A SIMPLE PROOF OF THE MODULAR IDENTITY FOR THETA FUNCTIONS

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Report No. 0114 (July 2001)
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To A.C.M. van Rooij on occasion of his 65th birthday

Abstract. The modular identity arises in the theory of theta functions in one complex variable. It states a relation between theta functions for parameters \( \tau \) and \(-1/\tau\) situated in the complex upper half plane. A standard proof uses Poisson summation and hence builds on results from Fourier theory. This paper presents an elementary proof using only a uniqueness property and the simple heat equation.

1. The \( \theta \) function

Let \( \mathbb{H} \subset \mathbb{C} \) denote the upper half plane of all complex numbers with a positive imaginary part. The following series converges locally uniformly in \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \) and hence defines a holomorphic function on \( \mathbb{C} \times \mathbb{H} \):

\[
\theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{2\pi ikz + \pi ik^2 \tau}
\]

This function is often called the \( \theta_3 \) function of Jacobi (some texts use \( q = e^{\pi i \tau} \) or replace \( 2\pi iz \) by \( z \)). For \( z = 0 \) it is also called Ramanujan’s theta function. It satisfies the shift relations in \( z \)

\[
\theta(z + 1, \tau) = \theta(z, \tau)
\]

and

\[
\theta(z + \tau, \tau) = e^{-2\pi iz - \pi i \tau} \theta(z, \tau)
\]

that can easily be verified from its definition. The following heat equation is also apparent from the definition of \( \theta \):

\[
\frac{d^2 \theta}{dz^2} = 4\pi i \frac{d \theta}{d\tau}.
\]

Let \( \Lambda(\tau) = \mathbb{Z} + \mathbb{Z} \tau \) be the lattice spanned by 1 and \( \tau \). For fixed parameter \( \tau \) the function \( \theta \) in \( z \) is the only entire function satisfying (1.1) and (1.2) up to complex multiples. This follows from the following theorem:

**Theorem 1.1.** If \( f(z) \) is an entire function on \( \mathbb{C} \) satisfying the shift relations (1.1) and (1.2) then either \( f \) vanishes identically or all its roots equal \((\tau + 1)/2\) modulo the lattice \( \Lambda(\tau) \).
Suppose $f$ does not vanish identically. The shift relations for $f$ imply
\[
\frac{f'(z+1)}{f(z+1)} = \frac{f'(z)}{f(z)} \quad \text{and} \quad \frac{f'(z + \tau)}{f(z + \tau)} = \frac{f'(z)}{f(z)} - 2\pi i.
\]
For $b \in \mathbb{C}$ define a closed fundamental domain $P \subset \mathbb{C}$ by
\[
P = \{ b + x + y\tau \mid x, y \in [0,1] \}.
\]
The number of roots of $f$ on $P$ and the sum of its roots on $P$ can be computed by the integrals
\[
\frac{1}{2\pi i} \oint_{\partial P} \frac{f'(z)}{f(z)} dz
\]
and
\[
\frac{1}{2\pi i} \oint_{\partial P} \frac{zf'(z)}{f(z)} dz
\]
respectively. By varying the number $b$ we may assume that $f$ has no roots on $\partial P$ so both integrals are well defined. Using the shift relations for $f$ the first integral evaluates to 1, showing that $f$ has only one root on $P$. The second integral evaluates to a value equal to $(\tau + 1)/2$ modulo the lattice $\Lambda(\tau)$. This proves the theorem. □

**Corollary 1.2.** If $f$ is as theorem 1.1, then $f(z) = c \cdot \theta(z, \tau)$ for some constant $c \in \mathbb{C}$.

Also by theorem 1.1 we find that $\theta(0, \tau)f(z) - f(0)\theta(z, \tau)$ must vanish identically as it vanishes at $z = 0$ as well as at $(\tau + 1)/2$. □

2. The modular identity

We are already in a position to prove the modular identity (2.3) for $\theta$. A very accessible treatment of this identity using Poisson summation can be found in [1]. For the proof given below, the heat equation suffices. Define an entire function $\vartheta$ by
\[
\vartheta(z) = e^{\pi i\tau z^2} \theta(\tau z, \tau).
\]
Then $\vartheta(z + 1) = \vartheta(z)$ and
\[
\vartheta(z - 1/\tau) = e^{-2\pi iz + \pi i/\tau} \vartheta(z).
\]
Hence $\vartheta(z) = c(\tau) \cdot \theta(z, -1/\tau)$ for some function $c$ on the upper half plane by corollary 1.2. Substituting $\tau = i$ and $z = 0$ shows that $c(i) = 1$. The heat equation for $\vartheta$ will produce a simple differential equation for $c$. Elementary computations show:

\[
\begin{align*}
(2.1) \quad \frac{d^2 \vartheta}{dz^2}(0) &= 2\pi i \vartheta(0, \tau) + \tau^2 \frac{d^2 \vartheta}{dz^2}(0, 0) = c(\tau)\frac{d^2 \vartheta}{dz^2}(0, -1/\tau) \\
(2.2) \quad \frac{d \vartheta}{d\tau}(0) &= \frac{d \vartheta}{d\tau}(0, \tau) = c'(\tau) \theta(0, -1/\tau) + c(\tau)\tau^{-2} \frac{d \vartheta}{d\tau}(0, -1/\tau).
\end{align*}
\]

Using the heat equation (1.3) on (2.1) yields
\[
\frac{1}{2\tau^{-1}} \theta(0, \tau) + \frac{d \theta}{d\tau}(0, 0) = c(\tau)\tau^{-2} \frac{d \theta}{d\tau}(0, -1/\tau)
\]
and combining this with (2.2) leads to
\[
\theta(0, \tau) = -2\tau c'(\tau) \theta(0, -1/\tau).
\]
However, substituting \( z = 0 \) in \( \vartheta(z) \) gives
\[
\theta(0, \tau) = \vartheta(0) = c(\tau) \theta(0, -1/\tau)
\]
and as \( \theta \) does not vanish at \( z = 0 \) we find
\[
-2\tau c'(\tau) = c(\tau).
\]
Together with \( c(i) = 1 \) we finally find
\[
c(\tau) = \frac{1}{\sqrt{-i\tau}}
\]
and thus the modular identity for the \( \theta \) function:
\[
\theta(z, -1/\tau) = \sqrt{-i\tau} e^{\pi i z^2} \theta(\tau z, \tau).
\]

References