THE DERKSEN INVARIANT Vs.
THE MAKAR-LIMANOV INVARIANT

Stefan Maubach

Report No. 0205 (March 2002)
The Derksen invariant Vs. the Makar-Limanov invariant

Stefan Maubach

April 22, 2002

Abstract

In this article an example is given of a ring (or variety) of which the Makar-Limanov invariant is trivial but the Derksen invariant is not.

1 Introduction

The Makar-Limanov invariant was introduced by Makar-Limanov in [?] to prove that the variety in $\mathbb{C}^4$ given by the equation $X^2Y + X + Z^2 + T^3$ is not isomorphic to $\mathbb{C}^4$. Later on, Derksen gave an alternative proof in [?] by introducing a different invariant. The general idea was that both invariants are kind of dual, in the sense that they can distinguish the same set of rings from the polynomial rings. However, this article gives an example in which this is not the case, thus stating that the invariants are clearly different. It is still an open question whether the Derksen invariant is actually stronger than the Makar-Limanov invariant, in the sense that it can distinguish more rings from polynomial rings.

2 Definitions and notations

In this section, $R$ denotes a commutative finitely generated $\mathbb{C}$-algebra and $\mathbb{N}$ the non-negative integers.

Definition 2.1. (i). A map $D : R \longrightarrow R$ is called a derivation if it satisfies the Leibniz rule: $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.

(ii). A derivation is called locally nilpotent if for each $a \in R$ there exists $n \in \mathbb{N}$ such that $D^n(a) = 0$. 

1
(iii). When $D$ is a locally nilpotent derivation, we denote by $R^D$ the kernel of the map $D$, i.e. $R^D := \{a \in R \mid D(a) = 0\}$.

(iv). $LND(R)$ is the set of all locally nilpotent derivations on $R$.

(v). $LND^*(R) := LND(R) \setminus \{0\}$ (notice the zero map “0” is actually a derivation).

(vi). $ML(R) := \cap_{D \in LND(R)} R^D$, the Makar-Limanov invariant of $R$. 1

(vii). $HD(R)$ is the $\mathbb{C}$-algebra generated by $\cup_{D \in LND^*(R)} R^D$.

Example 2.2. If $R = \mathbb{C}[X_1, \ldots, X_n]$ then $ML(R) = \mathbb{C}$, and in case $n \geq 2$ $HD(R) = R$. In case $n = 1$, $HD(R) = \mathbb{C}$ (a small exception).

Corollary 2.3. If $ML(R) \neq \mathbb{C}$ (i.e. $ML(R)$ is larger than $\mathbb{C}$) then $R$ is not a polynomial ring. If $\dim(R) \geq 2$ and $HD(R) \neq R$ then $R$ is not a polynomial ring.

3 A specific ring and its invariants

In this section we will give a ring whose Makar-Limanov invariant is trivial but its Derksen invariant is not.

Definition 3.1. Define the ideal $I := (X, Y) \subset \mathbb{C}[X, Y]$, and let

$$R := \mathbb{C}[X^2, X^3, Y^3, Y^4, Y^5, X^{1+i}Y^{1+j} \mid i, j \in \mathbb{N}] = \mathbb{C}[X^2, X^3, Y^3, Y^4, XY, X^2Y, XY^2, X^2Y^2, XY^3, X^2Y^3, XY^4, X^2Y^4]$$

(i.e. $R = \mathbb{C} \oplus \mathbb{C}X^2 \oplus \mathbb{C}XY \oplus I^3$).

Notice that $R$ is finitely generated, noetherian, and a domain.

Lemma 3.2. $ML(R) = \mathbb{C}$.

Proof. Let $D_1 := Y^3 \partial_X$ and $D_2 := X^2 \partial_Y$. These are locally nilpotent derivations on $R$, as can be easily checked. Then $R^{D_1} = R \cap \mathbb{C}[X, Y]^{D_1} \subseteq \mathbb{C}[X, Y]^{D_1} = \mathbb{C}[Y]$. Also $R^{D_2} = R \cap \mathbb{C}[X, Y]^{D_2} \subseteq \mathbb{C}[X, Y]^{D_2} = \mathbb{C}[X]$. Thus $\mathbb{C} \subseteq ML(\mathbb{C}) \subseteq R^{D_1} \cap R^{D_2} \subseteq \mathbb{C}[X, Y]^{D_1} \cap \mathbb{C}[X, Y]^{D_2} = \mathbb{C}[Y] \cap \mathbb{C}[X] = \mathbb{C}$. \hfill $\square$

1The original notation introduced by Makar-Limanov himself was $AK(R)$, “absolute kernel” and this notation is sometimes used too.

2This invariant is often denoted by “$D(R)$” but since $D$ is a very common notation for a derivation, the notation “$HD$” (for Harm Derksen) got into fashion.
In order to calculate $HD(R)$ we first show that every locally nilpotent derivation on $R$ actually comes from a locally nilpotent derivation on $\mathbb{C}[X,Y]$

**Lemma 3.3.** (i). The integral closure of $R$ in $\mathbb{C}[X,Y]$ is $\mathbb{C}[X,Y]$.
(ii). The integral closure of $R$ in $Q(R)$ (the fraction field of $R$) is $\mathbb{C}[X,Y]$.

**Proof.** (i) is easy, since the integral closure of the smaller ring $\mathbb{C}[X^2,Y^3]$ in $\mathbb{C}[X,Y]$ already is $\mathbb{C}[X,Y]$. (ii) $Q(R) = \mathbb{C}(X,Y)$. Let $a \in Q(R)$ be integral over $R$. Then surely $a$ is integral over $\mathbb{C}[X,Y]$. But $\mathbb{C}[X,Y]$ is a UFD and thus integrally closed in its fraction field i.e. $a \in \mathbb{C}[X,Y]$ already. Thus the integral closure of $R$ in $Q(R)$ is a subset of $\mathbb{C}[X,Y]$. Finally, since $Q(R) = \mathbb{C}(X,Y)$ and by part (i) we are done by part (i). □

Notice that if $D$ is a derivation (not necessarily locally nilpotent) on a domain $A$, then it extends uniquely to a derivation on the fraction field $Q(A)$ of $A$, by just forcing $D(a^{-1}b) = a^{-2}(aD(b) - D(a)b)$ for all $a \in A, b \in A\{0\}$.

**Theorem 3.4.** (Seidenberg) Let $A$ be a noetherian domain containing $\mathbb{Q}$, $K$ its quotient field and $\tilde{A}$ the integral closure of $A$ in $K$. Let $D$ be a derivation on $A$ and $\tilde{D}$ its unique extension to $K$. Then $\tilde{D}(\tilde{A}) \subseteq \tilde{A}$.

This is quoted literally from [?] prop. 1.2.15 page 17, but it is originally from [?]

**Theorem 3.5.** (Vasconcelos) Let $A \subseteq B$ be an integral extension where $B$ is a domain and $\mathbb{Q} \subseteq A$. If $D$ is a derivation on $B$ such that $DA \subseteq A$ and the restriction $D|_A$ of $D$ to $A$ is locally nilpotent, then $D$ is locally nilpotent.

This is quoted literally from [?] prop. 1.3.37 page 29, but it is originally from [?]

**Lemma 3.6.** If $D$ is a locally nilpotent derivation on $R$ then it extends uniquely to a locally nilpotent derivation on $\mathbb{C}[X,Y] \to \mathbb{C}[X,Y]$.

**Proof.** The integral closure of $R$ in $Q(R)$ is $\mathbb{C}[X,Y]$ (by ??). So by the above theorem of Seidenberg $D$ extends uniquely to $\mathbb{C}[X,Y]$. By the above theorem of Vasconcelos we see that $D$ is locally nilpotent. □

**Lemma 3.7.** If $f \in \mathbb{C}[X,Y]$ is a coordinate, $p(T) \in \mathbb{C}[T]$ and $p(f) \in R$ then $XY$ does not appear as a monomial in $p(f)$. 

3
Proof. \( f = f_0 + f_1 = f_0 + aX + bY + cX^2 + dXY + eY^2 + g \) for some 
\( g \in I^3 \) and \( a \neq 0 \) or \( b \neq 0 \). Now \( p(f) = q(f_1) \) for some \( q(T) \in \mathbb{C}[T] \).

\[
q(f_1) = \lambda_0 + \lambda_1 f_1 + \lambda_2 f_1^2 + \ldots + \lambda_n f_1^n \\
= \lambda_0 + \lambda_1(aX + bY + cX^2 + dXY + eY^2) + \\
\lambda_2(aX + bY + cX^2 + dXY + eY^2)^2 + g' \quad g' \in I^3
\]

and since \( a \neq 0 \) or \( b \neq 0 \) and \( q(f) \in R \) we must have \( \lambda_1 = 0 \). Thus

\[
q(f) = \lambda_0 + \lambda_2(aX + bY + cX^2 + dXY + eY^2)^2 + g' \quad g' \in I^3 \\
= \lambda_0 + \lambda_2(a^2X^2 + 2abXY + b^2Y^2) + g'' \quad g'' \in I^3
\]

but since in no element of \( R \) appears the monomial \( Y^2 \) and \( q(f) \in R \) we must have \( \lambda_2b^2 = 0 \) which implies \( 2\lambda_2ab = 0 \), which is the coefficient of \( XY \).

\[ \square \]

Lemma 3.8. Let \( D \in \text{LND}(R) \). Suppose there exists \( g \in R^D \) such that the coefficient of \( XY \) of \( g \) is nonzero (\( XY \) appears in \( g \)). Then \( D = 0 \).

Proof. We know by \( ?? \) that \( D \) can be extended as a locally nilpotent derivation to \( \mathbb{C}[X, Y] \). Suppose \( D \neq 0 \). Thus \( \mathbb{C}[X, Y]^D = \mathbb{C}[f] \) for some coordinate \( f \) by Rentschler’s theorem. Hence \( g = p(f) \in R^D \). But now by lemma \( ?? \), the coefficient of \( XY \) must be zero, a contradiction. Hence our assumption that \( D \) was nonzero was wrong, thus \( D = 0 \).

\[ \square \]

Lemma 3.9. \( \text{HD}(R) \neq R \)

Proof. If we show that \( XY \notin \text{HD}(R) \) then we are done. Suppose \( g_1, \ldots, g_n \in R \) are elements of kernels of nonzero locally nilpotent derivations such that \( XY = p(g_1, \ldots, g_n) \) for some \( p \in \mathbb{C}[T_1, \ldots, T_n] \). Then since \( g_i \in R \) we have that \( g_i = c_i + a_iX^2 + b_iXY + h_i \) for some \( a_i, b_i, c_i \in \mathbb{C}, h_i \in (X^3, X^2Y, XY^2, Y^3) \). We may assume that \( c_i = 0 \). Furthermore by lemma \( ?? \), \( b_i = 0 \). Let \( p' \) be the part of \( p \) which is linear. Now \( XY = p'(a_1X^2, \ldots, a_nX^2) + h' \) for some \( h' \in (X^3, X^2Y, XY^2, Y^3) \). This gives a contradiction.

\[ \square \]

Open question: Does \( \text{HD}(R) = R \) always imply \( \text{ML}(R) = \mathbb{C} \) (for rings of dimension \( \geq 2 \))?
4 Acknowledgements

The author would like to thank the organisers of the Workshop on Group actions on rational varieties (February 27- March 3 2002) at CRM in Montreal for an inspiring visit. Also he would like to express his gratitude to A. van den Essen and J.W.Bikker for some stimulating discussions.

Author’s address:

Department of Mathematics
University of Nijmegen
Toernooiveldt
6525 ED Nijmegen
The Netherlands

References


[3] L. Makar-Limanov, On the hypersurface \( x + x^2y + z^2 + t^3 = 0 \) in \( \mathbb{C}^4 \) or a \( \mathbb{C}^3 \)-like threefold which is not \( \mathbb{C}^3 \), Israel J. Math., 96(1996), 419-429
