1 \leq p \leq n - 1 \) or a \( p \times p \) matrix to, respectively, a vector of length \( n \), or an \( n \times n \) matrix. This is done as follows: let \( 1 \leq p \leq n - 1 \), \( c \in A[X]^p \), and \( T \in M_p(A[X]) \). Then \( \tilde{c}^n \) denotes the vector

\[ \tilde{c}^n = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in A[X]^n, \]

obtained by extending \( c \) by \( n - p \) zeros, and \( \tilde{T}^n \) denotes the matrix

\[ \tilde{T}^n = \begin{pmatrix} T \\ 0 \\ I_{n-p} \end{pmatrix} \in M_n(A[X]), \]

obtained by extending \( T \) with the \((n - p) \times (n - p)\) identity matrix. To simplify the notation we drop the superscript \( n \) and write \( \tilde{c} \) and \( \tilde{T} \), even sometimes when it is clear from the context that we mean \( \tilde{c}^{n-1} \), respectively, \( \tilde{T}^{n-1} \) instead of \( \tilde{c}^n \), respectively, \( \tilde{T}^n \).

Finally the adjoint of a matrix \( T \) is denoted by \( \text{Adj}(T) \) and if \( a_1, \ldots, a_p \) are elements of a (nonnecessary commutative) ring then \( \prod_{i=1}^p a_i \) denotes the element \( a_1 \cdots a_p \).

1.2. \( D_n(A) \) and the class \( H_n(A) \)

In [6] we introduced a new class of polynomial maps, denoted by \( H_n(A) \), and showed that for each \( H \in H_n(A) \) the Jacobian matrix \( JH \) is nilpotent and that the polynomial map \( F = X + H \) is invertible over \( A \) with \( \det(JF) = 1 \).

Let us recall the definition of \( H_n(A) \).

**Definition 1.1.** First if \( n = 1 \) we define \( H_1(A) = A \). If \( n \geq 2 \) we define \( H_n(A) \) inductively as follows: Let \( H \in A[X]^n \). Then \( H \in H_n(A) \) if and only if there exist \( T \in M_n(A) \), \( c \in A^n \), and \( H_* \in H_{n-1}(A[X]) \) such that

\[ H = \text{Adj}(T) \ast \begin{pmatrix} H_* \\ 0 \end{pmatrix}_{|T \ast X} + c. \quad (1) \]
The main aim of this section is to give an explicit description of the elements of $H_n(A)$. Therefore we introduce some useful objects.

**Definition 1.2.** Let $n \geq 2$. Then $D_n(A)$ is the set of $(2n-1)$-tuples

$$(T, c) := (T_2, \ldots, T_n, c_1, \ldots, c_n),$$

where $T_i \in M_i(A), T_i \in M_i(A[X_{i+1}, \ldots, X_n])$ for all $2 \leq i \leq n-1$, $c_n \in A^n = M_n, (A)$ and $c_i \in D_i(A[X_{i+1}, \ldots, X_n])$ for all $1 \leq i \leq n-1$.

If $n \geq 3$ we get a natural map $\pi: D_n(A) \to D_{n-1}(A[X_n])$ defined by

$$\pi((T_2, \ldots, T_n, c_1, \ldots, c_n)) = (T_2, \ldots, T_{n-1}, c_1, \ldots, c_{n-1}).$$

Instead of $\pi((T, c))$ we often write $(T', c')$.

**Definition 1.3.** Let $n \geq 2$ and $0 \leq p \leq n-2$. Then

$$E_{n, p}: D_n(A) \to A[X]^n$$

is given by

1. $E_{n, 0}(T, c) := A d_j(T_n) * c_{n-1}/T_n * X$ for all $(T, c) \in D_n(A)$.
2. If $n \geq 3$ and $1 \leq p \leq n-2$, then inductively (with respect to $n$)

$$E_{n, p}(T, c) := A d_j(T_n) * \left( E_{n-1, p-1}(T', c') \right)_{|T_n * X}$$

Instead of $E_{n, p}(T, c)$ we simply write $E_{n, p}(T, c)$.

Now we are able to give the main result of this section.

**Proposition 1.4.** Let $n \geq 2$ and $H \in A[X]^n$. Then $H \in H_n(A)$ if and only if there exists $(T, c) \in D_n(A)$ such that

$$H = \sum_{p=0}^{n-2} E_{n, p}(T, c) + c_n.$$

**Proof.** The proof is by induction on $n$. The case $n = 2$ is obvious, so let $n \geq 3$. Then

$$H = A d_j(T_n) * \left( H_* \right)_{|T_n * X} + c_n,$$

where $T_n \in M_n(A), c_n \in A^n, H_* \in H_{n-1}(A[X_n])$. So by the induction hypothesis we have

$$H_* = \sum_{p=0}^{n-3} E_{n-1, p}(T^*, c^*) + c_{n-1}.$$
for some \((T^*, c^*) \in \mathbb{D}_{n-\mathcal{A}(A,X_n)}\). Put \((T, c) := (T^*, T_n, c^*, c_n)\) and observe that \((T, c) \in \mathbb{D}_{n}(A)\) and \((T', c') = (T^*, c^*)\). So
\[
H = \sum_{p=0}^{n-3} \text{Adj}(T_n) * E_{n-1,p}(T', c') |_{T_n} X + \text{Adj}(T_n) * \left( \begin{array}{c} c_{n-1}^* \\ 0 \end{array} \right) |_{T_n} X + c_n
\]
\[
= \sum_{p=1}^{n-2} E_{n,p}(T, c) + E_{n,0}(T, c) + c_n
\]
\[
= \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n.
\]

**Proposition 1.5.** Let \(n \geq 2\), \(0 \leq p \leq n - 2\), and \((T, c) \in \mathbb{D}_{n}(A)\). Then
\[
E_{n,p}(T, c) = \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T) * \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T) * X.
\]

**Proof.** The proof is by induction on \(p\). The case \(p = 0\) is obvious. So let \(p \geq 1\). Then
\[
E_{n,p}(T, c) = \text{Adj}(T_n) * \left( E_{n-1,p-1}(T', c') \right) |_{T_n} X
\]
\[
= \text{Adj}(T_n) * \left[ \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) |_{T_n} X \right]
\]
\[
\times \left( \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) * X \right) |_{T_n} X
\]
(by the induction hypothesis)
\[
= \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) |_{T_n} X * T_n * X
\]
\[
= \text{Adj}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) * \tilde{c}_{n-p-1}(\tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1}) * X.
\]

**Example 1.6.** Consider the polynomial map \(F := X + H: \mathbb{C}^4 \to \mathbb{C}^4\), where \(H\) equals
\[
\begin{pmatrix}
-X_2X_3^2 - e_4X_3X_4 - \frac{m_2^3}{g_4^2}X_2X_3X_4 - g_4X_2X_3X_4 - k_2X_3^2 - \frac{m_2^2}{g_4^2}X_2X_3X_4 - m_2X_2X_3^2 - m_2X_2X_3^2 \\
-X_2X_3^2 - e_4X_3X_4 + g_4X_2X_3X_4 - k_2X_3^2 + m_2X_2X_3^2 + g_4^2X_2X_3^2 \\
\frac{1}{3}X_4^3 \\
0
\end{pmatrix}
\]
and $e_3, k_3, e_4, g_4, k_4, m_4 \in \mathbb{C}$ and $g_4 \neq 0$. This $F$ is invertible. In fact if we take $P = P^{-1} = (X_4, X_3, X_2, X_1)$, we have that $PFP$ is one of the eight representatives of the cubic homogeneous maps in dimension 4 as given by Hubbers [7] and also published in [4, Theorem 2.10].

Now consider the following element $(T, c)$ of $\mathbb{D}_4(\mathbb{C})$, where

$$T = \begin{pmatrix} 1 & 0 \\ g_4^2 X_3 & g_4 X_4 + m_4 X_3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$c = \begin{pmatrix} -1 \\ g_4^2 X_2 \end{pmatrix}, \quad \begin{pmatrix} -X_3^2 (e_4 X_4 + k_4 X_3) \\ -X_3^2 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 X_4^3 \end{pmatrix}$$

Our claim is that

$$H = \sum_{p=0}^{2} E_{4,p}(T, c) + c_4.$$

To prove this we will compute $E_{4,0}$, $E_{4,1}$, and $E_{4,2}$ by the method of Proposition 1.5. Note that $c_4 = 0$. Since $T_4 = T_3 = I_4$, $E_{4,0}$ and $E_{4,1}$ are easy:

$$E_{4,0} = \text{Adj}(T_4) \ast \tilde{c}_3^{T_4 \ast X} = \tilde{c}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 X_4^3 \end{pmatrix},$$

$$E_{4,1} = \text{Adj}(T_3 \triangle T_4) \ast \tilde{c}_2^{(T_3 \triangle T_4) \ast X} = \tilde{c}_2 = \begin{pmatrix} X_3^2 (e_4 X_4 + k_4 X_3) \\ -X_3^2 X_4^2 - e_3 X_4 X_3^2 - k_3 X_3^3 \\ 0 \\ 0 \end{pmatrix}.$$
Before we compute $E_{4,2}$ we present the following identities:

\[
\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4 = \tilde{T}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
g_4^2 X_3 & g_4 X_4 + m_4 X_3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]

\[
\text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) = \begin{pmatrix}
g_4 X_4 + m_4 X_3 & 0 & 0 & 0 \\
-g_4^2 X_3 & 1 & 0 & 0 \\
0 & 0 & g_4 X_4 + m_4 X_3 & 0 \\
0 & 0 & 0 & g_4 X_4 + m_4 X_3
\end{pmatrix};
\]

\[
(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast X = \begin{pmatrix}
X_3 \\
g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3 \\
X_3 \\
X_4
\end{pmatrix};
\]

\[
\tilde{c}_1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast X = \begin{pmatrix}
-X_1 X_3 - \frac{1}{g_4} X_2 X_4 - \frac{m_4}{g_4} X_2 X_3 \\
0 \\
0 \\
0
\end{pmatrix};
\]

and finally

\[
E_{4,2} = \text{Adj}(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast \tilde{c}_1(\tilde{T}_2 \triangle \tilde{T}_3 \triangle T_4) \ast X
\]

\[
\begin{pmatrix}
-(X_4 + \frac{m_4}{g_4} X_3) & (g_4 X_1 X_3 + X_2 X_4 + \frac{m_4}{g_4} X_2 X_3) \\
X_3(g_4^2 X_1 X_3 + g_4 X_2 X_4 + m_4 X_2 X_3) \\
0 \\
0
\end{pmatrix}.
\]

It is easy to verify that $H = E_{4,0} + E_{4,1} + E_{4,2} + c_4$, which was our claim.

2. **NICE DERIVATIONS**

Let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra and $D$ a subset of $\text{Der}_A(B)$. By $B^D$ we denote the set of all $b \in B$ such that $d(b) = 0$ for all $d \in D$. 
**Definition 2.1.** Let \( D \subseteq \operatorname{Der}_A(B) \) be a finite subset and \( \tau \in \operatorname{Der}_A(B) \).

1. We say that \( \tau \) is derived from \( D \) in at most one step if \( \tau \) is of the form \( \tau = \sum_{d \in D} b_d d \), where \( b_d \in B^0 \) for all \( d \in D \).

2. Let \( m \geq 2 \). We say that \( \tau \) is derived from \( D \) in at most \( m \) steps if there exists a sequence of finite subsets

\[
D = D_0, D_1, D_2, \ldots, D_m
\]

of \( \operatorname{Der}_A(B) \) such that \( \tau \in D_m \) and all elements of \( D_i \) are derived from \( D_{i-1} \) in at most one step, for all \( 1 \leq i \leq m \). If furthermore the elements of \( D \) satisfy \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( \tau \) is called nice of order \( \leq m \), with respect to \( x_1, \ldots, x_n \) and \( D \).

**Proposition 2.2.** The notation is as in Definition 2.1. If \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D \) and all \( i \), then \( d_1 d_2(x_i) = 0 \) for all \( d_1, d_2 \in D_m \) and all \( i \). In particular \( d^2(x_i) = 0 \) for every nice derivation.

**Proof.** We use induction on \( m \). The case \( m = 0 \) is obvious since \( D_0 = D \). Now let \( m \geq 1 \). Then \( d_1 = \sum_{d \in D_{m-1}} b_d d \), \( d_2 = \sum_{d' \in D_{m-1}} b_{d'} d' \), with \( b_{d'}, b_{d'} \in B^{D_{m-1}} \). Then

\[
d_1 d_2(x_i) = \sum_{d, d'} b_d (b_{d'}) d(x_i) + \sum_{d, d'} b_d b_{d'} d d'(x_i). \tag{2}
\]

Now observe that \( d(b_{d'}) = 0 \) since \( b_{d'} \in B^{D_{m-1}} \) and \( d \in D_{m-1} \). Finally the induction hypothesis gives \( d d'(x_i) = 0 \) for all \( d, d' \in D_{m-1} \) and all \( i \), so (2) implies \( d_1 d_2(x_i) = 0 \).

We demonstrate these aspects by the so-called Winkelmann derivation. See [11].

**Example 2.3.** Let \( \tau = (1 + X_4 X_2 - X_5 X_3) \partial_{X_1} + X_2 \partial_{X_2} + X_4 \partial_{X_4} \) a derivation on \( B := A[X_1, X_2, X_3, X_4, X_5] \). Let \( D = \{ \partial_{X_1}, \partial_{X_2}, \partial_{X_4} \} \). Then \( \tau \) is nice of order 2 with respect to \( X_1, X_2, X_3, X_4, X_5 \), and \( D \). To show that this is true, we present a sequence of finite subsets of \( \operatorname{Der}_A(B) \),

\[
D = D_0, D_1, D_2.
\]

Take \( D_1 := \{ \partial_{X_1}, X_5 \partial_{X_2} + X_4 \partial_{X_4} \} \) and \( D_2 := \{ \tau \} \). Note that in Definition
2.1 it is not demanded that the set $D_i$ of this sequence is a subset of $D_{i+1}$.

The only demand is that each $D_i$ is a finite subset of $\text{Der}_A(B)$. Since $X_4, X_3 \in B^D$ it follows immediately that $\partial X_4$ and $X_2 \partial X_3 + X_4 \partial X_1$ are derived from $D$ in one step. And from $1 + X_4 X_2 - X_2 X_3 \in B^D$, it follows that $\tau$ is derived from $D_1$ in one step. Obviously we have $d_1 d_2 (X_i) = 0$ for all $d_1, d_2 \in D$ and hence with Proposition 2.2 also $\tau^2 (X_i) = 0$.

3. Derivations Associated with Polynomial Maps

The main aim of this section is to show that for each $0 \leq p \leq n-2$ the polynomial map $X + E_{n,p}(T, c)$ (where $(T, c) \in D_n(A)$) is of the form $\exp(d)$, for some nice $A$-derivation $d$ of $A[X]$. Observe that $d$ is locally nilpotent if $d$ is nice with respect to $X_1, \ldots, X_n$ since $d^2(X_i) = 0$ for all $i$, by Proposition 2.2.

In order to prove this result (see Theorem 3.3), we need to generalise some of the notions of Sect. 1 to arbitrary finitely generated $A$-algebras. So let $B := A[x_1, \ldots, x_n]$ be a finitely generated $A$-algebra, and let $\varphi: A[X_1, \ldots, X_n] \rightarrow B$ be the $A$-ring homomorphism defined by $\varphi(X_i) = x_i$ for all $i$. For each $p, q \geq 1$ consider the natural extension

$$\varphi: M_{p,q}(A[X_1, \ldots, X_n]) \rightarrow M_{p,q}(B).$$

Then for each $(T, c) \in D_n(A)$ we define

$$E_{n,p}(T, c)(x) := \varphi(E_{n,p}(T, c)) \in B^n.$$

Now let $(\partial_1, \ldots, \partial_n)$ be an $n$-tuple of $A$-derivations of $B$. With each vector $b = (b_1, \ldots, b_n)' \in B^n$ we associate the following $A$-derivation of $B$:

$$D(b; \partial_1, \ldots, \partial_n) := b_1 \partial_1 + \cdots + b_n \partial_n \in B^n.$$

To formulate the next lemma we need some more notation: Let $(T, c) \in D_n(A)$. Put

$$(x_1', \ldots, x_n') := T_n \ast (x_1, \ldots, x_n),$$

$$(\partial_1', \ldots, \partial_n') := (A d(T_n)) \ast (\partial_1, \ldots, \partial_n),$$

$$x^n := (x_1', \ldots, x_{n-1}),$$

$$(T^n, c^n) := (T' (X_n = x_n'), c' (X_n = x_n')) \in D_{n-1}(A[x_n']).$$
Lemma 3.1. Let $n \geq 3$ and $1 \leq p \leq n - 2$. Then
\[ D(E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n) = D(E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1}). \]

Proof.
\[
D(E_{n,p}(T, c)(x); \partial_1, \ldots, \partial_n) \\
= (E_{n,p}(T, c)(x))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\
= \left( \left( E_{n-1,p-1}(T', c')(x') \right)^t \begin{pmatrix} 0 \end{pmatrix} \right) \star (\text{Adj}(T_n))^t \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \\
= \left( \left( E_{n-1,p-1}(T'', c'')(x'') \right)^t \begin{pmatrix} 0 \end{pmatrix} \right) \star \begin{pmatrix} \partial'_1 \\ \vdots \\ \partial'_{n-1} \end{pmatrix} \\
= D(E_{n-1,p-1}(T'', c'')(x''); \partial'_1, \ldots, \partial'_{n-1}). \]

Lemma 3.2. The notation is as above. Let $a \in A$ and let $\partial_1, \ldots, \partial_n$ be $A$-derivations of $B$ such that $\partial_j(x_i) = a\delta_{ij}$ for all $i, j$. Then
\[
\partial'_j(x'_i) = a \det(T_n) \delta_{ij}
\]
for all $i, j$.

Proof. Denote the $i$th column of $\text{Adj}(T_n)$ by $(t_{11}^{x_i}, \ldots, t_{nn}^{x_i})^t$ and the $j$th row of $T_n$ by $(t_{j1}, \ldots, t_{jn})$. Then
\[
\partial'_j(x'_i) = \left( \sum_{s=1}^n t_{is}^{x_i} \delta_s \right) \left( \sum_{s=1}^n t_{js} x_s \right) \\
= \sum_{s=1}^n a t_{si} t_{js} \\
= a(T_n \star \text{Adj}(T_n))_{ji} \\
= a \det(T_n) \delta_{ij}. \]

\[
\boxed{}
\]
Now we are able to prove:

**Theorem 3.3.** Let \( \partial_1, \ldots, \partial_n \) be \( A \)-derivations on \( A[x_1, \ldots, x_n] \) such that there exists an element \( a \in A \) such that \( \partial_i(x_j) = a \delta_{ij} \) for all \( i, j \). Let \( (T, c) \in D_n(A) \). Then the \( A \)-derivation \( d := D(E_{n, p}(T, c)(x_1, \ldots, x_n) \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 := \{ \partial_1, \ldots, \partial_n \} \), for all \( n \geq 2 \) and all \( 0 \leq p \leq n - 2 \).

**Proof.** 1. The hypotheses on the \( \partial_i \) imply that \( dd'(x_i) = 0 \) for all \( d, d' \in D_0 \) and all \( i \).

2. First we consider the case \( p = 0 \). Then
\[
E_{n, 0}(T, c) = A d(T_n) \ast \tilde{c}_{n-1} \ast x_1.
\]
So
\[
d = (\tilde{c}_{n-1} \ast x_1) \ast (A d(T_n))^i \begin{pmatrix}
\partial_1 \\
\vdots \\
\partial_n
\end{pmatrix}.
\]
Write \( \tilde{c}_{n-1} = (\gamma_1(x_n), \ldots, \gamma_{n-1}(x_n), 0) \). Then the definition of \( x_n' \) and the \( \partial_i' \) imply that
\[
d = (\gamma_1(x_n'), \ldots, \gamma_{n-1}(x_n'), 0) \ast (\partial_1', \ldots, \partial_n') = \sum_{i=1}^{n-1} \gamma_i(x_n') \partial_i'.
\]
Put \( D_1 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and observe that \( D_1 \subset \text{Der}_A(B) \) and that each element of \( D_1 \) is derived from \( D_0 \) in at most one step. Finally since \( \partial_i'(x_n') = 0 \) for all \( 1 \leq i \leq n - 1 \) (by Lemma 3.2) we get that \( \gamma_i(x_n') \in B^{D_1} \) for all \( 1 \leq i \leq n - 1 \). So (3) implies that \( d \) is derived from \( D_1 \) in at most one step. Consequently \( d \) is derived from \( D_0 \) in at most two steps. So \( d \) is nice with respect to \( x_1, \ldots, x_n \) and \( D_0 \) by case 1.

3. Now we prove the theorem by induction on \( n \). If \( n = 2 \), then \( p = 0 \) and we are in case 2. So let \( n \geq 3 \). By case 2 we may assume that \( p \geq 1 \). Then by Lemma 3.1 we have
\[
d = D(E_{n-1, p-1}(T^n, c^n)(x^n) ; \partial_1', \ldots, \partial_{n-1}')
\]
with \( (T^n, c^n) \in D_{n-1}(A[x_n^n]) \). By Lemma 3.2 we can apply the induction hypothesis to the ring \( A[x_n^n] \) and the \( (n - 1) \)-tuple of \( A[x_n^n] \)-derivations \( \partial_1', \ldots, \partial_{n-1}' \) on the \( A[x_n^n] \)-algebra \( B' := A[x_n^n][x_1^n, \ldots, x_n^n] \). So the \( A[x_n^n] \)-derivation \( d \) on \( B' \) is nice with respect to \( D_0 := \{ \partial_1', \ldots, \partial_{n-1}' \} \) and \( x_1^n, \ldots, x_n^n \). So there exists a sequence
\[
D_0, D_1', \ldots, D_m'
\]
of finite subsets of $\text{Der}_{A}(B')$ such that $d \in D_{m}'$ and $D_{i}'$ is derived from $D_{i-1}'$ in at most one step for all $1 \leq i \leq m$. Now observe that $D_{0}' \subset \text{Der}_{A}(B)$ and that $B' \subset B$ since by definition obviously $x'_i \in B$ for all $i$. Consequently if $d'$ is an $A[x'_i]$-derivation of $B'$ derived from $D_0'$ in at most one step, then $d' \in \text{Der}_{A}(B)$. Hence $D_{i}' \subset \text{Der}_{A}(B)$. Arguing in a similar way we conclude by induction on $i$ that $D_{i}' \subset \text{Der}_{A}(B)$ for all $0 \leq i \leq m$. Since as remarked in case 2 above, all elements of $D_{0}' (= D_{1}$ in case 2) are derived from $D_{0}$ in at most one step we deduce that $d$ is derived from $D_{0}$ in at most $m + 1$ steps. Just define $D_i := D_{i-1}'$ for all $1 \leq i \leq m + 1$. Hence $d$ is nice with respect to $x_1, \ldots, x_n$ and $D_0$ by 1.

**Corollary 3.4.** Let $(T, c) \in D_n(A)$ and $0 \leq p \leq n - 2$. Put

$$D := D \left( E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right).$$

Then $D$ is nice with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$. Furthermore we have $\exp(D) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D) = X - E_{n,p}(T, c)$.

**Proof.** The first part is an immediate consequence of Theorem 3.3. Furthermore $D^2(X_i) = 0$ by Proposition 2.2. So $\exp(D)(X) = X + E_{n,p}(T, c)$ and the inverse map is given by $\exp(-D)(X) = X - E_{n,p}(T, c)$.

### 4. The Main Theorem

In this section we show that for every $H \in H_n(A)$ the polynomial map $F = X + H$ is a product of $n$ polynomial automorphisms of the form $\exp(D)$, where each $D$ is a nice derivation on $A[X]$. More precisely

**Theorem 4.1.** Let $F = X + H$, where $H = \sum_{p=0}^{n-2} E_{n,p}(T, c) + c_n$, for some $(T, c) \in D_n(A)$. Then

$$F = \exp \left( D \left( c_n, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right)$$

$$\times \prod_{p=0}^{n-2} \exp \left( D \left( E_{n,p}(T, c); \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right).$$
Proof. Observe that
\[ \exp \left( -D \left( c_n, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n} \right) \right) \circ F = \sum_{p=0}^{n-2} E_n, p (T, c). \]

So the case \( n = 2 \) follows from Corollary 3.4. Hence we may assume that \( n \geq 3 \). Now Theorem 4.1 follows directly from Proposition 4.2 below and Corollary 3.4.

Proposition 4.2. Let \( n \geq 3, 0 \leq p \leq n - 3, \) and \( (T, c) \in D_n(A) \). Then
\[ \exp (-D(E_n, p(T, c))) \circ \left( X + \sum_{q=p}^{n-2} E_{n, q}(T, c) \right) = X + \sum_{q=p+1}^{n-2} E_{n, q}(T, c). \]

Proof. Put \( G := \exp (-D(E_n, p(T, c))) \). So \( G = X - E_n, p(T, c) \) (by Corollary 3.4). Hence if we put
\[ U := T_{n-p} \triangle \cdots \triangle T_{n-1} \triangle T_n \]
then by Proposition 1.4 we get
\[ G = X - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast X}. \]

So if we put
\[ f := X + \sum_{q=p}^{n-2} E_{n, q}(T, c) \]
then
\[ G \circ f = f - \text{Ad}(U(f)) \ast \tilde{c}_{n-p-1|U(f) \ast f}. \]

Since \( U(f) = f \) (by Corollary 4.4 below, with \( j = 0 \)) we get
\[ G \circ f = f - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast f}. \]

Now observe that each component of \( \tilde{c}_{n-p-1|U \ast f} \) belongs to \( A[X_{n-p}, \ldots, X_n] \) and that for each \( i \geq n - p \) \( (U \ast f)_i = (U \ast X)_i \) (by Lemma 4.3 below). So \( \tilde{c}_{n-p-1|U \ast f} = \tilde{c}_{n-p-1|U \ast X} \) and hence
\[ G \circ f = f - \text{Ad}(U) \ast \tilde{c}_{n-p-1|U \ast X} = f - E_n, p(T, c) \]
(by Proposition 1.4)
LEMMA 4.3. Let \( n \geq 3, 0 \leq p \leq n - 2, 0 \leq j \leq p, \) and \((T, c) \in \mathbb{D}_n(A)\). 

Put \( f := X + \sum_{q=p}^{n-2} E_{n,q}(T, c) \). Then

\[
\left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) * f \right]_i = \left[ \left( \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \right) * X \right]_i
\]

for all \( i \geq n - p + j \).

Proof. Put \( U := \tilde{T}_{n-p+j} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \). It suffices to show that for each \( q \geq p \)

\[
\left[ U * E_{n,q}(T, c) \right]_i = 0
\]

for all \( i \geq n - p + j \). So let \( q \geq p \). Then \( q \geq p - j \).

1. We first treat the case that \( q = p - j \). Then \( j = 0 \) and \( q = p \). Consequently \( U = \tilde{T}_{n-p} \triangle \cdots \triangle \tilde{T}_{n-1} \triangle T_n \), \( E_{n,q}(T, c) = E_{n,p}(T, c) \), and hence by Proposition 1.4

\[
U * E_{n,q}(T, c) = U * A \text{adj}(U) * \tilde{c}_{n-p-1;y} \mathcal{X}
\]

\[
= \det(U) * \tilde{c}_{n-p-1;y} \mathcal{X}
\]

Since the last \( p + 1 \) coordinates of \( \tilde{c}_{n-p-1} \) are zero, we obtain that

\[
\left[ U * E_{n,q}(T, c) \right]_i = 0
\]

for all \( i \geq n - p \), which proves the case that \( q = p - j \).

2. Now assume that \( q \geq p - j + 1 \). So \( n - q \leq n - p + j - 1 \). Put \( V := \tilde{T}_{n-q} \triangle \cdots \triangle \tilde{T}_{n-p+j-1} \). Then by Proposition 1.4 we can write

\[
E_{n,q}(T, c) = A \text{adj}(V \triangle U) * \tilde{c}_{n-q-1;y} \mathcal{X}
\]

\[
= \text{adj}(V_{U \mathcal{X}}) \mathcal{X} * \tilde{c}_{n-q-1;y} \mathcal{X}
\]

Consequently

\[
U * E_{n,q}(T, c) = \det(U) * A \text{adj}(V_{U \mathcal{X}}) \mathcal{X} * \tilde{c}_{n-q-1;y} \mathcal{X}
\]

Note that \( V \), and hence \( V_{U \mathcal{X}} \), is of the form \( \bar{B} \) for some \( B \in M_{n-p+j-1}(A[X]) \). Furthermore \( (\tilde{c}_{n-q-1}; 0) \mathcal{X} = 0 \) if \( i \geq n - q \), which implies that \( (\tilde{c}_{n-q-1}; 0) \mathcal{X} = 0 \) if \( i \geq n - p + j \) (since \( n - p + j > n - q \)). Now the desired result (4) follows from (5). \qed
**Corollary 4.4.** The notation is as in Lemma 4.3. Then

\[
\left( \tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n \right)(f) = \tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n.
\]

*Proof.* The proof is by induction on \( N := p - j \). If \( N = 0 \) the result is obvious. So let \( N \geq 1 \). Then

\[
\left( \tilde{T}_{n-p+j} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n \right)(f) = \tilde{T}_{n-p+j}(\tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n)(f) * f = \left( \tilde{T}_{n-p+j+1} \Delta \cdots \Delta \tilde{T}_{n-1} \Delta T_n \right)(f) \]

by the induction hypothesis. Finally observe that the matrix elements of \( T_{n-p+j} \) depend only on \( X_1, \ldots, X_n \). The result follows immediately from Lemma 4.3 (with \( j + 1 \) instead of \( j \)).

---

**5. STABLY TAMENESS**

With Theorem 4.1 we are now able to prove the stably tame generators conjecture for all maps in our class \( H_n(A) \), and, we will also show that this result is “sharp”: We give an example of an element of our class which is not tame, so in general we cannot get a better result than this stable tameness.

First let us recall the conjecture (it has already been mentioned in [1–4] and [8]):

**Conjecture 5.1.** For every invertible polynomial map \( F: k^n \to k^n \) over a field \( k \) there exist \( t_1, \ldots, t_m \) such that

\[
F^{[m]} = (F, t_1, \ldots, t_m): k^{n+m} \to k^{n+m}
\]

is tame, i.e., \( F \) is stably tame.

**Theorem 5.2.** Let \( F = X + H \) with \( H \in H_n(A) \). Then \( F \) is stably tame.

To do this we use the following result due to Martha Smith [10]:

**Proposition 5.3.** Let \( D \) be a locally nilpotent derivation of \( A[X] \). Let \( a \in \ker(D) \). Extend \( D \) to \( A[X][t] \) by setting \( D(t) = 0 \). Note that \( tD \) is locally nilpotent. Define \( \rho \in \text{Aut}_A A[X][t] \) by \( \rho(X_i) = X_i, \ i = 1, \ldots, n, \) and \( \rho(t) = t + a \). Then

\[
(\exp(aD), t) = \rho^{-1} \exp(-tD) \rho \exp(tD).
\]
**Corollary 5.4.** Let $D$ and $a$ be as in Proposition 5.3. If $D$ is conjugate by a tame automorphism to a triangular derivation, then $(\exp(aD), t)$ is tame.

**Lemma 5.5.** Let $\tau$ be a nice derivation of order $m$ with respect to $X_1, \ldots, X_n$ and $D := (\partial/\partial X_1, \ldots, \partial/\partial X_n)$ on $A[X]$. Then $\exp(\alpha\tau)$ is stably tame for all $\alpha \in \ker(\tau)$.

**Proof.** We use induction on $m$. Consider the case that $m = 1$. Then $\tau = \sum_{d \in D} b_d d$ with $b_d \in A[X]^D = \bigcap_{d \in D} \ker(d) = A$. Hence $\tau(X_i) \in A$ and clearly $\tau$ is on triangular form. So now we can apply Corollary 5.4 and find that $\exp(\alpha\tau)$ is stably tame.

Now consider the case $m > 1$. We may assume that for all nice derivations $\sigma \in \text{Der}_A(A[X])$ of order $m - 1$ with respect to $D$ and $X_1, \ldots, X_n$ and for any commutative ring $A$ we have that $\exp(\alpha\sigma)$ is stably tame for all $\alpha \in \ker(\sigma)$. Let $\tau$ be nice of order $m$. Define $\rho$ and extend $\tau$ to $A[X][t]$ as in Proposition 5.3 (in fact we extend all derivations of $D_i$ to $A[X][t]$ in this way). Now from

$$(\exp(\alpha\tau), t) = \rho^{-1} \exp(-\alpha\tau) \rho \exp(\alpha\tau)$$

it follows that it suffices to see that $\exp(\alpha\tau)$ is stably tame. Now we see that $\alpha\tau = \sum_{d \in D_{m-1}} t b_d d$ with $b_d \in A[X][t]^{D_{m-1}}$. But from this it follows that

$$\exp(\alpha\tau) = \exp\left( \sum_{d \in D_{m-1}} t b_d d \right) = \prod_{d \in D_{m-1}} \exp(t b_d d).$$

This last equation follows from Proposition 1.5. Obviously it suffices to prove that each $\exp(t b_d d)$ is stably tame to conclude that $\exp(\alpha\tau)$ is stably tame. But $d$ is a nice derivation of order $m - 1$, $b_d \in \ker(d)$, and hence we can apply the induction hypothesis to the ring $A[t]$ and find that $\exp(t b_d d)$ is stably tame and hence $\exp(\alpha\tau)$ is stably tame.

**Proof of Theorem 5.2.** Now if we look at Theorem 4.1 we see that each $F = X + H$ with $H \in \mathcal{H}_c(A)$ can be written as the product of a finite number of $\exp(a_i D_i)$s, where each $D_i$ is a nice derivation with respect to $X_1, \ldots, X_n$ and $(\partial/\partial X_1, \ldots, \partial/\partial X_n)$ and $a_i \in \ker(D_i)$. Applying Lemma 5.5 $n$ times gives us the desired result: $F$ is stably tame.

**Remark 5.6.** Note that we do not give an indication of the value of $m$ in Conjecture 5.1. As can be seen from the proof above, this $m$ can be very high. At the highest level we have $n \exp(a_i D_i)$s, but each of these factors can give rise to a great number of extra variables, depending on the “order of niceness” of each $D_i$.
To conclude this paper we show that in general the automorphisms \( F = X + H \) with \( H \in H_2(A) \) need not be tame. Actually, this idea was already presented by Nagata [9].

**Example 5.7.** Let \( A \) be a domain, but not a principle ideal domain. Let \( a, b \in A \) such that \( Aa + Ab \) is not a principal ideal. Let \( f(T) \in A[T] \) with \( \deg(f) \geq 2 \) and let \( F = X + H \) with

\[
H = \begin{pmatrix}
bf(aX_1 + bX_2) \\
-af(aX_1 + bX_2)
\end{pmatrix}
\]

Since \( H \in H_2(A) \) \( F \) is an automorphism of \( A[X_1, X_2] \). However, it is shown in [9] that \( F \) is not tame.

**References**

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