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SOME EXTENSIONAL
TERM MODELS FOR
COMBINATORY LOGICS
AND
\( \lambda \)-CALCULI

H. P. Barendregt
SOME EXTENSIONAL TERM MODELS
FOR COMBINATORY LOGICS
AND \( \lambda \) - CALCULI

PROEFSCHRIFT

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door

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dit proefschrift kwam tot stand mede
onder leiding van de lector
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FOR YOU
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Introduction

This thesis consists of two parts which are, physically, separated into the present 'proefschrift' and a supplementary part II published in Barendregt [1971]. The subjects studied are formal properties of extensions of combinatory logic and of the λ-calculus, and (formal) relations between them. But neither the choice of problems nor even of the techniques introduced in the proofs can be properly understood without a description of the notions which we intend to study. Such a description will be given in this introduction. The summaries of the text which follow the introduction not only quote the main formal results but interpret them in terms of the aims described in the introduction. Strictly speaking, we do not only summarize results actually stated in the text, but include background material and, sometimes, alternative proofs.

Rules: the intended interpretations. Whatever other interesting models the formal theories considered here may have, in particular the projective limits recently introduced by Scott [1970], the original intention of the founders of our subject was to study rules; or, in other words, to study the old-fashioned notion of 'function' in the sense of definition. In contrast to Dirichlet's notion (of graph, that is the set of pairs of argument and associated value) the older notion referred also to the process of stepping from argument to value, a process coded by a definition. Generally we think of such definitions as given by words in ordinary English, applied to arguments also expressed by words (in English); or, more specifically, we may think of the
definitions as programmes for machines, applied to, that is operating on, such programmes. In both cases we have to do with a **type free** structure, permitting self application, a feature which is often held to be 'responsible' for the contradictions in the first formulations of set theory (by Frege). Similarly the first formulation of the λ-calculus turned out to be inconsistent (paradox of Kleene-Rosser [1935]). In contrast the first formulation of combinatory logic was consistent (but not some of its later extensions as shown by the paradox of Curry [1942]).

Evidently, the type free character as such is not problematic. We have not only the examples informally described in the preceding paragraph, but also many familiar and natural structures in elementary algebra where an object acts both as argument and as function. Specifically, in the theory of semigroups an element \( a \) determines the function with the action

\[
x \rightarrow ax
\]

(and in group theory an 'element' determines its co'set').

It seems plausible that the contradictions are connected with the aim of providing foundations for the whole of mathematics. Before we distinguish between different meanings of such general concepts as 'set' or 'rule' employed in different areas of mathematics, it is tempting to put down axioms and rules, some of which are valid for one meaning, some for another. In this way contradictions are liable to arise.

Be that as it may, here we do not propose to use combinatory logic or the λ-calculus as a foundation for the whole of mathematics; but rather in the study of those parts of the
subject which actually present themselves as being about rules.
A paradigm of such parts is numerical arithmetic which is
'about' rules in the literal sense that
\[ 5 + 7 = 2 + 10 \]

asserts:
The LHS and RHS reduce to the same numeral, that is to the same
normal (or canonical) form when the computation rules for
addition are applied.

From this point of view, perhaps the single most important feature
of a language (for a theory of rules) is that each term should
code its own reduction procedure; not necessarily a deterministic
one, but a class of 'equivalent' ones. Thus implicit in an inter-
pretation or 'model' of the language is an immediate reduction
relation or 'multiplication table'. In fact, in CL or the \( \lambda \)-
calculus the intended reduction procedure is only implicit, that
is we may assign a specific procedure to each term metamathematic-
ally, but the intended immediate reduction relation cannot be
expressed in the purely equational languages of CL or the \( \lambda \)-
calculus. Moreover, throughout this thesis (usually in connection
with certain 'conservative extension results') we introduce
additional symbols for reduction relations and axioms which make
more explicit the intended meaning of the formalism. By the way
our use of such additional 'structure' is parallel to the use in
set theory of ordinals and their order, and of the relation
between sets \( x \) and ordinals \( \alpha \) \( (\alpha \) is the rank of \( x ) \); the only
difference is that in the case of set theory, at least in the
presence of the axiom of foundation, the extensions considered
are not merely conservative but definitional (familiar from
Term Models. In the light of the preceding paragraph, one fundamental role of terms is clear: they are the objects on which we, literally, operate and they are the objects which code the operations. Another role, to be distinguished from that of terms-as-elements-of-the-intended-reduction-relation, is the use of terms in a formal theory of this relation, as is clear from the assertion about numerical arithmetic mentioned earlier on. The difference is particularly striking for terms of the formal theory which contain variables, when we think of the reduction relation as being a relation between closed terms (also called: interior of the term models considered below).

The reason for speaking of term models, which suggests model theory, is this. Whatever our specific intended interpretations may be, our formal theory can also be interpreted in traditional model theoretic style. In fact, some of the more general results are consequences of such superficial syntactic features as these: When regarded as theories on 'standard' formalization, CL is a purely equational system and, a fortiori, axiomatized by universal formulae, and so is the \( \lambda \)-calculus (if for each term \( \lambda x t \) we associate a function symbol \( f_t \) with \( n \) arguments, if \( \lambda x t \) contains \( n \) free variables and the axiom \( f_t a = [x/a]t^* \), where \( t^* \) results from \( t \) by replacing the \( \lambda \)-expressions in \( t \) by the associated function symbols).
By what has just been said we do not think of these applications of model theory as profound but rather as separating general (some would say 'trivial') properties from the specific properties of our intended interpretation. Adapting a phrase from Bourbaki, model theory provides here the 'hygiene' of proof theory (used for establishing the more specific properties).

A typical example of such uses of elementary model theory occurs in the analysis of extensionality. This property (of a term model) cannot be expressed equationally, but needs the logically complicated form,

$$(\forall x \ E_1) \rightarrow E_2$$

for suitable equations $E_1$ and $E_2$. Hence it cannot be expressed directly in CL. However, as far as equational consequences of (this axiom of) extensionality are concerned, they can also be generated by the purely equational rule of extensionality. Thus, in contrast to the defects of equational theories, mentioned above, for expressing the intended reduction relation, these theories are adequate for formulating and solving problems of extensionality, in consequence of the following quite general 'model theoretic' result: (For arbitrary quantifier-free $A_1$ and $A_2$, in place of equations $E_1$ and $E_2$, Shepherdson [1965] finds a more delicate rule to replace axioms $\forall x \ A_1 \rightarrow A_2$.)

Let $S$ be any set of atomic formulae of a predicate logic, closed under logical deduction (that is substitution of a term of the language for variables), and under the rule: derive $P_0^{(i)}$ for $P_1^{(i)}, \ldots, P_k^{(i)}$ $(i \in I)$ where the $P$'s are all atomic.

Then the term model of the theory (in which $P$ is interpreted as $S \vdash P$), satisfies the axioms corresponding to the rules:
∀X(P₁ ∧...∧ Pₖ) → P₀(ι).

Proof. Suppose that ∀X(P₁ ∧...∧ Pₖ) is satisfied in the term model, then P₁(ι),...,Pₖ(ι) hold for all terms ι, hence in particular for ι = x. Thus P₁,...,Pₖ all hold in that model, hence P₁,...,Pₖ are provable in S. Hence by the rule P₀ is provable in S and therefore satisfied in the model.

Thus, in particular, if k = 1, P₁ is the equation Mx = M'x and P₀ is M = M', the result mentioned above follows.

Such hygienic uses of model theory are to be distinguished from model theoretic constructions formulated by use of sophisticated notions studied in mathematical practice and (hence) well-understood; for example the use of ultraproducts. Here we may, so to speak, look at the constructions and read off properties of these models which are not at all evident from the axioms. Particularly when consistency, that is the existence of suitable models, is involved, it is not at all necessary that the constructions provide all models of the theory considered: it is more important that we have 'enough' models (for some specific purpose) and that they be manageable. Similarly Scott's lattice theoretic models are useful despite the fact established in §3.2 that not all extensional models of CL are among them.

But it is of interest to observe that the particular model left out from Scott's collection of models, is very natural from a computational view. In this model the so called fixed-points which all act in the same computational way are identified.
To conclude this discussion of the role of model theory in our study of CL and the \( \lambda \)-calculus, we may perhaps compare it with the use of non-standard models of arithmetic (as developed at the present time). We establish consistency results by analyzing computation procedures and then restate them as properties of suitable term models. Similarly, one establishes consistency results for formal arithmetic by proof theory or recursion theory (Gödel's incompleteness results) and may then apply the completeness theorem for predicate logic to infer the existence of some non-standard model of arithmetic with required properties. Probably it is fair to say that the known non-standard models are not of intrinsic interest. In contrast, our formulations for term models are directly relevant to what we are talking about since, as we have said already, we study operations on terms and operations coded by terms.

Finally a comment on versions of the \( \lambda \)-calculus or CL with types (cf. Sanchis [1967]). Here the situation is much simpler. In terms of models, the theory with types can be immediately interpreted in familiar mathematical terms. One model consists of the collections of all set-theoretic functions of finite type (over an infinite domain). For the more interesting models HRO and in extensional version HEO see Troelstra [1971]. In terms of computation, it is shown in Sanchis [1967] that in the theories with types all terms have a normal form, contrary to the situation in the type free theory.
This normalizability of all terms of the typed \( \lambda \)-calculus has also an interesting 'negative' consequence for definability problems, specifically of recursion operators (for any given types). In the type free calculus we have an \( R \) with the property that, for (variable) \( M, N \) and each numerical \( n \)

\[
\begin{align*}
RMN_0 &= N \\
RMN_{n+1} &= M(RMN_n)n
\end{align*}
\]

In contrast, there is a specific type \( \tau \) such that no term of the typed \( \lambda \)-calculus satisfies (*) for \( M, N \) of relevant types.

(A refinement is possible showing that (*) do not hold for certain closed \( M, N \).)

The proof uses the following facts.

1. The normalizability of the terms in the typed \( \lambda \)-calculus can be established by the principles of first order arithmetic. Consequently there is a provably total valuation function \( v \), assigning to each (Gödel number of a) term, the (Gödel number of) its normal form.

2. By 1. all number theoretic functions which are (locally) definable are primitive recursive in \( v \) because if the term \( F \) defines \( f \), then the Gödel number of the term \( F_n \) is \( \phi(n) \) for a primitive recursive function \( \phi \), and from \( v(\phi(n)) \) it is possible to recover \( f(n) \) primitive recursively.

Corollary. Not all functions provably recursive in first order arithmetic are represented in the typed \( \lambda \)-calculus.

3. Let \( f_\theta \) be a provably total function, with Gödel number \( e_\theta \), which is not primitive recursive in \( v \). Such an \( f_\theta \) can be defined by the use of recursion operators. For, \( f_\theta \) being
provably total we have
\[ \forall x \exists y \, T(e_0, x, y) \] in intuitionistic arithmetic.

Hence by the dialectica interpretation of Gödel [1958] there exists a term \( t \) in the typed \( \lambda \)-calculus with \( R \) such that
\[ T(e_0, n, t(n)) \] is valid for all \( n \in \omega \).
Hence \( f \) is represented by \( \lambda n. u(t(n)) \) where \( u \) represents the \( U \) of Kleene's normal form.

Now we discuss in more detail the subjects treated in the text.

**Consistency.** The first formulation of the \( \lambda \)-calculus being inconsistent, a revised formulation (the \( \lambda I \)-calculus) was proved consistent by Church and Rosser [1936]. More informatively their theorem establishes the uniqueness of the normal forms and the fact that the normal form of a normalizable term \( M \) can be found simply by reducing \( M \). This reduction can be made deterministic, by the standardization theorem, cf. Curry, Feys [1958] Ch 4E.

However, the fine-structure (i.e. whether it is from the human point of view the shortest possible reduction) of the standard reduction is not discussed.

Finally, cf. remark 1.2.18, there is now a simple proof of the Church-Rosser theorem by Martin-Löf [1971]. This will be given in appendix II.

**The relation between the \( \lambda \)-calculus and CL.** Several possibilities of mapping \( \lambda \)-terms into CL (see 1.4.6) have been discussed (cf. Curry, Feys [1958] Ch 6A). But, without extensionality, they do not preserve the set of provable equations. They do, if
extensionality is included but not the provable reductions nor the normal forms. On the other hand the translations do preserve application and therefore by 3.2.20 the solvability of closed terms. The importance of this is seen below. In particular, in the extensional case consistency results can be transferred. In this way the \( \omega \)-consistency of the \( \lambda \)-calculus follows from the corresponding result for CL.

\[\text{\( \lambda \)-definability.} \]

We begin with the \( \lambda \)-calculus since, traditionally, \( \lambda \)-definability and not combinatory definability is treated. A number theoretic partial function \( f \) is said to be \( \lambda \)-definable if there is a term \( F \) such that \( \lambda \vdash F_\mathbf{n} = m \iff f(n) = m \) and \( F_\mathbf{n} \) has no normal form if \( f(n) \) is undefined. (Here \( \mathbf{n} \) is the \( \mathbf{m} \)th numeral). We changed this definition, requiring not only \( \lambda \vdash F_\mathbf{n} = m \iff f(n) = m \), but also that \( F_\mathbf{n} \) be unsolvable if \( f(n) \) is undefined. In this case we say that \( F \) strongly defines \( f \).

The concept of strong \( \lambda \)-definability has several advantages.

(i) If \( f_1, f_2 \) are defined by terms \( F_1, F_2 \), then it is \textit{not} true that \( f_1 \circ f_2 \) is defined by \( \lambda x. F_1(F_2(x)) \). For example let \( f_1 \) be the constant zero function and let \( f_2 \) be everywhere undefined. Then \( f_1 \circ f_2 \) is totally undefined but \( \lambda x. F_1(F_2(x)) = \lambda x. 0 \) represents the constant zero function.

Hence it is not immediate that the \( \lambda \)-definable functions are closed under composition.

Use of strong definability is made by representing the composition \( f_1 \circ f_2 \) by \( F = \lambda x. (F_2 x I F_1(F_2 x)) \). If \( f_2(n) \) is undefined \( F_2 \mathbf{n} \) is unsolvable and hence \( F_\mathbf{n} \) is unsolvable, and if \( f_2(n) \) is defined \( F_2 \mathbf{n} I \) is essentially the same as I.
ii) Traditionally the $\lambda$-definability of the partial recursive functions was proved by use of Kleene's normal form theorem. The representation thus obtained is not intensional with respect to definitional equality. Using strong $\lambda$-definability we give a representation of the partial recursive functions preserving their definition trees. This is not to be regarded as a mere technical improvement but simply central to the objects which are here intended. In this context, see Kearns [1969] who uses an extension of CL to give a faithful representation of the computations of a Turing machine.

iii) Application to undecidability results, see below.

In contrast to the representation of say the primitive recursive functions in the predicate calculus, their representation in the $\lambda$-calculus is not global, that is, their defining recursion equations are not derivable for the representing terms with a free variable, but only for each numerical instance.

For example in the extensional case (this is not essential) the term $F$ with $Fxy = xy$ represents exponentiation since

$$\lambda + \text{ext } \vdash \text{pm} = \text{m}^\text{n}.$$ 

Hence the function $\text{i}^X$ is represented by $G$ with $Gx = x\text{i}$. Now we have for all numerals $\lambda + \text{ext } \vdash G\text{n} = 1$, but not $\lambda + \text{ext } \vdash Gx = 1$ since $\lambda \vdash G(K0) = K01 = 0$.

(It should be admitted that $F$ is not the standard representation of exponentiation as this will be done in §1.3, but similar examples can be given there.) It is unlikely that there exists a global representation of the primitive recursive functions at all, since in contrast to the representation in predicate calculus, all models of the $\lambda$-calculus must contain 'non-
standard' elements (such as the element denoted by K).

Undecidability results. As was mentioned above, the translation going from the λ-calculus to CL preserves solvability of closed terms. Thus undecidability results about the λ-calculus as in §1.3 transfer to CL, without any need for a parallel development of CL-definability.

We note that the role of unsolvable closed terms in the λK-calculus is similar to the role of those closed terms in the λI-calculus which have no normal form. In particular

i) In the λK-calculus it is consistent to equate all unsolvable terms (see §3.2) and in the λI-calculus it is consistent to equate all terms without a normal form (see part II).

ii) For some purposes unsolvable terms can replace familiar arguments involving negation. See for example 3.2.19 where it is proved that Con, the set of equations consistent with the λ-calculus is complete \( \Pi_1^0 \).

An example of an undecidability result which does not need strong representability is 1.3.17 for which we give here an alternative proof (cf. Smullyan [1961]).

Let A,B be disjoint r.e. sets, recursively inseparable. Define

\[
  f(x) = \begin{cases} 
    0 & \text{if } x \notin A \\
    1 & \text{if } x \in B \\
    \top & \text{else}
  \end{cases}
\]

then \( f \) is partial recursive. Let \( f \) be defined by the term \( F \).

If \( T \) were a recursive consistent extension of the λ-calculus, then \( C = \{ n \mid T \vdash \neg \exists n = \bot \} \) would be a recursive separation of \( A \) and \( B \).
A minimal extensional term model. In Chapter II we define a model of CL which is minimal in two respects. Firstly its domain consists of the closed terms only, which are, as was stated in the introduction, principal object of our study. Secondly the model is minimal with respect to the equality relation, that is to the set of terms that are equated. It is clear, that the model is generated by the ω-rule described in §2.1. Formally the theory enriched with the ω-rule is a strengthening of the rule of extensionality: to infer $M = M'$ we do not require a derivation of $Mx = M'x$ with variable $x$, but only derivations of $MZ = M'Z$ for all closed terms $Z$ (with no uniformity on these derivations).

Two questions, so to speak, at opposite poles are

a) Is the theory consistent?

b) Is the theory conservative over the λ-calculus + ext?

Ad a) In §2.2, 2.3 it is shown by transfinite induction, that the extension of CL (or the λ-calculus) by the ω-rule is consistent.

Ad b) In §2.5 it is established for equations $M = M'$ where $M$ and $M'$ are not universal generators that the ω-rule is conservative. We prove this by showing the existence of variable like closed terms $E$ such that $ME = M'E = M = M'$.

This result includes the known special case (a consequence of the theorem of Böhm [1968], cf. §2.1) of equations between normal terms since they are not universal generators.

In this connection it is to be understood that our use of closed terms which are not in normal form is not haphazard: as we see it, different meanings, that is programmes, are to be given to
certain (different) terms which have no normal form; for example if \( M \equiv \lambda x. x\Omega \) and \( N \equiv \lambda x. x\bar{\Omega} \), where \( \Omega \) does not have a normal form, then \( M, N \) have no normal form but \( MK = 0 \) and \( NK = 1 \).

Unsolvable terms. These terms were already mentioned in '\( \lambda \)-definability' above, in connection with the definability of functions, where they were needed to provide particularly hereditarily undefined values. From the computational point of view this means that such terms do not do much. In accordance with this we now consider the possibility of putting them equal and establish (3.2.16) that this can be done consistently, even in the presence of extensionality (added in print: or the \( \omega \)-rule). It is the term model of this theory that is not among Scott's collection of models mentioned above. We note that in general the addition of extensionality not only adds new theorems (cf. the remark following 1.1.16) but can be problematic. In particular in 3.2.24 we provide a consistent extension of CL which becomes inconsistent when extensionality is added.

Recursion theoretic structures for the \( \lambda K \)- and the \( \lambda I \)-calculus. By a recursion theoretic structure we mean here a combinatorial structure where the domain consists of \( \omega \cup \{\ast\} \) and the application operator is interpreted as Kleene brackets, i.e.

\[
\text{\( n.m = \{n\}(m) = \varepsilon (y \in T(n,m,z)). \ast \) has to be treated as the undefined element, it serves to make the application operator total, with the additional property that \( \{n\}(\ast) = (\ast) \) (cf. the theory of Wagner and Strong (Strong [1968])).
\]

This kind of construction can of course be made more general. In stead of Kleene's brackets which come from his particular
equation calculus for computing recursive functions from recursion equations, an other equational calculus (and other numberings of equations) may be considered. In this way one discovers exactly which properties of numberings and equation calculi are relevant to our subject.

Perhaps the most essential difference between the $\lambda I$-calculus and the $\lambda K$-calculus is that the former is 'more' systematic or deterministic, not allowing short-cuts. All programmes that are 'represented' in a term of the $\lambda I$-calculus have to be performed, while in the $\lambda K$-calculus some subprogrammes can be skipped. For example we reduce $KMN$ to $M$ without looking at the value of $N$. (This may be compared to a 'stupid' evaluation of $0.(2^{15} + 3^{6})$ in numerical arithmetic where we evaluate $2^{15} + 3^{6}$ and a short cut using the reduction $0.x \rightarrow 0$; actually the $\lambda I$-calculus itself does 'stupid' numerical computations.)

Because of this systematic feature of the $\lambda I$-calculus we can find a recursion theoretic model for it or better for its combinatory equivalent, described in Rosser [1936], here called $C_{LI}$. It is like $CL$ but with primitive constants $I, J$ and the axioms $IM = M$ and $JMNLP = MN(MPL)$. For suitable numbers $i,j$ the structure $\oplus = \omega \cup \{\ast\}, i, j, \ast$ is a model for $CL_{I}$. In part II of our thesis we will prove that

\[ M \text{ has no } CL_{I} \text{ normal form } \iff n_{R}(M) = \ast. \]

The proof uses a formalization of the description of the recursive functions as is in Kleene [1959]. This implies that it is consistent with $CL_{I}$ to equate all terms without a normal form.
The recursion theoretic structure obtained by use of Kleene's particular equation calculus is not a model for CL. Though there are numbers $i, k, s \in \omega$ which satisfy $i \cdot x = x$, $k \cdot x \cdot y = x$ and $s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$ for all $x, y, z \in \omega$ no $k$ satisfies $k \cdot x \cdot y = x$ for $y = *$.

However the structure $\mathcal{G}' = \langle \omega \cup \{\ast\}, i, k, s, \ast \rangle$ realizes the language of CL, but is not a model of it, and satisfies an analogue to (1) namely

(2) $M$ has no CL-normal form $\iff R'(M) = \ast$.

Considering finally the canonical mapping: $M \rightarrow M^{WS}$ of the language CL into the language of Wagner and Strong, we find a term $M$ in Part II such that

$M$ has no CL (and hence no CL$^I$) normal form but $M^{WS} = \ast$ is not a theorem of the theory of Wagner-Strong.

Since however properties (1) and (2) hold for our intended objects of study, we should wish to extend the theory of Wagner-Strong; specifically to extend their language. Since the proofs of (1) and (2) for $\mathcal{G}$ and $\mathcal{G}'$ required an analysis of computations, the extended language should refer to the latter, in particular to length of computations (as pointed out to us by C.Gordon).
§1.1. The $\lambda$-calculus.

The $\lambda$-calculus is a theory studied thoroughly in the thirties (by Church, Kleene, Rosser and others). It has been designed to describe a class of functions $V$, where the domain of all functions is $V$ itself. Therefore the objects we consider are at the same time function and at the same time argument. (A similar situation we have in most set theories: there the objects are at the same time set and at the same time element.) Hence we have the feature that a function can be applied to itself. In the usual conception of a function in mathematics, for example in Zermelo Fraenkel set theory, this is impossible (because of the axiom of foundation).

The $\lambda$-calculus defines (or better represents) a class of (partial) functions ($\lambda$-definable functions) which turns out to be the class of (partial) recursive functions. In fact the equivalence between the Turing computable functions and the $\mu$-recursive functions was proved via the $\lambda$-definable functions: Kleene [1936] proved: the $\mu$-recursive functions are exactly the $\lambda$-definable functions. Turing [1937] proved: the computable functions are exactly the $\lambda$-definable functions.

In this sense the $\lambda$-calculus played a central role in the early investigations of the theory of the recursive functions. The consistency of the $\lambda$-calculus was proved by Church and
Rosser [1936]. Because some theories related to the \(\lambda\)-calculus turned out to be inconsistent (paradox of Kleene-Rosser [1935], paradox of Curry [1942]), we see that this consistency proof is not a luxury. In contrast to most mathematical theories, and like set theory, the \(\lambda\)-calculus was initiated before any models were known. In group theory for example many concrete groups were known long before formal group theory was developed. Not until the end of 1969 however, were the first models in ordinary mathematical terms for the \(\lambda\)-calculus constructed (Scott [1970]).

In the \(\lambda\)-calculus we have the fundamental operation of application. The application of a function \(f\) to \(a\) will be written as \(fa\).

Apart from this application we have an abstraction operator \(\lambda\)

The intuitive meaning of \(\lambda x \ldots\) is the function which assigns \(\ldots\) to \(x\). Its use is illustrated by the following formula (not a formula of the \(\lambda\)-calculus by the way)

\[(\lambda x \cdot x^2 + 2x + 1)^3 = 16.\]

Now we give a formal description of the \(\lambda\)-calculus.

1.1.1 Definition

We define the following language \(L_\lambda\):

Alphabet: \(a, b, c, \ldots\) variables

\(\lambda, (,)\) improper symbols

\(=\) equality

Terms: Terms are inductively defined by

1) a variable is a term

2) if \(M, N\) are terms, then \((MN)\) is a term
3) If $M$ is a term, then $(\lambda x M)$ is a term ($x$ is an arbitrary variable).

Formulas: If $M, N$ are terms then $M = N$ is a formula.

1.1.2 Definition

An occurrence of a variable $x$ in a term is called bound if this $x$ is "in the scope of $x$". Otherwise we call this occurrence of $x$ free.

$BV(M)$ (FV($M$)) is the set of all variables in $M$ that occur in $M$ as a bound (free) variable.

$BV$ and $FV$ can be defined inductively as follows:

\[
\begin{align*}
BV(x) &= \emptyset \\
BV(MN) &= BV(M) \cup BV(N) \\
BV(\lambda xM) &= BV(M) \cup \{x\} \\
FV(x) &= \{x\} \\
FV(MN) &= FV(M) \cup FV(N) \\
FV(\lambda xM) &= FV(M) - \{x\}.
\end{align*}
\]

A variable can occur free and bound in the same term (e.g. $((\lambda x(xy))x)$).

Note that, as is common, we use for equality in the object-language and in the metalanguage the same symbol "$=$". When we need to distinguish between them we write (as Curry does) "$\equiv$" for equality in the metalanguage. This is mainly the case when we mean syntactic identity.
1.1.3 Notation

i) \( M_1 M_2 \ldots M_n \) stands for \(((\ldots (M_1 M_2) \ldots ) M_n)\) (association to the left).

ii) \( \lambda x_1 x_2 \ldots x_n \cdot M \) stands for \((\lambda x_1 (\lambda x_2 \ldots (\lambda x_n M) \ldots ))\)

iii) \([x/N]M\) stands for the result of substituting \(N\) in those places of \(x\) in \(M\) which are free:

\[
\begin{align*}
[x/N]x &= N \\
[x/N]y &= y \\
[x/N](M_1 M_2) &= ([x/N]M_1)([x/N]M_2) \\
[x/N](\lambda x M) &= \lambda x M \\
[x/N](\lambda y M) &= \lambda y ([x/N]M).
\end{align*}
\]

In the above \(x \neq y\).

1.1.4 Definition

The \(\lambda\)-calculus is the theory in \(L_\lambda\) defined by the following axioms and rules:

I

1. \(\lambda x M = \lambda y [x/y] M\) if \(y \notin \text{FV}(M) \cup \text{BV}(M)\)

2. \((\lambda x M) N = \[x/N] M\) if \(\text{BV}(M) \cap \text{FV}(N) = \emptyset\).

II

1. \(M = M\)

2. \(M = N\) if \(N = M\)

3. \(M = N, N = L\) if \(M = L\)

4. \(M = M', M = M', M = M'\) if \(2M = 2M', M'Z = M'Z\)

In the above \(M, M', N, L, Z\) denote arbitrary terms and \(x, y\) denote arbitrary variables.

We say that \(...2\) is a direct consequence of \(...1\) if \(\frac{...1}{...2}\).
1.1.5 Remarks

1. Axiom I 1. allows us the change of bound variables (like \[ \int_0^1 x^2 \, dx = \int_0^1 y^2 \, dy \].

If \( M = N \) is provable using only axiom I 1. (and the rules of II) then we say that \( M \) is a vari-ant of \( N \) (notation \( M \equiv_a N \)).

If \( M \equiv_a N \) then only the bound variables are renamed. Note that \( M \equiv_a N \) is a statement of the metalanguage.

Axiom I 2. expresses the essential feature of the \( \lambda \) operator. The Axiom and rules of II express that \( = \) is an equality.

2. If we would drop the restriction in I 1. we would get

\[ (\lambda b \cdot ab) = (\lambda b \cdot bb) \]

which is in conflict with our intuitive interpretation of \( \lambda \).

If we would drop the restriction in I 2. we would get

a) \( (\lambda a(\lambda b \cdot ba))b = \lambda b \cdot bb \)
b) \( (\lambda a(\lambda c \cdot ca))b = \lambda c \cdot cb \)

which is also undesirable. In (a) there is a difference between the "local" variable \( b \) and the "global" variable \( b \).

The restriction in I 2. is just to prevent confusion of variables.

3. If we would restrict the definition of terms as follows:

3') If \( M \) is a term and if \( x \in \text{FV}(M) \), then \( \lambda xM \) is a term, then we get a restricted form of the \( \lambda \)-calculus, the so called \( \lambda I \)-calculus. Our form of the \( \lambda \)-calculus is called the \( \lambda K \)-calculus because we can define a term \( K = \lambda ab \cdot a \), with the property \( KMN = M \), which is impossible in the \( \lambda I \)-calculus.

In the absence of \( K \) many theorems are a bit harder to prove; see Church [1941] who treats the \( \lambda I \)-calculus.
4. Axioms I.1. and I.2. are called α-resp. β-reduction.

5. \( \lambda \vdash M = N \) means that \( M = N \) is a provable formula of the \( \lambda \)-calculus.

1.1.6 Examples

Define \( I = \lambda a.a \)

\( K = \lambda a.b.a \)

\( S = \lambda a.b.c.a(c(b)) \)

Then \( \lambda \vdash IM = M \)

\( \lambda \vdash KN = M \)

\( \lambda \vdash SNL = ML (NL) \)

A nice exercise is the following.

1.1.7 Theorem (fixed point theorem)

For every term \( M \) there exists a term \( \Omega \) such that \( \lambda \vdash M\Omega = \Omega \).

Proof.

Define \( \omega = (\lambda x. M(xx)) \) with \( x \notin \text{FV}(M) \)

\( \Omega = \omega \omega \)

Then \( \lambda \vdash \Omega = (\lambda x. M(xx))\omega = M(\omega \omega) = M\Omega \).

This fixed point theorem, so simple to prove, is strongly related to the fixed point theorem in recursion theory (recursion theorem). In 1.1.7 we have proved more than we formulated. The fixed point can be found in a uniform way.

1.1.8 Corollary

There exists a term \( FP \) such that for every term \( M \) we have

\( \lambda \vdash M(FPM) = FPM \)

Proof.

Define \( FP = \lambda a.((\lambda b.a(bb))(\lambda a.b(aa))) \)
The Russell paradox and Gödel's self-referencing sentence can be considered as applications of FP. Therefore in Curry Feys [1958] such a term FP is called a paradoxical combinator.

1.1.9 Corollary
There exists a term M such that \( \lambda Mx = M \). Such a term is called a fixed point.

Proof.
Define \( M = FP K \) (where K is defined as in 1.1.6).
Then \( \lambda Mx = KMx = M \).

1.1.10 Definition (See also 1.2.6.)
1) A term \( M \) is in normal form if it has no part of the form \((\lambda xP)Q\).
2) A term \( M \) has a normal form if there exists a term \( M' \) which is in normal form such that \( \lambda M = M' \). In this case we say that \( M' \) is a normal form of \( M \).

In the next § we state a theorem (1.2.9) which has as consequence:

1.1.11 Theorem
If \( M \) has a normal form, then this normal form is unique up to \( \alpha \)-reduction (i.e. change of bound variables).

Examples.
1. \((\lambda a.a)(\lambda b.bb)\) is not in normal form but has the normal form \( \lambda b.bb \).
2. \(c(\lambda a.a)(\lambda b.bb)\) is in normal form.

In § 1.2 we will give examples of terms that do not have normal forms.
In the λ-calculus we can represent the natural numbers:

1.1.12 **Definition**

\[ \begin{align*}
0 &= \lambda ab \cdot b \\
1 &= \lambda ab \cdot ab \\
2 &= \lambda ab \cdot a(ab) \\
3 &= \lambda ab \cdot a(a(ab)) \\
&\text{etc.}
\end{align*} \]

Hence \( n = \lambda ab \cdot a(...(ab)... = \lambda ab \cdot a^n b \) \( n \) times

Note that \( n \) is in normal form and has the property:

1.1.13

\[ \lambda \vdash nfx = f^n x. \]

1.1.14 **Definition**

Let \( f : \omega^2 \rightarrow \omega \) be a partial function.

\( f \) is called \( \lambda \)-definable iff there exists a term \( F \) such that

\[ \lambda \vdash F k_1 \ldots k_n = m \quad \text{if} \quad f(k_1, \ldots, k_n) = m \]

\( F k_1 \ldots k_n \) has no normal form if \( f(k_1, \ldots, k_n) \) is undefined.

In this case we say that \( F \) defines \( f \).

If \( F \) defines \( f \), but we no longer know \( f \), can we recover \( f \) from \( F \)? The following gives an affirmative answer.

1.1.15 **Theorem**

If \( F \) defines \( f \) then

\[ \lambda \vdash F k_1 \ldots k_n = m \iff f(k_1, \ldots, k_n) = m. \]

**Proof.**

\( \ast \) By definition.
Suppose \( \lambda \vdash F_{k_1} \ldots k_n = m \). Then \( F_{k_1} \ldots k_n \) has a normal form so \( f(k_1, \ldots, k_n) \) is defined, say

\[ f(k_1, \ldots, k_n) = m'. \]

Hence \( \lambda \vdash F_{k_1} \ldots k_n = m' \).

By 1.1.11 it follows that \( m \equiv_a m' \).

Hence \( m = m' \) and therefore \( f(k_1, \ldots, k_n) = m \). \( \Box \)

Now we consider the following rule of extensionality:

1.1.16

\begin{align*}
\text{ext} \quad & Mx = Nx & x \notin \text{FV}(MN) \\
\Rightarrow & M = N
\end{align*}

ext is not provable in the \( \lambda \)-calculus:

Let \( M = \lambda a \cdot ba \)

\( N = b \).

Then \( \lambda \vdash Ma = Na \), but not \( \lambda \vdash M = N \) because \( M \) and \( N \) are distinct normal forms. So we may take ext as an extra axiom for the \( \lambda \)-calculus.

\( \lambda + \text{ext} \vdash M = N \) means that \( M = N \) is a provable formula of the \( \lambda \)-calculus + extensionality.

1.1.17 Theorem

\( \lambda + \text{ext} \vdash (\lambda x(Mx)) = M \) if \( x \notin \text{FV}(M) \)

(In the literature, this is called \( \eta \)-reduction.)

Proof.

\( (\lambda x(Mx))x = Mx, \quad x \notin \text{FV}(M) \).

Hence by extensionality

\( (\lambda x(Mx)) = M \). \( \Box \)
Note that extensionality follows in turn from 1.1.17:

1.1.18 Theorem

If we assume 1.1.17 as an extra axiom for the λ-calculus, then we can prove extensionality.

Proof.

Let \( Mx = Nx \quad x \notin FV(MN) \).

Then \( \lambda x(Mx) = \lambda x(Nx) \) by rule II.4.

Hence \( M = N \) by 1.1.17.

1.1.19 Remark

Often in the literature the λ-calculus we described is called the \( \lambda\beta \)-calculus. (Because its essential axiom is 1.2.: \( \beta \)-reduction.) The λ-calculus with 1.1.17 as an extra axiom is called the \( \lambda\beta\eta \)-calculus. When it is needed to stress that we are working with the \( \lambda\text{K} \)-calculus, we speak about the \( \lambda\text{K}\beta \)-calculus and the \( \lambda\text{K}\beta\eta \)-calculus.

Of course we have also the \( \lambda\text{I}\beta \)-calculus and the \( \lambda\text{I}\beta\eta \)-calculus.

§1.2. The Church-Rosser theorem. The consistency of the λ-calculus.

Since we do not have a negation in the λ-calculus we cannot define the concept of a contradiction. Therefore we define the consistency as follows.

1.2.1 Definition

The λ-calculus is consistent if we cannot prove \( a = b \).

(If \( a = b \) were provable every formula would be provable.)

As was remarked in the introduction some theories related to the λ-calculus turned out to be inconsistent. Fortunately the λ-
calculus itself is consistent as was proved by Church and Rosser [1936].

We will now give an idea of how this was done. Let us go back to the informal discussion in the beginning of §1. We had the expression \((\lambda x.x^2+2x+1)^3\). After computing we replace this expression by 16. No one would replace 16 by the more complicated \((\lambda x.x^2+2x+1)^3\). Hence we can assign a certain asymmetry to the relation =.

This will be expressed by

\((\lambda x.x^2+2x+1)^3 \triangleright 16\).

\(\triangleright\) is called reduction.

Note that reduction is a stronger relation than equality. I.e. if M reduces to N then M is equal to N. Now we will describe an extension of the \(\lambda\)-calculus in which we formalize this reduction relation.

1.2.2 Definition

We define the following language \(L^\triangleright\).

The alphabet of \(L^\triangleright\) consists of that of \(L^=\) together with the symbol "\(\triangleright\)".

The terms of \(L^\triangleright\) are the same as those of \(L^=\).

The formulas of \(L^\triangleright\) are defined as follows:

If \(M, N\) are terms of \(L^\triangleright\), then \(M = N\) and \(M \triangleright N\) are formulas.

1.2.3 Definition

In the language \(L^\triangleright\) we define an extension of the \(\lambda\)-calculus by the following axioms and rules (see appendix):
I 1. \( \lambda xM \triangleright \lambda y[x/y]M \) if \( y \notin FV(M) \) (\( \alpha \)-reduction)
2. \((\lambda xM)N \triangleright [x/N]M \) if \( BV(M) \cap FV(N) = \emptyset \) (\( \delta \)-reduction)

II Same as in 1.1.4. (These state that "=" is an equality.)

III 1. \( M \triangleright M \)
2. \( M \triangleright N \), \( N \triangleright L \) 
   \[ \frac{}{M \triangleright L} \]
3. \( M \triangleright M' \), \( M \triangleright M' \), \( M \triangleright M' \)
   \[ \frac{ZM \triangleright ZM', \quad M \triangleright M', \quad \lambda xM \triangleright \lambda xM'}{ZM \triangleright ZM', \quad M \triangleright M', \quad \lambda xM \triangleright \lambda xM'} \]
4. \( M \triangleright N \)
   \[ \frac{M \triangleright N}{M = N} \]

Again in the above \( M, M', N, L, Z \) denote arbitrary terms and \( x,y \) arbitrary variables.

If we want to include extensionality we add the axiom

1.2.4
1. \( \lambda x(Mx) \triangleright M \) if \( x \notin FV(M) \) (\( \eta \)-reduction)

It is easy to see that this extended \( \lambda \)-calculus (with or without extensionality) is in fact a conservative extension of the \( \lambda \)-calculus (with or without extensionality) considered in §1.1.

For this reason we also write for the extended \( \lambda \)-calculus
\[ \lambda \vdash M = N \] and \( \lambda + \text{ext} \vdash M = N. \)

Henceforth, if we refer to the \( \lambda \)-calculus, we mean the extended \( \lambda \)-calculus.

Note that if we have \( \lambda + \text{ext} \vdash M \triangleright N \), \( \eta \)-reduction may be used.

With the help of \( \triangleright \) we can express an important property of normal forms:

1.2.5 Lemma

If \( M \) is in normal form and \( \lambda \vdash M \triangleright N \) then \( M \equiv_{\alpha} N. \)
Proof.

By induction to the length of proof of $\lambda \vdash M \Rightarrow N$. 

Remark. It is not true that if $\forall N (\lambda \vdash M \Rightarrow N = M \equiv_{\alpha} N)$, then $M$ is in normal form. Consider for example $(\lambda a.aa)(\lambda a.aa)$.

1.2.6 Definition

If we consider the $\lambda$-$\eta$-calculus we can define

A term is in $n$-normal form if it has no part of the form $\lambda x(Px)$. The concept of normal form as defined in 1.1.8 is then called $\beta$-normal form.

A term is in $\beta\eta$-normal form if it is both in $\beta$- and in $n$-normal form.

A term $M$ has a $\beta\eta$-normal form if there exists a term $M'$ which is in $\beta\eta$-normal form such that $\lambda + ext \vdash M = M'$.

If we just speak of normal form, we mean $\beta$-normal form.

Analogous to 1.2.5 we have

1.2.7 Lemma

If $M$ is in $\beta\eta$-normal form and if $\lambda + ext \vdash M \Rightarrow N$ then $M \equiv_{\alpha} N$.

1.2.8 Lemma

If $M$ has a $\beta$-normal form, then $M$ has a $\beta\eta$-normal form.

Proof.

Let $M'$ be a $\beta$-normal form of $M$. Then $\lambda \vdash M = M'$, hence a fortiori $\lambda + ext \vdash M = M'$.

By applying 1.3 ($n$-reduction) a finite number of times to $M'$, we obtain a $M''$ which is in $n$-normal form and $\lambda + ext \vdash M = M''$. Because $n$-reductions do not introduce subterms of the form
$(\lambda x P)Q$, \(M''\) will be in $\beta\eta$-normal form, hence it is a $\beta\eta$-normal form of \(M\).

In Curry Hindley Seldin [1917] Ch.11 E, lemma 13.1 it is proved that the converse of 1.2.8 also holds.

Now we state without proof:

1.2.9 Theorem (Church-Rosser theorem [1936])
If \(\lambda \vdash M = N\) then there exists a term Z such that \(\lambda \vdash M \triangleright Z\) and \(\lambda \vdash N \triangleright Z\).

See Mitschke [1970] for an elegant proof.

Mitschke's proof is still rather long, but much simpler than the original one. In §1.5 we will use Mitschke's ideas to prove a Church-Rosser theorem for combinatory logic.

1.2.10 Corollary
If \(M, N\) are terms both in normal form and \(M \not\equiv_a N\), then \(\lambda \not\vdash M = N\).

Proof.
Suppose \(\lambda \vdash M = N\), then by 1.2.8 there exists a \(Z\) such that \(\lambda \vdash M \triangleright Z, \lambda \vdash N \triangleright Z\).
\(M, N\) are in normal form, hence by 1.2.5 \(M \equiv_a Z\) and \(N \equiv_a Z\).
But then \(M \equiv_a N\) contradiction.

From this 1.1.11 readily follows:

1.1.11 Theorem
If \(M\) has a normal form, then this normal form is unique up to $\alpha$-reduction.
Proof.

If $\lambda \vdash M = N_1$, $\lambda \vdash M = N_2$ with $N_1, N_2$ in normal form, then $\lambda \vdash N_1 = N_2$. Hence by 1.2.10, $N_1 \equiv_a N_2$. 

1.2.11 Corollary

The $\lambda$-calculus is consistent.

Proof.

Because the terms $a, b$ are in normal form and $a \not\equiv_a b$ we have $\lambda \not\equiv a = b$ by 1.2.10. 

Also the question of the consistency of the $\lambda$-calculus with extensionality arises. In Curry Feys [1958] Ch.4D an extension of the Church-Rosser theorem is proved for the $\lambda$-calculus with $n$-reduction. We state this here without proof.

1.2.12 Theorem

If $\lambda + \text{ext} \vdash M = N$, then there exists a term $Z$ such that $\lambda + \text{ext} \vdash M \rightarrow Z$ and $\lambda + \text{ext} \vdash N \rightarrow Z$.

Analogous to corollaries 1.2.10, 1.1.11 and 1.2.11 we have:

1.2.13 Corollary

If $M, N$ are terms, both in $\beta_n$-normal form and $M \not\equiv_a N$, then $\lambda + \text{ext} \not\equiv a M = N$.

1.2.14 Corollary

If $M$ has a $\beta_n$-normal form, then this $\beta_n$-normal form is unique up to $a$-reduction.

1.2.15 Corollary

The $\lambda$-calculus with extensionality is consistent.
In chapter II we will prove a theorem stronger than 1.2.15.

Now we state a corollary of the Church-Rosser theorem with the help of which we can show that some terms have no normal form:

1.2.16 Corollary
If $M$ has the normal form $N$, then $\lambda \vdash M \Rightarrow N$.

Proof.
If $\lambda \vdash M = N$ and $N$ is in normal form, then by 1.2.8 there exists a $Z$ such that $\lambda \vdash M \Rightarrow Z$ and $\lambda \vdash N \Rightarrow Z$, hence by 1.2.5. $N \equiv_{\alpha} Z$.

So we have $\lambda \vdash M \Rightarrow N$.

1.2.17 Examples of terms without normal form.
From 1.2.16 follows that the term $\omega_1 \omega_2$ with $\omega_2 = \lambda a \cdot aa$ has no normal form, because $\omega_1 \omega_2$ reduces only to itself and it is not in normal form. When we reduce $\omega_3 \omega_3$ with $\omega_3 = \lambda a \cdot aaa$, then the result will grow larger and larger. Along this line we will prove in chapter II, the following extreme result:

There exists a term $M$ such that $\forall N \exists M' \lambda \vdash M \Rightarrow M'$ and $N$ is sub-term of $M'$. Such a term is called a universal generator.

1.2.18 Remarks 1)

1. Apart from the proofs already mentioned

   two abstract forms of the Church-Rosser

1) (Added in proof.) Recently a very simple proof of the Church-Rosser theorem is given by Martin-Löf [1971]. See appendix II.
Theorem were proved in Newman [1942] and Curry [1952] which were supposed to imply the original theorem as a corollary. However, as was pointed out by Schroer (see Rosser [1956]) and Newman [1952], it turned out that these general theorems did not have as corollary the original theorem.

Two adequate abstract forms of the Church-Rosser theorem were proved by Schroer [1965] and Hindley [1969]. Mitschke [1970] gave a new proof of the Church-Rosser theorem, which is conceptually simpler than the original one. In §1.5 we will use Mitschke's ideas to give a proof of the Church-Rosser theorem for combinatory logic.

Finally there is another proof of the Church-Rosser theorem by Rosen [1971]. He proves also an abstract form of the Church-Rosser property, which is applicable to the $\lambda$-calculus. But this application needs a construction similar to that of Mitschke [1970] and this makes the proof rather long. On the other hand, Rosen's general theorem is also applicable to other fields (like the McCarthy calculus of recursive definitions). See also Curry, Feys [1958] Ch 4 S and Kleene [1962] for historical remarks.

2. Before the Church-Rosser theorem was proved the consistency of combinatory logic was shown already by Curry [1930]. It was shown in Rosser [1935] that this implies the consistency of the $\lambda$-calculus.

In part II of our thesis a very simple consistency proof for combinatory logic, due to Scott, is given.
§1.3. The \( \lambda \)-definability of the partial recursive functions.

Undecidability results.

The (partial) functions \( f \) considered in this § are all number theoretic (i.e. \( f : \omega^n \rightarrow \omega \)).

1.3.1 Definition

We define some standard functions:

1) \( \pi^f_i(x_1, \ldots, x_n) = x_i \) are the projection functions.

2) \( \text{S}^+(x) = x+1 \) is the successor function.

3) \( \text{Z}(x) = 0 \) is the zero function.

1.3.2 Definition

The partial recursive functions can be defined as the smallest class \( \mathcal{R} \) of partial functions such that

1) \( \pi^f_i \in \mathcal{R} \)

2) \( \text{S}^+ \in \mathcal{R} \)

3) \( \text{Z} \in \mathcal{R} \)

4) If \( g, h_1, \ldots, h_m \in \mathcal{R} \) and \( f \) is defined by
   \[ f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n)) \]
   then \( f \in \mathcal{R} \) (\( \mathcal{R} \) is closed under substitution).

5) If \( g, h \in \mathcal{R} \), \( g \) and \( h \) are total and \( f \) is defined by
   \[ f(0, x) = g(x) \quad (\text{where } x = x_1, \ldots, x_n) \]
   \[ f(k+1, x) = h(f(k, x), k, x) \]
   then \( f \in \mathcal{R} \) (\( \mathcal{R} \) is closed under recursion).

6) If \( g \in \mathcal{R} \), \( g \) is total and \( f \) is defined by
   \[ f(x) = \mu y[g(x, y) = 0] \]
   (i.e. the least \( y \) such that \( g(x, y) = 0 \); if there does not
exist such a \( y \), then \( \mu y [g(x, y) = 0] \) is undefined.

then \( f \in A \) (\( A \) is closed under minimalisation).

The primitive recursive functions are defined by 1) - 5) only.
We write \( f(\bar{x}) \downarrow \) if \( f(\bar{x}) \) is defined and \( f(\bar{x}) \uparrow \) if \( f(\bar{x}) \) is undefined.

In order to show that the partial recursive functions are \( \lambda \)-definable, we will have to show a slightly stronger fact.

1.3.3 Definition
Let \( f \) be a partial function which is \( \lambda \)-definable by a term \( F \).
We say that \( F \) strongly defines \( f \) iff
\[
f(k_1, \ldots, k_n) \downarrow \iff \forall z_1 \ldots z_s \ Fk_1 \ldots k_n z_1 \ldots z_s \ \text{has no normal form.}
\]
In this case \( f \) is called strongly \( \lambda \)-definable.

Note 1. Our concept of strongly \( \lambda \)-definable should not be confused with that of Curry Hindley and Seldin [1971] Ch 13 A. Our notion of \( \lambda \)-definability corresponds to their strong definability.

2. If a total function is \( \lambda \)-definable it is automatically strongly \( \lambda \)-definable.

Now we will prove that the partial recursive functions are all strongly \( \lambda \)-definable. In order to do this we have to show that:

1) \( U^0_i \), \( Z \), \( S^+ \) are strongly \( \lambda \)-definable.

2) The strongly \( \lambda \)-definable functions are closed under substitution, recursion and minimalisation.

1.3.4 Lemma
The functions \( U^0_i \), \( S^+ \), \( Z \) are \( \lambda \)-definable. (Hence they are strongly \( \lambda \)-definable.)
Proof.

1) $U^n_1$ is defined by $U^n_1 = \lambda a_1, \ldots, a_n \cdot a^+_1:

\lambda \vdash U^n_1 k_1 \ldots k_n = (\lambda a_1 \ldots a_n \cdot a^+_1) k_1 \ldots k_n =
\lambda \vdash (\lambda a_2 \ldots a_n \cdot a^+_1) k_2 \ldots k_n =
\vdots
\lambda \vdash (\lambda a_{i+1} \ldots a_n \cdot a^+_1) k_{i+1} \ldots k_n =
\vdots = k_i$

2) $Z$ is defined by $Z = \lambda a \cdot 0$:

$\lambda \vdash Z k = 0$.

3) $S^+$ is defined by $S^+ = abc \cdot b(abc)$:

$\lambda \vdash S^+ n = \lambda b c \cdot b(abc) = \lambda b c \cdot b(b^+c)$

$= \lambda b c \cdot b^{n+1} c = n+1.$

1.3.5 Lemma

The strongly $\lambda$-definable functions are closed under substitution
(i.e. if $g, h_1, \ldots, h_n$ are $\lambda$-definable and $f(x) = g(h_1(x), \ldots, h_m(x))$ (where $x = x_1, \ldots, x_n$), then $f$ is $\lambda$-definable).

Proof.

Let $g, h_1, \ldots, h_m$ be strongly $\lambda$-defined by $G, H_1, \ldots, H_m$ respectively. Then $f$ is defined by

$F = \lambda a_1 \ldots a_n \cdot (H_1 a_1 \ldots a_n) \ldots (H_m a_1 \ldots a_n) G(H_1 a_1 \ldots a_n) \ldots \ldots (H_m a_1 \ldots a_n)$

For suppose that $f(k_1, \ldots, k_n) = k$.

Then $\forall i \exists n_i \ h_i(k_1, \ldots, k_n) = n_i$, therefore $\lambda \vdash H_1 k, \ldots, H_m k = m_i$.

Furthermore $g(n_1, \ldots, n_m) = k$. Thus

$\lambda \vdash \Phi k_1 \ldots k_n = (H_1 k_1 \ldots k_n I) \ldots (H_m k_1 \ldots k_n I) G(H_1 k_1 \ldots k_n) \ldots \ldots (H_m k_1 \ldots k_n)$

$= (n_1 I) \ldots (n_m I) G n_1 \ldots n_m$
Suppose \( f(k_1, \ldots, k_n) \). Then there are two cases:

1. \( \forall i \exists n_i \ h_i(k_1, \ldots, k_n) = n_i \).
   Then \( g(n_1, \ldots, n_m) \) and
   \[
   \lambda \vdash Fk_1 \ldots k_n = Gn_1 \ldots n_m
   \]
   Hence \( \forall z_1 \ldots z_s \ (H_{k_1} \ldots k_n)z_1 \ldots z_s \) has no normal form,
   because \( G \) strongly defines \( g \).

2. \( h_1(k_1, \ldots, k_n) \) for some \( i \).
   Then \( \forall z_1 \ldots z_s \ (H_{i}k_1 \ldots k_n)z_1 \ldots z_s \) has no normal form.
   Hence \( \forall z_1 \ldots z_s \ (F_{k_1} \ldots k_n)z_1 \ldots z_s \) has no normal form.
   (For this case it was needed to introduce the notion of strong definability.)

In the \( \lambda \)-calculus it is possible to define ordered pairs.

1.3.6 **Definitions**

\[ [M, N] = \lambda z \cdot zMN, \text{ where } z \notin FV(MN) \]

\[ K = \lambda a \cdot a, \quad \tau_1 = \lambda a \cdot aK \]

\[ K' = \lambda a \cdot b, \quad \tau_2 = \lambda a \cdot aK' \]

Then we have

\[ \lambda \vdash KMN = M \text{ hence } \lambda \vdash [M, N]K = M, \text{ thus } \lambda \vdash \tau_1[M, N] = M \text{ and} \]

\[ \lambda \vdash K'MN = N \text{ hence } \lambda \vdash [M, N]K' = N, \text{ thus } \lambda \vdash \tau_2[M, N] = N. \]

Hence \( [M, N] \) is an ordered pair function with projections \( \tau_1, \tau_2 \).

Note that \( M = [\tau_1 M, \tau_2 M] \) is not provable.

1.3.7 **Theorem** (Bernays)

The strongly \( \lambda \)-definable functions are closed under recursion.
Because in 1.3.2. 5) only total functions are considered we do not need to bother about strong definability.

Let $f$ be defined by

$$f(0) = n_0$$

$$f(n+1) = g(f(n), n)$$

and suppose $g$ is $\lambda$-defined by $G$.

(For the sake of simplicity we treat the case that $f$ has no additional parameters. The proof in the general case is analogous).

We will prove that $f$ is $\lambda$-definable.

Consider $M = \lambda a[O^+(t, a), G(t, a)]$. Then $M$ has the property

$$\lambda \vdash M[n, f(n)] = [S^+n, G f(n) n] = [n_1, f(n_1)]$$

Now we have $\lambda \vdash [0, f(0)] = [0, n_0]$.

$$\lambda \vdash [1, f(1)] = M[0, n_0]$$

$$\lambda \vdash [2, f(2)] = M[1, f(1)] = M^2[0, n_0]$$

$$\cdots$$

$$\lambda \vdash [n, f(n)] = M^n[0, n_0] = n M[0, n_0]$$

Hence

$$\lambda \vdash f(n) = \tau_z[n, f(n)] = \tau_z[n M[0, n_0]]$$

Therefore $f$ can be $\lambda$-defined by

$$F = \lambda a \cdot \tau_z(a M[0, n_0])$$

By 1.3.4, 1.3.5 and 1.3.7 we can already state:

1.3.8 All primitive recursive functions are $\lambda$-definable.

1.3.9 Theorem

The strongly $\lambda$-definable functions are closed under minimalisation.
(This theorem was first proved for the \( \lambda I \)-calculus in Kleene [1934]. We give here a simplification for the \( \lambda K \)-calculus due to Turing [1937a]).

**Proof.**

Let \( f(x) = \text{uy}(g(x,y) = 0) \) where \( g \) is a total \( \lambda \)-definable function. (Again for simplicity we suppose that \( f \) has only one argument.) We will prove that \( f \) is strongly \( \lambda \)-definable.

Let \( g'(x,y) = \text{sg}(g(x,y)) \) where \( \text{sg}(0) = 0 \) and \( \text{sg}(n+1) = 1 \).

From 1.3.8 and 1.3.5 it follows that \( g' \) is \( \lambda \)-definable, say by \( G' \). Hence

\[
G'k n = \begin{cases} 0 & \text{if } g(k,n) = 0 \\ 1 & \text{if } g(k,n) \neq 0.
\end{cases}
\]

By the fixed point theorem there exists a term \( M \) such that

\[
\lambda \vdash M = \lambda ab . G'ab(\lambda c \text{ Ma}(S+c))b
\]

(Define \( N = \lambda m \text{ ab . G'ab(\lambda c \cdot ma(S+c))b} \) and take \( M = \text{FPN} \).)

Define \( F = \lambda a \text{ Ma 0} \).

Then \( F \) strongly defines \( f \):

\[
\lambda \vdash Fk = Mk 0 = G'k 0(\lambda c \cdot Mk(S+c))0 = 0
\]

\[
= (\lambda c \cdot Mk(S+c))0
\]

\[
= Mk 1 = G'k 1(\lambda c \cdot Mk(S+c))1 = 1
\]

\[
= (\lambda c \cdot Mk(S+c))1
\]

\[
= Mk 2
\]

\[
\vdots
\]

If \( f(k) \dagger \) then \( \forall y \ g'(k,y) = 1 \), hence \( \forall Z_1 \ldots Z_g \ Fk Z_1 \ldots Z_g \) has no normal form, as is readily seen.

From 1.3.4, 1.3.5, 1.3.7 and 1.3.9 we get:

1.3.10 Theorem

All partial recursive functions are \( \lambda \)-definable.
Remark. The converse of 1.3.10: All $\lambda$-definable functions are partial recursive, is intuitively clear from Church thesis. For a proof see Kleene [1936].

We now state some corollaries to 1.3.10 which concern undecidability.

Let $\gamma$ denote the Gödel number of a term or formula in some Gödelisation. For the definition of some notions in recursion theory the reader is referred to Rogers [1967] Ch 7.

With the following we answer a question of Mostowski.

1.3.11 Theorem

The set \{ $\gamma$ | $\lambda \vdash M = N$ \} is 1-complete and hence creative.

Proof.

Let $X$ be an arbitrary r.e. set.

Define $f(x) = \begin{cases} 0 & \text{if } x \in X \\ \uparrow & \text{else} \end{cases}$

Then $f$ is partial recursive. Let $f$ be defined by $F$.

Then

\[ k \in X \iff f(k) = 0 \iff \lambda \vdash F_k = 0 \]

Hence $X \preceq \{ \gamma | \gamma \vdash \lambda \vdash M = N \}$ via the function $h(k) = \gamma F_k = 0$.

1.3.12 Corollary

The $\lambda$-calculus is undecidable.

Grzegorczyk [1970] has proved even that the $\lambda$-calculus is essentially undecidable (see 1.3.17).
1.3.13 Lemma (Kleene [1936])

There exists a term $E$ such that

\[ \forall M \forall n \in \text{FV}(M) = \emptyset \rightarrow \exists n \lambda \vdash E_n = M. \]


1.3.14 Lemma

For any terms $M_0, M_1$ there exists a term $M$ such that

\[ \lambda \vdash M 0 = M_0 \]
\[ \lambda \vdash M 1 = M_1. \]

Proof.

Define $M = \lambda a \cdot a((KM_1)M_0)$ (where $K$ is defined as in 1.3.6).

Then $\lambda \vdash M 0 = 0 (KM_1)M_0 = M_0$
\[ \lambda \vdash M 1 = 1 (KM_1)M_0 = KM_1M_0 = M_1. \]

1.3.15 Theorem (Scott [1963])

Let $A$ be a set of terms such that

1) $A$ is not trivial (i.e. $M_0 \in A$ and $M_1 \notin A$ for some $M_0, M_1$).
2) If $M \in A$ and $\lambda \vdash M = M'$, then $M' \in A$.

Then $A^- = \{M^- | M \in A\}$ is not recursive.

(Compare this theorem to the theorem of Rice [1953]).

Proof.

Suppose $A^-$ is recursive and let $M_0 \in A$ and $M_1 \notin A$. By 1.3.14 there exists a term $M$ such that $\lambda \vdash M 0 = M_0$ and $\lambda \vdash M 1 = M_1$.

Define $B = \{n | M(En) \notin A\}$, with $E$ as in 1.3.13.

Then $B$ is recursive, hence there exists a term $c_B$ such that

\[ \lambda \vdash c_B^0 = 0 \quad \text{if} \quad n \in B \quad \text{and} \]
\[ \lambda \vdash c_B^1 = 1 \quad \text{if} \quad n \notin B. \]

We can assume that $\text{FV}(c_B) = \emptyset$. 
Let \( n_0 \) be such that \( \lambda \vdash E_{n_0} = \mathcal{C}_B \).

Then
\[
\begin{align*}
n_0 \in B &\implies \lambda \vdash E_{n_0} = 0 \implies \lambda \vdash M(E_{n_0}E_0) = M_0 \\
&\implies M(E_{n_0}E_0) \in A \implies n_0 \notin B \\
n_0 \in B &\implies \lambda \vdash E_{n_0}E_0 = 1 \implies \lambda \vdash M(E_{n_0}E_0) = M_1 \\
&\implies M(E_{n_0}E_0) \notin A \implies n_0 \in B.
\end{align*}
\]

This is a contradiction.

1.3.15 Corollary (Church [1936])
The set \( \{M | M \text{ has a normal form} \} \) is not recursive.

Proof.
Take \( A = \{M | M \text{ has a normal form} \} \) in 1.3.15.

1.3.17 Corollary (Grzegorczyk [1970])
The \( \lambda \)-calculus is essentially undecidable (i.e. has no decidable consistent extension).

Proof.
Suppose \( T \) is a decidable consistent extension of the \( \lambda \)-calculus (i.e. the set of theorems of \( T \) is recursive).

Define \( A = \{M | T \vdash M = 0 \} \). Then \( \sim A \) is recursive because \( T \) is.

But \( A \) satisfies 1) and 2) of 1.3.15. Hence, \( \sim A \) is not recursive.

Contradiction.

Remark. 1.3.12, 1.3.15 and 1.3.17 were proved independently.

§1.4. Combinatory logic.
Combinatory logic is a theory closely related to the \( \lambda \)-calculus. Considerable parts of it were developed by Curry. See Curry Feys [1958] and Curry Hindley Seldin [1971] for an extensive treatment of the subject.
Combinatory logic is intended to be a foundation for mathematical logic. Therefore it includes "illative" notions corresponding to concepts like equality, quantification etcetera. However, we will be concerned only with the combinatorial part of combinatory logic or as Curry calls it: "pure combinatory logic". For illative combinatory logic the reader is referred to the above standard texts.

1.4.1 Definition
We define the following language \( L^h \):

Alphabet: \( a, b, c, \ldots \) variables
- \( I, K, S \) constants
- \( (,) \) improper symbols
- \( = \) equality
- \( \supset \) reduction

Terms: Terms are inductively defined by
1) any variable or constant is a term
2) if \( M, N \) are terms, then \( (MN) \) is a term.

Formulas: if \( M, N \) are terms, then \( M = N \) and \( M \supset N \) are formulas.
Again \( M_1M_2\cdots M_n \) stands for \( ((\ldots (M_1M_2)\ldots )M_n) \).

To be explicit terms of CL are called sometimes CL-terms, terms of the \( \lambda \)-calculus are called \( \lambda \)-terms.

1.4.2 Definition
Combinatory logic (CL) is the theory defined in \( L^h \) by the following axioms and rules (see appendix).
I
1. IM \rightarrow M
2. KMN \rightarrow M
3. SMNL \rightarrow ML(NL)

II
Same as in 1.2.3.

III
1. Same as in 1.2.3
2. Same as in 1.2.3.
3. \frac{MM > M'}{2M > 2M'} , \frac{M > M'}{MZ > M'Z}
4. Same as in 1.2.3.

In the above M,M',N,L,Z denote arbitrary terms.

Notation. We write x \in M if x occurs in the term M.

As in §1.1. we can adjoin to CL extensionality.

1.4.3
\text{ext} \quad \text{Mx=Nx} , \quad x \in MN
\quad M=N

CL (+ ext) \vdash \ldots \quad \text{means that} \quad \ldots \quad \text{is provable in} \quad CL (+ ext).

For example

1.4.4
\text{CL} + \text{ext} \vdash S(S(KS)K)(KI) = I

Curry proved in his thesis that from a finite number of such theorems as 1.4.4, extensionality is provable. These axioms are called the combinatory axioms. See Curry, Feys [1958] Ch 6 C.

We will show that the \lambda-calculus and combinatory logic are interpretable in each other.

1.4.5 Theorem
There exists a mapping \phi: CL \rightarrow \lambda \text{ (i.e. from the set of terms of L_H into the set of terms of L_K)} such that
1. CL (+ ext) ⊢ M = N ⇔ λ (+ ext) ⊢ ϕ(M) = ϕ(N)
2. CL ⊢ M ⊴ N ⇔ λ ⊢ ϕ(M) ⊴ ϕ(N)

Proof.
Define \( ϕ(x) = x \) for any variable \( x \).
\[
\begin{align*}
ϕ(I) &= I \quad (≡ \ λa·a) \\
ϕ(K) &= K \quad (≡ \ λab·a) \\
ϕ(S) &= S \quad (≡ \ λabc·ac(bc)) \\
ϕ(MN) &= ϕ(M)ϕ(N)
\end{align*}
\]
Then it is clear that \( ϕ \) has the required properties.

Remark. The converse of 1. and 2. do not hold. For example:
\( λ \vdash ϕ(SKK) = ϕ(I) \) but CL \( \not\vdash SKK = I \)
as follows from the Church-Rosser theorem for CL (see §1.5.).

In order to obtain a reverse interpretation, we need first a simulation of the \( λ \)-operator in CL.

1.4.6 Definition
For any term \( M \) of \( L_H \) and any variable \( x \) we define inductively a term \( λ^\ast x M \).
\[
\begin{align*}
λ^\ast x x &= I \\
λ^\ast x M &= KM \quad \text{if } x \not\in M \\
λ^\ast x (MN) &= S(λ^\ast x M)(λ^\ast x N)
\end{align*}
\]
Remarks.
1. This method of defining \( λ \)-terms with \( I, K \) and \( S \) comes from Schönfinkel [1924], although it is not generally formulated there.
2. It is possible to formulate CL with K and S alone. In that case we define

\[ \lambda x \ f = SKK \]

because CL ⊢ SKK \( x = x \)

1.4.7 **Lemma**

\( \lambda^* \) has the following properties:

1) \( x \) does not occur in \( \lambda^*x \) \( M \)

2) \( \lambda^*x \ M = \lambda y[x/y]M \) if \( y \notin M \)

3) CL ⊢ (\( \lambda x \) \( M \))\( N = [x/N]M \)

4) \( \lambda^*[y/N]M = [y/N]\lambda^*x \ M \) if \( x \neq y \) and \( x \notin N \).

**Proof.**

The proof uses induction on the complexity of \( M \) and is in all the four cases similar.

As an example we prove 4).

\( M = x \) Then \( \lambda^*[y/N]M = \lambda^*x \ x = I \)

and \( [y/N]\lambda^*x \ M = [y/N]I = I \).

\( x \in M \) Then \( \lambda^*[y/N]M = K[y/N]M = [y/N]KM \)

and \( [y/N]\lambda^*x \ M = [y/N]KM \).

\( M = M_1M_2 \) and \( x \in M \).

Then \( \lambda^*[y/N]M = \lambda^*[y/N]M_1M_2 \)

\[ = \lambda^*[y/N]M_1[y/N]M_2 \]

\[ = \lambda^*[y/N]M_1([y/N]M_2) \]

\[ = \lambda^*[y/N]M_1([y/N]M_2) \] by the induction hypothesis

\[ = [y/N] \lambda^*[M_1([y/N]M_2)] \]

\[ = [y/N] \lambda^* x \ M \]

\( \square \)
1.4.8 **Lemma**

\[ M = N \]

\[ \lambda^* M = \lambda^* N \]

**Proof.**

Let \( \text{CL + ext} \vdash M = N \)

Then \( \text{CL + ext} \vdash (\lambda^* M)x = M = N = (\lambda^* N)x \)

Hence by ext

\( \text{CL + ext} \vdash \lambda^* M = \lambda^* N \)

1.4.9 **Theorem**

There exists a mapping \( \psi: \lambda \rightarrow \text{CL} \) such that

\[ \lambda + \text{ext} \vdash M = N \Rightarrow \text{CL + ext} \vdash \psi(M) = \psi(N) \]

**Proof.**

Define \( \psi \) inductively:

\[ \psi(x) = x \text{ for any variable } x \]

\[ \psi(MN) = \psi(M)\psi(N) \]

\[ \psi(\lambda x M) = \lambda^* x \psi(M). \]

**Sublemma**

\[ \psi([x/N]M) \equiv [x/\psi(N)] \psi(M) \text{ if } \text{BV}(M) \cap \text{FV}(N) = \emptyset. \]

**Proof.**

Induction on the structure of \( M \). (Use: \( x \in \text{FV}(M) \iff x \in \psi(M) \)

and 1.4.7.4.) Now suppose \( \lambda + \text{ext} \vdash M = N \). We will show by induction on the length \( l \) of the proof of \( M = N \) that:

1. \( \text{CL + ext} \vdash \psi(M) = \psi(N) \).

\[ l = 0 \]

\( M = N \) is axiom of the \( \lambda \)-calculus.

**case 1.** \( M \equiv N \), then \( \psi(M) \equiv \psi(N) \),

hence \( \text{CL} \vdash \psi(M) = \psi(N) \).

**case 2.** \( M \equiv \lambda x P \) and \( N \equiv \lambda y[x/y]P \), with \( y \not\in \text{FV}(P) \).
Then $\psi(M) \equiv \lambda x \psi(P)$ and
$\psi(N) \equiv \lambda y[x/y] \psi(P)$ by the sublemma.
Hence by 1.4.7.2) $\mathsf{CL} \vdash \psi(M) = \psi(N)$.

**case 3.** $M \equiv (\lambda x P)Q$ and $N \equiv [x/Q]P$, with
$\mathsf{BV}(P) \cap \mathsf{FV}(Q) = \emptyset$.
Then $\mathsf{CL} \vdash \psi(M) = (\lambda x \psi(P))\psi(Q) = [x/\psi(Q)]\psi(P)$
by 1.4.7. 3)
and $\mathsf{CL} \vdash \psi(N) = \psi([x/Q]P) = [x/\psi(Q)]\psi(P)$ by the
sublemma, hence $\mathsf{CL} \vdash \psi(M) = \psi(N)$.

$1 \leq k$ and the theorem holds for $1' < k$.

$M = N$ is the consequence of a rule of inference. Because of
1.4.8 the rules of inference for the $\lambda$-calculus + ext and $\mathsf{CL} +$
ext are the same, hence (1) follows immediately by the induction
hypothesis.

**Remarks**

1. It is not true that
$\lambda \vdash M = N \Rightarrow \mathsf{CL} \vdash \psi(M) = \psi(N)$.
For example $\lambda \vdash \lambda a((\lambda b \cdot b)a) = \lambda a \cdot a$.
But $\mathsf{CL} \not\vdash S(KI)I = I$ as follows from §1.5.

2. Also we do not have $\lambda \vdash M \triangleright N \Rightarrow \mathsf{CL} \vdash \psi(M) \triangleright \psi(N)$.
Because of this peculiarity, a special theory of **strong reduc-
ibility** (notation: $\triangleright$) for $\mathsf{CL}$ is developed.
Then we have $\lambda \vdash M \triangleright N \Rightarrow \psi(M) \triangleright \psi(N)$ in $\mathsf{CL}$.
(See Curry Feys [1958] Ch 6 F.)
In order to distinguish it from $\triangleright$, $\triangleright$ is called **weak**
reduction.
1.4.10 Theorem
1) \( \lambda \vdash \phi(\psi(M)) \Rightarrow M \)
2) \( \text{CL + ext} \vdash \psi(\phi(M)) = M \)

Proof.
1) Induction on the structure of \( M \).
Prove first \( \lambda \vdash \phi(\lambda x M) \Rightarrow \lambda x \phi(M) \) for \( M \) a term of \( \text{CL} \).
2) Induction on the structure of \( M \).
The essential step is to show that
\[
\text{CL + ext} \vdash S = \lambda x y z . x z (y z) \quad \text{and}
\text{CL + ext} \vdash K = \lambda x y . x.
\]

Remark. It is not true that \( \text{CL} \vdash \psi(\phi(M)) = M \).
Take for example \( M = K \).
Then \( \text{CL} \not\vdash S(KK)I = K \) as follows from §1.5.

1.4.11 Corollary
The \( \lambda \)-calculus + ext and CL + ext are equivalent:
1) \( \text{CL + ext} \vdash \psi(\phi(M)) = M \)
2) \( \lambda + \text{ext} \vdash \phi(\psi(M)) = M \)
3) \( \lambda + \text{ext} \vdash M = N \iff \lambda + \text{ext} \vdash \phi(M) = \phi(N) \)
4) \( \lambda + \text{ext} \vdash M = N \iff \text{CL + ext} \vdash \psi(M) = \psi(N) \)

Proof.
1) and 2) follow directly from 1.4.10.
3) \( \text{CL + ext} \vdash M = N \implies \lambda + \text{ext} \vdash \phi(M) = \phi(N) \) by 1.4.5
   \[ \implies \text{CL + ext} \vdash \psi(\phi(M)) = \psi(\phi(N)) \] by 1.4.9
   \[ \implies \text{CL + ext} \vdash M = N \] by 1).  \( \Box \)
4) Similar proof as 3).  \( \Box \)
Because of this equivalence the names for the $\lambda$-calculus and combinatory logic are sometimes interchanged. Scott [1970] in fact constructs a model for combinatory logic + extensionality.

1.4.12 Definition
1) A term $M$ of CL is in normal form if it has no part of the form $IM$, $KMN$ or $SMNL$.
2) A term $M$ has a normal form if there exists a term $M'$ in normal form such that $CL \vdash M = M'$.

Remark. The terms in normal form can be defined inductively as follows:
1) I, K and S are in normal form.
2) If $M$ is in normal form, then $KM$ and $SM$ are in normal form.
3) If $M$ and $N$ are in normal form then $SMN$ is in normal form.

Similarly to 1.2.5 we have

1.4.13 Theorem
If $M$ is in normal form and if $CL \vdash M \geq N$, then $M \equiv N$.

As in §1.1 we can represent the natural numbers in CL. Then we can define similar to definition 1.1.14 the concept of CL-definability. Analogous to §1.3 it is possible to prove that the partial CL-definable functions are exactly the partial recursive functions. See Curry Hindley Seldin [1971] Ch 13 for details.
§1.5. The Church-Rosser property for combinatory logic à la Mitschke.

In this § we will use the ideas of Mitschke [1970] to give a proof of the Church-Rosser property for combinatory logic (with weak reduction).

In Hindley [1977] this will be proved as an application of an abstract Church-Rosser theorem proved in Hindley [1969].

(Added in print. We have realized too late that for the combinatory equivalent of the λ-calculus the Church-Rosser property was already proved in Rosser [1935] T 12, p.144. This proof carries over immediately to CL (cf.Curry, Hindley, Seldin [1971]). Compare Rosser's proof with that of Martin-Löf in appendix II.)

The theorem we are about to prove is:

1.5.1 Theorem (Church-Rosser property for CL)

If CL ⊢ M = N, then there exists a term Z such that

CL ⊢ M ⊳ Z and CL ⊢ N ⊳ Z.

In order to show this we have to define several auxiliary languages and theories.

1.5.2 Definition

CL' is an extension of CL where the alphabet of CL' has the additional symbol "⊳" (one step reduction); the language of CL' has the same terms as CL; CL' has as extra formulas M ⊳ N (for arbitrary terms M and N).

CL' has the following axioms and rules (see appendix):

I 1. IM ⊳ M
    2. KMN ⊳ M
    3. SMNL ⊳ ML(NL)

II Same as in 1.4.2.

III Same as in 1.4.2.
Again $M, M', N, L, Z$ denote arbitrary terms.

1.5.3 Lemma

$CL'$ is a conservative extension of $CL$. Hence in order to prove 1.5.1 it is sufficient to prove the Church-Rosser property for $CL'$.

Proof.

First show that $CL' \vdash M \Rightarrow N \Rightarrow CL \vdash M \Rightarrow N$.

Then the rest follows easily.

Now we will introduce a theory $CL^*$ which plays the same role as $\lambda^*$ in Mitschke [1970].

1.5.4 Definition

We define the following language $L_H^*$.

Alphabet: The alphabet of $CL'$ together with the extra symbol ",".

Terms: Terms are inductively defined by

1) Any variable or constant is a term.
2) If $M$ and $N$ are terms, then $(MN)$ is a term.
3) If $M$, $N$ and $L$ are terms, then $S(M,N,L)$ is a term.

Formulas: If $M$ and $N$ are terms, then

$M \Rightarrow N$, $M \Rightarrow N$ and $M = N$ are formulas.
1.5.5 Definition

CL* is the theory with the language \( L_H^* \) defined by the following axioms and rules (see appendix)

I  1. \( IM \Rightarrow M \)
    2. \( KMN \Rightarrow M \)

II  3. \( SMNL \Rightarrow S(M,N,L) \)

III Same as in 1.4.2 II

IV Same as in 1.4.2 III

\[ \begin{align*}
1. & \quad M \Rightarrow M' \\
2. & \quad S(M,N,L) \Rightarrow S(M',N,L) \\
3. & \quad S(M,N,L) \Rightarrow S(M',N,L) \\
4. & \quad S(M,N,L) \Rightarrow S(M,N,L')
\end{align*} \]


The essential axiom of this theory is I 3. It freezes the action of S.

Our method of proving 1.5.1 is the following:
First, we (almost) prove the Church-Rosser property for CL*;
then with the help of a homomorphism argument, we obtain the corresponding result for CL', and hence, by 1.5.3 for CL.

1.5.6 Lemma

CL' \( \vdash M \Rightarrow N \) there exists a term L with exactly one occurrence of a variable x and terms N, N' such that M \( \equiv [x/N]L \), M' \( \equiv [x/N']L \) and N \( \Rightarrow N' \) is an axiom of CL'.
Proof.

\[ \rightarrow \text{Induction on the length of proof of } M \Rightarrow M'. \]
\[ \Rightarrow \text{Induction on the structure of } L. \]

1.5.7 \textbf{Lemma}

If \( CL^* \vdash M_1 \Rightarrow_1 M_2 \) and \( CL^* \vdash M_3 \Rightarrow_1 M_4 \), then there exists a term \( M_6 \), such that

\[ CL^* \vdash M_2 \Rightarrow_1 M_6 \] \( \text{and} \) \( CL^* \vdash M_3 \Rightarrow_1 M_6 \) (see fig.1)

\[ \begin{array}{c}
    M_1 \\
    \downarrow 1 \\
    M_2 \\
    \downarrow 1 \\
    M_3 \\
    \downarrow 1 \\
    M_4
\end{array} \quad \text{Figure 1} \]

Proof.

First we consider the possibility that \( M_1 \Rightarrow_1 M_2 \) is an axiom.

(By repeated use of 1.5.6 we obtain the subcases.)

\begin{itemize}
  \item \textbf{case 1.} \( M_1 \equiv IM, \; M_2 \equiv M \)
    
    \text{subcase 1.1.} \( M_3 \equiv M_1 \) or \( M_3 \equiv M_2 \). Take \( M_6 \equiv M_2 \).
    
    \text{subcase 1.2.} \( M_3 \equiv IM' \) and \( CL^* \vdash M \Rightarrow_1 M' \). Take \( M_6 \equiv M' \).
  \item \textbf{case 2.} \( M_1 \equiv KMN, \; M_2 \equiv M \)
    
    \text{subcase 2.1.} \( M_3 \equiv M_1 \) or \( M_3 \equiv M_2 \). Take \( M_6 \equiv M_2 \).
    
    \text{subcase 2.2.} \( M_3 \equiv KM'N \) and \( CL^* \vdash M \Rightarrow_1 M' \). Take \( M_6 \equiv M' \).
    
    \text{subcase 2.3.} \( M_3 \equiv KMN' \) and \( CL^* \vdash M \Rightarrow_1 M' \). Take \( M_6 \equiv M \).
  \item \textbf{case 3.} \( M_1 \equiv SMNL, \; M_2 \equiv S(M,N,L) \)
    
    \text{subcase 3.1.} \( M_3 \equiv M_1 \) or \( M_3 \equiv M_2 \). Take \( M_6 \equiv M_2 \).
    
    \text{subcase 3.2.} \( M_3 \equiv SM'NL \) and \( CL^* \vdash M \Rightarrow_1 M' \).
    
    \text{Take} \( M_6 \equiv S(M',N,L) \)
    
    \text{subcases 3.3,3.4} \( M_3 \equiv SM'NL \) or \( M_3 \equiv SMNL' \)
    
    Similar to subcase 3.2.
  \item \textbf{case 4.} \( M_1 \equiv M_2 \). Take \( M_6 \equiv M_2 \).
\end{itemize}
Now we have proved that the lemma for $M_1 \Rightarrow M_2$ is an axiom. Since $CL^* \vdash M_1 \Rightarrow M_2$ it follows from 1.5.6 that $M_1 \equiv \ldots M \ldots, M_2 \equiv \ldots M' \ldots$ and $M \Rightarrow M'$ is an axiom of $CL^*$. 

case 1. $M_3 \equiv M_1$ or $M_3 \equiv M_2$. Take $M_4 \equiv M_2$.

case 2. $M_1 \equiv \ldots N \ldots$, $M_3 \equiv \ldots N' \ldots$ and $N \Rightarrow N'$ is an axiom of $CL^*$.

subcase 2.1. $M$ and $N$ are disjoint subterms of $M_1$.

Then $M_1 \equiv \ldots M \ldots N \ldots$, $M_2 \equiv \ldots M' \ldots N \ldots$ and $M_3 \equiv \ldots M \ldots N' \ldots$. Take $M_4 \equiv \ldots M' \ldots N' \ldots$.

subcase 2.2. $N$ is a subterm of $M$.

Then $M \Rightarrow M'$ is an axiom, and $CL^* \vdash M \Rightarrow M''$ by the reduction $CL^* \vdash N \Rightarrow N'$. Hence there exists a term $M'''$ such that $CL^* \vdash M' \Rightarrow M'''$ and $CL^* \vdash M'' \Rightarrow M'''$. Take $M_4 \equiv \ldots M''' \ldots$.

subcase 2.3. $M$ is a subterm of $N$.

Analogous to subcase 2.2.

1.5.8 Lemma

If $CL^* \vdash M_1 \Rightarrow M_2$ and $CL^* \vdash M_1 \Rightarrow M_3$ then there exists a term $M_4$ such that

\[ (*) \quad CL^* \vdash M_2 \Rightarrow M_4 \quad \text{and} \quad CL^* \vdash M_3 \Rightarrow M_4. \]

Proof.

Note that $CL^* \vdash M_1 \Rightarrow M_2 \iff \exists N_1 \ldots N_k \quad CL^* \vdash M_1 \equiv N_1 \Rightarrow N_2 \Rightarrow \ldots \Rightarrow N_k \equiv M_2$. 

The same holds for $\text{CL}^* \vdash M_1 \succ M_3$.

Then repeated use of 1.5.7 yields ($\ast$). (See fig. 2.)

Figure 2.

Remark. 1.5.7 does not hold for $\text{CL}'$, but only

$\text{CL}' \vdash M_1 \succ_1 M_2$ and $\text{CL}' \vdash M_1 \succ_1 M_3$.

$\text{CL}' \vdash M_2 \succ_1 M_4$ and $\text{CL}' \vdash M_3 \succ_1 M_4$ for some term $M_4$.

Therefore an analogue of 1.5.8 for $\text{CL}'$ is much harder to prove.

From 1.5.8 we can easily derive the Church-Rosser property for $\text{CL}^*$. However we do not need to do so.

1.5.9 Definition

We define inductively a mapping $\theta: \text{CL}^* \to \text{CL}'$ (in fact from the set of terms of $\text{CL}^*$ into the set of terms of $\text{CL}'$):

\[
\begin{align*}
\theta(c) &= c & \text{if } c \text{ is a variable or constant} \\
\theta(MN) &= \theta(M)\theta(N) \\
\theta(S,M,N,L) &= \theta(M)\theta(L)\theta(N)\theta(L)
\end{align*}
\]

It is clear that if $M$ is a term of $\text{CL}'$, then $\theta(M) = M$.

1.5.10 Lemma

1) $\text{CL}^* \vdash M \succ_1 N \Rightarrow \text{CL}' \vdash \theta(M) \succ \theta(N)$

2) $\text{CL}^* \vdash M \succ N \Rightarrow \text{CL}' \vdash \theta(M) \succ \theta(N)$

3) $\text{CL}^* \vdash M = N \Rightarrow \text{CL}' \vdash \theta(M) = \theta(N)$

Proof.

In all cases the result follows by induction on the length of,
proof in CL*.

1.5.11 Lemma

If $CL' \vdash M_1 \supset M_2$ then there exists a term $M^*_2$ of $CL*$ such that $CL^* \vdash M_1 \supset M^*_2$ and $\theta(M^*_2) = M_2$ (see fig. 3).

Proof.

Induction on the length of proof of $M_1 \supset M_2$. Suppose first that $M_1 \supset M_2$ is an axiom of $CL'$.

- case 1, 2. $M_1 \equiv I M$ or $M_1 \equiv K M N$. Take $M^*_2 \equiv M$.
- case 3. $M_1 \equiv S M N L$, $M_2 \equiv M L(N L)$. Take $M^*_2 \equiv S(M, N, L)$.
- case 4. $M_1 \equiv M_2$. Take $M^*_2 \equiv M_2$.

Suppose now that $M_1 \supset M_2$ is $Z M_1 \supset Z M_2$ and is a direct consequence of $M_1 \supset M_2$.

By the induction hypothesis there exists a term $M^*_2$ such that $CL^* \vdash M_1 \supset M^*_2$ and $\theta(M^*_2) = M_2$. Then take $M^*_2 \equiv Z M_2^*$.

The case that $M_1 \supset M_2$ is $M_1 Z \supset M_2 Z$ is treated analogously.

1.5.12 Definition

We write $M \succ^* N$ for

$\exists M^*, N^* \quad CL^* \vdash M^* \succ N^* \quad \text{and} \quad \theta(M^*) = M, \theta(N^*) = N$.

1.5.13 Main Lemma

Suppose $M \succ^* N$.

Then there exists a term $N^*_1$ such that $CL^* \vdash M \succ N^*_1 \quad \text{and} \quad \theta(N^*_1) = N$ (see fig. 4.)
We will postpone the proof of the main lemma until 1.5.18.

1.5.14 Lemma

If $CL' \vdash M_1 \gg M_2$ and $CL' \vdash M_1 \gg M_3$ then there exists a term $M_4$ such that

$CL' \vdash M_2 \gg M_4$ and $M_3 \gg M_4$, and hence by 1.5.10 2)

$CL' \vdash M_3 \gg M_4$.

Proof.

$CL' \vdash M_1 \gg M_2 \iff \exists N_1, \ldots, N_k \; CL' \vdash M_1 \equiv N_1 \gg N_2 \gg \cdots \gg N_k \equiv M_2$.

We prove the lemma by induction on $k$.

If $k = 1$ then $M_1 \equiv M_2$ and we can take $M_4 \equiv M_2$, then $M_3 \gg M_4$ follows from 1.5.11.

Suppose now that $\exists N_1, \ldots, N_{k+1} \; CL' \vdash M_1 \equiv N_1 \gg \cdots \gg N_{k+1} \equiv M_2$.

By the induction hypothesis, there exists a term $N'_k$ such that

$CL' \vdash M_2 \gg N'_k$ and $N_k \gg N'_k$.

By 1.5.13 there exists a $N''_k$ such that

$CL' \vdash N_k \gg N''_k$ and $\Theta(N''_k) = N'_k$ (see fig. 5).

By 1.5.11 there exists a $N^*_k$ such that

$CL' \vdash N_k \gg N^*_k$ and $\Theta(N^*_k) = N^*_k$ (see fig. 5).

Now it follows from 1.5.8 that there exists a $N'^*_k$ such that

$CL' \vdash N'^*_k \gg N'^*_k$ and $CL' \vdash N'^*_k \gg N'^*_k$.
Take $M_4 \equiv \theta(N_{k+1}^*)$, then $CL' \vdash N_1^* \geq M_4$, by 1.5.10 2), and hence $CL' \vdash M_2 \geq M_4$ and $M_3 \not\geq M_4$.

1.5.15 Lemma

If $CL' \vdash M_1 \geq M_2$ and $CL' \vdash M_1 \geq M_3$ then there exists a $M_4$ such that $CL' \vdash M_2 \geq M_4$ and $CL' \vdash M_3 \geq M_4$.

Proof.

$CL' \vdash M_1 \geq M_2 \iff \exists N_1 \ldots N_k \quad CL' \vdash M_1 \equiv N_1 \geq \ldots \geq N_k \equiv M_2$.

The result follows easily from 1.5.14, using induction to $k$.

1.5.16 Theorem (Church-Rosser property for $CL'$)

If $CL' \vdash M = N$, then there exists a term $Z$ such that

$CL' \vdash M \geq Z$ and $CL' \vdash N \geq Z$.

Proof.

Induction on the length of proof of $M = N$. 
case 1. \( M \neq N \). Take \( Z = M \).

case 2. \( M = N \) is a direct consequence of \( N = M \). By the induction hypothesis there exists a \( Z \) such that

\[ CL' \vdash N \succ Z \quad \text{and} \quad CL' \vdash M \succ Z. \]

case 3. \( M = N \) is a direct consequence of \( M = L \) and \( L = N \).

By the induction hypothesis there exists terms \( Z_1, Z_2 \) such that

\[ CL' \vdash M \succ Z_1, \quad CL' \vdash L \succ Z, \]
\[ CL' \vdash L \succ Z_2 \quad \text{and} \quad CL' \vdash N \succ Z_2 \]

(see fig. 6).

\[ \begin{array}{c}
\text{M} \\
\downarrow Z_1 \\
\downarrow Z \\
\uparrow Z_2 \\
\text{L} \\
\text{N}
\end{array} \]

By 1.5.15 there exists a \( Z \) such that

\[ CL' \vdash Z_1 \succ Z \quad \text{and} \quad CL' \vdash Z_2 \succ Z, \]

hence \( CL' \vdash M \succ Z \) and \( CL' \vdash N \succ Z \).

case 4. \( M = N \) is \( Z_1 M' = Z_1 N' \) (or \( M' Z_1 = N' Z_1 \)) and is a direct consequence of \( M' = N' \).

By the induction hypothesis there exists a term \( Z' \) such that

\[ CL' \vdash M' \succ Z' \quad \text{and} \quad CL' \vdash N' \succ Z'. \]

Take \( Z = Z_1 Z' \) (resp. \( = Z' Z_1 \)).

case 5. \( M = N \) is a direct consequence of \( M \succ N \). Take \( Z = N \).

1.5.17 Corollary

The Church-Rosser property for \( CL \) (see 1.5.1) holds.

Proof.

This follows from 1.5.16 by 1.5.3.

In the remainder of this section we will prove the main lemma. In order to do this, we introduce a new theory \( CL' \).
1.5.18 Definition

We define the following language \( L^*_H \).

**Alphabet:** The alphabet for \( L^*_H \), with an extra symbol "_".

**Terms** are inductively defined by

1) \( S(M, N, L) \) is a term.

2) \( S(M, N, L) \) is a term.

3) \( S(M, N, L) \) is a term.

4) If \( S(M, N, L) \) is a term, then \( S(M, N, L) \) is a term.

Formulas are defined in the same way as in 1.5.4.

1.5.19 Definition (see appendix)

\( CL^*_H \) is a theory in \( L^*_H \) defined by the same axioms and rules as \( CL^*_H \) except that

I 3 is replaced by

I 3'. \( SMNL \Rightarrow S(M, N, L) \)

and there are the additional rules:

IV 5.

\[
\begin{align*}
M & \Rightarrow M' \quad N \Rightarrow N' \\
S(M, N, L) & \Rightarrow S(M', N, L) \\
L & \Rightarrow L'
\end{align*}
\]

1.5.20 Definition

We define inductively two mappings \( \varnothing \) and \( |\ldots| : CL^*_H \rightarrow CL^*_H \) as follows:

\[
\begin{align*}
\varnothing(c) &= c & \text{if } c \text{ is a variable or constant} \\
\varnothing(MN) &= \varnothing(M)\varnothing(N) \\
\varnothing(S(M, N, L)) &= \varnothing(M)\varnothing(L)\varnothing(N)\varnothing(L) \\
\varnothing(S(M, N, L)) &= S(\varnothing(M), \varnothing(N), \varnothing(L)) \\
|c| &= c & \text{if } c \text{ is a variable or constant} \\
|M| &= |M| |N|
\end{align*}
\]
\[ |S(M, N, L)| = S(|M|, |N|, |L|) \]
\[ |S(M, N, L)| = |S(M, N, L)| \]

|M| is apart from the underlining the same as M.

1.5.21 Lemma
1) \( \Theta(\Theta(M)) = \Theta(|M|) \) for all terms \( M \) of \( \text{CL}^* \).
2) \( \Theta(M) = \Theta(M) \) if \( M \) is a term of \( \text{CL}^* \).
Proof.
Induction on the structure of \( M \).

1.5.22 Lemma
If \( \text{CL}^* \vdash M_1 \gg M_2 \) or \( \text{CL}^* \vdash M_1 \gg M_2 \), then there exists a term \( M'_2 \) of \( \text{CL}^* \) such that \( |M'_2| = M_2 \) and \( \text{CL}^* \vdash M_1 \gg M'_2 \).
Proof.
Induction on the length of proof of \( M_1 \gg M_2 \) or \( M_1 \gg M_2 \).

1.5.23 Lemma
If \( \text{CL}^* \vdash M_1 \gg M_2 \) or \( \text{CL}^* \vdash M_1 \gg M_2 \), then \( \text{CL}^* \vdash \Theta(M_1) \gg \Theta(M_2) \).
Proof.
Induction on the length of proof of \( M_1 \gg M_2 \) or \( M_1 \gg M_2 \).

Now we are able to prove the main lemma.
1.5.13 Main Lemma.
If \( M \gg N \), then there exists a term \( N^* \) such that
\( \text{CL}^* \vdash M \gg N^* \) and \( \Theta(N^*) = N \).
Proof.
Since \( M \gg N \), there are terms \( M_1, N_1 \) of \( \text{CL}^* \) such that
\( \text{CL}^* \vdash M_1 \gg N_1 \) and \( \Theta(M_1) = M, \Theta(N_1) = N \).
By 1.5.22 there exists a term $N'_1$ such that $|N'_1| = N_1$, and $CL^* \vdash M_1 \succ N'_1$. 
By 1.5.23 it follows that $CL^* \vdash \emptyset(M_1) \succ \emptyset(N'_1)$. 
Take $N^* = \emptyset(N'_1)$. 
By 1.5.21 2) we have $\emptyset(M_1) = \emptyset(M) = M$, hence $CL^* \vdash M \succ N^*$. 
By 1.5.21 1) we have $\emptyset(N^*) = \emptyset(\emptyset(N'_1)) = \emptyset(|N'_1|) = \emptyset(N_1) = N$. 

Remarks.

The idea of using $CL^*$, $\emptyset$ and the main lemma is taken from Mitschke [1970]. (He formulated them for the $\lambda$-calculus.) The proof of the main lemma is new.

More extensive use of auxiliary theories like $CL^*$ will be made several times in the sequel.
CHAPTER II

The \(\omega\)-rule for combinatory logic and \(\lambda\)-calculus

\section{The \(\omega\)-rule.}

\subsection{The \(\omega\)-rule.}

\subsubsection{Definition}

A term \(M\) of the \(\lambda\)-calculus is called \textit{closed} if \(\text{FV}(M) = \emptyset\).
A term \(M\) of \(\text{CL}\) is called \textit{closed} if no variable occurs in \(M\).

\subsubsection{Definition}

We can extend the \(\lambda\)-calculus or \(\text{CL}\) with the following rule, which we call the \(\omega\)-rule,

\[
\begin{align*}
\omega\text{-rule} & \quad \frac{M = N}{M = N}
\end{align*}
\]

We write \(\lambda \omega\) or \(\text{CL} \omega\) for the \(\lambda\)-calculus or \(\text{CL}\) extended with the \(\omega\)-rule.

It is clear that the \(\omega\)-rule implies extensionality. For this reason it does not matter whether we consider \(\lambda \omega\) or \(\text{CL} \omega\), because the \(\lambda\)-calculus and \(\text{CL}\) are equivalent when we have extensionality (1.4.11). In general we will formulate and prove results about the \(\omega\)-rule in \(\text{CL}\). However when it is easier or even necessary we do this in the \(\lambda\)-calculus.

\subsubsection{Definition}

1. \(\text{CL}\) is \(\omega\)-consistent if \(\text{CL} \omega\) is consistent.
2. \(\text{CL}\) is \(\omega\)-complete if the \(\omega\)-rule is derivable in \(\text{CL} + \text{ext}\) (i.e. if \(\text{CL} + \text{ext} \vdash M = N\) for all closed \(Z\) => \(\text{CL} + \text{ext} \vdash M = N\)).
In a personal communication professor Curry suggested the possibility that CL is \( \omega \)-complete.

In this chapter we will prove:

1. CL is \( \omega \)-consistent (§§2.2, 2.3).
2. The \( \omega \)-rule is derivable in \( \lambda + \text{ext} \) for a large class of terms \( M, N \) (in fact for all terms which are not universal generators) (§§2.4, 2.5).

It is still an open question whether CL is \( \omega \)-complete 1).

As a corollary to the following theorem of Böhm [1968] which we state here without a proof, we can show that the \( \omega \)-rule holds in \( \lambda + \text{ext} \) for terms having a normal form. This was suggested to us by R. Hindley.

2.1.4 Theorem (Böhm [1968]. Cf. Curry, Hindley, Seldin [1971] Ch.11 F.)

Let \( M, N \) be closed \( \lambda \)-terms in \( \beta \eta \)-normal form such that \( M \not\equiv_\alpha N \).

Then there exists closed terms \( Z_1, \ldots, Z_n \) (\( n \geq 1 \)) such that for variables \( x, y \)

\[
\lambda \vdash M Z_1 \ldots Z_n xy = x \quad \text{and} \quad \lambda \vdash M Z_1 \ldots Z_n xy = y.
\]

2.1.5 Corollary

For closed \( \lambda \)-terms \( M, N \) which have a \( \beta \)-normal form the \( \omega \)-rule is provable i.e. if \( \lambda + \text{ext} \vdash MZ = NZ \) for all closed \( Z \), then

\( \lambda + \text{ext} \vdash M = N \).

------------------------

1) However there is a rumour that CL is not \( \omega \)-complete.
Proof.

By 1.2.8 it follows that both $M, N$ have a $\beta n$-normal form

hence $\lambda + ext \vdash M \rightarrow M_1$, $\lambda + ext \vdash N \rightarrow N_1$ where $M_1, N_1$ are closed and in normal forms.

Suppose that $M_1 \not\equiv_a N_1$.

Then by the theorem of Böhm it follows that there exists closed terms $Z_1, ..., Z_n$ such that

$$\lambda \vdash M_1 Z_1 ... Z_n xy = x$$

and

$$\lambda \vdash N_1 Z_1 ... Z_n xy = y.$$ 

From the assumption $\lambda + ext \vdash M Z = N Z$ for all closed $Z$ it follows that $\lambda + ext \vdash M_1 Z_1 = N_1 Z_1$. Hence $\lambda + ext \vdash x = M_1 Z_1 ... Z_n xy = N_1 Z_1 ... Z_n xy = y$ which contradicts the consistency of $\lambda + ext$.

Hence $M_1 \not\equiv_a N_1$, therefore $\lambda \vdash M_1 = N_1$, and hence $\lambda + ext \vdash M = N$. 

Corollary 2.1.5 will be included in the result of §2.5.

A priori we cannot state a similar result for CL because the theorem of Böhm does not extend to CL:

Let $M = S[K(SII)][K(SII)]$

$$N = K$$

$M$ and $N$ are closed terms both in normal form. But then the conclusion of 2.1.4 does not hold, because $CL \vdash MZ = (SII)(SII)$ for all $Z_1$, hence $M Z_1 ... Z_n$ does not have a normal form for all $Z_1 ... Z_n$. 
§2.2. The \( \omega \)-consistency of combinatory logic.

In order to prove the \( \omega \)-consistency of CL we introduce a theory CL\( \omega \)' which is a conservative extension of CL\( \omega \). In the object language of CL\( \omega \)' itself something like the length of a proof in CL\( \omega \)' is formulated. Because the \( \omega \)-rule is an infinitary rule, this length can be a transfinite (however countable) ordinal. Ordinals will be denoted by \( \alpha, \beta \) ... etc.

2.2.1 Definition

CL\( \omega \)' has the following language: Alphabet =
Alphabet\( _{CL} \) \( \cup \{ \equiv_\alpha | \alpha \) countable\( \} \cup \{ \vdash_\alpha | \alpha \) countable\( \} \).
The terms are those of CL.
Formulas: If \( M, N \) are terms, then
\( M \vdash N, M \equiv_\alpha N, M \equiv_\alpha N \) and \( M \equiv_\alpha N \) are formulas.

2.2.2 Definition

CL\( \omega \)' has the following axioms and rules (see appendix).

I  
Same as in 1.4.2.

II  
1. \( M \equiv_\alpha M \),  
\( M \equiv_\alpha M \), \( M \equiv_\alpha M \)

2. \( M \equiv_\alpha N \),  
\( M \equiv_\alpha N \), \( M \equiv_\alpha N \)

\[ \begin{align*}
   M \equiv_\alpha N \quad &\text{and} \quad N \equiv_\alpha M
\end{align*} \]

3. \( M \equiv_\alpha N, N \equiv_\alpha L \)
\[ M \equiv_\alpha L \]

4. \( M \equiv_\alpha M', M \equiv_\alpha M' \)
\[ \begin{align*}
   2M \equiv_\alpha 2M', MZ \equiv_\alpha M'Z
\end{align*} \]

5. \( M \equiv_\alpha M', \alpha \equiv_\alpha' \), \( M \equiv_\alpha M' \)
\[ M \equiv_\alpha M' \]

\[ \begin{align*}
   M \equiv_\alpha M
\end{align*} \]
III 1. M $\triangleright$ M

2. M $\triangleright$ N, N $\triangleright$ L

3. $\frac{M \triangleright M'}{ZM \triangleright ZM'}$, $\frac{M \triangleright M'}{NZ \triangleright M'Z}$

4. $\frac{M \triangleright M'}{M \equiv \alpha M'}$, $\frac{M \sim \alpha M'}{M = M'}$

IV $\omega'$-rule

In the above M, M', N, L, Z are arbitrary terms, and $\alpha, \alpha'$ arbitrary countable ordinals.

The intuitive interpretation of

- $M =_\alpha N$ is: M = N is provable using the $\omega$-rule at most $\alpha$ times.
- $M \sim_\alpha N$ is: M = N is provable without use of transitivity.
- $M \equiv_\alpha N$ is: M = N follows directly from the $\omega$-rule (or is provable in CL in case $\alpha = 0$).

2.2.3 *Lemma*

$CL \vdash M \triangleright N \iff CL_\omega \vdash M \triangleright N \iff CL_\omega' \vdash M \triangleright N.$

*Proof.*

Induction on the length of proof of $M \triangleright N$.

2.2.4 *Lemma*

$CL_\omega' \vdash M = N \iff \exists \alpha \ CL_\omega' \vdash M =_\alpha N.$

*Proof.* Trivial

2.2.5 *Lemma*

$CL_\omega' \vdash M = N \iff CL_\omega' \vdash M = N$

*Proof.*

Show by induction (on the length of proof):
1. \( \text{CL} \vdash M = N \Rightarrow \exists \alpha \text{ CL} \vdash M = _\alpha N \) and

2. \( \text{CL} \vdash M = _\alpha N \Rightarrow \text{CL} \vdash M = N \)

\( \text{CL} \vdash M \sim _\alpha N \Rightarrow \text{CL} \vdash M = N \)

\( \text{CL} \vdash M \approx _\alpha N \Rightarrow \text{CL} \vdash M = N \)

Then the result follows from 2.2.4.

2.2.3 and 2.2.5 state that \( \text{CL} \vdash \) is a conservative extension of \( \text{CL} \).

2.2.6 Lemma

\( \text{CL} \vdash M = N \iff \text{CL} \vdash M = _0 N \)

Proof.

Show by induction

1. \( \text{CL} \vdash M = N \Rightarrow \text{CL} \vdash M = _0 N \) and

2. \( \text{CL} \vdash M = _0 N \Rightarrow \text{CL} \vdash M = N \)

\( \text{CL} \vdash M \sim _0 N \Rightarrow \text{CL} \vdash M = N \)

\( \text{CL} \vdash M \approx _0 N \Rightarrow \text{CL} \vdash M = N \)

2.2.7 Lemma

\( \text{CL} \vdash M = _\alpha N \iff \exists \beta_1, \ldots, \beta_k \exists \beta_1, \ldots, \beta_k N_k \leq \alpha \)

\( \text{CL} \vdash M \sim _\beta N_1 \sim _\beta N_2 \sim \ldots \sim _\beta N_k \approx N \)

Proof.

= Trivial.

\( \Rightarrow \) Induction on the length of proof of \( M \leq _\alpha N \).

2.2.8 Lemma

1) If \( \text{CL} \vdash M \sim _\alpha N \) and \( M, N \) are closed, then \( \exists M', N', Z \) closed.

\( \{ \text{CL} \vdash ZM' = _\alpha M, \text{CL} \vdash ZN' = _\alpha N \text{ and } \text{CL} \vdash M' \approx _\alpha N' \} \).

2) \( \text{CL} \vdash M \approx _\alpha N \ ) \alpha \neq 0 \iff \forall Z \text{ closed } \exists \beta < \alpha \text{ CL} \vdash M \approx _\beta NZ. \)
Proof.

1. Induction on the length of proof of $M \sim_\alpha N$ (as an illustration we give the full proof):

   **case 1.** $M \sim_\alpha N$ is an instance of II 1.
   
   Take $M' = N' = M (\equiv N)$ and $Z = I$.

   **case 2.** $M \sim_\alpha N$ is a direct consequence of $N \sim_\alpha M$.
   
   By the induction hypothesis there are $N', M', Z$ closed such that
   
   $\text{CL}_\omega' \vdash ZN' = 0, N, \text{CL}_\omega' \vdash ZM' = M$ and $\text{CL}_\omega' \vdash N' \equiv_\alpha M'$.
   
   This is what we had to prove.

   **case 3.** $M \sim_\alpha N$ is $ZM_1 \sim \alpha Z_1 N_1$ (hence $Z_1$ is closed) and is a direct consequence of $M_1 \sim_\alpha N_1$.
   
   By the induction hypothesis there are closed $M_1', N_1', Z_0$ such that
   
   $\text{CL}_\omega' \vdash Z_0 M_1' = 0, M_1, \text{CL}_\omega' \vdash Z_0 N_1' = 0, N_1$ and $\text{CL}_\omega' \vdash M_1' \equiv_\alpha N_1'$.

   Define $Z = \lambda a. Z_1(Z_0 a)$, then the conclusion holds for $M_1', N_1', Z$ as follows from 1.4.7.

   **case 4.** $M \sim_\alpha N$ is $M_1 Z_1 \sim_\alpha N_1 Z_1$. This case is treated analogous to case 3.

   **case 5.** $M \sim_\alpha N$ is a direct consequence of $M \equiv_\alpha N$. Take $M' = M$, $N' = N$ and $Z = I$.

2. Induction on the length of proof of $M \equiv_\alpha N$.

2.2.9 Main Lemma

If $\text{CL}_\omega' \vdash ZM \geq K$ and $\text{CL}_\omega' \vdash M \equiv_\alpha N$, where $\alpha \not\equiv 0$ and $M, N$ and $Z$ are closed, then

$\exists \beta < \alpha [\text{CL}_\omega' \vdash ZNKK = \beta K]$.

We will carry out the proof of the main lemma in §2.3.
2.2.10 Notation

\[ MK_n \] stands for \( M K \cdots K \) \( n \) times

Note that \( K_n \) is not a term, because \( MK_\cdots K \) stands for \((\ldots (MK)K)\ldots K\ldots \).

2.2.11 Lemma

If \( CL\alpha \vdash M = K \) and \( M \) is closed, then

\[ \exists n \in \omega \; CL \vdash MK_{2n} \to K \]

Proof.

Induction on \( \alpha \). Because we will make use of a double induction we call the induction hypothesis with respect to this induction the \( \alpha \)-ind.hyp.

**case 1.** \( \alpha = 0 \). Then \( CL\alpha \vdash M = K \) implies that \( CL \vdash M = K \) by 2.2.6, hence \( CL \vdash M \to K \) by 1.5.1. So we can take \( n = 0 \).

**case 2.** \( \alpha > 0 \). From 2.2.7 it follows that

\[ CL\alpha \vdash M = K \iff \exists M_1 \cdots M_k \exists \beta_1, \ldots, \beta_k < \alpha \]

\[ CL\alpha \vdash \sim \beta_1 M_1 \sim \beta_2 M_2 \cdots \sim \beta_k M_k \to K. \]  

We can suppose that the \( M_i \), \( i = 1, \ldots, k \) are all closed. Because if they were not, we could substitute some constant for the free variables of the \( M_i \) and then also \( (**) \) would hold.

Now we prove with induction on \( k \) that \( (**) \rightarrow (*) \).

The induction hypothesis w.r.t. this induction is called the \( k \)-ind.hyp.

If \( k = 0 \) then there is nothing to prove, so suppose that \( k > 0 \).
subcase 2.1. $\beta_k < a$.

Then $\text{CL} \omega' \vdash M_{k-1} \sim \beta M_k \geq K$ with $\beta = \beta_k$.

Therefore $\text{CL} \omega' \vdash M_{k-1} \sim \beta K$.

Hence by the $\alpha$-ind.hyp.

$\exists n \in \omega \text{ CL} \vdash M_{k-1}2^n \geq K$, because we assumed that $M_{k-1}$ is closed.

Thus

$\exists n \in \omega \text{ CL} \vdash M_{k-1}2^n \geq K$, which is

$\exists n, n' \in \omega \text{ CL} \vdash M_{k-1}2^n \geq K$, hence by the $k$-ind.hyp.

$\exists n, n' \in \omega \text{ CL} \vdash M_{k-1}2^n \geq K$, which is

$\exists n, n' \in \omega \text{ CL} \vdash M_{k-1}2^n \geq K$.

\[ \text{subcase 2.2. } \beta_k = a. \]

Then $\text{CL} \omega' \vdash M_{k-1} \sim \sim \beta M_k \geq K$.

By 2.8.1 it follows that there are $M'_{k-1}, M'_k, z$ such that

$\text{CL} \omega' \vdash ZM_{k-1} \equiv_\beta M_{k-1} \sim_\beta M_k$ and

$\text{CL} \omega' \vdash M_{k-1} \equiv_\alpha M'_k$.

Hence by 2.2.6 and 1.5.1 it follows that

$\text{CL} \omega' \vdash ZM_k \geq K$.

By the main lemma 2.2.9 it follows that

$\exists \beta < \alpha [\text{CL} \omega' \vdash ZM_{k-1}KK \equiv_\beta K]$, thus

$\exists \beta < \alpha [\text{CL} \omega' \vdash M_{k-1}KK \geq K]$. Hence by the $\alpha$-ind.hyp. $\exists n \in \omega \text{ CL} \vdash M_{k-1}KK2^n \geq K$, thus

$\exists n \in \omega \text{ CL} \vdash M_{k-1}KK2^n \geq K$.

Therefore by the $k$-ind.hyp. we have

$\exists n, n' \in \omega \text{ CL} \vdash M_{k-1}KK2^n \geq K$ i.e.

$\exists n, n' \in \omega \text{ CL} \vdash M_{k-1}KK2^n \geq K$.  \(\Box\)
2.2.12 Corollary
If \( \text{CL} \vdash M = K \), \( M \) is closed, then \( \exists n \in \omega \) \( \text{CL} \vdash \text{MK}_n = K \).

Proof.
This follows immediately from 2.2.11 by 2.2.4 and 2.2.5.

2.2.13 Theorem
CL is \( \omega \)-consistent.

Proof.
Suppose \( \text{CL} \omega \) were inconsistent, then
\( \text{CL} \omega \vdash KK = K \). Therefore by 2.2.12
\( \exists n \in \omega \) \( \text{CL} \vdash \text{KK}_n = K \), Hence \( \text{CL} \vdash KK = K \).
This contradicts the Church-Rosser theorem for CL.

Theorem 2.2.13 implies in particular that \( \lambda \omega \), CL + ext and the \( \lambda \)-calculus + ext are consistent.

§2.3. The theory CL.
The most convenient way to carry out the proof of the main lemma 2.2.9, is to develop a new theory CL.
The intuitive idea behind the proof is the following.
Definition. An occurrence \( M' \) of a subterm of \( M \) is said to be active if this occurrence is in a part \( (M'N) \) of \( M \), otherwise it is passive.
In the theory CL we keep track of the occurrences of the residuals of \( M \) in the reduction \( ZM \rightarrow K \) by underlining them. Then we substitute \( N \) for the underlined subterms (this is done by \( \phi_N \) of 2.3.8) and we obtain a reduction of \( ZN \). When an occurrence of a
residual of M is active, we omit the underlining, because sub-
terms like MN sometimes have to be evaluated. (This is the
essence of axiom VI). This is not in conflict with the substitut-
ion of N for the underlined subterms, because by 2.2.8.2) it fol-
lows that if \( M \approx N \), active occurrences of M in the reduction
\( ZM \not\geq K \) can be replaced by N up to equality of a lower level
(i.e. \( \beta < \alpha \)).

If in the reduction of \( ZM \) to K it happens that all the residuals
of M are active sooner or later, we are done, because then
\( ZN =_N K \). In the opposite case K is a residual of M, hence \( M \not\geq K \)
and \( ZN =_N N \). Therefore
\[ ZNNK =_N NNNK =_N MKK \not\geq K, \text{ with } \beta, \beta' < \alpha. \]

2.3.1 Definition

\( CL \) is a theory defined in the following language:

Alphabet\(_{CL} = \) Alphabet\(_{CL} \cup \{ _-, = \}. \)

Simple terms are defined inductively by

1) Any variable or constant is a simple term.

2) If M, N are simple terms, then (MN) is a simple term.

Terms are defined inductively by

1) Any simple term is a term.

2) If M is a simple term, then \( M \) is a term.

3) If M, N are terms, then (MN) is a term.

Formulas: If M, N are terms, then

\( M \not\geq N, M \geq N, M = N \) and \( M = N \) are formulas.

Remark: The simple terms of \( CL \) are exactly the terms of \( CL \).
2.3.2 Definition

CL has the following axioms and rules (see appendix):

I  Same as in 1.5.2.

II Same as in 1.5.2.

III Same as in 1.5.2.

IV  Same as in 1.5.2 plus \( M >_1 M' \)

\[ M >_1 M' \]

V 1. \( M = M \)

2. \( M = M' \)

\[ M' = M \]

3. \( M = N, N = L \)

\[ M = L \]

4. \( M = M' \)

\[ 2M = 2M' \]

\[ MZ = M'Z \]

5. \( M = M \)

VI  \( MN >_1 MN \).

In the above the restrictions on the terms are clear.

Remarks: Axiom VI is essential for CL as will become clear later on.

If \( CL \vdash M = M' \), then \( M \) and \( M' \) are, except for the underlining, equal.

2.3.3 Lemma

1) \( CL \vdash M >_1 M' \) \iff \( CL \vdash M > M' \) if \( M, M' \) are simple terms

2) \( CL \vdash M >_1 M' \) \iff \( CL \vdash M > M' \) if \( M, M' \) are simple terms

3) \( [CL \vdash M >_1 M' \text{ and } N \text{ sub } M'] \ implies \exists N[N \text{ sub } M \text{ and } CL \vdash N >_1 N'] \).

\( N \text{ sub } M \) means that \( N \) is a subterm of \( M \).
Proof.
1) Induction on the length of proof of $M \models M'$.
2) Immediate.
3) Immediate using 2) and 1).

It follows from 2.3.3. 1) that $\text{CL}$ is a conservative extension of $\text{CL}$.

2.3.4 Lemma

$\text{CL} \vdash M \models M' \iff \exists N_1 \ldots N_k \quad \text{CL} \vdash M \equiv N_1 \implies \ldots \implies N_k \equiv M'$.

Proof. Immediate.

2.3.5 Lemma

If $\text{CL} \vdash ZM \models M'$, where $Z$ is simple, and $N$ sub $M'$, then

(*) $\text{CL} \vdash M \models N$.

Proof. By 2.3.4 $\text{CL} \vdash ZM \models M' \iff \exists N_1 \ldots N_k \quad \text{CL} \vdash ZM \equiv N_1 \implies \ldots \implies N_k \equiv M'$.

From lemma 2.3.3 it follows by induction on $k$ that (*) holds.

2.3.6 Lemma

Let $M, M', N$ be terms such that

1) $M$ and $M'$ are simple,
2) $\text{CL} \vdash M \models M'$ and
3) $\text{CL} \vdash M \equiv N$,

then there exists a term $N'$ such that

4) $\text{CL} \vdash N \models N'$ and
5) $\text{CL} \vdash M' \equiv N'$ (see figure 7).
Proof.

Induction on the length of proof of $M \supset M'$ with the use of axiom VI and the sublemmas:

$$\text{ If } \text{ CL } \vdash M = N \iff M \equiv N \text{ or } [M \equiv M_1 \text{ and CL } \vdash M_1 = N] \text{ or } [N \equiv N_1 \text{ and CL } \vdash M = N_1] \text{ or } [M \equiv M_1 N_2 \text{ and } N \equiv N_1 N_3 \text{ and } \text{ CL } \vdash M_1 = N_1 \text{ and CL } \vdash M_2 = N_2].$$

2.3.7 Definition

Let $A$ be a simple term of CL.

We define a mapping $\phi_A : \text{CL} \rightarrow \text{CL}$ (in fact from the set of terms of CL onto the set of terms of CL).

$\phi_A(c) = c$ if $c$ is a constant or variable.

$\phi_A(MN) = \phi_A(M) \phi_A(N)$

$\phi_A(M) = A$

2.3.8 Lemma

If $\text{ CL } \vdash \{VI\} \vdash M \supset M'$, then $\text{ CL } \vdash \phi(N)(M) \supset \phi(N)(M')$ for simple terms $N$.

Proof.

Immediate.

2.3.9 Lemma

If $\text{ CL } \vdash ZM \supset M'$, where $Z$ is simple, $Z, M, M'$ are closed and $\text{ CL}_\omega \vdash M \equiv_N N$, then $\exists \beta < \alpha \text{ CL}_\omega' \vdash \phi_{\beta}(ZM) = \beta \phi_{\alpha}(M')$.

Proof.

Suppose $\text{ CL } \vdash ZM \supset M'$ and $\text{ CL}_\omega' \vdash M \equiv_N N$.

Then $\exists N_1 \ldots N_k \text{ CL } \vdash ZM \equiv N_1 \supset \ldots \supset N_k \equiv M'$. 
We claim that

\( (*) \forall i \leq k. \exists \beta \leq \alpha. \text{ CL} \models \phi_N(N_i) = \beta \phi_N(N_{i+1}) \)

We will prove this with induction on the length of proof of \( N_i \models N_{i+1} \).

**Case 1.** \( N_i \models N_{i+1} \) is an axiom.

**Subcase 1.1.** \( N_i \models N_{i+1} \) is not an instance of axiom VI.

Then it follows from 2.3.8 and 2.2.6 that

\( \text{CL} \models \phi_N(N_i) = \phi_N(N_{i+1}) \).

**Subcase 1.2.** \( N_i \models N_{i+1} \) is an instance of axiom VI, say \( M_1, M_2 \models M_1, M_2 \).

Then we have to show that

\( (** \exists \beta \leq \alpha. \text{ CL} \models \phi_N(M_1) = \beta \phi_N(M_2) \) because \( M_1 \) is simple and hence \( \phi_N(M_1) = M_1 \).

Since \( \text{CL} \models \exists M \models M_1, M_2 \) it follows from 2.3.5.

Hence since \( \text{CL} \models M \models N \), it follows from 2.2.8, 2) and 2.2.6 that

\( \exists \beta \leq \alpha. \text{ CL} \models \phi_N(M_1) = \beta \phi_N(M_2) = \phi_N(M_2) \).

This implies (**).

**Case 2.** \( N_i \models N_{i+1} \) is \( ZM_1 \models ZM_2 \) and is a direct consequence of

\( M_1 \models M_2 \). By the induction hypothesis

\( \exists \beta \leq \alpha. \text{ CL} \models \phi_N(M_1) = \beta \phi_N(M_2) \) hence

\( \text{CL} \models \phi_N(Z) \phi_N(M_1) = \beta \phi_N(Z) \phi_N(M_2) \) which is

\( \text{CL} \models \phi_N(N_i) = \beta \phi_N(N_{i+1}) \).

**Case 3.** \( N_i \models N_{i+1} \) is \( M_1, Z \models M_2, Z \). This case is analogous to case 2.

Now we have established (*). Let \( \beta = \text{Max}\{\beta_0, \ldots, \beta_k\} \) then \( \beta < \alpha \) and \( \text{CL} \models \phi_N(Z) = \beta \phi_N(N_i) = \beta \phi_N(N_{i+1}) = \beta \phi_N(M') \). \( \Box \)
Now we are able to prove the main lemma.

2.2.9 Main Lemma
If $\text{CLw}' \vdash ZM \geq K$ and $\text{CLw}' \vdash M \equiv N$, where $\alpha \neq 0$ and $M$, $N$ and $Z$ are closed, then $\exists \beta < \alpha [\text{CLw}' \vdash \text{ZNKK} = \beta K]$.

Proof.
If $\text{CLw}' \vdash ZM \geq K$, then by 2.2.3 $\text{CL} \vdash ZM \geq K$, hence by 2.3.3 $\text{CL} \vdash ZM \geq K$ and therefore by 2.3.6 $\text{CL} \vdash ZM \geq K'$ with $\text{CL} \vdash K' = K$ hence $K' \equiv K$ or $K' \equiv K$.

case 1. $K' \equiv K$.
By 2.3.9 it follows that $\exists \beta < \alpha \text{CLw}' \vdash \text{ZN} = \beta K$, hence $\exists \beta < \alpha \text{CLw}' \vdash \text{ZNKK} = \beta \text{K}$.

case 2. $K' \equiv K$.
Then $\text{CL} \vdash ZM \geq K$ hence by 2.3.5

(1) $\text{CL} \vdash M \geq K$

Again by 2.3.9 we have $\exists \beta < \alpha \text{CLw}' \vdash \text{ZN} = \beta N$. Hence

(2) $\text{CLw}' \vdash \text{ZNKK} = \beta \text{NK}$.

Because $\text{CLw}' \vdash M \equiv N$, it follows from 2.2.8.2) that

(3) $\exists \beta' < \alpha \text{CLw}' \vdash \text{NKK} = \beta' \text{MK}$.

From (1), (2) and (3) it follows that

$\exists \beta, \beta' < \alpha \text{CLw}' \vdash \text{ZNKK} = \beta \text{NK} = \beta' \text{MK} \geq \text{KK} \geq K$.

Hence $\exists \beta'' < \alpha \text{CLw}' \vdash \text{ZNKK} = \beta'' K$. [Q.E.D.]

§2.4. Universal generators.
The motivation of the contents of this § will be stated in 2.4.6.

2.4.1 Definition
The $\lambda$-family of a term $M$ of the $\lambda$-calculus, notation $\mathcal{F}_\lambda(M)$ is the following set of $\lambda$-terms
\( J_\lambda (M) = \{ N \mid \exists M' \quad \lambda \vdash M \supset M' \quad \text{and} \quad N \subseteq M' \} \).

Analogously we define \( J_{\text{CL}} \) and \( J_{\lambda + \text{ext}} \).

2.4.2 Definition

U is a universal generator (u.g.) for the \( \lambda \)-calculus if

\( J_\lambda (U) \) consists of all closed \( \lambda \)-terms.

Analogously we define the universal generators for CL.

Remark: If U is a universal generator for the \( \lambda \)-calculus then

\( J_\lambda (U) \) even consists of all \( \lambda \)-terms, since every \( \lambda \)-term \( M \) is sub-

term of a closed \( \lambda \)-term (take the closure \( \lambda x_1 \ldots x_n . M \)).

2.4.3 Lemma

There exists a closed term \( E \) such that

\[ \forall M [ FV(M) = \emptyset \Rightarrow \exists n \quad \lambda \vdash E_n \supset M ] \].

Proof.

This follows from inspection of the proof of 1.3.13.

See for details Barendregt [1970].

2.4.4 Theorem

There exists a closed universal generator for the \( \lambda \)-calculus.

Proof.

We give a modification of our original construction, due to Scott.

Let \( E \) be as in lemma 2.4.3, let \( \ldots, \ldots \) be the pairing

function as in 1.3.6 and let \( S^* \) be the \( \lambda \)-defining term of the
successor function \((1.3.4.3)\).

Define \(A = \lambda b n.[E_n b(\bar{S}^m n)]\).

\[ B = \text{FP } A \quad \text{(see 1.1.8)} \]

Then \(\lambda A n > A B n > [E_n, B n+1]\). Hence

\[ \lambda A n > [E_0, B_0] > [E_0, [E_1, B_2]] > \]

\[ > [E_0, [E_1, [E_2, B_3]]] > \ldots \]

Because \(E\) enumerates all closed terms, \(B_0\) is a universal generator. Since \(E\) is closed, \(B_0\) is closed too.

The above considerations also hold for CL. In particular

Kleene's \(E\) (2.4.3) is given for CL by a term \(GD^{-1}\) in Curry, Hindley, Seldin [1971], Ch 13. The name \(GD^{-1}\) is used because it is the inverse of the Gödel numbering.

Therefore we have

2.4.5 Theorem

There exists a universal generator for CL.

2.4.6 Remark

The motivation for the introduction of universal generators is the following:

In the next \(\S\) we will prove that, if \(M\) and \(N\) are not u.g.'s and if \(\lambda + \text{ ext } \vdash MZ = NZ\) for all closed \(Z\), then \(\lambda + \text{ ext } \vdash M = N\).

At the moment of discovery of this theorem, we were still unaware of the existence of u.g.'s. We hoped to prove that they did not exist in order to obtain, as corollary, the \(\omega\)-completeness of the \(\lambda\)-calculus.

However we subsequently found a proof of the existence of u.g.'s. The proof of the existence of u.g.'s was presented first, because
the above mentioned theorem is easier to prove with their application.

§2.5. The provability of the \( \omega \)-rule for non universal generators.

In this § we will prove a result on partial \( \omega \)-completeness. We present the proof for the \( \lambda \)-calculus because there extensionality can be axiomatized by \( n \)-reduction for which the Church-Rosser property holds (1.2.11). We do not know whether a similar result holds for CL, but probably we can prove it using strong reduction.

2.5.1 Definition
A \( \lambda \)-term \( Z \) is said to be of order 0 if there is no term \( P \) such that \( \lambda \vdash Z \geq (\lambda x P) \).

2.5.2 Lemma
Let \( Z \)' be of order 0, then:
1) For no term \( P \) we have \( \lambda + \text{ext} \vdash Z \geq \lambda x P \)
2) If \( \lambda + \text{ext} \vdash Z \geq Z' \), then \( Z' \) is of order 0
3) If \( \lambda + \text{ext} \vdash ZM \geq N \), then there exist terms \( Z', M' \) such that
   \[ N \equiv Z'M', \lambda + \text{ext} \vdash Z \geq Z' \text{ and } \lambda + \text{ext} \vdash M \geq M' \]
4) For all terms \( M, ZM \) is of order 0.

Proof.
For this proof let us call a term of the first kind if it is a variable, of the second kind if it is of the form \( (MN) \) and of the third kind if it is of the form \( (\lambda x M) \).

1) Suppose \( \lambda + \text{ext} \vdash Z \geq \lambda x P \) for some \( P \). By Curry,Feys [1958] Ch 4D, theorem 2 pg 132, it follows that there exists a term \( Z' \) such that \( \lambda \vdash Z \geq Z' \) and \( \lambda + \text{ext} - I \vdash Z' \geq (\lambda x P) \) (i.e.
without using $\beta$-reduction). Because $Z$ is of order 0, $Z'$ is of the first or of the second kind. $Z'$ cannot be a variable because $\lambda + \text{ext} \vdash Z' \gg \lambda x \, P$. Hence $Z'$ is of the second kind. By induction on the length of proof in $\lambda + \text{ext} - I \ 2$ of a reduction $M \gg N$ we can show that if $M$ is of the second kind, then $N$ is of the second kind. This would imply that $\lambda x \, P$ is of the second kind; a contradiction.

2) Immediate, using 1).

3) By induction on the length of proof of $ZM \gg N$ using 2).

4) By 3) it follows that if $\lambda + \text{ext} \vdash ZM \gg N$, then $N$ is of the second kind. Hence $ZM$ is of order 0.

2.5.3 Examples

1. Any variable is of order 0.

2. $\Omega_2 = \omega_2 \omega_2$ with $\omega_2 = (\lambda a \cdot aa)$ is of order 0.

Terms of order 0 behave in some sense like variables. Namely, if $\lambda + \text{ext} \vdash MZ \gg L$ where $Z$ is of order 0, we can substitute $x$ for the residuals of $Z$ in this reduction and we obtain $\lambda + \text{ext} \vdash Mx \gg L'$.

Because $\Omega_2$ is at the same time closed and of order 0 it will play an important role in connection with the $\omega$-rule. If $\lambda + \text{ext} \vdash MZ = NZ$ for all closed $Z$ we have in particular $\lambda + \text{ext} \vdash M\Omega_2 = N\Omega_2$. We hoped that this would imply $\lambda + \text{ext} \vdash Mx = Nx$, by substituting everywhere $x$ for $\Omega_2$ in the proof. The problem is that there is a difference between variables and terms of order 0. In a reduction, variables can never be generated whereas closed terms can.

Therefore we have to find a term of order 0 $Z_0 \in \mathcal{F}_\lambda(M) \cup \mathcal{F}_\lambda(N)$
(see 2.4.1). This is only possible if M and N are not universal
generators. Then it follows from $\lambda + \text{ext} \vdash MZ = NZ$ that
$\lambda + \text{ext} \vdash Mx = Nx$ and hence $\lambda + \text{ext} \vdash M = N$.
In order to follow the residuals of a subterm in a reduction
we again make use of the underlining technique.

An outline of what happens is the following (see fig. 8).
In 2.5.4 - 2.5.14 we define and develop a theory $\lambda$.
In 2.5.15 - 2.5.16 we consider a mapping $\phi_x$ which replaces a
term of order 0 by a variable $x$ as is mentioned above.
In 2.5.21 - 2.5.24 we define the concept of closed terms which
are variable like and prove their existence.

Then, to prove the main result, we assume that $\lambda + \text{ext} \vdash ME = NE$.
It follows from the Church-Rosser theorem that for some term $L^*$,
$\lambda + \text{ext} \vdash ME \supset L$ and $\lambda + \text{ext} \vdash NE \supset L$. From this it follows, by
the results of the theory $\lambda$ that $\lambda + \text{ext} \vdash ME \supset L'$ and
$\lambda + \text{ext} \vdash NE \supset L''$. The main difficulty is then to prove that
$L' \equiv L''$. If we have $L' \equiv L''$, then it follows by a homomorphism
argument that $\lambda + \text{ext} \vdash Mx = \phi_x(L') \equiv \phi_x(L'') = Nx$.
In order to prove that $L' \equiv L''$, we need proposition 2.5.20, a
statement about $\phi_{\lambda+\text{ext}}(ME)$ and lemma 2.5.27, a statement about
variable like terms.

§2.5

Figure 8
2.5.4 Definition

We will define a theory \( \lambda + \text{ext} \) formulated in the following language (see appendix).

Alphabet \( \lambda + \text{ext} = \text{Alphabet}_\lambda \cup \{>, \_\} \)

Simple terms of the theory \( \lambda \) are exactly the terms of the \( \lambda \)-calculus.

Terms are defined inductively by

1) Any simple term is a term.
2) If \( M \) is a simple term and \( \text{FV}(M) = \emptyset \), then \( M \) is a term.
3) If \( M, N \) are terms, then \( (MN) \) is a term.
4) If \( M \) is a term, then \( (\lambda x \, M) \) is a term (\( x \) is an arbitrary variable).

Formulas: if \( M, N \) are terms, then

\( M \gg N, M \gg N, M = N \) and \( M \neq N \) are formulas.

A term of the theory \( \lambda + \text{ext} \) is called \( \lambda \)-term.

The operations \( \text{BV}, \text{FV} \) and \( [x\backslash N] \) can be extended to \( \lambda \)-terms in the obvious way.

(Note that: \( \text{BV}(M) = \text{BV}(M), \text{FV}(M) = \emptyset \) and \( [x\backslash N] M = M \).

2.5.5 Definition

The relation "...is subterm of ..." is defined in such a way that only \( M \) is a subterm of \( M \). To be explicit:

\( \text{Sub}(x) = \{x\} \) for any variable \( x \).

\( \text{Sub}(MN) = \text{Sub}(M) \cup \text{Sub}(N) \cup \{MN\} \)

\( \text{Sub}(\lambda x \, M) = \text{Sub}(M) \cup \{\lambda x \, M\} \)

\( \text{Sub}(M) = \{M\} \)

\( N \text{ sub } M \iff N \in \text{Sub } M \)
2.5.6 Definition

We define the theory $\lambda + \text{ext}$ by the following axioms and rules (see appendix).

I 1. $\lambda x \ M \succ_i \lambda y[\chi \ y] M$ if $y \notin \text{FV}(M)$
2. $(\lambda x \ M)N \succ_i [\chi \ M] N$ if $\text{BV}(M) \cap \text{FV}(N) = \emptyset$
3. $\lambda x(Mx) \succ_i M$ if $x \notin \text{FV}(M)$

II Same as in 1.1.4.

III Same as in 1.2.3.

IV 1. $M \succ_i M$
2. $M \succ_i M'$
3. $ZM \succ_i ZM'$
4. $M \succ_i M'$
5. $M \succ_i M'$

V 1. $M = M$
2. $M = N$
3. $N = M$
4. $M = M'$
5. $M = M'$

In the above the restriction on the terms is clear.

Note that we do not have a counterpart for axiom VI of 2.3.2.

$\lambda$ is the theory which results from the above axioms and rules, omitting I 3.

2.5.7 Lemma

1) $\lambda + \text{ext} \vdash M \succ M'$ if $M, M'$ are simple terms
2) $\lambda + \text{ext} \vdash M \succ M'$ if $M, M'$ are simple terms
3) $[\lambda + \text{ext} \vdash M \succ M'$ and $N \subseteq M'] = [\exists N \ N \subseteq M$ and $\lambda + \text{ext} \vdash N \succ N'].$
1) Induction on the length of proof of $M \geq M'$

2) Immediate

3) Immediate, using 1) and the following sublemma

   \[ N \subseteq [x \land Q] P \iff N \subseteq P \text{ or } N \subseteq Q. \]

The proof of the sublemma proceeds by induction on the structure of $P$.

**2.5.8 Lemma**

\[ \lambda + \text{ext } \vdash M \geq M' \iff \exists N_1 \ldots N_k \quad \lambda + \text{ext } \vdash M \equiv N_1 \geq \ldots \geq N_k \equiv M'. \]

**Proof.**

Immediate.

**2.5.9 Lemma**

\[ \{ \lambda + \text{ext } \vdash M \geq M' \text{ and } N \subseteq M' \} \Rightarrow \{ \exists N \quad N \subseteq M \text{ and } \lambda + \text{ext } \vdash N \geq N' \}. \]

**Proof.**

By 2.5.8

\[ \lambda + \text{ext } \vdash M \geq M' \iff \exists N_1 \ldots N_k \quad \lambda + \text{ext } \vdash M \equiv N_1 \geq \ldots \geq N_k \equiv M'. \]

From lemma 2.5.7, 3) it follows by induction on $k$ that the conclusion holds.

**2.5.10 Lemma**

\[ \{ \lambda \vdash \text{L } = \text{ M } \Rightarrow [L \equiv M \text{ or } L \equiv M^\prime ] \}, \quad L \equiv M \]

1) \[ \lambda \vdash L = M \iff [L \equiv M \text{ or } L \equiv M'] \quad L \equiv M \]

2) \[ \lambda \vdash L = \lambda x M \iff [\exists M' \quad L \equiv \lambda x M' \text{ and } \lambda \vdash M = M'] \text{ or } L \equiv \lambda x M \]

3) \[ \lambda \vdash L = MN \iff [\exists M'N' \quad L \equiv M'N' \text{ and } \lambda \vdash M = M', \lambda \vdash N = N'] \text{ or } L \equiv MN \]

4) \[ \lambda \vdash M = M' \text{ and } \lambda \vdash N = N' \quad \lambda \vdash [x \land N] M = [x \land N'] M'. \]

**Use**

\[ \lambda \vdash M = M' \Rightarrow |M| \equiv |M'|. \]

\[ \ldots \]

\[ \text{the in } \lambda \]
Proof.

1) Induction on the proof of \( L = M \), making for the induction hypothesis the statement slightly stronger.

\[ [\lambda \vdash L = M \text{ or } \lambda \vdash M = L] \iff [L \equiv M \text{ or } L \equiv M] \]

2) Similarly, making use of 1).

4) Induction on the structure of \( M \), making use of 1), 2) and 3).

2.5.11 Lemma

Let \( M, N \) be simple terms such that \( M \gg_1 N \) is an axiom of \( \lambda + \text{ext} \) but not an instance of I 2.

Let \( \lambda \vdash M = M' \). Then \( \exists N'[\lambda + \text{ext} \vdash M' \gg N' \text{ and } \lambda \vdash N = N'] \).

(See fig. 9.)

![Figure 9](image_url)

Proof.

By distinguishing cases and using the previous lemma.

2.5.12 Lemma

Let \( M, N \) be simple terms such that \( M \gg_1 N \) is an instance of axiom I 2. Let \( \lambda \vdash M = M' \), where \( M' \) is such that if \( Z \) sub \( M' \), then \( Z \) is of order 0.

Then \( \exists N'[\lambda + \text{ext} \vdash M' \gg N' \text{ and } \lambda \vdash N = N'] \) (see fig. 9).

Proof.

Let \( M \gg N \) be \((\lambda x)Q \gg [x\backslash Q]P\).

As \( \lambda \vdash (\lambda x)Q \equiv M' \) we can distinguish by 2.5.10 several cases.
case 1. $M' \equiv (\lambda x \, P)Q$. Take $N' \equiv [x\backslash Q]P$.

case 2. $M' \equiv M''Q'$ with $\lambda \vdash \lambda x \, P = M''$ and $\lambda \vdash Q = Q'$.

subcase 2.1. $M'' \equiv \lambda x \, P'$ with $\lambda \vdash P = P'$.

Take $N' \equiv [x\backslash Q']P'$, then the result follows from 2.5.10,4).

subcase 2.2. $M'' \equiv \lambda x \, P$. This case cannot occur, because then $\lambda x \, P$ sub $M'$, and $\lambda x \, P$ is not of order 0, contrary to our assumption. ☐

2.5.13 Lemma
Let $M, N$ be simple terms such that $\lambda \vdash M \Rightarrow N$.

Let $\lambda \vdash M = M'$, where $M'$ is such that if $Z$ sub $M'$, then $Z$ is of order 0. Then $\exists N'(\lambda + \vdash M' \Rightarrow N', \lambda \vdash N = N'$ and $[Z \text{ sub } N' \Rightarrow Z \text{ is of order } 0]$) (see fig. 9).

Proof.

Induction on the length of proof of $M \Rightarrow N$.
If $M \Rightarrow N$ is an axiom we are done by 2.5.11 or 2.5.12, since by lemma's 2.5.9 and 2.5.12, it follows from the assumptions that $Z$ is of order 0 if $Z$ sub $N'$. (We need this fact for the induction step in rule III 3 (transitivity).)

2.5.14 Proposition
Let $\lambda + \vdash MZ \Rightarrow L$, where $M, Z$ and $L$ are simple and $Z$ is of order 0. Then $\exists L'(\lambda + \vdash MZ \Rightarrow L', \lambda \vdash L = L'$ and $[Z' \text{ sub } L' \Rightarrow \lambda + \vdash Z \Rightarrow Z'])$ (see fig. 10).

Figure 10
2.5.15 Definition

Let $x$ be any variable. We define a mapping $\phi_x : \lambda \to \lambda$ (i.e. from the set of $\lambda$-terms into the set of $\lambda$-terms) as follows:

\[
\phi_x(y) = y
\]

\[
\phi_x(MN) = \phi_x(M)\phi_x(N)
\]

\[
\phi_x(\lambda y M) = \lambda y \phi_x(M)
\]

\[
\phi_x(M) = x
\]

2.5.16 Lemma

If $\lambda \vdash M \to N$ and if $x$ is a variable not occurring in this proof, then $\lambda \vdash \phi_x(M) \to \phi_x(N)$.

Proof.

Induction on the length of proof of $M \to N$, using the following sublemma:

If $z \neq x$, then $\phi_z([x\setminus N]M) = \phi_z(M)$.

The proof of the sublemma proceeds by induction on the structure of $M$.

2.5.17 Lemma

Let $M, N$ be simple and $x \not\in \text{FV}(M)$.

If $\lambda \vdash Mx \to N$, then $\exists M'$ simple $[x \not\in \text{FV}(M')]$, $\lambda \vdash M \to M'$ and $\lambda \vdash M'x \to N$.

Proof.

Because $\lambda \vdash Mx \to N$ we have by 2.5.7.1 and 2.5.8 that

$\exists N_1 \ldots N_k \quad \lambda \vdash Mx \equiv N_1 \supset \ldots \supset N_k \equiv N$.

If all $N_i$, $i < k$ are of the form $P_x$ with $x \not\in \text{FV}(P)$, then we are
Otherwise let \( N_{i+1} \) be the first term not of the form \( P_x \) with \( x \not\in \text{FV}(P) \).

Then \( N_1 \) is of the form \( (\lambda z N'_1)x \).

By \( \alpha \)-reduction this reduces to \( (\lambda x [z/x]N'_1)x \) which is \( (\lambda x N_{i+1})x \).

Hence \( \lambda + \text{ext} \vdash Mx \geq (\lambda z N'_1)x \geq (\lambda x N_{i+1})x \geq (\lambda x N)x \geq N \).

So we can take \( M' \equiv \lambda x N \).

2.5.18 Lemma

1) Let \( L, L' \) be \( \lambda \)-terms such that \( \phi_x(L) = \phi_x(L') \) where \( x \not\in \text{FV}(LL') \). Let \( Z \) be a simple term such that \( Z \subseteq L \).

Then \( Z \subseteq L' \).

2) If \( M, N \) are simple terms, then we have

\[ \phi_x([x/N]\ M) = \phi_x(M) = M. \]

Proof.

Induction on the structure of \( L \) resp. \( M \).

2.5.19 Lemma

Let \( \lambda + \text{ext} \vdash M_A \geq L \) and \( Z \subseteq L \), where \( M, A \) and \( Z \) are simple and \( A \) is closed. Let \( L \) satisfy: \( A' \subseteq L = A \).

Then \( Z \subseteq \mathcal{F}_{\lambda + \text{ext}}(M) \).

Proof.

By lemma 2.5.16 we have \( \lambda + \text{ext} \vdash Mx \geq \phi_x(L) \) where \( x \) does not occur in the proof of \( M_A \geq L \). Hence by lemma 2.5.17 there exists a simple term \( M' \) such that \( \lambda + \text{ext} \vdash M \geq M' \) and \( \lambda + \text{ext} \vdash M'x \geq \phi_x(L) \). Hence \( \lambda + \text{ext} \vdash M'A \geq L \).

Suppose now \( Z \subseteq L \) and \( Z \) simple. By distinguishing the different possibilities for the proof of \( M'A \geq L \) we can then
show that $Z \subseteq M'$ hence $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$. 

2.5.20 Proposition

Let $\lambda + \text{ext} \vdash MA \supseteq L$ and $Z \subseteq L$, where $M, A$ and $Z$ are simple and $A$ is closed.

Then $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$.

Proof.

Let $x$ not occur in the proof of $\lambda + \text{ext} \vdash MA \supseteq L$.

Then by 2.5.16 we have $\lambda + \text{ext} \vdash Mx \supseteq \phi_x(L)$. Hence $\lambda + \text{ext} \vdash MA \supseteq [x\backslash A] \phi_x(L) \equiv L'$, say. By 2.5.18 it follows that $Z \subseteq L \Rightarrow Z \subseteq L'$ for simple terms $Z$. Furthermore, $L'$ satisfies the assumptions of 2.5.19.

Hence if $Z \subseteq L$ and $Z$ is simple, then $Z \subseteq L'$ and therefore $Z \in \mathcal{F}_{\lambda+\text{ext}}(M)$ by 2.5.19.

2.5.21 Definition

1) A term $M$ is called an $\Omega_2$-term if $M$ is of the form $\Omega_2 M'$.

2) A subterm occurrence $Z$ of $M$ is called non-$\Omega_2$ in $M$ if $Z$ has no $\Omega_2$ subterm and $Z$ is not a subterm of an $\Omega_2$ subterm of $M$.

3) A term $U$ is called a hereditarily non-$\Omega_2$ universal generator if $U$ is a closed u.g. and if $\lambda + \text{ext} \vdash U \supseteq U'$, then there is a subterm occurrence $Z$ of $U'$ which is a u.g. and which occurs non-$\Omega_2$ in $U'$.

Example: Only the second occurrence of $Z$ in the term $x(\Omega_2(MZ))Z$ is non-$\Omega_2$ (if $Z$ does not have an $\Omega_2$ subterm).

2.5.22 Lemma

If $U$ is a hereditarily non-$\Omega_2$ u.g. and if $\lambda + \text{ext} \vdash U \supseteq U'$, then $U'$ is a u.g. which is not an $\Omega_2$-term.
By definition it follows that some subterm $Z$ of $U'$ is a u.g. Then $U'$ itself is a u.g. That $U'$ is not an $U_2$-term follows from the fact that $Z$ occurs non-$U_2$ in $U'$.

2.5.23 Proposition

There exists a hereditarily non-$U_2$ universal generator.

Proof.

We introduce ordered triples as follows

$[M, N, L] = \lambda z. MNL$.

Define $A = \lambda b_0, E_n, b(S^+ n)$, where $E$ and $S^+$ are as in 2.4.4, $B = FPA$ and $U = B_0$.

We will prove that $U$ is a hereditarily non-$U_2$ u.g.

As in 2.4.4 we see that $U$ is a closed u.g.:

$\lambda U \equiv B_0 \Rightarrow AB_0 \Rightarrow [B_0, E_0, B_1] \Rightarrow [B_0, E_0, [B_0, E_1, B_2]] \Rightarrow \ldots$

Let us define $U \geq_k U'$ to mean

$\exists N_1, \ldots, N_k \lambda + \text{ext } U \equiv \exists N_1 \Rightarrow \ldots \Rightarrow N_k \equiv U'$.

(Here we need $X$ only to express one step reduction $\Rightarrow$.)

Suppose now that $\lambda + \text{ext } U \equiv U'$. Then for some $k$ we have $U \geq_k U'$.

With induction on $k$ we can show that $U'$ is of the form

1) $A^P B_0$ (remember that $M^P N = M(\ldots(M(MN))\ldots)$

or 2) $A^P (\lambda n[U^n, \ldots, \ldots]) \overline{\otimes}$ where

$B \geq_k, \lambda n[U^n, \ldots, \ldots]$ and $U \geq_k U''$ with $k', k'' < k$,

or 3) $[U'', \ldots, \ldots]$ where $U \geq_k, U''$ with $k' < k$.

Now we prove with induction on $k$ that if $U \geq_k U'$, then there exists a subterm occurrence $Z_{U'}$ of $U'$ which is a u.g. and
is non-$\Omega_2$ in $U'$:
If $U'$ is of the form (1) we take $Z_U^f = A^{P\bar{B}0}$.
If $U'$ is of the form (2) we take $Z_U^f = Z_{U''}$.
Finally if $U'$ is of the form (3) we take $Z_U^f = Z_{U''}^f$.

2.5.24 Definition
A term $E$ is called variable like if $E \equiv \Omega_2 U$, where $U$ is a
hereditarily non-$\Omega_2$ universal generator.

2.5.25 Definition
Let $L, L'$ be $\Lambda$-terms such that $L$ is simple and $\Lambda \vdash L = L'$. Then
$L$ and $L'$ are equal except for the underlining and we can give
the following informal definitions:

1) If $Z'$ is a subterm occurrence of $L'$, then there is a unique
subterm occurrence $Z$ of $L$ which corresponds to $Z'$, such
that $\Lambda \vdash Z = Z'$.

Instead of giving a formal definition we illustrate this
concept with an example.
Let $L \equiv S(KS)(SKK)$ and $L' \equiv S(KS)(SKK)$, then $\Lambda \vdash L = L'$.
$S$ corresponds to $S$, $KS$ corresponds to $KS$ and $(SKK)$ corresponds
to $(SKK)$.

2) Let $L''$ be another $\Lambda$-term with $\Lambda \vdash L = L''$. Then we say that
$L''$ has more line than $L'$, notation $L' \subset L''$, if for all sub-
term occurrences $Z'$ of $L'$ there is a subterm occurrence $Z''$
of $L''$ such that $Z'$ sub $Z''$ where $Z', Z''$ are the subterm
occurrences of $L$ corresponding to $Z'$, $Z''$ respectively.

For example, let $L'' \equiv S(KS)(SKK)$ then $L' \subset L''$ where $L'$ is as
in the above example.
3) Let Z be a subterm occurrence of L.
   Z is exactly underlined in L' if Z is a subterm occurrence of L' and Z corresponds to Z.

4) Let Z be a subterm occurrence of L.
   Z is underlined in L' if Z is a subterm of Z₁(sub L) which is exactly underlined in L'.

For instance the first occurrence of K in L of the above example is underlined in L'.

5) Let Z be a subterm occurrence of L.
   Z has some line in L' if Z is underlined in L' or if there is a subterm occurrence Z₁ of Z which is exactly underlined in L'.

For instance SKK sub L has some line in L' in the above example.

2.5.26 Lemma
Let L, L', L" be λ-terms such that L is simple and L \models L' = L = L"  
1) If L' C L" and L" C L', then L' \models L".
2) If for all subterm occurrences Z of L', the corresponding subterm occurrence Z of L is underlined in L", then L' C L".
3) If Z is a subterm occurrence of L such that there is a corresponding subterm occurrence Z' of L' which is simple, then Z has some line in L'.

Proof.
This is clear from the definitions.
2.5.27 Lemma

Let \( L, L' \) be \( \lambda \)-terms such that \( L \) is simple and \( \lambda \vdash L = L' \). Let \( \Xi \) be a variable like \( \lambda \)-term.

Suppose that

1) If \( Z \) is a subterm occurrence of \( L \) which is exactly underlined in \( L' \), then \( Z \) is an \( \Omega_2 \)-term.

2) If \( Z \) is a subterm occurrence of \( L \) which is a u.g. then \( Z \) has some line in \( L' \).

Suppose further that \( \lambda + \text{ext} \vdash \Xi \supset E' \) and \( E' \) is a subterm occurrence of \( L \).

Then \( E' \) is underlined in \( L' \).

Proof.

\( \Xi \) is variable like, hence \( \Xi \equiv \Omega_2 U \), where \( U \) is a hereditarily non-\( \Omega_2 \) universal generator.

Since \( \Omega_2 \) is of order 0 it follows from 2.5.2.3) that \( E' \equiv \Omega_2 U' \), where \( \lambda + \text{ext} \vdash U \supset U' \).

Since \( U \) is a hereditarily non-\( \Omega_2 \) u.g. there is a subterm occurrence \( Z \) of \( U' \) which is a u.g. and a non-\( \Omega_2 \) subterm occurrence of \( U' \) (see fig.11). By our assumption 2), \( Z \) has some line in \( L' \). The possibility that some subterm occurrence \( Z_1 \) of \( Z \) is exactly underlined in \( L' \) is excluded, since by 1) then \( Z_1 \) would be an \( \Omega_2 \)-term whereas \( Z \) is a non-\( \Omega_2 \) subterm occurrence of \( L \).

Therefore \( Z \) is underlined in \( L' \), i.e. there is a subterm occurrence \( Z_2 \) of \( L \) which corresponds to \( Z_2 \) sub \( L' \) and such that \( Z \) sub \( Z_2 \).

We claim that \( \Omega_2 U' \) sub \( Z_2 \) (see fig.11).

First note that, since \( Z_2 \) sub \( L' \), it follows from 1), that \( Z_2 \) is an
Hence since $Z$ is a non-$\Omega_2$ subterm occurrence of $U'$, $\Omega_2$ is not a subterm of $U'$. Therefore $U'$ sub $Z_2$, since subterms are either disjoint or comparable with respect to the relation sub.

Since by 2.5.22 $U'$ is not an $\Omega_2$-term $U'$ is a proper subterm of $Z_2$.

Hence indeed $\Omega_2 U'$ sub $Z_2$.

Therefore $E' \equiv \Omega_2 U'$ is underlined in $L'$.

2.5.28 Theorem

Let $M,N$ be $\lambda$-terms which are not universal generators and let $E$ be a variable like $\lambda$-term.

If $\lambda + \text{ext} \vdash ME = NE$, then $\lambda + \text{ext} \vdash Mx = Nx$ for some variable $x \in \text{FV}(MN)$.

Proof.

It follows from the Church-Rosser theorem 1.2.11 and the assumption $\lambda + \text{ext} \vdash ME = NE$, that there exists a term $L$ such that $\lambda + \text{ext} \vdash ME \geq L$ and $\lambda + \text{ext} \vdash NE \geq L$.

Since $E \equiv \Omega_2 U$ it follows from 2.5.3 and 2.5.2.4) that $E$ is of order $0$. Hence from 2.5.14 it follows that there are terms $L', L''$ such that $\lambda + \text{ext} \vdash ME \geq L'$, $\lambda + \text{ext} \vdash NE \geq L''$ and $\lambda \vdash L' = L = L''$ (see fig. 12).
Now we claim that \( L' \not\equiv L'' \).

In order to prove this, it is sufficient to show that \( L' \subseteq L'' \), since by symmetry argument then also \( L'' \subseteq L' \) and hence by 2.5.26.1) \( L' \equiv L'' \).

We will show that for every subterm occurrence \( Z' \) of \( L' \), \( Z' \) is underlined in \( L'' \), where \( Z' \) is the subterm occurrence of \( L \) corresponding to \( Z' \). Then it follows by 2.5.26.2) that \( L' \subseteq L'' \).

Suppose therefore that \( Z' \) is a subterm occurrence of \( L' \). By 2.5.14 it follows that \( \lambda + \text{ext} \not\vdash \exists > Z' \).

We verify the conditions 1) and 2) of 2.5.27 for \( L', L'' \).

1) If \( Z \) is a subterm occurrence of \( L \) which is exactly underlined in \( L'' \), then \( Z \) sub \( L'' \), hence it follows by 2.5.14, that \( \lambda + \text{ext} \not\vdash \exists > Z \), hence \( Z \) is an \( \Omega_1 \)-term.

2) If \( Z \) is a subterm occurrence of \( L \) which is a u.g. then \( Z \not\in \lambda + \text{ext} \)(otherwise \( N \) would be a u.g.). Hence by 2.5.20 \( Z \) is not the corresponding subterm occurrence of a simple subterm of \( L'' \).

Therefore \( Z \) has some line in \( L' \), by 2.5.26.3).

Therefore it follows from 2.5.27 that \( Z' \) is underlined in \( L'' \). Hence we have proved that \( L' \equiv L'' \).

Let \( x \) be a variable not occurring in the reductions represented in fig. 12. Then it follows from 2.5.16 that
\[ \lambda + \text{ext} \vdash Mx = \phi_x(M) \supset \phi_x(L') \]
\[ \lambda + \text{ext} \vdash Nx = \phi_x(N) \supset \phi_x(L''). \]
Hence \( \lambda + \text{ext} \vdash Mx = Nx \) since \( \phi_x(L') \equiv \phi_x(L'') \).

Remark. We also have

Let \( M, N \) be \( \lambda \)-terms which are not u.g.'s and let \( \Sigma \) be a variable like \( \lambda \)-term.
If \( \lambda \vdash ME = NE \), then \( \lambda \vdash Mx = Nx \) for some variable \( x \in FV(MN) \).

2.5.29 Theorem
Let \( M, N \) be \( \lambda \)-terms which are not universal generators. Then the \( \omega \)-rule for \( M \) and \( N \) is derivable in the \( \lambda \)-calculus with extensionality.

Proof.
Suppose that \( \lambda + \text{ext} \vdash MZ = NZ \) for all closed \( Z \).
Then \( \lambda + \text{ext} \vdash ME = NE \), for variable like terms \( \Sigma \), since they are closed.
Hence by 2.5.28 it follows that \( \lambda + \text{ext} \vdash Mx = Nx \) for some variable \( x \in FV(MN) \).
Therefore, by extensionality, \( \lambda + \text{ext} \vdash M = N \).
Chapter III
Consistency results and term models

§3.1. Modeltheoretic notions for combinatory logic and some of its extensions.

A combinatory structure is an algebraical structure for a reduct of the language of CL, in which we drop the relation $\geq$. $=$ is always interpreted as the real equality.

A combinatory structure is called trivial if its domain consists of a single element.

A combinatory model is a non trivial combinatory structure $\mathfrak{C} = \langle C, i, k, s, \cdot \rangle$ such that $i \cdot x = x$, $k \cdot x \cdot y = x$ and $s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$ for all $x, y, z \in C$ (as usual we associate to the left).

A combinatory structure $\mathfrak{C}$ assigns homomorphically to each closed CL-term $M$ an element of $C$ which we will denote by $\#_C(M)$.

If $\mathfrak{C}$ is a combinatory model, its interior $\mathfrak{C}^\circ$ is by definition the restriction of $\mathfrak{C}$ to $C^\circ = \{ x \in C \mid x = \#_C(M) \text{ for some closed CL-term } M \}$.

A combinatory model is called hard if it coincides with its own interior.

A combinatory model is an extensional model if it satisfies the axiom of extensionality i.e. if $\forall x, y \in C \ [ \forall z \in C \ (x \cdot z = y \cdot z) \rightarrow x = y ]$.

Note that the axiom of extensionality cannot be expressed in CL, since CL has no logical connectives.

A combinatory model $\mathfrak{C}$ is an $\omega$-model if it satisfies the axiom corresponding to the $\omega$-rule, i.e. if
\[ \forall x, y \in C \left( \forall z \in C^0 (x \cdot z = y \cdot z) \Rightarrow x = y \right). \]

It is clear that an \( \omega \)-model is extensional.

From the completeness of predicate logic it follows that for every consistent extension of CL we can define a canonical model. Since the language of CL is logic free, this model is a particularly simple one, namely a term model.

3.1.1 Definition

Let \( \mathcal{A} \) be a consistent extension of CL (in the same language). The term model of \( \mathcal{A} \) consists of all CL-terms (closed and open) where terms that are provably equal in \( \mathcal{A} \) are equated and application is defined as juxta position.

Hence the term model consists of the set of terms with the minimal equality which satisfies \( \mathcal{A} \). The non-triviality of the term model follows by the consistency of \( \mathcal{A} \).

An extensional model or an \( \omega \)-model can be obtained as term model of \( CL + \text{ext} \) resp. \( CL_\omega \).

The restriction of an \( \omega \)-model to its interior is again an \( \omega \)-model. But the restriction of an extensional model to its interior is not necessarily extensional.

The notion of \( \omega \)-completeness should be distinguished from a stronger one. Let us call an consistent extension \( \mathcal{A} \) of CL strongly \( \omega \)-complete, if all extensional models of \( \mathcal{A} \) are in fact \( \omega \)-models. Strong \( \omega \)-completeness implies \( \omega \)-completeness, as follows by considering the term model of \( \mathcal{A} + \text{ext} \).

The converse is not necessarily true.

The rumour mentioned in the footnote on page 49 was not quite
justified. Apparently Jacopini [1971] has proved that CL is not strongly $\omega$-complete. Hence the question of the $\omega$-completeness of CL still remains open.

3.1.2 Theorem (Grzegorczyk [1971])
There is no recursive model for CL.

Proof.
If $\mathcal{C}$ would be a recursive model, then $\text{Th}(\mathcal{C})$ would form a consistent recursive extension of CL, contradicting 1.3.17. \[\square\]

Since it is not clear how to interpret the $\lambda$-operation in a model we restricted ourselves to models of combinatory logic. It is nevertheless possible to define $\lambda$-abstraction in a model. This is done in the later versions of Scott [1970].

§3.2. Term models

In this § we will answer negatively the question whether Scott's lattice theoretic method provides us with all extensional models for CL. We do this by equating all the unsolvable CL-terms and obtain an extensional term model in which there is only one fixed-point (an element $x$ such that for all $y$ $x \cdot y = x$ is called a fixed-point). In Scott's models there are at least two fixed-points.

Further it is shown that $\text{Con}$, the set of equations that can be added consistently to the $\lambda$-calculus, is complete $\Pi^0_1$ (after Gödelization). This is not an immediate consequence of the fact that $\lambda$-calculus is a complete $\Pi^0_1$ theory, since there is no negation. Unsolvable terms will play the role of negation. Finally we construct a term model for CL which cannot be embedded into nor mapped onto an extensional model.
3.2.1 Definition
1. A CL-term $M$ is called CL-solvable if $\exists N_1 \ldots N_k \quad \text{CL} \vdash MN_1 \ldots N_k = K$.
2. A $\lambda$-term $M$ is called $\lambda$-solvable if $\exists N_1 \ldots N_k \quad MN_1 \ldots N_k$ has a $\beta$-normal form.
3. A $\lambda$-term $M$ is called $\lambda\eta$-solvable if $\exists N_1 \ldots N_k \quad MN_1 \ldots N_k$ has a $\beta\eta$-normal form.

In 3.2.20, we will give an alternative characterization of solvable terms.

3.2.2 Lemma
Let $Z$ be a $\lambda$-term. Then $Z$ is $\lambda$-solvable iff $Z$ is $\lambda\eta$-solvable.

Proof.

By the remark following 1.2.8.

3.2.3 Theorem
1) If $\text{CL} \vdash ZM = K$, then $M$ is CL-solvable or $\text{CL} \vdash Zx = K$ for any variable $x$.
2) If $ZM$ has a $\beta\eta$-normal form $N$, then $M$ is $\lambda\eta$-solvable or $\lambda \ + \ \text{ext} \vdash Zx = N$ for any variable $x$.

The proof which, in the CL case, makes use of an auxiliary theory $\text{CL}'$ similar to $\text{CL}$ is postponed until §3.3.
3.2.4 Definition (Morris)
Let $M, M'$ be $\lambda$-terms.

$M \leq M'$ if $\forall Z \left[ ZM \text{ has a } \beta\eta\text{-normal form } \Rightarrow \lambda \ast \text{ ext } \vdash ZM = ZM' \right]$.

In his thesis Morris [1968] proved

3.2.5 Theorem
If $\lambda \ast \text{ ext } \vdash MA = A$, then $\text{FP } M \leq A$.

Hence $\text{FP } M$ is the minimal fixedpoint (in the sense of $C$).

3.2.6 Theorem
If $M$ is an unsolvable $\lambda$-term (i.e. if $M$ is not $\lambda$-solvable) then $M \leq M'$ for all terms $M'$. Hence $M$ is minimal (in the sense of $C$) in the set of all terms.

Proof.
This follows immediately from 3.2.3 2) and lemma 3.2.2.

As many fixedpoints are unsolvable, e.g. $\text{FP K, FP S}$ etc.,
3.2.6 is for those terms a sharpening of 3.2.5.

3.2.7 Definition
$K_{CL} = \{ M = M' \mid M, M' \text{ are unsolvable } CL\text{-terms} \}$
$K_{\lambda} = \{ M = M' \mid M, M' \text{ are unsolvable } \lambda\text{-terms} \}$.

We will prove now that $CL + K_{CL}$ is consistent. The most convenient way to prove this is to develop a theory $CL^+$ which is a conservative extension of $CL + K_{CL}$. $CL^+$ will play the same role as $CL\omega'$ in the consistency proof of $CL + \omega\text{-rule}$.
3.2.8 Definition

CL+ has the following language:
Alphabet = Alphabet_{CL} U \{=,\sim\}.
The terms are those of CL.
Formulas: If M,N are terms, then
M \succ N, M = N, M \equiv N and M \sim N are formulas.

3.2.9 Definition

CL+ has the following axioms and rules (see appendix).

I  Same as in 1.4.2.
II 1. M = M  M = M  M \sim M
  2. M = N  M = N  M \sim N
  3. M = N, N = L  M = L
  4. M = M', M = M'  M \sim M', M \sim M'
III 1. Same as in 1.4.2
  2. Same as in 1.4.2
  3. Same as in 1.4.2
  4. M \succ M'  M \equiv M'  M \sim M'
IV  M \equiv M' if M,M' are unsolvable terms.

Now we proceed as in §2.2 to prove that CL + \mathcal{K}_{CL} is consistent.
If the proofs are similar to those in §2.2 or easy, we omit them.
3.2.10 Lemma
1) $\text{CL} \vdash M \supset N \iff \text{CL} + \text{CL} \vdash M \supset N \iff \text{CL}^+ \vdash M \supset N$.
2) $\text{CL} + \text{CL} \vdash M = N \iff \text{CL}^+ \vdash M = N$.
Hence $\text{CL}^+$ is a conservative extension of $\text{CL} + \text{CL}$.

3.2.11 Lemma
$\text{CL}^+ \vdash M = N \iff \exists N_1 \ldots N_k \text{ CL}^+ \vdash M \sim N_1 \sim \ldots \sim N_k \equiv N$.

3.2.12 Lemma
If $\text{CL}^+ \vdash M \sim N$, then $\exists M', N', Z$
\[ (\text{CL}^+ \vdash ZM' \supset M, \text{CL}^+ \vdash ZN' \supset N \text{ and } \text{CL}^+ \vdash M' \equiv N') \].

3.2.13 Lemma
If $\text{CL}^+ \vdash M \equiv N$, then either 1) $M, N$ are unsolvable or 2) $\text{CL}^+ \vdash M \supset N$ or $\text{CL}^+ \vdash N \supset M$.

3.2.14 Theorem
If $\text{CL}^+ \vdash M = K$, then
\[ (*) \text{ CL} \vdash M = K. \]

Proof.
From 3.2.11 it follows that $\text{CL}^+ \vdash M = K$
\[ (**) \exists N_1 \ldots N_k \text{ CL}^+ \vdash M \sim N_1 \sim \ldots \sim N_k \equiv K. \]
By induction on $k$ we prove that $(**)$ $\Rightarrow$ $(*)$.
If $k = 0$, then there is nothing to prove. So suppose that $k > 0$.
Since $\text{CL}^+ \vdash M_{k-1} \sim M_k$ it follows by 3.2.12 that $\exists M_{k-1}', M_k', Z$
such that
\[ \text{CL}^+ \vdash ZM_{k-1}' \supset M_{k-1}', \text{CL}^+ \vdash ZM_k' \supset M_k \text{ and CL}^+ \vdash M_{k-1}' \equiv M_k'. \]
By 3.2.13 we can distinguish the following cases:
case 1. \( M'_{k-1}, M'_k \) are unsolvable.

then, since \( \text{CL}^+ \vdash ZM' \geq M_k \equiv K \) it follows from 3.2.10 and 3.2.3 1) that
\( \text{CL} \vdash Zx = K \) for any \( x \).

Hence \( \text{CL} \vdash ZM'_{k-1} = K \) and therefore
\( \text{CL}^+ \vdash M_{k-1} \geq K \) as follows from
\( \text{CL}^+ \vdash ZM'_{k-1} = M'_{k-1} \), 3.2.10 and 1.5.1.

Thus \( \text{CL}^+ \vdash M \sim M_1 \sim \ldots \sim M_{k-1} \geq K \), hence by the induction hypothesis
\( \text{CL} \vdash M = K \).

case 2. \( \text{CL}^+ \vdash M'_{k-1} \geq M'_k \) or \( \text{CL}^+ \vdash M'_k > M'_{k-1} \).

In both cases we have
\( \text{CL} \vdash M_{k-1} = ZM'_{k-1} = ZM'_k = M_k \equiv K \) by 3.2.10.

Hence \( \text{CL}^+ \vdash M_{k-1} \geq K \) by 1.5.1 and 3.2.10.

Then \( \text{CL} \vdash M = K \) follows as above from the induction hypothesis.

3.2.15 Corollary
\( \text{CL} + K_{\text{CL}} \) is consistent.

Proof.

Suppose that \( \text{CL} + K_{\text{CL}} \vdash KK = K \), then by 3.2.10 \( \text{CL}^+ \vdash KK = K \)
and hence by 3.2.14 \( \text{CL} \vdash KK = K \). This contradicts the Church-Rosser theorem for CL.

3.2.16 Remark
In the same way we can prove \( \text{Con}(\lambda + K_{\lambda}) \). With a more involved argument we can show that \( \text{Con}(\text{CL} + K_{\text{CL}} + \text{ext}) \) and
\( \text{Con}(\lambda + K_{\lambda} + \text{ext}) \). The idea is to use the language of CL\( m' \)
(where \( \approx_M, \sim_a, =_a \) are only used for finite \( a \)) and to add the rule
The consistency of CL + $\mathcal{K}_{CL} + \text{ext}$ is not automatically a consequence of the consistency of CL + $\mathcal{K}_{CL}$, as will be seen in 3.2.24 where it is shown that Con(CL + $\mathcal{M}$) $\Rightarrow$ Con(CL + $\mathcal{M}$ + ext), where $\mathcal{M}$ is a set of equations, does not hold in general.

3.2.17 Remark
Let us call an element $x$ of a combinatory model a fixed-point if $xy = x$ holds for all $y$ in the model.
In every combinatory model $\mathcal{C}$, $\mathcal{K}_C(\text{FP } K)$ is a fixed-point. Since fixed-points in a term model correspond to unsolvable terms, it is clear that in the term model of CL + $\mathcal{K}_{CL} (+ \text{ext})$ there is only one fixed-point.
In Scott's lattice theoretic models there are always at least two fixed-points: all the elements of the initial lattice $D_0$ become fixed-points in the limit $D_\omega$. See Scott [1970], p.41 theorem 2.14.
Hence Scott's method does not provide us with all models for CL + ext. This was suggested to us by professor Gross. His suggestion had inspired us to prove 3.2.15.
With the help of Con($\lambda + \mathcal{K}_\lambda$) we can classify in the Kleene-Mostowski hierarchy the set of equations which can be added consistently to the $\lambda$-calculus.

3.2.18 Lemma
$K = \text{FP } K$ is not consistent with CL or the $\lambda$-calculus.
Proof (for CL).

Let \( Z = \text{FP} \cdot K \), then \( CL \vdash Z = KZ \).

Hence \( CL + K = Z \vdash KIMN = KKIMN \), thus \( CL + K \vdash Z \leq N = M \).

3.2.19 Theorem

\( \text{Con}^\lambda = \{ M = N \mid \text{Con}(\lambda + M = N) \} \) is a complete II^\_ set.

(As in §1.3. \( \ldots \) denotes the Gödel number of \( \ldots \) in some Gödelisation.)

Proof.

\( \text{Con}_\lambda \) is complete II^\_ set. Let \( X \) be an arbitrary II^\_ set. Let \( Y = \omega - X \), the complement of \( X \). Then \( Y \) is r.e.

Define \( f(n) = \begin{cases} 0 & \text{if } n \in Y \\ \uparrow & \text{else} \end{cases} \)

then \( f \) is partial recursive.

Hence there is a \( \lambda \)-term \( F \) which strongly defines \( f \). Note that if \( f(n) \uparrow \), then by 1.3.3 \( F_n \) is unsolvable. Then

\[
\begin{align*}
\text{if } n \in Y & \implies f(n) = 0 \implies \lambda \vdash F_n = 0 \implies \lambda \vdash F_n \leq K \implies \\
& \implies \text{FP} \cdot K \not \in \text{Con}_\lambda \quad \text{by lemma 3.2.18} \implies \\
\text{if } n \notin Y & \implies f(n) \uparrow \implies F_n \text{ is unsolvable} \implies \\
& \implies \text{FP} \cdot K \not \in \text{Con}_\lambda \quad \text{by 3.2.16 since FP K is unsolvable}.
\end{align*}
\]

Hence

\[
\begin{align*}
\text{if } n \in X & \implies n \notin Y \implies \text{FP} \cdot K \not \in \text{Con}_\lambda \implies \\
& \implies X \leq \text{Con}_\lambda \quad \text{via the function} \h(n) = \text{FP} \cdot K \not \in \text{Con}_\lambda, i.e. \text{Con}_\lambda \text{ is complete II^\_.}
\end{align*}
\]
3.2.20 Remark
The same result holds for $\text{Con}_{\lambda+\text{ext}}$ and also for $\text{Con}_{\text{CL}+\text{ext}}$. In the CL-case this is proved using the strong definability (in our sense) in CL of the partial recursive functions. This CL-definability is essentially proved in Curry, Hindley, Seldin [1971], Ch 13 A.

The proof of 3.2.19 suggests the following result (from which 3.2.19 follows more directly).

3.2.20 Theorem
Let $M$ be a closed term of CL or the $\lambda$-calculus.
1) $M$ is CL-$(\lambda\eta^-, \lambda^-)$ solvable $\iff \exists N_1 \ldots N_k \lambda + \text{ext} \vdash MN_1 \ldots N_k = K$ is provable in CL$(\lambda + \text{ext}, \lambda)$.
2) $M$ is CL-$(\lambda\eta^-, \lambda^-)$ solvable $\iff M = \text{FP} K$ is inconsistent with CL $(\lambda + \text{ext}, \lambda)$.

Proof.
1) i) For CL this is just the definition.
   ii) $\implies$: If $M$ is $\lambda\eta$-solvable, then 
       $\exists N_1 \ldots N_k \lambda + \text{ext} \vdash MN_1 \ldots N_k = N$, where $N$ is in $\lambda\eta$-normal form.
       By a theorem of Bohm [1968] there exists closed terms $P_1, \ldots, P_m, Z_1, \ldots, Z_n$ such that for $N' \equiv [x_1/P_1] \ldots [x_m/P_m]N$
       (here $\text{FV}(N) = \{x_1, \ldots, x_m\}$)
       we have
       $\lambda + \text{ext} \vdash N'Z \ldots Z_n xy = x$.
       Define $N'_1 \equiv [x_1/P_1] \ldots [x_m/P_m]N_1$, then
       $\lambda + \text{ext} \vdash MN'_1 \ldots N'_k = N'$. 
Hence $\lambda + \text{ext} \vdash \prod_{1}^{i} N_{k} Z_{k} Z_{p} K K = K$.

By definition.

iii) If $M$ is $\lambda$-solvable, then by 3.2.2 $M$ is $\lambda\eta$-solvable and therefore by ii) $\exists N_{1} \ldots N_{k} \lambda + \text{ext} \vdash \prod_{1}^{i} N_{k} = K$.

Since the $\omega$-rule implies the rule of extensionality, we have

$\exists N_{1} \ldots N_{k} \lambda_{\omega} \vdash \prod_{1}^{i} N_{k} = K$.

Hence by the analogue of 2.2.12 (which follows from the CL $\Rightarrow \lambda$-calculus translation 1.4.11) for the $\lambda$-calculus

$\exists N_{1} \ldots N_{k} \exists n \in \omega \lambda \vdash \prod_{1}^{i} N_{k} K_{2n} = K$.

By definition.

2) Let $M$ be $\lambda\eta$-solvable, then by 1)

$\exists N_{1} \ldots N_{k} \vdash \prod_{1}^{i} N_{k} = K$.

Hence $\lambda + M = FP K \vdash K = \prod_{1}^{i} N_{1} \ldots N_{k} = FP K N_{1} \ldots N_{k} = FP K$ (remember that $\lambda + \vdash FP K x = FP K$).

This yields according to 3.2.18 a contradiction.

If $M$ would be unsolvable, then $M = FP K$ were consistent with $\lambda +$ by 3.2.15 and 3.2.16 (FP K is unsolvable).
3.2.21 Definition
Let $M, N$ be CL-terms.

$M$ and $N$ are separable if $\exists Z [\text{CL} \vdash ZM = K \text{ and } \text{CL} \vdash ZN = KK]$.

Trivial is

3.2.22

$\text{Con(CL + } M = N) \nleftrightarrow M \text{ and } N \text{ are not separable}$

but

3.2.23 Theorem

The converse of 3.2.22 is not true.

Proof.
Let $M = FP K$ and $N = K$. Then, by 3.2.18, not $\text{Con(CL} + M = N)$.

But $M$ and $N$ are not separable, for suppose $\text{CL} \vdash ZM = K$, then

by 3.2.3 $\text{CL} \vdash Zx = K$ since $M = FP K$ is unsolvable.

Hence also $\text{CL} \vdash ZN = K$, which implies that $\text{CL} \vdash ZN = KK$ is impossible.

The rest of this § is devoted to establishing the following theorem.

3.2.24 Theorem

There is a set $\mathcal{M}$ of equations such that

$\text{Con(CL + } \mathcal{M}) \text{ but not } \text{Con(CL + } \mathcal{M} + \text{ext)}$.

If $\mathcal{M}$ is such a set of equations, then the term model of $\text{CL} + \mathcal{M}$ can neither be embedded in nor mapped homomorphically onto an extensional model.

We will show by a Church-Rosser technique

\[\]
that CL + K = Ω₂(KI) + KK = Ω₂(SK) is consistent (here we use par abus de langage Ω₂ as an abbreviation for SII(SII)). If we add extensionality however, this theory becomes inconsistent, since CL + ext ⊢ KI = SK.

3.2.25 Definition

CL" is a theory with the same language as CL' (see appendix). CL" is defined by the following axioms and rules (see appendix)

I  Same as in 1.5.2.

II Same as in 1.4.2.

III Same as in 1.4.2.

IV 1. M ≅ M
2. M ≅ M', N ≅ N' \(\frac{MN ≅ M'N'}{M ≅ N'}\) (1)
3. M ≅ M', N ≅ N' \(\frac{M ≅ N'}{M' ≅ N'}\)

V 1. Ω₁(KI) ≅ K where CL" ⊢ Ω₁ ≅ Ω₂
2. Ω₁(SK) ≅ KK where CL" ⊢ Ω₁ ≅ Ω₂

In the above M, M', N, L denote arbitrary terms and Ω₂ = SII(SII).

3.2.26 Lemma

CL" ⊢ M = N ⇐⇒ CL + K = Ω₂(KI) + KK = Ω₂(SK) ⊢ M = N.

Proof.

Show by induction

1. CL" ⊢ M ≅ N ⇒ CL + ≅ ⊢ M = N
   CL" ⊢ M ≅ N ⇒ CL + ≅ ⊢ M = N
   CL" ⊢ M = N ⇒ CL + ≅ ⊢ M = N

2. CL + ≅ ⊢ M = N ⇒ CL" ⊢ M = N

where ≅ = \{K = Ω₁(KI), KK = Ω₂(SK)\}. 
3.2.27 Lemma

\[ \text{CL}'' \vdash M \succ N \iff \exists N_1 \ldots N_k \quad \text{CL}'' \vdash M \succ N_1 \succ \ldots \succ N_k \succ N. \]

Proof.

Induction on the length of proof of \( M \succ N \).

3.2.28 Lemma

If \( \text{CL}'' \vdash MN \succ L \), then

1) \( L \equiv M'N' \) and \( \text{CL}'' \vdash M \succ M' \), \( \text{CL}'' \vdash N \succ N' \) or
2) \( M \equiv I \) and \( L \equiv N \) or
3) \( M \equiv KM_1 \) and \( L \equiv M_1 \) or
4) \( M \equiv SM_1M_2 \) and \( L \equiv M_1N(M_2N) \) or
5) \( M \equiv \Omega_1 \), where \( \text{CL}'' \vdash \Omega_2 \succ \Omega_2' \), \( N \equiv KI \) and \( L \equiv K \) or
6) \( M \equiv \Omega_2' \), where \( \text{CL}'' \vdash \Omega_2 \succ \Omega_2' \), \( N \equiv SK \) and \( L \equiv KK \).

Proof.

Induction on the length of proof of \( MN \succ L \).

3.2.29 Lemma

If \( \text{CL}'' \vdash \Omega_2 \succ \Omega_2' \), then \( \text{CL} \vdash \Omega_2 \succ \Omega_2' \), hence \( \Omega_2' \) is not of the form \( I, K, KM, S, SM_1 \) or \( SM_1M_2 \).

Proof.

By induction on the length of proof of \( \Omega_2 \succ \Omega_2' \) one shows that \( \Omega_2 \) is of the form \( I^n(SII)[I^m(SII)] \). Hence \( \text{CL} \vdash \Omega_2 \succ \Omega_2' \).

3.2.30 Lemma

If \( \text{CL}'' \vdash M_1 \succ M_2 \) and \( \text{CL}'' \vdash M_1 \succ M_3 \), then there exists a term \( M_4 \) such that \( \text{CL}'' \vdash M_2 \succ M_4 \) and \( \text{CL}'' \vdash M_3 \succ M_4 \). (See fig.1, p.38.)

Proof.

Induction on the length of proof of \( M_1 \succ M_2 \).
case 1. $M_1 \Rightarrow M_2$ is an axiom.

subcase 1.1. $M_1 \Rightarrow M_2$ is $IM \Rightarrow IM$.

By 3.2.28 and 3.2.29 it follows that either

a) $M_3 \equiv IM'$ with $CL'' \vdash M \Rightarrow M'$, then we
can take $M_4 \equiv M'$, or

b) $M_3 \equiv M$, then we can take $M_4 \equiv M$.

subcase 1.2,1.3. $M_1 \Rightarrow M_2$ is $KMN \Rightarrow M$ or $SMNL \Rightarrow ML(NL)$.

Analogous to subcase 1.1.

subcase 1.4. $M_1 \equiv M_2$. Then we can take $M_4 \equiv M_3$.

subcase 1.5. $M_1 \Rightarrow M_2$ is $\Omega_1^2(KI) \Rightarrow K$.

By 3.2.28 and 3.2.29 it follows that either

a) $M_3 \equiv K$, then we can take $M_4 \equiv K$, or

b) $M_3 \equiv \Omega_3^2(KI)$ with $CL'' \vdash \Omega_2^2 \Rightarrow \Omega_2^2$,
hence we can take $M_4 \equiv \Omega_2^2$.

subcase 1.6. $M_1 \Rightarrow M_2$ is $\Omega_1^2(SK) \Rightarrow \Omega_1^2$.

Analogous to subcase 1.5.

case 2. $M_1 \Rightarrow M_2$ is $MN \Rightarrow M''N''$ and is a direct consequence of

$CL'' \vdash M \Rightarrow M''$ and $CL'' \vdash N \Rightarrow N''$.

If $M_1 \Rightarrow M_3$ is an axiom, then we are done by case 1.  
Otherwise $M_1 \Rightarrow M_3$ is $MN \Rightarrow M''N''$ and is a direct consequence of $CL'' \vdash M \Rightarrow M''$ and $CL'' \vdash N \Rightarrow N''$.

By the induction hypothesis there exist $M''''N''''$ such that

$CL'' \vdash M' \Rightarrow M''$, $CL'' \vdash M'' \Rightarrow M'''$ and the same for $N$.

Hence we can take $M_4 \equiv M''''N''''$.

3.2.31 Lemma

If $CL'' \vdash M_1 \Rightarrow M_2$ and $CL'' \vdash M_1 \Rightarrow M_3$, then there exists a term $M_4$ such that $CL'' \vdash M_1 \Rightarrow M_4$.  

\[\Box\]
3.2.32 Theorem (Church-Rosser theorem for CL")
If CL" \vdash M = N, then there exists a term Z such that
CL" \vdash M \geq Z and CL" \vdash N \geq Z.

Proof.
Induction on the length of proof of M = N (as 1.5.16 follows from 1.5.15).

3.2.33 Corollary
1) CL" is consistent.
2) CL + K = \Omega_2(KI) + KK = \Omega_2(SK) is consistent.
3) Conjecture 3.2.24 is false.

Proof.
1) If CL" \vdash K = KK, then there would be a Z such that
CL" \vdash K \geq Z and CL" \vdash KK \geq Z, a contradiction.
2) This follows immediately from 1) and 3.2.26.
3) Since CL + ext \vdash KI = SK,
CL + K = \Omega_2(KI) + KK = \Omega_2(SK) + ext \vdash K = \Omega_2(KI) = \Omega_2(KS) = KK.
Hence: CL + \text{ext} is consistent \Rightarrow CL + \text{ext} is consistent.
3.3.1 Definition

CL' is a theory with the same language as CL.

We define a mapping \(|\cdot| : CL' \rightarrow CL\) as follows

\[|c| = c\] if \(c\) is a constant or variable

\[|MN| = |M||N|\]

\[|M| = M\]

CL' is defined by the following axioms and rules (see appendix).

I, II, III, IV and V are as in 2.3.2.

VI \(MN > M|N|\)

In the above the restrictions on the terms are clear.

Axiom VI is essential for CL'; compare it with axiom VI for CL.

3.3.2 Definition

Let \(M, M'\) be CL-terms.

\(M'\) is in the CL-solution of \(M\), notation \(M > M'\) if

\[\exists N_1 \ldots N_k CL \vdash MN_1 \ldots N_k > M' \quad (k = 0 \text{ is allowed}).\]

Note that \(>\) is transitive. \(M\) is CL solvable iff \(M > K\).

Now we proceed as in §2.3. When the proofs are similar to those in §2.3 we omit them.

3.3.3 Lemma

1) \(CL' \vdash M > M' \iff CL \vdash M > M'\) if \(M, M'\) are simple terms

2) \(CL' \vdash M > M' \iff CL' \vdash M > M'\) if \(M, M'\) are simple terms

3) \([CL' \vdash M > M'\) and \(N' \text{ sub } M'\) \(\rightarrow \exists N[N \text{ sub } M \text{ and } N > N']\]

3.3.4 Lemma

\(CL' \vdash M > M' \iff \exists N_1 \ldots N_k CL' \vdash M \equiv N_1 > \ldots > N_k \equiv M'\).
3.3.5 Lemma

([CL'] ⊢ M ≻ M' and N' sub M') ⇒ ∃N [N sub M and N ≻ N']

3.3.6 Lemma

Let M, M', N be terms such that
1) M and M' are simple
2) CL ⊢ M ≻ M'
3) CL' ⊢ M = N,
then there exists a term N' such that
4) CL' ⊢ N ≻ N'
5) CL' ⊢ M' ≻ N' (see fig. 7, page 60).

3.3.7 Definition

Let M be a CL-term. An x substitution of M is the result of replacing some occurrences of x in M by other terms.

3.3.8 Lemma

If CL ⊢ M ≻ M' and N is an x substitution of M, then there exists a term N' which is an x substitution of M' such that CL ⊢ N ≻ N'.

Proof.
Induction on the length of proof of M ≻ M'.

3.3.9 Lemma

If CL' ⊢ M ≥_x M', then CL ⊢ ϕ_x(M) ≥ M", where M" is an x substitution of ϕ_x(M') (ϕ_x is defined in 2.3.7 with A = x).

Proof.
Induction on the length of proof of M ≥_x M'.

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case 1. $M \succsim M'$ is an axiom.

subcase 1.1. $M \succsim M'$ is not an instance of axiom VI 1 or 2.

Then since $CL - \{VI\} = CL' - \{VI\}$ it follows from 2.3.8 that $CL \vdash \phi_x(M) \succsim \phi_x(M')$.

Hence we can take $M'' \equiv \phi_x(M')$.

subcase 1.2. $M \succsim M'$ is an instance of axiom VI, say $M_1 \succsim M_2 \succsim M_1 \succsim M_2$.

Then $\phi_x(M) \equiv \phi_x(M_2)$ and $\phi_x(M_1) \equiv x$.

Hence we can take $M'' \equiv \phi_x(M_2)$ which is an $x$ variant of $\phi_x(M')$.

case 2. $M \succsim M'$ is $ZM \succsim ZM'$ and is a direct consequence of $M_1 \succsim M_1'$.

By the induction hypothesis we have $CL \vdash \phi_x(M_1) \succsim M''_1$, where $M''$ is an $x$ substitution of $\phi_x(M')$.

Therefore

$CL \vdash \phi_x(M) \equiv \phi_x(Z)\phi_x(M_1) \succsim \phi_x(Z)M_1''$.

Hence we can take $M'' \equiv \phi_x(Z)M_1''$

since this is an $x$ substitution of $\phi_x(M') \equiv \phi_x(Z)\phi_x(M_1')$.

case 3. $M \succsim M'$ is $M_1 \succsim M_1'Z$. This case is analogous to case 2.

3.3.10 Lemma

Let $M, M'$ be $CL'$-terms such that $M'$ is simple and $CL' \vdash M \succsim M'$.

Let $x \in M'$.

Then $CL \vdash \phi_x(M) \succsim M'$.

Proof.

Suppose $CL' \vdash M \succsim M'$, then $\exists N_1 \ldots N_k$ $CL' \vdash N_1 \succsim \ldots \succsim N_k \equiv M'$.

Hence $\exists N_1 \ldots N_k$ $CL' \vdash M \equiv N_k \succsim \ldots \succsim N_1 \equiv M'$. 


With induction on \( i < k \) we will prove \( \text{CL} \vdash \varphi_x(N_i) \geq M' \).

If \( i = 1 \) we are done.

By lemma 3.3.9 it follows that \( \text{CL} \vdash \varphi_x(N_{i+1}) \geq N'_i \) where \( N'_i \) is an \( x \) substitution of \( \varphi_x(N_i) \). Since by induction hypothesis \( \text{CL} \vdash \varphi_x(N_i) \geq M' \) there exists by lemma 3.3.8 a \( M'' \) which is an \( x \) variant of \( M' \) such that \( \text{CL} \vdash N'_i \geq M'' \). But \( x \notin M' \), hence \( M'' \equiv M' \).

Therefore \( \text{CL} \vdash \varphi_x(N_{i+1}) \geq N'_i \geq M'' \equiv M' \).

Now we are able to prove the CL part of theorem 3.2.3.

3.2.3 **Theorem**

If \( \text{CL} \vdash ZM = K \), then \( M \) is Cl-solvable or \( \text{CL} \vdash Zx = K \) for any variable \( x \).

**Proof.**

If \( \text{CL} \vdash ZM = K \), then \( \text{CL} \vdash ZM \geq K \) by 1.5.1, hence by 3.3.6 \( \text{CL}' \vdash ZM \geq K' \) with \( \text{CL}' \vdash K' = K \), hence \( K' \equiv K \) or \( K' \equiv K \).

- **case 1.** \( K' \equiv K \). Thus \( \text{CL}' \vdash ZM \geq K \)
  
  By 3.3.10 it follows that
  
  \( \text{CL} \vdash \varphi_x(ZM) \geq K \) i.e. \( \text{CL} \vdash Zx \geq K \).

- **case 2.** \( K' \equiv K \). Thus \( \text{CL}' \vdash ZM \geq K \).
  
  Hence by 3.3.5 \( M \geq K \) i.e. \( M \) is CL solvable.
Appendix I

Survey of the theories used in the text.

This appendix presents a full description of the theories considered.

In order to facilitate the locating of those descriptions we list them here with a reference to the place where they were introduced and their page in the appendix.

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The $\lambda$-calculus

Language

Alphabet: $a, b, c, \ldots$ variables

$\lambda, (, )$ improper symbols

$= \quad$ equality

$\rightarrow \quad$ reduction

Terms: Terms are defined inductively by

1) Any variable is a term.
2) If $M, N$ are terms, then $(MN)$ is a term.
3) If $M$ is a term, then $(\lambda x M)$ is a term

Formulas: If $M, N$ are terms, then

$M = N$ and $M \rightarrow N$ are formulas.

To be able to formulate the axioms we define inductively:

The set of free variables of a term:

$FV(x) = \{x\}$

$FV(MN) = FV(M) \cup FV(N)$

$FV(\lambda x M) = FV(M) - \{x\}$

The set of bound variables of a term:

$BV(x) = \emptyset$

$BV(MN) = BV(M) \cup BV(N)$

$BV(\lambda x M) = BV(M) \cup \{x\}$

Substitution of a term $N$ in the free occurrences of the variable $x$ in $M$:

$[x/N]x = N$

$[x/N]y = y$

$[x/N](M_1 M_2) = [x/N]M_1 ([x/N]M_2)$

$[x/N](\lambda x M) = \lambda x M$

$[x/N](\lambda y M) = \lambda y ([x/N]M)$

In the above $x$ is an arbitrary variable and $y$ is a variable different from $x$.

Note. In the text $[x/N]$ sometimes was confused with $[x\backslash N]$
The λ-calculus (+ extensionality, + w-rule)

**Axioms and rules**

**I** 1. \( \lambda x M \geq \lambda y[x/y]M \) if \( y \notin \text{FV}(M) \)

2. \((\lambda x M)N \geq [x/N]M \) if \( \text{BV}(M) \cap \text{FV}(N) = \emptyset \).

**II** 1. \( M = M \)

2. \( M = N \quad N = M \)

3. \( M = N, N = L \quad M = L \)

4. \( M = M', M = M', \frac{2M = 2M'}{MZ = M'Z} \quad \frac{M = M'}{\lambda x M = \lambda x M'} \)

**III** 1. \( M \geq M \)

2. \( M \geq N, N \geq L \quad M \geq L \)

3. \( M \geq M', M \geq M', \frac{2M \geq 2M'}{MZ \geq M'Z} \quad \frac{M \geq M'}{\lambda x M \geq \lambda x M'} \)

4. \( M \geq M' \quad M = M' \)

In \( \lambda \) + ext we add

**I** 3. \( \lambda x(Mx) \geq M \quad \text{if } x \notin \text{FV}(M) \)

In \( \lambda w \) we add

w-rule

\[
\frac{MZ = NZ \quad \text{for all } Z \text{ with } \text{FV}(Z) = \emptyset}{M = N}
\]

In the above \( M, M', N, L \) and \( Z \) denote arbitrary terms and \( x \) and \( y \) arbitrary variables.
Combinatory logic (CL)

Language

Alphabet: $a, b, c, \ldots$ variables
$I, K, S$ constants
$(, )$ improper symbols
$= \quad$ equality
$\Rightarrow$ reduction

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If $M, N$ are terms, then $(MN)$ is a term.

Formulas: If $M, N$ are terms, then
$M = N$ and $M \Rightarrow N$ are formulas.

$M_1 M_2 \ldots M_n$ stands for $(\ldots (M_1 M_2) \ldots M_n)$
CL (+ extensionality, + w-rule)

**Axioms and rules**

I
1. IM > M
2. IMN > M
3. SMNL > ML(NL)

II
1. M = M
2. M = N
   N = M
3. M = N, N = L
   M = L
4. M = M', ZM = ZM', M = M', MZ = M'Z

III
1. M > M
2. M > N, N > L
   M > L
3. M > M', ZM > ZM', M > M', MZ > M'Z
4. M > M'
   M = M'

In CL + ext we add

 ext Mx = M'x
     M = M' if x \in FV(MM')

In CLw we add

 w-rule MZ = NZ for all Z without free variables
     M = N

In the above M, M', N, L and Z denote arbitrary terms.
**CL'**

**Language**

*Alphabet:* a, b, c, ... variables  
I, K, S  constants  
( , )  improper symbols  
=  equality  
\(\Rightarrow\)  reduction  
\(\Rightarrow_1\)  one step reduction

**Terms:** Terms are defined inductively by
1) Any variable or constant is a term.
2) If M, N are terms, then (MN) is a term.

**Formulas:** If M, N are terms, then
M = N, M \(\Rightarrow\) N and M \(\Rightarrow_1\) N are formulas.

\(M_1M_2...M_n\) stands for \((...(M_1M_2)...M_n)\)
Axioms and rules

I
1. $M = M$
2. $KNM = M$
3. $SMNL = M(LN)$

II
1. $M = M$
2. $M = N$
   \[ N = M \]
3. $M = N$, $N = L$
   \[ M = L \]
4. $M = \alpha'$
   \[ ZM = \alpha'Z \]
   \[ M = \alpha'Z \]

III
1. $M > M$
2. $M > N$, $N > L$
   \[ M > L \]
3. $M > \alpha'$
   \[ ZM > \alpha'Z \]
   \[ M > \alpha'Z \]
4. $M > \alpha'$
   \[ M = \alpha' \]

IV
1. $M \geq M$
2. $M \geq \alpha'$
   \[ ZM \geq \alpha'Z \]
   \[ M \geq \alpha'Z \]
3. $M \geq \alpha'$
   \[ M \geq \alpha' \]

In the above $M, \alpha', N, L$ and $Z$ denote arbitrary terms.
Language
Alphabet: a, b, c, ... variables
I, K, S constants
(, ), improper symbols
= equality
\Rightarrow \quad \text{reduction}
\Rightarrow_1 \quad \text{one step reduction}

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If M, N are terms, then (MN) is a term.
3) If M, N and L are terms, then
   S(M, N, L) is a term.

Formulas: If M, N are terms, then
M = N, M \geq N and M \Rightarrow N are formulas.

M_1 M_2 \ldots M_n stands for (\ldots(M_1 M_2)\ldots M_n)
CL*

Axioms and rules

I  1. IM \succ M
   2. KMN \succ M
   3. SMNL \succ S(M,N,L)

II  1. M = M
    2. M = N
       N = M
    3. M = N, N = L
       M = L
    4. M = M', M = M'
       ZM = ZM', NZ = M'Z

III  1. M > M
     2. M > N, N > L
        M > L
     3. M > M', M > M'
        ZM > ZM', NZ > M'Z

IV  1. M \succ M
     2. M \succ M', M \succ M'
        ZM > ZM', NZ > M'Z
     3. M \succ M'
        M \succ M'
     4. \frac{M \succ M', S(M,N,L) \succ S(M',N,L)}{S(M,N,L) \succ S(M',N,L')}
     \frac{\frac{N \succ N', S(M,N,L) \succ S(M,N',L)}{L \succ L'}}{S(M,N,L) \succ S(M,N,L')}

In the above M,M',N,N',L,L' and Z denote arbitrary terms.
Language
Alphabet: $a, b, c, \ldots$ variables
$I, K, S$ constants
$(, ), \, , , \ldots$ improper symbols
$=$ equality
$\succ$ reduction
$\succ_1$ one step reduction

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If $M, N$ are terms, then $(MN)$ is a term.
3) If $M, N$ and $L$ are terms, then
   $S(M, N, L)$ and $S(M, N, L)$ are terms.

Formulas: If $M, N$ are terms, then
   $M = N, M \succ N$ and $M \succ_1 N$ are formulas.

$M_1M_2\ldots M_n$ stands for $\ldots (M_1M_2)\ldots M_n$
Axioms and rules (We give an equivalent version which is slightly different from the original one.)

I
1. IM >_1 M
2. KMN >_1 M
3. SMNL >_1 S(M,N,L)

II
1. M = M
2. M = N, N = M
3. M = N, N = L
   \[ M = L \]
4. M = M', M = M'
   \[ 2M = 2M', MZ = M'Z \]

III
1. M > M
2. M > N, N > L
   \[ M > L \]
3. M > M', M > M'
   \[ 2M > 2M', MZ > M'Z \]

IV
1. M > M
2. M > M', M > M'
   \[ 2M > 2M', MZ > M'Z \]
3. M > M', M > M'
   \[ N > N' \]
4. S(M,N,L) > S(M',N,L)
   \[ S(M,N,L) > S(M',N,L) \]
   \[ L > L' \]
   \[ S(M,N,L) > S(M,N,L') \]
   \[ S(M,N,L') > S(M,N,L) \]
5. M > M'
   \[ N > N' \]

In the above M, M', N, N', L, L' and Z denote arbitrary terms, except in IV 5, where M, M' denote terms of the form S(P,Q,R).
Language

Alphabet: a, b, c, ... variables
I, K, S constants
(, ) improper symbols
= equality
\geq reduction
\equiv_\alpha, \sim_\alpha special equalities, for every countable ordinal \alpha.

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If M, N are terms, then (MN) is a term.

Formulas: If M, N are terms, then
M = N, M \geq N, M \equiv_\alpha N, M \sim_\alpha N
and M =_\alpha N are formulas.

\underbrace{M_1M_2...M_n} stands for (\ldots(M_1M_2)...M_n)
Axioms and rules

I
1. IM ⊳ M
2. XMN ⊳ M
3. SMNL ⊳ ML(NL)

II
1. M =_a M
2. M =_a N
3. M =_a N, N =_a L
4. M =_a M', M =_a M'
5. M =_a M', α < α', M =_a M'

III
1. M ⊳ M
2. M ⊳ N, N ⊳ L
3. M ⊳ M', M ⊳ M'
4. M ⊳ M', M =_α M', M =_α M'

IV
∀Z closed 3B<α MZ =_β NZ

In the above M,M',N,L and Z denote arbitrary terms, and α,α' arbitrary countable ordinals.
Language
Alphabet: a, b, c, ... variables
I, K, S constants
(, ), _ improper symbols
= equality
\rightarrow reduction
\rightarrow_1 one step reduction
= intrinsic equality

Simple terms: Simple terms are defined inductively by
1) Any variable or constant is a simple term.
2) If M, N are simple terms, then (MN) is a simple term.

Terms: Terms are defined inductively by
1) Any simple term is a term.
2) If M is a simple term, then M is a term.
3) If M, N are terms, then (MN) is a term.

Formulas: If M, N are terms, then
M = N, M \rightarrow N, M \rightarrow_1 N and M = N are formulas.

M_1 M_2 \ldots M_n stands for (. . . (M_1 M_2) \ldots M_n)
CL

Axioms and rules

I 1. IM ⊳ M
   2. 'KMN ⊳ M
   3. SMNL ⊳ M(LNL)

II 1. M = M
    2. M = N
       N = M
    3. M = N, N = L
       M = L
    4. M = M', M = M'
       2M = 2M', MZ = M'Z

III 1. M ⊳ M
    2. M ⊳ N, N ⊳ L
       M ⊳ L
    3. M ⊳ M'
       2M ⊳ 2M'
       MZ ⊳ M'Z
    4. M ⊳ M'
       M = M'

IV 1. M ⊳ M
    2. M ⊳ M', M ⊳ M'
       2M ⊳ 2M'
       MZ ⊳ M'Z
    3. M ⊳ M'
       M ⊳ M'
    4. M ⊳ M'
       M ⊳ M'

V 1. M = M
    2. M = N
       N = M
    3. M = N, N = L
       M = L
    4. M = M', M = M'
       2M = 2M'
       MZ = M'Z
    5. M = M

VI MN ⊳ MN

In the above M, M', N, L and Z denote arbitrary terms except in IV 4, V 5 and VI where M, M' denote simple terms.
Language

Alphabet: a, b, c, ..., variables
   λ, (, ),_ improper symbols
   =, = equality, resp.intrinsic equality
   ⇒, ⇒ reduction, resp. one step reduction

Simple terms: Simple terms are defined inductively by
1) Any variable is a simple term.
2) If M, N are terms, then (MN) is a term.
3) If M is a term, then λxM is a term
   (x is an arbitrary variable).
The set of free variables of a simple term is
   inductively defined by
   FV(x) = {x}
   FV(MN) = FV(M) ∪ FV(N)
   FV(λxM) = FV(M) - {x}

Terms: Terms are defined inductively by
1) Any simple term is a term.
2) If M is a simple term and if FV(M) = ∅,
   then M is a term.
3) If M, N are terms, then (MN) is a term.
4) If M is a term, then λxM is a term.

Formulas: If M, N are terms then
   M = N, M ⇒ N, M ⇒ N and M = N are formulas.
FV, BV, [x/N] are defined inductively by
   FV(x) = {x}
   FV(MN) = FV(M) ∪ FV(N)
   FV(λxM) = FV(M) - {x}
   FV(M) = ∅
   BV(x) = ∅
   BV(MN) = BV(M) ∪ BV(N)
   BV(λxM) = BV(M) ∪ {x}
   BV(M) = BV(M)

[x/N]x = N
[x/N]y = y
[x/N](M, N) = ([x/N]M, [x/N]N)
[x/N](λxM) = λxM
[x/N](λyM) = λy[x/N]M
[x/N]M = M

In the above x is an arbitrary variable and y is a variable different from x.
\[ \lambda + \text{ext.} \]

**Axioms and rules**

**I**
1. \( \lambda x M >_{1} \lambda x[x/y] M \) if \( y \notin \text{FV}(M)^{'} \) \( \forall \lambda \in \text{FV}(M) \)
2. \( (\lambda x M) N >_{1} [x/N] M \) if \( \text{BV}(M) \cap \text{FV}(N) = \emptyset \)
3. \( \lambda x (Mx) >_{1} M \) if \( x \notin \text{FV}(M) \).

**II**
1. \( M = M \)
2. \( M = N \)
   \( N = M \)
3. \( M = N, N = L \)
   \( M = L \)
4. \( M = M', M = M', M = M \)
   \( 2M = 2M', MZ = M'Z, \lambda x M = \lambda x M' \)

**III**
1. \( M >_{1} M \)
2. \( M >_{1} N, N >_{1} L \)
3. \( M >_{1} M', M >_{1} M', M >_{1} M \)
   \( 2M >_{1} 2M', MZ >_{1} M'Z, \lambda x M >_{1} \lambda x M' \)
4. \( M >_{1} M', M = M \)

**IV**
1. \( M >_{1} M \)
2. \( M >_{1} M', M >_{1} M', M >_{1} M \)
   \( 2M >_{1} 2M', MZ >_{1} M'Z, \lambda x M >_{1} \lambda x M' \)
3. \( M >_{1} M', M >_{1} M' \)
   \( M >_{1} M' \)

**V**
1. \( M = M \)
2. \( M = N \)
   \( N = M \)
3. \( M = N, N = L \)
   \( M = L \)
4. \( M = M', M = M', M = M \)
   \( 2M = 2M', MZ = M'Z, \lambda x M = \lambda x M' \)
5. \( M = M \)

In the above, \( M, M', N, L \) and \( Z \) denote arbitrary terms except in the last item of IV 2 and in V 5 where \( M, M' \) denote simple terms.
Language
Alphabet: a, b, c, ... variables
I, K, S constants
(, ) improper symbols
= equality
> reduction
=, ~ special equalities

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If M, N are terms, then (MN) is a term.

Formulas: If M, N are terms, then
M = N, M \geq N, M \equiv N and M \sim N are formulas.

\prod_{i=1}^{n} M_i \text{ stands for } \ldots (M_iM_2) \ldots M_n
Axioms and rules
I 1. IM > M
2. KMN > M
3. SMNL > ML(NL)

II 1. M = M  M = M  M ~ M
2. M = N  M = N  M ~ N
   N = M  N = M  N ~ M
3. M = N, N = L
   M = L
   4. M = M', M = M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'
      M = M', M = M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'
      M = M', M = M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'

III 1. M > M
2. M > N, N > L
   M = L
   3. M > M', M > M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'
      M = M', M = M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'
      M = M', M = M'
      2M = 2M'  2M = 2M'  2M = 2M'  2M = 2M'

IV  M = M' if M, M' are unsolvable terms.

In the above M, M', N, L and Z denote arbitrary terms.
Language
Alphabet: a, b, c, ..., variables
I, k, S, constants
(, ) improper symbols
= equality
\geq reduction
\rightarrow one step reduction

Terms: Terms are defined inductively by
1) Any variable or constant is a term.
2) If M, N are terms, then (MN) is a term.

Formulas: If M, N, are terms, then
M = N, M \geq N and M \rightarrow N are formulas.

M_1 M_2 \ldots M_n stands for (...)M_1 M_2 \ldots M_n
\Omega_2 stands for SII(SII).
Axiomes and rules

I
1. IM \rightarrow M
2. KMN \rightarrow M
3. SMLN \rightarrow ML(NL)

II
1. M = M
2. M = N
   N = M
3. M = N, N = L
   M = L
4. M = M', M = M'
   2M = 2M', M = M'
   MZ = M'Z

III
1. M \rightarrow M
2. M \rightarrow N, N \rightarrow L
   M \rightarrow L
3. M \rightarrow M'
   2M \rightarrow 2M', MZ \rightarrow M'Z
4. M \rightarrow M'
   M = M'

IV
1. M \rightarrow M'
2. M \rightarrow M', N \rightarrow M'
   MN \rightarrow M'M'
3. M \rightarrow M'
   M \rightarrow M'

V
1. \Omega_2(KI) \rightarrow KK where CL'' \vdash \Omega_2 \rightarrow \Omega_2'
2. \Omega_2(SK) \rightarrow KK where CL'' \vdash \Omega_2 \rightarrow \Omega_2'.

In the above M,M',N,L and Z denote arbitrary terms, and \Omega_2 \equiv SII(SII).
Language
Alphabet: \( a, b, c, \ldots \) variables
\( I, K, S \) constants
( , ), _ improper symbols
\( = \) equality
\( \succ \) reduction
\( \succ_1 \) one step reduction
\( = \) intrinsic equality

Simple terms: Simple terms are defined inductively by
1) Any variable or constant is a simple term.
2) If \( M, N \) are simple terms, then \( MN \) is a simple term.

Terms: Terms are defined inductively by
1) Any simple term is a term.
2) If \( M \) is a simple term, then \( M \) is a term.
3) If \( M, N \) are terms, then \( MN \) is a term.

Formulas: If \( M, N \) are terms, then
\( M = N, M \succ N, M \succ_1 N \) and \( M = N \) are formulas.

\( M_1 M_2 \ldots M_n \) stands for \(( (M_1 M_2) \ldots ) M_n\)

To be able to formulate the axioms we define a mapping \( \left| \ldots \right| \): Terms \( \rightarrow \) Simple terms.
\( |c| = c \) if \( c \) is a constant or variable
\( |MN| = |M| |N| \)
\( |M| = M \)
Axioms and rules

I 1. $IM \Rightarrow M$
2. $KMN \Rightarrow M$
3. $SHNL \Rightarrow ML(NL)$

II 1. $M = M$
2. $M = N$
   $N = M$
3. $M = N$, $N = L$
   $M = L$
4. $M = M'$, $M = M'$
   $ZH = ZH'$, $HZ = H'Z$

III 1. $M > M$
2. $M > N$, $N > L$
   $M > L$
3. $M > M'$, $M > M'$
   $ZH > ZH'$, $HZ > H'Z$
4. $M > M'$
   $M = M'$

IV 1. $M > M$
2. $M > M'$
   $MZ > ZM'$
3. $M > M'$
   $M > M$
4. $M > M'$
   $M > M$

V 1. $M = M$
2. $M = N$
   $N = M$
3. $M = N$, $N = L$
   $M = L$
4. $M = M'$, $M = M'$
   $ZH = ZM'$, $HZ = H'Z$
5. $M = M$

VI $MN \Rightarrow M|N|

In the above $M, M', N, L$ and $Z$ denote arbitrary terms except in IV 4, V 5 and VI where $M, M'$ denote simple terms.
Appendix II

The Church-Rosser theorem for the $\lambda$-calculus \\
a la Martin-Löf

This appendix contains a proof of the Church-Rosser theorem \\
recently discovered by Martin-Löf [1971].

This proof is strikingly simple compared to those mentioned in \\
1.2.18.

The idea of the proof arose from cut elimination properties of \\
certain formal systems. In fact the Church-Rosser theorem is a \\
kind of cut elimination theorem, the transitivity of $=$ in the \\
$\lambda$-calculus corresponding to the cut.

The trick is to define a relation $\triangleright$ between terms in such a way \\
that

1) The transitive closure of $\triangleright$ is the (classical) reduction \\
relation ($\Rightarrow$).

2) If $M_1 \triangleright M_2$, $M_1 \triangleright M_3$, then there exists a term $M_4$ such that \\
$M_2 \triangleright M_4$ and $M_3 \triangleright M_4$.

From 1) and 2) the analogue of 2) for $\rightarrow$ can be derived.

From this the Church-Rosser theorem easily follows.

Definition 1.

$\lambda'$ is a theory formulated in the following language:

Alphabet: $a,b,c,...$ variables \\
$\lambda$, ( , ) improper symbols \\
$=$ equality \\
$\Rightarrow$ reduction \\
$\triangleright$ one step reduction
Terms: The terms are defined as in the λ-calculus (1.1.1).

Formulas: If M, N are terms, then M = N, M \succ N and M \succ_i N are formulas.

As in 1.1.2 we define BV(M), FV(M) and [x/N]M.

Definition 2.

\( \lambda' \) is defined by the following axioms and rules.

I. 1. \( \frac{M \succ N', y \notin FV(M') \cup BV(M')}{(\lambda x M) \succ \lambda y [x/y] M'} \) if \( y \notin FV(M') \cup BV(M') \)

2. \( \frac{M \succ_1 M', N \succ_1 N'}{(\lambda x M) N \succ_1 [x/N] M'} \) if \( BV(M') \cap FV(N') = \emptyset \)

II. 1. \( M \succ_1 M \)

2. \( \frac{M \succ_1 M', N \succ_1 N'}{MN \succ_1 MN'} \)

3. \( \frac{M \succ_1 M'}{\lambda x M \succ_1 \lambda x M'} \)

4. \( \frac{M \succ_1 M'}{M \succ_1 M'} \)

III. 1. \( \frac{M \succ N, N \succ L}{M \succ L} \)

2. \( \frac{M \succ M', M \succ M', M \succ M'}{Z M \succ Z M', M \succ M', M \succ M'} \)

3. \( \frac{M \succ M'}{M = M'} \)

IV. 1. \( \frac{M = N}{N = M} \)

2. \( \frac{M = N, N = L}{M = L} \)

3. \( \frac{M = M', M = M', M = M'}{Z M = Z M', M = M', M = M'} \)

In the above M, M', N, L and Z denote arbitrary terms and x, y denote arbitrary variables.
Lemma 3.
\[ \lambda' \vdash M \triangleright N \iff \exists N_1 \ldots N_k \lambda' \vdash M \equiv N_1 \rightarrow \ldots \rightarrow N_k \equiv N. \]

Proof.
\[ \Rightarrow: \text{Immediate.} \]
\[ \Leftarrow: \text{Induction on the length of proof of } \lambda' \vdash M \triangleright N. \]

Lemma 4.
\[ \lambda' \vdash M \triangleright N_1 \Rightarrow \lambda' \vdash M \triangleright N \]
\[ \lambda' \vdash M \triangleright N \Rightarrow \lambda' \vdash M \triangleright N \]
\[ \lambda' \vdash M = N \Rightarrow \lambda' \vdash M = N \]

Proof.
In all cases induction on the length of proof.

Lemma 5.
\[ \text{If } \lambda' \vdash M \triangleright N, \text{ then there exists a term } M' \text{ such that } \]
\[ \lambda' \vdash M \triangleright M' \text{ and } N \equiv \lambda x M' \text{ or } N \equiv \lambda y [x/y] M' \text{ with } y \notin \text{FV}(M'). \]

Proof.
Induction on the length of proof of \( M \triangleright M' \) using the sublemma:
\[ \forall x \in \text{FV}(N), x \notin \text{BV}(M') \]
If \( x \neq y \) then \( [x/N_1][y/N_2]M \equiv [y/[x/N_1]N_2K[x/N_1]M] \).

The proof of the sublemma proceeds by induction on the structure of \( M \).

Lemma 6.
1) If \( \lambda' \vdash \lambda x M \triangleright N \), then there exists a term \( M' \) such that
\[ \lambda' \vdash M \triangleright M' \text{ and } N \equiv \lambda x M' \text{ or } N \equiv \lambda y [x/y] M' \text{ with } y \notin \text{FV}(M'). \]
2) If \( \lambda' \vdash M_1 M_2 \triangleright N \), then there exist \( M'_1, M'_2 \) such that
\[ \lambda' \vdash M_1 \triangleright M'_1 \text{ and } N \equiv M'_1 M'_2 \text{ or } M_1 = (\lambda x M'_1) \text{ and } N \equiv [x/M'_1] M''_1 \]
where \( \lambda' \vdash M'_1 \triangleright M''_1 \) and \( \lambda' \vdash M_2 \triangleright M'_2 \).
Proof.

Induction on the length of proof.

Lemma 7.

If $\lambda' \vdash M_1 \succ M_2$ and $\lambda' \vdash M'_1 \succ M'_3$, then there exists a term $M_4$ such that $\lambda' \vdash M_2 \succ M_4$ and $\lambda' \vdash M_3 \succ M_4$.

Proof.

Induction on the sum of lengths of proof of $M_1 \succ M_2$ and $M_2 \succ M_3$.

case 1. $M_1 \succ M_2$ is an axiom. Then $M_1 \equiv M_2$ and we can take $M_4 \equiv M_3$.

case 2. $M_1 \succ M_2$ is $\lambda x M \succ \lambda y [x/y] M'$ where $y \notin \text{FV}(M')$ and is a direct consequence of $M \succ M'$.

By lemma 6 it follows that

$$M_3 = \lambda y' [x/y'] M''$$

where $\lambda \vdash M \succ M''$ and $y' \notin \text{FV}(M'')$ or $y' = x$.

By the induction hypothesis there exists a $M'''$ such that $\lambda' \vdash M' \succ M'''$ and $\lambda' \vdash M'' \succ M'''$. Then we can take $M_4 = \lambda y'' [x/y''] M'''$ with $y'' \notin \text{FV}(M''')$.

case 3. $M_1 \succ M_2$ is $\lambda x M \succ \lambda x M'$ and is a direct consequence of $M \succ M'$. Analogously to case 2 we can find the required term $M_4$.

case 4. $M_1 \succ M_2$ is $(\lambda x M) N \succ (x/N) M'$ and is a direct consequence of $M \succ M'$, $N \succ N'$.

By lemma 6 we can distinguish the following subcases.

subcase 4.1. $M_3 = (\lambda y [x/y] M'') N''$, where $\lambda' \vdash M \succ M''$, $\lambda' \vdash N \succ N''$.

By the induction hypothesis there exist terms $M''$, $N''$ such that $\lambda' \vdash M' \succ M''$. 

\[ \lambda' \vdash M'' \gg_{1} M'', \lambda' \vdash N' \gg_{1} N''. \] and
\[ \lambda' \vdash N'' \gg_{1} N'''. \]

Then by lemma 5 we can take \( M_{4} \equiv [x/N'']M''' \).

**subcase 4.2.** \( M_{3} \equiv [x/N''']M'' \) with \( \lambda' \vdash M \gg_{1} M'' \), \( \lambda' \vdash N \gg_{1} N'' \).

By the induction hypothesis there exist terms \( M''', N''' \) such that \( \lambda' \vdash M' \gg_{1} M''' \) etc.

Then by lemma 5 we can take \( M_{4} \equiv [x/N']M''' \).

**case 5.** \( M_{1} \gg M_{2} \) is \( MN \gg M'N' \) and is a direct consequence of \( M \gg_{1} M', N \gg_{1} N' \).

By lemma 6 we can distinguish the following subcases.

**subcase 5.1.** \( M_{3} \equiv M''N'' \) with \( \lambda' \vdash M \gg_{1} M'', \lambda' \vdash N \gg_{1} N'' \).

By the induction hypothesis there exist terms \( M''', N''' \) such that \( \lambda' \vdash M' \gg_{1} M''' \) etc.

Then we can take \( M_{4} \equiv M'''N''' \).

**subcase 5.2.** \( M_{1} \gg M_{3} \) is \( (\lambda x_{M_{1}})N \gg [x/N']M_{1}' \) and is a direct consequence of \( M_{1} \gg_{1} M_{1}' \), \( N \gg_{1} N'' \).

This case is analogous to subcase 4.1.

**Lemma 8.**

If \( \lambda' \vdash M_{1} \gg M_{2} \) and \( \lambda' \vdash M_{1} \gg M_{3} \), then there exists a term \( M_{4} \) such that \( \lambda' \vdash M_{2} \gg M_{4} \) and \( \lambda' \vdash M_{3} \gg M_{4} \).

**Proof.**

By lemma 3 \( \lambda' \vdash M_{1} \gg M_{2} \iff \exists N_{1}, \ldots, N_{k} \lambda' \vdash M_{1} \equiv N_{1} \gg_{1} \cdots \gg_{1} N_{k} \equiv N \) and similarly for \( \lambda' \vdash M_{1} \gg M_{3} \).

By repeated use of lemma 7 (see figure 2, page 40) it follows that the conclusion holds.
Lemma 8.

If \( \lambda' \vdash M = N \), then there exists a term \( Z \) such that
\( \lambda' \vdash M \geq Z \) and \( \lambda' \vdash N \geq Z \).

Proof.

Induction on the length of proof of \( M = N \), using lemma 8 in the case of transitivity of \( = \). \( \Box \)

Theorem 10. (Church-Rosser theorem)

If \( \lambda \vdash M = N \), then there exists a term \( Z \) such that
\( \lambda \vdash M \geq Z \) and \( \lambda \vdash N \geq Z \).

Proof.

This follows immediately from lemma 9 and lemma 4. \( \Box \)

Remark.

In the same way we can prove the Church-Rosser theorem for \( \lambda + \text{ext} \) by adding to \( \lambda' \) the rule

\[
\frac{M \geq M'}{\lambda x(Nx) \geq_m M'}
\]
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Samenvatting

Dit proefschrift houdt zich bezig met de combinatorische logica, niet als basis voor de rest van de wiskunde, maar als formeel systeem voor de bestudering van berekeningsprocedures.

Hoofdstuk I geeft een overzicht en uitbreiding van reeds bekend materiaal.

In Hoofdstuk II wordt de \( \omega \)-regel ingevoerd en met behulp van transfinite inductie bewezen dat de uitbreiding van de combinatorische logica met de \( \omega \)-regel consistent is. Verder wordt de existentie van universele generatoren bewezen. Voor de termen die geen universele generatoren zijn, geldt dat de \( \omega \)-regel een afgeleide regel is.

In Hoofdstuk III worden een aantal andere consistentie resultaten bewezen, waardoor verschillende niet elementair equivalent modellen van de combinatorische logica verkregen worden.

In de bewijzen van de hierboven vermelde resultaten wordt meestal gebruik gemaakt van conservatieve uitbreidingen van de combinatorische logica. Hierbij speelt een nieuwe bewijstechniek een belangrijke rol, te weten de methode van het onderlijnen. Deze methode formaliseert het begrip residu en vermijdt aldus de anders nogal omslachtige argumenten.
Curriculum vitae

Op 18 december 1947 werd ik geboren te Amsterdam.


Na in 1965 het eindexamen gymnasium-B gehaald te hebben, liet ik mij inschrijven als student in de wis- en natuurkunde aan de Rijksuniversiteit Utrecht.

Onder leiding van Dr. D. van Dalen legde ik mij toe op de grondslagen van de wiskunde. Daarnaast volgde ik een college van Prof. J. J. de Jongh in Nijmegen en nam ik aldaar deel aan het interuniversitaire seminarium grondslagen van de wiskunde dat samen met de Rijksuniversiteit Utrecht gegeven werd.

In 1968 kreeg ik van de Polska Akademia Nauk een studiebeurs voor een maand studie onder leiding van Prof. Mostowski in Warschau.

Van 1966 tot 1969 was ik verbonden als slagwerker aan Dansgroep Pauline de Groot, hetgeen mij financieel in staat stelde als student deel te nemen aan diverse internationale wiskunde congres.

In december 1968 legde ik het doctoraal examen wiskunde met groot bijvleugel wiskunde af.

Sinds 1969 ben ik verbonden aan de Centrale Interfaculteit van de Rijksuniversiteit Utrecht als wetenschappelijk medewerker in de wijsbegeerte van de wiskunde.

Door de bestudering van het proefschrift van Goodman 'The interpretation of intuitionistic arithmetic in a theory of constructions' in een interuniversitaire werkgroep samen met de afdeling grondslagen van de wiskunde van de Gemeente Universiteit van Amsterdam werd mijn belangstelling gewekt voor de combinatorische logica. Deze belangstelling werd verder gestimuleerd door de hoogleraren Curry, Scott en Grzegorczyk.

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STELLINGEN

I
De wijze waarop Rosenbloom het extensionaliteits principe formuleert geeft aanleiding tot verwarring, met name bij Rosenbloom zelf.

Rosenbloom: The elements of mathematical logic, blz.112.

II
Ten onrechte schrijft Goodman aan zijn abstractie operator λ zekere eigenschappen met betrekking tot gedefinieerdheid toe.

Goodman: Intuitionistic arithmetic as a theory of constructions, section 8.

III
In de intuitionistische theorie der gelijkheid is het axioma
\[ \neg \forall z \ (z \neq x \lor z \neq y) \rightarrow x = y \]
echt sterker dan het stabiliteits axioma
\[ \neg \neg x = y \rightarrow x = y. \]

IV
Door de axioma's van Kearns betreffende de discriminatoren in combinatorische logica iets voorzichtiger te formuleren, is het mogelijk dat de reductie- en de gelijkheidsrelatie ook rechtsmonotoon zijn.

Kearns: Combinatory logic with discriminators,
Het begrip 'sterk definitioneel gelijk', zoals Tait dit in heeft gevoerd, is niet helemaal adequaat. De moeilijkheid is op te lossen door een variant van Curry's sterke reductie relatie in te voeren.


Curry's opvatting, dat de combinatorische logica een prelogica vormt die de grondslag vormt voor alle formele systemen, gaat voorbij aan de moeilijkheden in de analyse van het iteratie proces.

Curry en Feys: Combinatory logic, Introduction.

Bij de vraag of post- ook propterhypnotisch gedrag is, gaat het er niet om of de proefpersoon toneel speelt, beleefd is, bedriegt of wat dan ook. Relevant is alleen van welke stimuli alternatief gedrag afhankelijk is.

De molens in Nederland draaien met hun wieken tegen de wijzers van de klok. Dit is een gevolg van de omstandigheid dat er in Nederland meer ruimende dan krimpende wind voorkomt: wanneer de molenaar tijdens werkzaamheden tengevolge van deze ruimende wind moet kruien gaat dit lichter dan met krimpende wind in verband met de gyroscopische werking van het wiekenkruis.

H.P.Barendregt 16 juni, 1971