THE BOREL HIERARCHY AND
THE PROJECTIVE HIERARCHY
IN INTUITIONISTIC MATHEMATICS

Wim Veldman

Report No. 0103 (March 2001)
0 Introduction

0.1 E. Borel, H. Lebesgue, R. Baire, N. Lusin, A. Souslin and others who studied
Borel sets and later also analytic and projective sets, did so partly for philosophical
reasons. They found it difficult to attach a sense to the expression: "all subsets of the
set \( \mathbb{R} \) of real numbers" and decided to focus attention on certain classes of definable
subsets of the continuum. L.E.J. Brouwer, when developing his intuitionistic math­
ematics, had similar but also more fundamental concerns. He came to see that the
meaning of mathematical statements is only to be found in their constructive content.
Mathematicians should acknowledge this and adapt and refine their language. Fur­
ther reflection on how to understand the continuum then led him to enunciate some
revolutionary axioms. We want to find out what becomes of the questions raised by Borel and Lebesgue if one takes Brouwer's point of view.

0.2 A subset of the set $\mathbb{R}$ of real numbers will be called *positively Borel* if and only if it is obtained from open subsets of $\mathbb{R}$ by the repeated use of the operations of countable intersection and countable union.

Given any sequence $K_0, K_1, K_2, \ldots$ of classes of subsets of $\mathbb{R}$, we let its product $\prod_{n \in \mathbb{N}} K_n$ be the class consisting of all sets of the form $\bigcap_{i \in \mathbb{N}} X_i$ where each set $X_i$ belongs to some $K_n$, and we let its sum $\sum_{n \in \mathbb{N}} K_n$ be the class consisting of all sets of the form $\bigcup_{i \in \mathbb{N}} X_i$ where each set $X_i$ belongs to some $K_n$.

We call a class of subsets of $\mathbb{R}$ a *canonical class (of positively Borel subsets)* if and only if it is obtained from the class of the open subsets of $\mathbb{R}$ and the class of the closed subsets of $\mathbb{R}$ by the repeated use of the operations of countable product and countable sum.

The Borel Hierarchy Theorem proved by Borel and Lebesgue is the statement that no canonical class exhausts the collection of all positively Borel subsets of $\mathbb{R}$.

0.3 We do not include the operation of taking the complement of a given set among the generating operations of the class of positively Borel sets. This economy seems harmless from a classical point of view but intuitionistically it is not. The well-known and beautiful new application of Cantor's diagonal argument by which Borel and Lebesgue prove their Hierarchy Theorem is no longer conclusive.

Each canonical class is shown by them to contain a so-called *universal element*. Forming the diagonal set of the universal element they produce a member of the class the complement of which is not a member of the class.

This complement is not, unfortunately, a positively Borel set.

0.4 One might question the decision to consider positively Borel sets only and to shut out the operation of taking the complement as a generating operation. A reason for doing so, however, is that the set-theoretical operation of taking the complement and the logical operations of negation and implication are not that well understood. G. Gentzen, for one, was of this persuasion and argued from it that the consistency of the formal system of intuitionistic arithmetic is not to be accepted without further proof. Also, a theorem in constructive mathematics is generally judged the more useful and beautiful, the less negations it contains. G.F.C. Griss and D. van Dantzig, under the influence of Brouwer's ideas, distrusted negation and emphasized the importance of so-called *affirmative* mathematics.

0.5 It is possible to prove the Hierarchy Theorem for positively Borel sets?

Surprisingly, this question is ignored by P. Martin-Löf in [14] and by E. Bishop and
D. Bridges in [1], although also these authors seem to hold the opinion that, in constructive mathematics, the best Borel sets are positively Borel sets. Brouwer, much earlier studying positively Borel sets of the second level only, sought and found a countable union of closed sets that is not a countable intersection of open sets, and also a countable intersection of open sets that is not a countable union of closed sets, see [2] and [6]. Brouwer’s example of a set of the first kind needs some emendation. Also, his proof of its correctness is based upon the Fan Theorem but an elementary argument suffices. (The Fan Theorem is acceptable both from a classical and an intuitionistic point of view, although its intuitionistic proof has been sometimes disputed.) In his proof of the existence of a set of the second kind, Brouwer is making a classically unacceptable assumption: the Continuity Principle. The use of this principle seems unavoidable. Brouwer did not carefully sort out his assumptions, and left this task for others, notably G. Kreisel and S.C. Kleene, see [10] and [12]. Brouwer’s treatment of the second level of the Borel Hierarchy is important as it suggests how to formulate the Hierarchy Theorem affirmatively, that is, without using negation, and how to prove it.

0.6 Brouwer’s Continuity Principle plays a crucial role in the proof of most of the results of this paper. Once we agree to accept and use it we enter a new world and discover many facts for which there does not exist a classical counterpart. The principle entails for instance that the union of the two closed sets [0, 1] and [1, 2] is not a countable intersection of open subsets of $\mathbb{R}$. One may also infer that there are unions of three closed sets different from every union of two closed sets. These observations are the tip of an iceberg. The intuitionistic Borel Hierarchy shows off an exquisite fine structure.

0.7 We prefer to study subsets of Baire space $\mathcal{N}$ rather than subsets of the set $\mathbb{R}$ of real numbers. Results about subsets of $\mathcal{N}$ translate easily into corresponding results about subsets of $\mathbb{R}$. Given subsets $X, Y$ of $\mathcal{N}$ we say that $X$ reduces to $Y$, notation: $X \preceq Y$, if and only if there exists a continuous function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $X$ if and only if $f(\alpha)$ belongs to $Y$. The Hierarchy Theorem may be restated in terms of this reducibility relation, as follows: for every positively Borel subset $P$ of $\mathcal{N}$ there exists a positively Borel subset $Q$ of $\mathcal{N}$ such that $Q$ does not reduce to $P$.

0.8 The contents of this paper are as follows. First, we set out the axioms of intuitionistic analysis. We then discuss the second level of the hierarchy. We go on to consider the notion of reducibility in the class of countable subsets of $\mathcal{N}$ and find the Cantor-Bendixson hierarchy re-emerging in various new ways. In this context we find occasion to introduce the notions Perhaps and Almost. We extend our study into the wider realm of countable unions of closed sets and are amply rewarded. We then
prove the Borel Hierarchy Theorem, and show that the fine structure discovered at
the second level is also present at higher levels. We conclude with some remarks on
projective sets. Apart from the Introduction, the paper is divided into nine Sections.
The titles of the remaining nine Sections are as follows.

1. The axioms and their plausibility.
2. The second step, and our first one.
3. Rediscovering, perhaps, the Cantor-Bendixson hierarchy.
4. Perhaps and Almost.
5. Finite unions of closed sets.
6. Forming limits and finding more hierarchies.
7. The Borel Hierarchy Theorem.
8. The never-ending productivity of disjunction.
9. On (strictly) analytic, co-analytic and projective sets.

0.9 I dedicate this paper to the memory of Johan J. de Jongh. He introduced me to
Brouwer’s intuitionistic mathematics and asked the question that led to all further
ones. As much a philosopher as a mathematician, he hoped to gain insight from
precisely and carefully proved mathematical results and also expected that sensible
mathematical questions may arise from philosophical reflection. I was made wiser at
the seminar on intuitionistic mathematics he conducted in the seventies in Nijmegen.
In particular, Wim Gielen’s attempts to justify and possibly extend the principles of
intuitionistic mathematics were influential and now and then resound in Section 1.
I am indebted also to the other participants of this seminar, among them Harrie de
Swart and Jo Gielen.

My later students and, in particular, my Ph.D. students Tonny Hurkens and Frank
Waaldijk, by their enthusiasm and sometimes critical interest, helped me to sustain
my belief that intuitionistic mathematics is a worthwhile cause. The notion Perhaps,
considered in Sections 3 and 4, grew out of a study of Frank Waaldijk’s notion of
“weak stability”, see [25].

0.10 I thank the referee of an earlier version of this paper for spotting several inac­
curacies. His comments made me write this second version.

1 The axioms and their plausibility

We explain our point of view and list the assumptions to be used.

1.1 We are contributing to intuitionistic analysis. The logical constants have their
constructive meaning and we follow the rules of intuitionistic logic. In particular, a
disjunctive statement \( A \lor B \) is considered proven only if either \( A \) or \( B \) is proven and
a proof of an existential statement \( \exists x \in V[A(x)] \) has to provide one with a particular element \( x_0 \) from the set \( V \) and a proof of the corresponding statement \( A(x_0) \).

Intuitionistic mathematics distinguishes itself from other varieties of constructive mathematics by its conception of the set of all infinite sequences of natural numbers. This set is not a set in the sense of classical set theory. One does not call it into existence by bringing together its already existing elements. It could be described tentatively as a kind of frame on which all kinds of projects for constructing infinite sequences of natural numbers may be executed, or, more poetically, as a loom on which all kinds of tapestry may be woven.

The seemingly more simple set of the natural numbers also has to be handled with care. In our view it is a never finished project for producing one by one the natural numbers, 0, 1, 2, ....

Just as the canonical infinite sequence 0, 1, 2, ... every infinite sequence \( \alpha \) of natural numbers is incomplete all the time and growing step by step as it elements are produced one by one, \( \alpha(0), \alpha(1), \alpha(2), ... \)

The course of the infinite sequence is sometimes dictated by a finitely described algorithm that we keep evaluating. Brouwer’s imagination did go further. He contrived, for instance, the following project \( \alpha \): for each \( n \), \( \alpha(n) = 0 \) if at the moment I want to decide on the value of \( \alpha \) in \( n \) I have found a proof of Riemann’s hypothesis and \( \alpha(n) = 1 \) if not. The sequence \( \alpha \) is thus made to depend on my future experience as a creating mathematical subject. One may be puzzled and even annoyed by this proposal, and remark that unlike an ordinary definition, it does not settle unambiguously the successive values of the sequence. The creating subject still has many decisions to take, for instance, how to count its time. Brouwer, disregarding such problems, and radically pursuing his line of thought envisaged the even more embarrassing possibility of not prescribing anything and allowing the creating subject to choose the successive values of \( \alpha \) wholly to its own liking. There is then, as far as we know, no “rule” or “secret plan” governing the development of the sequence, but we, the creating subject, are determined to keep our promise to continue the project and to deliver a next value whenever invited to do so. In such circumstances we never have more information about the infinite sequence than its first finitely many values.

Brouwer did not believe that one may distinguish clearly between algorithmic sequences and non-algorithmic ones. There are sequences that fall between the two stools. Nothing forbids that one starts building a sequence by free choices and then, at some moment, decides to fix its further course by a description in finitely many words.

What is more, we decide to consider infinite sequences from the extensional point of view and deliberately disregard their origin. Every infinite sequence of natural numbers comes into being in many different ways and always, also if it is given by an algorithm, may be imagined to be the result of a free step-by-step construction.

In the following, we endeavour to justify some of the axioms of intuitionistic analysis
1.2 We present four axioms of countable choice.

We let $\mathbb{N}$ be the set of natural numbers and $\mathcal{N}$ the set of all infinite sequences of natural numbers. We use $m, n, \ldots$ as variables over $\mathbb{N}$, and $\alpha, \beta, \ldots$ as variables over $\mathcal{N}$. $\mathcal{N}$ is sometimes called *Baire space*. *Cantor space* $\mathcal{C}$ is the set of all $\alpha$ in $\mathcal{N}$ that only assume the values 0,1.
1.2.1 First Axiom of Countable Choice:

For every binary relation \( R \) on \( \mathbb{N} \), if for every \( m \) there exists \( n \) such that \( mRn \), then there exists \( \alpha \) such that, for every \( m \), \( mR(\alpha(m)) \).

We accept this axiom for the following reason.
Suppose we are able to calculate, given any natural number \( m \), a natural number \( n \) suitable for \( m \), that is, such that \( mRn \). We then are sure to be able to carry through the project of constructing step by step an infinite sequence \( \alpha \) such that, for every \( m \), \( \alpha(m) \) is suitable for \( m \). We first choose \( \alpha(0) \), then \( \alpha(1) \), \ldots. We do not feel the need to formulate a rule that predicts the choices we will make. Observe that we can not, like non-intuitionistic mathematicians, define \( \alpha \) by saying: let \( \alpha(m) \) be the least \( n \) such that \( mRn \). One may be unable to find the least such \( n \), for instance, if one knows \( 0R1 \) but cannot decide if \( 0R0 \) or not.

1.2.2 Before we introduce a second axiom of countable choice we agree on some notations. \( \mathbb{N}^* \) is the set of all finite sequences of natural numbers. We let \( \langle \rangle \) be a fixed bijective mapping from \( \mathbb{N}^* \) onto \( \mathbb{N} \). Such a function is called a coding of the set of finite sequences of natural numbers: \( \langle a_0, a_1, \ldots, a_{k-1} \rangle \) is the code number of the finite sequence \( (a_0, a_1, \ldots, a_{k-1}) \). We assume that the empty sequence is coded by the number 0 and that for each finite sequence \( (a_0, a_1, \ldots, a_{k-1}) \), for every \( i < k \), the code number \( \langle a_0, a_1, \ldots, a_{k-1} \rangle \) is greater than \( a_i \). We let * denote concatenation: * is a binary function on \( \mathbb{N} \) such that, for all \( m, n \), \( m \) is an initial part of \( n \) if and only if there exists \( p \) such that \( n = m * p \); and \( n \) is an immediate successor of \( m \) if and only if there exists \( p \) such that \( n = m * (p) \).

We also define another function, called \( J \), from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \): for all \( m, n : J(m, n) := \langle m \rangle * n \). It is easy to see that \( J \) is a bijective mapping from \( \mathbb{N} \times \mathbb{N} \) onto \( \mathbb{N} \setminus \{0\} \). We let \( K, L \) be the inverse functions of \( J \), that is, \( K \) and \( L \) are functions from \( \mathbb{N} \setminus \{0\} \) to \( \mathbb{N} \) and for each \( m, n \neq 0 : J(K(m), L(m)) = m \).

\( J \) is a non-surjective pairing function on \( \mathbb{N} \).

We define, for all \( \alpha \), for all \( m, n \), \( \alpha^m(n) := \alpha(J(m, n)) \). \( \alpha^m \) is called the \( m \)-th subsequence of \( \alpha \). We also define, for all \( \alpha \), for all \( m, n \), \( \alpha^{m,n} := (\alpha^m)^n \).
1.2.3 Second Axiom of Countable Choice:

For every binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$, if for every $m$ there exists $\alpha$ such that $mR\alpha$, then there exists $\alpha$ such that, for every $m$, $mR(\alpha^m)$.

We accept this axiom for the following reason. Suppose we are able to calculate, given any natural number $m$, an infinite sequence $\alpha$ of natural numbers suitable for $m$, that is such that $mR\alpha$. We then form the project of building an infinite sequence $\alpha$ such that for every $m$, $\alpha^m$ will be suitable for $m$. We again construct this $\alpha$ step by step. The difficulty of the construction is only slightly greater than in the case of the First Axiom of Countable Choice. We have to start and keep going an infinite number of never finished constructions, one for $\alpha^0$, one for $\alpha^1$, ..., and so on. At each stage exactly one of these constructions is brought one step further: at stage $n$ one defines $\alpha^{K(n)}(L(n))$.

1.2.4 First Axiom of Dependent Choices:

For every subset $X$ of $\mathbb{N}$, for every binary relation $R \subseteq X \times X$, if for every $m$ in $X$ there exists $n$ in $X$ such that $mRn$, then for every $m$ in $X$ there exists $\alpha$ such that $\alpha(0) = m$ and, for every $n$, $\alpha(n)Ra(n + 1)$.

We accept this axiom for the following reason. Suppose we are given at least one element of $X$, say $m$. Also assume that for every element $p$ of $X$ we are able to calculate an element $n$ of $X$ suitable for $p$, that is, such that $pRn$. We then start building a sequence $\alpha$ step by step, first defining $\alpha(0) = m$, then finding $\alpha(1)$ suitable for $\alpha(0)$, then finding $\alpha(2)$ suitable for $\alpha(1)$, and so on.

The First Axiom of Dependent Choices will be used in the proof of Lemma 8.5.

1.2.5 Second Axiom of Dependent Choices:

For every subset $X$ of $\mathbb{N}$, for every binary relation $R \subseteq X \times X$, if for every $\alpha$ in $X$ there exists $\beta$ in $X$ such that $\alphaR\beta$, then for every $\alpha$ in $X$ there exists $\beta$ such that $\beta^0 = \alpha$ and, for every $n$, $\beta^nR\beta^{n+1}$.

We accept this axiom for the same reason as the previous one. We do not think it important that, this time, the objects to be chosen are infinite sequences of natural numbers rather than natural numbers. We state this axiom for completeness’ sake, as we will have no occasion to use it in this paper.

1.3 The following axiom is classically false. It makes that intuitionistic analysis is not a subsystem of classical analysis.

1.3.1 We define, given any $\alpha$ and any $n$, $\vec{\alpha}(n) := (\alpha(0), \alpha(1), \ldots, \alpha(n - 1))$. If confusion is unlikely to arise, we sometimes write $\vec{\alpha}n$ for $\vec{\alpha}(n)$.
We also define, given any $\alpha$ and any $s$, $s$ is an initial part of $\alpha$, or: $\alpha$ passes through $s$, if and only if, for some $n$, $\bar{\alpha}n = s$.

1.3.2 **Brouwer’s Continuity Principle:**

For every binary relation $R \subseteq \mathbb{N} \times \mathbb{N}$, if for every $\alpha$ there exists $m$ such that $\alpha R m$, then for every $\alpha$ there exist $m, n$ such that for every $\beta$, if $\bar{\alpha}n = \bar{\beta}m$, then $\beta R m$.

We accept this axiom for the following reason.

Suppose we are able to calculate, given any infinite sequence $\alpha$ of natural numbers, a natural number $m$ suitable for $\alpha$, that is, such that $\alpha R m$. We are attaching the strongest possible meaning both to the “for every $\alpha$” and to the “there exists”. Given any infinite sequence whatsoever from the wildly unsurveyable set $\mathbb{N}$ we know how to effectively discover a natural number suitable for it. In particular we can find a suitable number if the sequence is created step by step. A number $m$, suitable for an $\alpha$ that is given step by step will be found at some moment of time, and at that moment only finitely many values of $\alpha$, say $\alpha(0), \alpha(1), \ldots, \alpha(n - 1)$ will be known. The number $m$ will therefore suit every $\beta$ that has its first $n$ values the same as $\alpha$.

We repeat the remark we made at the end of Section 1.1: every $\alpha$, even an algorithmically given one, can be thought of as resulting from a free step-by-step-construction. Brouwer’s Continuity Principle will be used time and again.

1.3.3 Let $X$ be a subset of $\mathbb{N}$.

We let the closure of $X$, notation $\overline{X}$, be the set of all $\alpha$ in $\mathbb{N}$ such that for each $n$ there exists $\beta$ in $X$ passing through $\bar{\alpha}n$.

$X$ is weakly-closed if and only if $X$ coincides with its closure $\overline{X}$.

$X$ is a spread if and only if $X$ is weakly closed and, in addition, we may decide, for every natural number $s$, if $s$ contains an element of $X$ or not.

Deviating from Brouwer’s usage, we also want to call the empty set a spread.

1.3.4 We let $\text{Fun}$ be the set of all $\gamma$ such that, for every $\alpha$, there exists $n$ such that $\gamma(\bar{\alpha}n) \neq 0$. For every $\gamma$ in $\text{Fun}$, every $\alpha$, we let $\gamma(\alpha)$ be the natural number $p$ such that there exists $n$ such that $\gamma(\bar{\alpha}n) = p + 1$ and, for every $m < n$, $\gamma(\bar{\alpha}m) = 0$. In this way, every $\gamma$ in $\text{Fun}$ acts as a code for a continuous function from $\mathbb{N}$ to $\mathbb{N}$.

Observe that, if $\gamma$ belongs to $\text{Fun}$ and $\gamma(0) = 0$, then for each $n$, $\gamma^n$ belongs to $\text{Fun}$.

For every $\gamma$ in $\text{Fun}$ such that $\gamma(0) = 0$, for every infinite sequence $\alpha$ we define an infinite sequence $\gamma|\alpha$ as follows: for each $n$, $(\gamma|\alpha)(n) := \gamma^n(\alpha)$. In this way, every $\gamma$ in $\text{Fun}$ such that $\gamma(0) = 0$ acts as a code for a continuous function from $\mathbb{N}$ to $\mathbb{N}$.

1.3.5 Let $X \subseteq \mathbb{N}$ be a non-empty spread.

We intend to define an element $r_X$ of $\text{Fun}$ with the following two properties:
(i) \( r_X(0) = 0 \) and for each \( \alpha \), \( r_X|\alpha \) belongs to \( X \)
(ii) For each \( \alpha \) in \( X \), \( r_X|\alpha \) coincides with \( \alpha \).

\( r_X \) will be called the **canonical retraction** of \( \mathcal{N} \) onto \( X \).

In order to define \( r_X \) we first define \( \delta \) such that \( \delta(0) = 0 \) and for each \( s, n \), \( \delta(s \star \langle n \rangle) := \delta(s) \star \langle n \rangle \) if \( \delta(s) \star \langle n \rangle \) contains an element of \( X \), and \( \delta(s \star \langle n \rangle) := \delta(s) \star \langle p \rangle \) where \( p \) is the least natural number \( q \) such that \( \delta(s) \star \langle q \rangle \) contains an element of \( X \), if \( \delta(s) \star \langle n \rangle \) does not contain an element of \( X \). It is easy to see that \( \delta \) is well-defined and that for each \( s \), if \( s \) contains an element of \( X \), then \( \delta(s) = s \).

We now may determine \( r_X \) in such a way that for every \( \alpha \), every \( n \), \( r_X|\alpha \) passes through \( \delta(\alpha n) \).

Observe that for every \( \alpha, \beta, n \), if \( \alpha n = \beta n \), then \( r_X|\alpha n = r_X|\beta n \).

1.3.6 **Theorem:** *(Extension of Brouwer’s Continuity Principle to spreads)*

Let \( X \) be a non-empty spread and \( R \) a subset of \( X \times \mathbb{N} \).

If for every \( \alpha \) in \( X \) there exists \( m \) such that \( \alpha R m \), then for every \( \alpha \) in \( X \) there exist \( m, n \) such that for every \( \beta \) in \( X \), if \( \alpha n = \beta n \), then \( \beta R m \).

**Proof:** Observe that for every \( \alpha \) in \( \mathcal{N} \) there exists \( m \) such that \( (r_X|\alpha) R m \), and apply Brouwer’s Continuity Principle.

1.3.7 Let \( X \) be a spread. We let \( \text{Fun}^0_X \) be the set of all \( \gamma \) such that, for every \( \alpha \) in \( X \), there exists \( n \) such that \( \gamma(\alpha n) \neq 0 \). For every \( \gamma \) in \( \text{Fun}^0_X \), every \( \alpha \) in \( X \), we let \( \gamma(\alpha) \) be the natural number \( p \) such that there exists \( n \) such that \( \gamma(\alpha n) = p + 1 \), and for every \( m \lt n \), \( \gamma(\alpha m) = 0 \).

We let \( \text{Fun}^1_X \) be the set of all \( \gamma \) such that \( \gamma(0) = 0 \) and, for each \( n \), \( \gamma^n \) belongs to \( \text{Fun}^0_X \). For every \( \gamma \) in \( \text{Fun}^1_X \), for every \( \alpha \) in \( X \), we define the element \( \gamma|\alpha \) of \( \mathcal{N} \) as follows: for each \( n \), \( (\gamma|\alpha)(n) := \gamma^n(\alpha) \).

If \( \gamma \) belongs to \( \text{Fun}^1_X \), we say that \( \gamma \) is a function from \( X \) to \( \mathbb{N} \).

If \( \gamma \) belongs to \( \text{Fun}^0_X \) we say that \( \gamma \) is a function from \( X \) to \( X \).

In particular, an element of \( \text{Fun} \) will be called a function from \( \mathcal{N} \) to \( \mathbb{N} \), and an element \( \gamma \) of \( \text{Fun} \) such that \( \gamma(0) = 0 \) will be called a function from \( \mathcal{N} \) to \( \mathcal{N} \).

Suppose that \( Z \) is a subset of \( \mathcal{N} \) and \( \gamma \) is a function from \( X \) to \( \mathcal{N} \) such that for every \( \alpha \) in \( X \), \( \gamma|\alpha \) belongs to \( Z \). We then say that \( \gamma \) is a function from \( X \) to \( Z \).

1.4 The perception underlying the Continuity Principle may be given a more incisive formulation.

1.4.1 **First Axiom of Continuous Choice.**

For every binary relation \( R \subseteq \mathcal{N} \times \mathbb{N} \), if for every \( \alpha \) there exists \( m \) such that \( \alpha R m \), then there exists \( \gamma \) in \( \text{Fun} \) such that for every \( \alpha \), \( \alpha R(\gamma(\alpha)) \).
We accept this axiom for the following reason.
Suppose we are able to find, for every infinite sequence $\alpha$ a natural number $m$ \textit{suitable for} $\alpha$, that is, such that $\alpha R m$. We allow ourselves to construct the promised $\gamma$ step by step and consider the (code numbers of the) finite sequences of natural numbers one by one.
Each time we imagine the finite sequence as beginning an infinite sequence that is growing step by step, and we ask ourselves if, as such, it suffices for the determination of a natural number that suits this infinite sequence. If it does, we determine such a number, call it $p$, and let the value of $\gamma$ at (the code number of) the finite sequence be $p + 1$, if not, we let that value be 0. We may convince ourselves that for every $\alpha$, whether it is given by an algorithm or is constructed, more or less freely, step by step, there will exist $n$ such that $\gamma(\bar{\alpha} n) \neq 0$, by reasons similar to the ones that made us accept Brouwer’s Continuity Principle.
In this paper, the First Axiom of Continuous Choice is used only in the proof the Finite Borel Hierarchy Theorem, Theorem 7.6.3, but not in the proof of the general Borel Hierarchy Theorem, Theorem 7.7.9.
1.4.2 Second Axiom of Continuous Choice:

For every binary relation \( R \subseteq \mathcal{N} \times \mathcal{N} \),
if for every \( \alpha \) there exists \( \beta \) such that \( \alpha R \beta \),
then there exists \( \gamma \) in \( \text{Fun} \) such that \( \gamma(0) = 0 \) and \( \forall \alpha [\alpha R (\gamma|\alpha)] \).

This axiom implies the two Axioms of Countable Choice and the First Axiom of Continuous Choice. We accept it for the following reason.

Suppose we are able to find, for each infinite sequence \( a \), an infinite sequence \( m \) suitable for \( a \), that is, such that \( \alpha R m \). We construct the promised \( \gamma \) step by step, as follows: we require \( \gamma(0) := 0 \) and now define, inductively, for each \( a \), the numbers \( \gamma^0(a), \gamma^1(a), \ldots \) simultaneously.

Our definition will be such that for each \( n, a \), if \( \gamma^n(a) \neq 0 \), then \( n < a \). When considering the code number \( n \) of a finite sequence \( a \) of natural numbers we look for the largest \( n \leq a \) with the property that for every \( m < n \) there exists an initial part \( b \) of \( a \) such that \( \gamma^m(b) \neq 0 \). We call this number \( n_0 \). We imagine the finite sequence coded by \( a \) as beginning an infinite sequence \( \alpha \) that we are constructing step by step. We already managed to determine the first \( n_0 \) values of a sequence \( \beta \) suitable for \( \alpha \) and, as we are able to continue the project we started earlier, now ask ourselves if \( \alpha \) suffices to determine a next value. If so, we calculate this next value, call it \( p \) and define \( \gamma^{n_0}(a) := p + 1 \), if not, we define \( \gamma^{n_0}(a) := 0 \). For each \( m \neq n_0 \), we define \( \gamma^m(a) := 0 \).

The argument that this procedure guarantees: \( \gamma(0) = 0 \), \( \gamma \) belongs to \( \text{Fun} \) and \( \forall \alpha [\alpha R (\gamma|\alpha)] \) is similar to the argument given for the First Axiom of Continuous Choice and we do not spell it out.

The Second Axiom of Continuous Choice is used in order to prove the collapse of the projective hierarchy, see Theorem 9.16.

1.4.3 Theorem: (Extension of the Axioms of Continuous Choice to spreads).

Let \( X \) be a spread.

(i) For every binary relation \( R \subseteq X \times \mathbb{N} \),
if for every \( \alpha \) in \( X \) there exists \( m \) such that \( \alpha R m \), then there exists a function \( \gamma \) from \( X \) to \( \mathbb{N} \) such that for every \( \alpha \) in \( X \), \( \alpha R (\gamma|\alpha) \).

(ii) For every binary relation \( R \subseteq X \times \mathcal{N} \),
if for every \( \alpha \) in \( X \) there exists \( \beta \) such that \( \alpha R \beta \), then there exists a function from \( X \) to \( \mathcal{N} \) such that for every \( \alpha \) in \( X \), \( \alpha R (\gamma|\alpha) \).

Proof: The proof is similar to the proof of Theorem 1.3.6. □

1.5 We need something like countable ordinals and introduce stumps. We have taken the word “stump” from [4] but are giving it a slightly different meaning. For each \( n \), let \( n \) be the element of \( \mathcal{N} \) with the constant value \( n \).
1.5.1 The set $\text{Stp}$ of stumps is a subset of Baire space $\mathcal{N}$ and is defined as follows.

(i) $\bot$ is a stump. We sometimes call $\bot$ the empty stump.
(ii) For all $\beta$ in $\mathcal{N}$, if, for each $n$, $\beta^n$ is a stump, and $\beta(0) = 0$, then $\beta$ itself is a stump. We call the stumps $\beta^0, \beta^1, \ldots$ the immediate substumps of the stump $\beta$.
(iii) Clauses (i) and (ii) produce all stumps.

N. Lusin doubted the legitimacy of introducing a set in this way by an inductive definition and Brouwer occasionally expressed similar feelings. We accept the above definition without hesitation, and, as a consequence of (iii), recognize the possibility of giving proofs and constructing functions by transfinite induction on $\text{Stp}$.

For every stump $\beta$, we define the successor of $\beta$, notation: $\beta^+$ or $S(\beta)$, by: $(S(\beta))(0) = 0$ and for every $n$, $(S(\beta))^n = \beta$.

We define a sequence $0^*, 1^*, \ldots$ of stumps by induction, as follows. $0^* := \bot$ and for each $n$, $(n + 1)^* := S(n^*)$. Thus we obtain a natural embedding of the set $\mathbb{N}$ into the set $\text{Stp}$.

1.5.2 First Principle of Induction on the set $\text{Stp}$ of stumps:

For every subset $P$ of $\text{Stp}$, if the empty stump $\bot$ belongs to $P$, and every non-empty stump $\beta$ belongs to $P$ as soon as each one of its immediate substumps $\beta^0, \beta^1, \ldots$ belongs to $P$, then $P$ coincides with $\text{Stp}$.

1.5.3 For every $\beta$, for every $n$, we say that $n$ belongs to $\beta$ if and only if $\beta(n) = 0$. (We are interpreting $\beta$ as the (inverse) characteristic function of a subset of $\mathbb{N}$).

Let $\beta$ be a stump. The set of all finite sequences of natural numbers whose code number belongs to $\beta$ is more like a “stump” in the sense given to this word by Brouwer. We mention three important properties of this set.

(i) We may decide, for every finite sequence of natural numbers, if its code number belongs to $\beta$ or not.
(ii) Every initial part of a number belonging to $\beta$ belongs to $\beta$.
(iii) For every $\gamma$ in $\mathcal{N}$ we may calculate $n$ such that $\gamma n$ does not belong to $\beta$.

Observe that there is no finite sequence whose code number belongs to $\bot$. Observe that we may decide, for every stump $\beta$, if $\beta = \bot$ or not.

1.5.4 Principle of Stump Induction:

For every non-empty stump $\beta$, for every subset $P$ of $\mathbb{N}$, if, for every $m$, $m$ belongs to $P$ as soon as every immediate successor $m * (n)$ of $m$ belonging to $\beta$ belongs to $P$, then $0$ belongs to $P$. 13
We leave it to the reader to prove this principle by induction on the set $\text{Stp}$.

1.5.5 From now on we use $\sigma, \tau, \ldots$ as variables on the set $\text{Stp}$. We define binary relations $<, \leq$ on the set $\text{Stp}$ of stumps as follows: for every stump $\sigma$, $1 \leq \sigma$ and for no stump $\sigma$, $\sigma < 1$. Furthermore, for all stumps $\sigma, \tau$ such that $\tau \neq 1$, $\tau \leq \sigma$ if and only if, for each $n$, $\tau^n < \sigma$, and $\sigma < \tau$ if and only if, for some $n$, $\sigma \leq \tau^n$.

One may prove that the relations $<, \leq$ are transitive and that for all stumps $\sigma, \tau$, if $\sigma < \tau$, then $\sigma \leq \tau$. Another useful fact is that for all stumps $\sigma, \tau, \rho$, if $\sigma \leq \tau$ and $\tau < \rho$, then $\sigma < \rho$.

A subset $P$ of $\text{Stp}$ is called hereditary if and only if for every stump $\sigma$, $\sigma$ belongs to $P$ if every $t < \sigma$ belongs to $P$.

1.5.6 Second Principle of Induction on the set $\text{Stp}$ of stumps

Every hereditary subset of $\text{Stp}$ coincides with $\text{Stp}$.

The proof is straightforward. Observe that this principle does not imply that every inhabited set $P$ of stumps contains an element $\sigma$ that, for all $\tau$ in $P$, $\sigma \leq \tau$. It is not even true that every inhabited subset of $\{0, 1\}$ has a least element.

Lemma 6.7 enunciates a Principle of Double Induction on the set $\text{Stp}$ of stumps.

Theorem 6.5 enunciates a Principle of Induction on the $\text{Stp}^*$ of finite sequences of stumps.

1.6 We consider the assumption that underlies the famous Bar Theorem. It will play a role in this paper when we come to discuss the notion Almost in Section 4 and also figures in various results about (strictly) analytic and co-analytic sets in Section 9.

1.6.1 A subset $P$ of $\mathbb{N}$ will be called a bar if and only if for each $\alpha$ there exists $n$ such that $\overline{an}$ belongs to $P$.

A subset $P$ of $\mathbb{N}$ is called monotone if and only if for every $m$, if $m$ belongs to $P$, then every immediate successor $m + \langle n \rangle$ of $m$ belongs to $P$.

1.6.2 Brouwer’s Thesis:

For every subset $P$ of $\mathbb{N}$,

if $P$ is a bar then there exists a stump $\beta$ such that the set of all elements of $P$ belonging to $\beta$ is a bar.

Brouwer thought that his Thesis could be seen to be true by reflection on the possible structure of a (canonical) proof of the fact “for every $\alpha$ there exists $n$ such that $P(\overline{an})$”. We refrain from discussing his argument.

We warn the reader that our formulation of Brouwer’s Thesis does not literally occur in Brouwer’s writings.
1.6.3 Principle of Induction on Monotone Bars:

Let $P$ be a monotone bar. Let $Q$ be a subset of $\mathbb{N}$ such that $P \subseteq Q$ and
for every $m$, if every immediate successor $m \ast \langle n \rangle$ of $m$ belongs to $P$, then
$m$ belongs to $Q$. Then $0$ belongs to $Q$.

One easily proves this from Brouwer’s Thesis and the Principle of Induction on
Stumps.

1.6.4 Principle of Double Induction on Monotone Bars:

Let $P$ be a subset of $\mathbb{N}$ such that for every $\alpha, \beta$ there exists $n$ such that
$\langle \alpha n, \beta n \rangle$ belongs to $P$, and for every $s, t$, if $\langle s, t \rangle$ belongs to $P$ then, for
every $m$, $\langle s \ast \langle m \rangle, t \rangle$ and $\langle s, t \ast \langle m \rangle \rangle$ belong to $P$. Let $Q$ be a subset of
$\mathbb{N}$ such that $P \subseteq Q$ and for every $s, t$, if for every $m$ both $\langle s \ast \langle m \rangle, t \rangle$ and
$\langle s, t \ast \langle m \rangle \rangle$ belong to $Q$, then $\langle s, t \rangle$ belongs to $Q$. Then $\langle 0, 0 \rangle$ belongs to
$Q$.

The proof is left to the reader.

1.6.5 A famous consequence of Brouwer’s Thesis is the Fan Theorem.

A fan or finitary spread is a subset $F$ of Baire space $\mathcal{N}$ such that there exists $\beta$ with
the following two properties:
(i) for every $\alpha$, $\alpha$ belongs to $F$ if and only if, for each $n$, $\beta(\alpha n) = 0$, and
(ii) for each $n$ such that $\beta(n) = 0$ there exists $m$ such that for all $k$, if $\beta(n \ast \langle k \rangle) = 0$,
then $k < m$.

Let $X$ be a subset of $\mathcal{N}$ and let $P$ be a subset of $\mathbb{N}$. $P$ is called a bar in $X$ if for every
$\alpha$ in $X$ there exists $n$ such that $\alpha n$ belongs to $P$.

1.6.6 Fan Theorem:

Let $F \subseteq \mathcal{N}$ be a fan.

For every subset $P$ of $\mathbb{N}$,

if $P$ is a bar in $F$, then some finite subset of $P$ is a bar in $F$.

Brouwer used the Fan Theorem for proving that every real function defined on $[0, 1]$ is uniformly continuous on $[0, 1]$, see [3].

1.7 We show how one may introduce real numbers into intuitionistic analysis.

1.7.1 Let $\rho$ be an enumeration of the set $\mathbb{Q}$ of rational numbers.

Let $\alpha$ belong to $\mathcal{N}$.

$\alpha$ is called a real number if and only if, for each $n$, $\rho(\alpha(2n)) < \rho(\alpha(2n + 2)) <$
$\rho(\alpha(2n + 3)) < \rho(\alpha(2n + 1))$, and, for every $q, r$ in $\mathbb{Q}$, if $q < r$, then there exists $n$
such that either $\rho(\alpha(2n + 1)) < r$ or $q < \alpha(2n)$. 

15
α is called a canonical real number if, in addition, for each n, if \( \rho(K(n+1)) < \rho(L(n+1)) \), then either \( \rho(\alpha(2n+1)) < \rho(L(n+1)) \) or \( \rho(K(n+1)) < \rho(\alpha(2n)) \).

We denote the set of real numbers by \( \mathbb{R} \) and the set of canonical real numbers by \( \mathbb{C}_{rn} \).

1.7.2 Let \( \alpha, \beta \) be real numbers. \( \alpha \) is really-smaller than \( \beta \), notation \( \alpha <^* \beta \), if and only if there exists \( n \) such that \( \rho(\alpha(2n+1)) < \rho(\beta(2n)) \). \( \alpha \) is really-not-greater than \( \beta \), notation \( \alpha \leq^* \beta \) if and only if, for each \( n \), \( \rho(\alpha(2n)) < \rho(\beta(2n+1)) \). \( \alpha \) is really-apart from \( \beta \), notation \( \alpha \#^* \beta \), if and only if either \( \alpha <^* \beta \) or \( \alpha <^* \beta \).

\( \alpha \) really-coincides with \( \beta \), notation \( \alpha =^* \beta \) if and only if the assumption \( \alpha \#^* \beta \) leads to a contradiction.

We let the open real interval \( (\alpha, \beta) \) be the set of all real numbers \( \gamma \) such that \( \alpha <^* \gamma <^* \beta \). We let the closed real interval \( [\alpha, \beta] \) be the set of all real numbers \( \gamma \) such that \( \alpha \leq^* \gamma \leq^* \beta \).

Let \( A, B \) be subsets of \( \mathbb{R} \).

We say that \( A \) really coincides with \( B \) if and only if every element of \( A \) really coincides with an element of \( B \) and conversely, every element of \( B \) really coincides with an element of \( A \).

1.7.3 Observe that \( \mathbb{R} \) really-coincides with \( \mathbb{C}_{rn} \) and that \( \mathbb{C}_{rn} \), viewed as a subset of \( \mathcal{N} \), is a spread. This fact lies at the basis of the famous result that every real function is continuous, see [3] and [22].

2 The second step, and our first one

We introduce sets from the second level of the Borel Hierarchy and prove the corresponding case of the Hierarchy Theorem.

2.1 Let \( X \) be a subset of Baire space \( \mathcal{N} \).

\( X \) is basic open if and only if either \( X \) is empty or there exists \( s \) in \( \mathbb{N} \) such that \( X \) consists of all \( \alpha \) passing through \( s \).

\( X \) is open or \( \Sigma^0_1 \) if and only if there exists a sequence \( X_0, X_1, \ldots \) of basic open sets such that \( X = \bigcup_{n \in \mathbb{N}} X_n \).

\( X \) is closed or \( \Pi^0_1 \) if and only if there exists an open set \( Y \) such that \( X \) consists of all \( \alpha \) that do not belong to \( Y \).

Every closed subset of \( \mathcal{N} \) is weakly closed in the sense of Section 1.3.3, but not conversely. In general, a closed subset of \( \mathcal{N} \) is not a spread in the sense of Section 1.3.3, but every spread is a closed subset of \( \mathcal{N} \).

\( X \) is \( \Sigma^0_2 \) if and only if there exists a sequence \( X_0, X_1, \ldots \) of closed sets such that \( X = \bigcup_{n \in \mathbb{N}} X_n \).

16
$X$ is $\Pi_2^0$ if and only if there exists a sequence $X_0, X_1, \ldots$ of open sets such that $X = \bigcap_{n \in \mathbb{N}} X_n$.

2.2 Let $X$ be a subset of the set $\mathbb{R}$ of real numbers. $X$ is (really) basic open if and only if either $X$ is empty or there exists rational numbers $p, q$ such that $X$ is the open interval $(p, q)$, that is, $X$ consists of all $\alpha$ in $\mathbb{R}$ for which there exists $n$ such that $p < \rho(\alpha(2n)) < \rho(\alpha(2n + 1)) < q$.

The notions of a (really) $\Sigma_1^0$, (really) $\Pi_1^0$, (really) $\Sigma_2^0$, (really) $\Pi_2^0$ subset of $\mathbb{R}$ are defined as the corresponding notions for subsets of $\mathcal{N}$ in Section 2.1.

2.3 We let $\text{Rat}$ be the set of all real numbers $\alpha$ for which there exists a rational number $q$ such that, for every $n$, $\rho(\alpha(2n)) < q < \rho(\alpha(2n + 1))$.

We let $\text{PosIrr}$ be the set of real numbers $\alpha$ such that for every rational number $q$ there exists $n$ such that either $q < \rho(\alpha(2n))$ or $\rho(\alpha(2n + 1)) < q$.

$\text{Rat}$ is the set of all real numbers coinciding with a rational, and $\text{PosIrr}$ is the set of all positively irrational numbers.

Observe that every element of $\text{Rat}$ is really-apart from every element of $\text{PosIrr}$.

2.4 Theorem: For every sequence $X_0, X_1, \ldots$ of open subsets of $\mathbb{R}$, if $\text{Rat}$ is a subset of $\bigcap_{n \in \mathbb{N}} X_n$, then some element of $\text{PosIrr}$ belongs to $\bigcap_{n \in \mathbb{N}} X_n$.

Proof: Suppose that $\text{Rat}$ is a subset of $\bigcap_{n \in \mathbb{N}} X_n$.

We construct a real number $\alpha$ such that, for each $n$, the open interval $(\rho(\alpha(2n)), \rho(\alpha(2n + 1)))$ is a subset of $X_n$, and either $\rho(n) < \rho(\alpha(2n))$ or $\rho(\alpha(2n + 1)) < \rho(n)$. $\triangleright$

2.5 Theorem: For every sequence $X_0, X_1, \ldots$ of closed subsets of $\mathbb{R}$, if $\text{PosIrr}$ is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, then some element of $\text{Rat}$ belongs to $\bigcup_{n \in \mathbb{N}} X_n$.

Proof: We let $\text{Pir}$ be the set of all real numbers $\alpha$ such that for each $n$, either $\rho(n) < \rho(\alpha(2n))$ or $\rho(\alpha(2n + 1)) < \rho(n)$. Observe that $\text{Pir}$ is a subset of $\text{PosIrr}$ and that $\text{Pir}$, viewed as a subset of $\mathcal{N}$, is a spread. In addition, $\text{PosIrr}$ really-coincides with $\text{Pir}$. Suppose that $\text{PosIrr}$ is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, and let $\alpha_0$ be some element of $\text{Pir}$. Applying the Continuity Principle, we find $m, n$ such that for all $\alpha$ in $\text{Pir}$, if $\alpha(2m) = \alpha_0(2m)$ then $\alpha$ belongs to $X_n$. We conclude that every positively irrational number in the open interval $\rho(\alpha(2m - 2)), \rho(\alpha(2m - 1))$ belongs to the closed set $X_n$, therefore also every rational number in this interval belongs to $X_n$. $\triangleright$

2.6 Corollary: $\text{Rat}$ is $\Sigma_2^0$ but not $\Pi_2^0$, and $\text{PosIrr}$ is $\Pi_2^0$ but not $\Sigma_2^0$.

Proof: $\text{Rat}$ is a countable union of singletons. Every singleton is the set of all elements of $\mathbb{R}$ really-coinciding with a given real number and is a closed subset of $\mathbb{R}$.
Therefore $\text{Rat}$ is $\Sigma^0_2$, whereas, according to Theorem 2.4, $\text{Rat}$ is not $\Pi^0_2$.

$\text{PosIrr}$ is the intersection of countably many sets of the form $
\{a | a \in \mathbb{R} | \text{ for some } n, \text{ either } q < a(2n) \text{ or } a(2n + 1) < q\}$, where $q$ is a given rational number, and every such set is an open subset of $\mathbb{R}$. Therefore $\text{PosIrr}$ is $\Pi^0_2$, whereas, according to Theorem 2.5, $\text{PosIrr}$ is not $\Sigma^0_2$.

2.7 $\text{Rat}$ is a not too difficult example of a set that belongs to the class $\Sigma^0_2$ but not to the class $\Pi^0_2$.

Brouwer describes a more complicated example in [6]. He considers a set $\bigcup_{n \in \mathbb{N}} K_n$, where, for each $n$, $K_n$ is the set of all real numbers $\alpha$ for which there exists a sequence $\beta(0), \beta(1), \ldots$ such that for each $i$, $\beta(i)$ belongs to $\{0, 1, 2\}$, and for each $m$, the set $\{i | i < m \land \beta(i) = 1\}$ has at most $n$ members, and $\alpha$ really-coincides with $\sum_{i \in \mathbb{N}} \beta(i) \cdot 3^{-i-1}$, (so $\alpha$ belongs to the closed interval $[0, 1]$ and has a ternary expansion in which the number 1 occurs at most $n$ times). Brouwer assumes that every set $K_n$ is a closed subset of $\mathbb{R}$, but this is only true if $n = 0$. This oversight was observed and pointed out to me by the referee of an earlier version of this paper. We may correct it by replacing every set $K_n$ by its closure $\overline{K}_n$, that is the set of all real numbers $\alpha$ that may be obtained as the limit of a convergent sequence $x_0, x_1, \ldots$ where each $x_i$ is a rational number of the form $\sum_{i=0}^{m} b(i) \cdot 3^{-i-1}$ and each $b(i)$ belongs to $\{0, 1, 2\}$ and the set $\{i | i < m \land b(i) = 1\}$ has at most $n$ members. In general, an element of $\overline{K}_n$ does not have a ternary expansion and does not belong to $K_n$.

In his proof that the set $\bigcup_{n \in \mathbb{N}} \overline{K}_n$ is not $\Pi^0_2$, Brouwer unnecessarily applies the Fan Theorem.

Brouwer proposes the set $K_\infty$ as an example of a set that is $\Pi^0_2$ and not $\Sigma^0_2$. $K_\infty$ is the set of all numbers $\alpha$ in the closed interval $[0, 1]$ with a ternary expansion in which the number 1 occurs infinitely many times.

In proving this example correct, Brouwer uses the Continuity Principle, like we did in the proof of Theorem 2.5. Observe that $K_\infty$, like $\mathbb{R}$ itself, really-coincides with a spread.

2.8 We return to Baire space $\mathcal{N}$.

For all $\alpha, \beta$ in $\mathcal{N}$, we define: $\alpha$ is apart from $\beta$, or: $\alpha$ lies apart from $\beta$, notation: $\alpha \# \beta$, if and only if there exists $n$ such that $\alpha(n) \neq \beta(n)$. This constructive inequality relation is co-transitive, that is, for all $\alpha, \beta, \gamma$, if $\alpha \# \beta$, then either $\alpha \# \gamma$ or $\gamma \# \beta$.

Let $\gamma$ belong to $\mathcal{N}$. Recall from Section 1.3.7 that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ if and only if $\gamma(0) = 0$.

Let $X, Y$ be subsets of $\mathcal{N}$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$.

$\gamma$ maps $X$ into $Y$ if and only if for every $\alpha$ in $X$, $\gamma(\alpha)$ belongs to $Y$.

$\gamma$ reduces $X$ to $Y$ if and only if for every $\alpha$ in $X$ and only $\alpha$ in $Y$, that is for every $\alpha$, $\alpha$
belongs to $X$ if and only if $y|\alpha$ belongs to $Y$. 

$X$ is reducible to $Y$, or $X$ reduces to $Y$, notation $X \preceq Y$, if and only if some $\gamma$ in $\text{Fun}$ reduces $X$ to $Y$. 

If $\gamma$ reduces $X$ to $Y$, then $\gamma$ can be considered as an effective method to translate every question: “does $\alpha$ belong to $X$?” into a question: “does $\beta$ belong to $Y$?” This notion is called “Wadge-reducibility” in classical descriptive set theory. Its analogue in recursion theory is called “many-one-reducibility” or “$m$-reducibility”. We use the unadorned expression “reducible” as no other notion of reducibility figures in this paper.

2.9 We define subsets $A_2$ and $E_2$ of $\mathcal{N}$ as follows. 

$A_2$ is the set of all $\alpha$ such that for every $m$ there exists $n$ such that $\alpha^m(n) \neq 0$. 

$E_2$ is the set of all $\alpha$ such that for some $m$, for every $n$, $\alpha^m(n) = 0$. Observe that every element of $A_2$ is apart from every element of $E_2$.

2.10 Theorem: Let $X$ be a subset of $\mathcal{N}$. 

(i) $X$ belongs to $\Pi^0_3$ if and only if $X$ reduces to $A_2$. 

(ii) $X$ belongs to $\Sigma^0_2$ if and only if $X$ reduces to $E_2$. 

Proof: We leave the straightforward proof to the reader.

2.11 Theorem: Every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $E_2$ into $A_2$ also maps some element of $A_2$ into $A_2$.

Proof: Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ that maps $E_2$ into $A_2$. We now construct $\alpha$ such that both $\alpha$ and $\gamma|\alpha$ belong to $A_2$. First define $\alpha_0 := 0$. Then $\alpha_0$ belongs to $E_2$, therefore $\gamma|\alpha_0$ belongs to $A_2$. Calculate $n_0$ such that $(\gamma|\alpha_0)^0(n_0) \neq 0$. Also calculate $m_0$ such that for every $\beta$, if $\beta m_0 = \alpha_0 m_0$, then $(\gamma|\beta)^0(n_0) = (\gamma|\alpha_0)^0(n_0)$. Now define $\alpha_1$ such that $\alpha_1((0, m_0)) = 1$ and $\alpha_1 m_0 = \alpha_0 m_0$ and $(\alpha_1)^1 = 0$. Then $\alpha_1$ belongs to $E_2$, therefore $\gamma|\alpha_1$ belongs to $A_2$. Calculate $n_1$ such that $(\gamma|\alpha_1)^1(n_1) \neq 0$. Also calculate $m_1$ such that $m_1 > (0, m_0)$ and for every $\beta$, if $\beta m = \alpha_1 m_1$, then $(\gamma|\beta)^1(n_1) = (\gamma|\alpha_1)^1(n_1)$. 

Now define $\alpha_2$ such that $\alpha_2 m_1 = \alpha_1 m_1$ and $\alpha_2((1, m_1)) = 1$ and $(\alpha_2)^2 = 0$. Then $\alpha_2$ belongs to $E_2$, therefore $\gamma|\alpha_2$ belongs to $A_2$. Continuing in this way, we find two sequences $n_0, n_1, \ldots$ and $m_0, m_1, \ldots$ of natural numbers and a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$ such that $m_0 < m_1 < \ldots$ and for each $k$, $\alpha_{k+1} m_k = \alpha_k m_k$ and for each $k$, for each $\beta$, if $\beta m_k = \alpha_k m_k$, then $(\gamma|\beta)^k(n_k) \neq 0$ and if $\beta m_{k+1} = \alpha_{k+1} m_{k+1}$, then $\beta^k(m_k) = 1$. Consider the sequence $\alpha$ such that for each $k$, $\alpha m_k = \alpha_k m_k$ and observe: both $\alpha$ and $\gamma|\alpha$ belong to $A_2$. 

2.12 Theorem: There exists a function $f$ from $\mathcal{N}$ to $\mathcal{N}$ with the following properties:
(i) For every α, \( f|α \) belongs to \( A_2 \).
(ii) For every \( β \) in \( A_2 \) there exists \( α \) such that \( f|α = β \).
(iii) For every \( n, α \), there exists \( m \) such that for every \( β \), if \( β_m = \overline{(f|α)m} \) and \( β \) belongs to \( A_2 \), then there exists \( γ \) such that \( γn = \overline{αn} \) and \( γ = f|γ \).

**Proof:** Observe that, for every \( β \), \( β \) belongs to \( A_2 \) if and only if for each \( m \) there exists \( n \) such that \( β^m(n) \neq 0 \) if and only if there exists \( δ \) such that for each \( m \), \( β^m(δ(m)) \neq 0 \).

We now define a function \( f \) from \( N \) to \( N \) such that for every \( α \), \( (f|α)(0) = α^1(0) \) and for every \( m \), \( (f|α)^m(α^0(m)) = \text{Max}(1, α^{1\cdot m}(α^0(m))) \) and for every \( m, k \), if \( k \neq α^0(m) \), then \( (f|α)^m(k) = α^{1\cdot m}(k) \). One verifies easily that \( f \) has the promised properties. \( \square \)

2.13 Theorem: Every function from \( N \) to \( N \) that maps \( A_2 \) into \( E_2 \), also maps some element of \( E_2 \) into \( E_2 \).

**Proof:** Let \( γ \) be a function from \( N \) to \( N \) that maps \( A_2 \) into \( E_2 \). Let \( f \) be a function from \( N \) to \( N \) with the properties mentioned in Theorem 2.12. Observe that for each \( α \) we may calculate \( n \) such that \( (γ|(f|α))^n = 0 \). Applying the Continuity Principle we find \( m, n \) such that for every \( α \), if \( αm = 0m \) then \( (γ|(f|α))^n = 0 \). We calculate \( p \) such that for every \( β \), if \( βp = \overline{(f|0)p} \) and \( β \) belongs to \( A_2 \), then there exists \( α \) passing through \( 0m \) such that \( β = f|α \). We now define \( δ \) in such a way that \( δp = \overline{(f|0)p} \) and \( δ^p = 0 \). Observe that also \( (γ|δ)^n = 0 \), as for each \( q \) there exists \( α \) such that \( αq = δq \) and \( (f|α)^n = 0 \). Therefore both \( δ \) and \( γ|δ \) belong to \( E_2 \). \( \square \)

2.14 Corollary: The set \( E_2 \) belongs to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \). The set \( A_2 \) belongs to \( \Pi^0_2 \) but not to \( \Sigma^0_2 \).

**Proof:** Obvious. \( \square \)

3 Rediscovering, perhaps, the Cantor-Bendixson hierarchy

We study a special collection of enumerable sets and show how, in various ways, they form a hierarchy. We also introduce a binary operation, called Perhaps, on the class of subsets of \( N \).

3.1 For every \( s \in N \), every \( α \), we let \( s*α \) be the element of \( N \) that we obtain by putting the infinite sequence \( α \) behind the finite sequence (coded by) \( s \).

For every \( s \in N \), every subset \( X \) of \( N \) we let \( s*X \) be the set of all infinite sequences \( s*α \), where \( α \) belongs to \( X \). We now define, for every stump \( σ \), a subset \( CB_σ \) of \( N \) by means of the following inductive definition:
(i) \(CB_1 := \emptyset\)
(ii) For every non-empty stump \(\sigma\), \(CB_\sigma := \{0\} \cup \bigcup_{n \in \mathbb{N}} \bar{0}n \ast (1) \ast CB_\sigma^n\).

The letters \(CB\) have been chosen in honour of Cantor and Bendixson.

3.2 Let \(X\) be a subset of \(\mathcal{N}\). Recall that the closure of \(X\), notation \(\overline{X}\), is the set of all \(\alpha\) in \(\mathcal{N}\) such that for every \(n\) there exists an element of \(X\) passing through \(\bar{\alpha}n\). If we may decide, for every \(s\), if \(s\) contains an element of \(X\) or not, then \(\overline{X}\) is a closed subset of \(\mathcal{N}\) and a spread.

3.3 Theorem:
(i) For every stump \(\sigma\), \(\overline{CB_\sigma}\) is a spread.
(ii) For every non-empty stump \(\sigma\), if \(\overline{CB_\sigma}\) coincides with \(\overline{CB_\sigma}\), then \(CB_\sigma\) is a finite subset of \(\mathbb{N}\).

Proof: We leave the proof of (i) to the reader.
As to (ii), assume that \(\overline{CB_\sigma}\) coincides with \(\overline{CB_\sigma}\). Observe that now, for every \(\alpha\) in \(\overline{CB_\sigma}\), either \(\alpha = 0\), or, for some \(n\), \(\alpha(n) = 1\). \(\overline{CB_\sigma}\) is a spread containing \(0\).
Applying the Continuity Principle we find \(n\) such either every \(\alpha\) in \(\overline{CB_\sigma}\) passing through \(\bar{0}n\) coincides with \(0\), or every \(\alpha\) in \(\overline{CB_\sigma}\) passing through \(\bar{0}n\) is apart from \(0\).
Both alternatives are absurd as both \(0\) and \(\bar{0}n \ast (1) \ast 0\) belong to \(CB_\sigma\).

3.4 Let \(X\) be a subset of \(\mathcal{N}\).
\(X\) will be called enumerable if and only if either \(X\) is empty or there exists an enumeration of \(X\), that is, an element \(\alpha\) of \(\mathcal{N}\) such that \(X\) coincides with the set \(\{\alpha^0, \alpha^1, \ldots\}\).
Every enumerable subset of \(\mathcal{N}\) belongs to the class \(\Sigma^0_0\).
Given any subset \(X\) of \(\mathcal{N}\), we let the complement of \(X\), notation \(X^\sim\), be the set of all \(\alpha\) such that \(\alpha\) does not belong to \(X\), that is, the assumption that \(\alpha\) does belong to \(X\) leads to a contradiction.
Every subset \(X\) of \(\mathcal{N}\) forms part of its double complement \(X^{\sim\sim}\) but the converse is not generally true. On the other hand, for every subset \(X\) of \(\mathcal{N}\), \(X^{\sim\sim}\) coincides with \(X^\sim\).

3.5 Theorem:
(i) For every stump \(\sigma\), \(\overline{CB_\sigma}\) is an enumerable subset of \(\mathcal{N}\).
(ii) For every \(\sigma\), the set \(\overline{CB_\sigma}\) coincides with the set \((\overline{CB_\sigma})^{\sim\sim}\).

Proof: We leave the proof of (i) to the reader.
As to (ii), first remark that for every \(\sigma\), \((\overline{CB_\sigma})^{\sim\sim}\) is a subset of \(\overline{CB_\sigma}\), as for every \(\alpha\) in \((\overline{CB_\sigma})^{\sim\sim}\), for every \(n\), \(\bar{\alpha}n\) contains a member of \(CB_\sigma\).
In order to prove that for every \(\sigma\), the set \(\overline{CB_\sigma}\) forms part of the set \((\overline{CB_\sigma})^{\sim\sim}\) we use induction on the set of stumps.
Observe that the set $CB_1$ coincides with its closure $\overline{CB_1}$ and with its double complement $(CB_1)^\sim$.

Now assume that $\sigma$ is a non-empty stump and that, for each $n$, the set $\overline{CB_\sigma n}$ forms part of the set $(CB_\sigma)^\sim$. Let $\alpha$ belong to $\overline{CB_\sigma}$. We distinguish two cases. First case: $\alpha = \emptyset$. Then $\alpha$ belongs to $CB_\sigma$. Second case: $\alpha \neq \emptyset$. Calculate $n, \beta$ such that $\alpha = \overline{\emptyset m} * \{1\} * \beta$. Observe that $\beta$ belongs to $\overline{CB_\sigma n}$ and therefore also to $(CB_\sigma)^\sim$. Therefore $\alpha$ belongs to $(CB_\sigma)^\sim$.

We thus see that if either $\alpha = \emptyset$ or $\alpha \neq \emptyset$, then $\alpha$ belongs to $(CB_\sigma)^\sim$. We recall that $\neg(\alpha = \emptyset \vee \alpha \neq \emptyset)$ and conclude that $\alpha$ belongs to $(CB_\sigma)^\sim = (CB_\sigma)^\sim$. Therefore $CB_\sigma$ forms part of $(CB_\sigma)^\sim$.

3.6 Observe that from a classical point of view, the sets $CB_\sigma$ are closed subsets of $\mathbb{N}$.

We have just seen that, except for $CB_1$, the sets $CB_\sigma$ are not closed. Being countable, they do belong to the class $\Sigma^0_2$.

Let $X$ be a subset of $\mathbb{N}$ and $P$ a subset of $\mathbb{N}$. Recall that $P$ is called a bar in $X$ if and only if every element of $X$ has an initial part in $P$.

3.7 Theorem:

(i) For every stump $\sigma$, for every subset $P$ of $\mathbb{N}$, if $P$ is a bar in the countable set $CB_\sigma$, then some finite subset of $P$ is a bar in $CB_\sigma$.

(ii) For every stump $\sigma$, for every open subset $Y$ of $\mathbb{N}$, if the set $CB_\sigma$ forms part of $Y$, then its closure $\overline{CB_\sigma}$ forms part of $Y$.

(iii) For every non-empty stump $\sigma$, if the set $CB_\sigma$ belongs to the class $\Pi^0_2$, then $CB_\sigma$ is a finite subset of $\mathbb{N}$.

Proof: (i) We use induction on the set of stumps. The empty set is a bar in $CB_1$. Assume that $\sigma$ is a non-empty stump and the statement holds true for every set $CB_{\sigma n}$. Also assume that $P$ is a bar in $CB_\sigma$. Calculate $m$ such that $\overline{\emptyset m}$ belongs to $P$. Using the induction hypothesis, find finite subsets $P_0, \ldots, P_{m-1}$ of $P$ such that for each $j < m$, $P_j$ is a bar in the set of all $\alpha$ in $CB_\sigma$ passing through $\overline{\emptyset j} * \{1\}$. The set $Q := \{\overline{\emptyset m}\} \cup \bigcup_{j < m} P_j$ will satisfy our purposes.

(ii) Let $Y$ be an open subset of $\mathbb{N}$. Let $P$ be the set of all $s$ in $\mathbb{N}$ such that every $\alpha$ passing through $s$ belongs to $Y$. Suppose that $\sigma$ is a stump and $CB_\sigma$ forms part of $Y$. Then $P$ is a bar in $CB_\sigma$. Determine a finite subset $Q$ of $P$ such that $Q$ is a bar in $CB_\sigma$ and observe that $Q$ is also a bar in $\overline{CB_\sigma}$. Therefore $\overline{CB_\sigma}$ forms part of $Y$.

(iii) Assume that $\sigma$ is a non-empty stump and $CB_\sigma$ belongs to $\Pi^0_2$. Determine a sequence $G_0, G_1, \ldots$ of open subsets of $\mathbb{N}$ such that $CB_\sigma = \bigcap_{n \in \mathbb{N}} G_n$. According to (ii), $\overline{CB_\sigma} \subseteq \bigcap_{n \in \mathbb{N}} G_n$, therefore $CB_\sigma = \overline{CB_\sigma}$ and, by Theorem 3.3, $CB_\sigma$ is a finite subset of $\mathbb{N}$.

\[ \square \]
3.8 Let $X, Y$ be spreads and let $\gamma$ be a function from $X$ to $Y$.

$\gamma$ embeds $X$ into $Y$, or: $\gamma$ is an embedding of $X$ into $Y$ if and only if for all $\alpha, \beta$ in $X$, if $\alpha$ is apart from $\beta$, then $\gamma|\alpha$ is apart from $\gamma|\beta$. $X$ embeds into $Y$ if and only if some $\gamma$ embeds $X$ into $Y$.

Let $\beta$ belong to $\mathcal{N}$. $\beta$ is repetitive if and only if for each $m$ there exists $n$ such that $n > m$ and $\beta^n = \beta^m$.

We introduce the set $\text{Hrs}$ of hereditarily repetitive stumps.

$\text{Hrs}$ is a subset of $\text{Stp}$ and is given by the following inductive definition:

(i) $1$ is a hereditarily repetitive stump.
(ii) For all $\beta$, if, for each $n$, $\beta^n$ is a hereditarily repetitive stump, and $\beta$ is repetitive, and $\beta(0) = 0$, then $\beta$ itself is a hereditarily repetitive stump.
(iii) Clauses (i) and (ii) produce all hereditarily repetitive stumps.
3.9 Theorem:
(i) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \overline{CB_{\sigma}} \) embeds into \( \overline{CB_{\tau}} \), then \( \sigma \leq \tau \).
(ii) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma < \tau \), then \( CB_\sigma \) does not reduce to \( CB_\tau \).
(iii) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma \leq \tau \) then both \( \overline{CB_{\sigma}} \) embeds into \( \overline{CB_{\tau}} \) and \( CB_\sigma \) reduces to \( CB_\tau \).

Proof: (i) We use induction on the set of hereditarily repetitive stumps. For each hereditarily repetitive stump \( \sigma \) we define the proposition \( P(\sigma) \) as follows:

\[ P(\sigma) := \begin{cases} 
\sigma \text{ has the property } P \\
\text{for every hereditarily repetitive stump } \tau,
\end{cases} \]

if \( \overline{CB_{\sigma}} \) embeds into \( \overline{CB_{\tau}} \), then \( \sigma \leq \tau \).

Observe that \( P(1) \) is obviously true. Now assume that \( \sigma \) is a non-empty hereditarily repetitive stump and for each \( n \), \( P(\sigma^n) \).

Assume that \( \tau \) is a hereditarily repetitive stump and \( \gamma \) embeds \( \overline{CB_{\sigma}} \) into \( \overline{CB_{\tau}} \). Let \( m \) belong to \( \mathbb{N} \). Calculate \( p \) such that \( p > m \) and \( \sigma^p = \sigma^m \).

Observe that either \( \gamma|\overline{\{0^m \ast \{1\} \ast \emptyset\}} \) is apart from \( \emptyset \) or \( \gamma|\overline{\{0^p \ast \{1\} \ast \emptyset\}} \) is apart from \( \emptyset \). Let us assume the latter and calculate \( q \) such that \( \gamma|\overline{\{0^p \ast \{1\} \ast \emptyset\}} \) equals \( \overline{0^q \ast \{1\} \ast \alpha} \), for some \( \alpha \). Calculate \( r \) such that for every \( a \) in \( \overline{CB_{\tau}} \), if \( a \) passes through \( \overline{0^p \ast \{1\} \ast \emptyset} \), then \( \gamma|a \) passes through \( \overline{0^q \ast \{1\}} \). Let \( \delta \) be a function from \( CB_\sigma \) to \( \mathcal{N} \) such that, for every \( \alpha \) in \( CB_\sigma \), \( \gamma \) maps the sequence \( \overline{0^p \ast \{1\} \ast \overline{0^r \ast \alpha}} \) onto \( \overline{0^q \ast \{1\} \ast \delta|\alpha} \). Observe that \( \delta \) embeds \( CB_\sigma \) into \( CB_\tau \), therefore \( \sigma^m = \sigma^\delta \leq \tau^\delta \).

The case that \( \gamma|\overline{\{0^m \ast \{1\} \ast \emptyset\}} \) is apart from \( \emptyset \) is treated similarly.

We conclude \( \forall m \exists q|\sigma^m \leq \tau^q \), therefore \( \sigma \leq \tau \).

Therefore every hereditarily repetitive stump has the property \( P \).

(ii) We again use induction on the set of hereditarily repetitive stumps. For each hereditarily repetitive stump \( \sigma \) we define the proposition \( Q(\sigma) \) as follows:

\[ Q(\sigma) := \begin{cases} 
\sigma \text{ has the property } Q \\
\text{for every hereditarily repetitive stump } \tau,
\end{cases} \]

if \( \sigma < \tau \), then \( CB_\sigma \) does not reduce to \( CB_\tau \).

\( Q(1) \) is obvious. Assume that \( \sigma \) is a non-empty hereditarily repetitive stump and, for each \( n \), \( Q(\sigma^n) \). Also assume that \( \tau \) is a hereditarily repetitive stump and \( \sigma < \tau \) and \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( CB_\tau \) to \( CB_\sigma \). Calculate \( q \) such that \( \sigma^q = \sigma^\gamma \). We claim that \( \gamma \) maps \( \overline{\emptyset q \ast \{1\} \ast \emptyset} \) onto \( \emptyset \). For suppose \( \gamma|\overline{\emptyset q \ast \{1\} \ast \emptyset} \) is apart from \( \emptyset \) and determine \( m \) such that \( \gamma|\overline{\emptyset q \ast \{1\} \ast \emptyset} \) equals \( \overline{0^m \ast \{1\} \ast \alpha} \), for some \( \alpha \). Calculate \( t \) such that for every \( \beta \), if \( \beta \) passes through \( \overline{\emptyset q \ast \{1\} \ast \emptyset} \), then \( \gamma|\beta \) passes through \( \overline{0^m \ast \{1\}} \). Construct a function \( \delta \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha \), \( \gamma|\overline{\emptyset q \ast \{1\} \ast \emptyset} \) equals \( \overline{0^m \ast \{1\} \ast \delta|\alpha} \) and observe that \( \delta \) reduces the set of all \( \alpha \) such that \( \overline{\emptyset q \ast \emptyset} \) belongs to \( CB_\tau \) to \( CB_\sigma \).

The set \( CB_\tau \) is easily seen to reduce to the set of all \( \alpha \) such that
\( t \alpha \) belongs to \( CB_{\tau^4} \) and therefore \( CB_{\tau^4} \) itself reduces to \( CB_{\tau^m} \). We now have a contradiction, as \( \sigma^m < \sigma \leq \tau^4 \), therefore \( \sigma^m < \tau^4 \), and, by the induction hypothesis, \( CB_{\tau^4} \) does not reduce to \( CB_{\tau^m} \).

Now \( \tau \) is hereditarily repetitive, so there exists a strictly increasing sequence \( q = q_0 < q_1 < \cdots \) such that for each \( n \), \( \tau^q = \tau^q \), and therefore \( \gamma \) maps \( \emptyset q_n \ast (1) \ast \emptyset \) onto \( 0 \). Consider the closure \( X \) of the set \( \{ \emptyset q_n \ast (1) \ast \emptyset \mid n \in \mathbb{N} \} \). Observe that \( X \) is a spread containing \( 0 \) and that \( \gamma \) maps every element of \( X \) onto \( \emptyset \), and thus into \( CB_\tau \), therefore, as \( \gamma \) reduces \( CB_\tau \) to \( CB_{\sigma} \), \( X \) is a subset of \( CB_{\tau} \), and every \( \alpha \) in \( X \) either coincides with \( 0 \) or is apart from \( 0 \). Applying the Continuity Principle we find \( m \) such that either every \( \alpha \) in \( X \) passing through \( \emptyset m \) coincides with \( 0 \) or every \( \alpha \) in \( X \) passing through \( \emptyset m \) is apart from \( 0 \), an obvious contradiction.

We conclude that \( CB_{\tau} \) does not reduce to \( CB_{\sigma} \).

Therefore every hereditarily repetitive stump has the property \( Q \).

(iii) The proof is left to the reader.

3.10 Some of the results we obtained thus far contrast starkly with some theorems in classical descriptive set theory: there, by a result of W. Wadge, see [11], p. 169, every set \( X \) that belongs to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \) is \( \Sigma^0_2 \)-complete, that is, every set belonging to \( \Sigma^0_2 \) reduces to \( X \). As a consequence, all sets that belong to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \) are of the same reducibility degree. Here, we find large hierarchies formed by sets from \( \Sigma^0_2 \setminus \Pi^0_2 \).

Let \( X \) be a subset of \( \mathcal{N} \). We let the Cantor-Bendixson-derivative of \( X \), notation \( X' \), be the set of all \( \alpha \) in \( \mathcal{N} \) such that for each \( n \) there exists an element of \( X \) passing through \( \overline{an} \), but apart from \( \alpha \).

Iterating this operation, we define, by induction on the set of stumps, for every subset \( X \) of \( \mathcal{N} \) and every stump \( \alpha \), another subset of \( \mathcal{N} \), called the \( \alpha \)-th Cantor-Bendixson derivative of \( X \), notation \( \text{Der} (\alpha, X) \), as follows:

(i) \( \text{Der} (1, X) := X \)
(ii) For every non-empty stump \( \sigma \), \( \text{Der} (\sigma, X) := \bigcap_{n \in \mathbb{N}} \text{Der} (\sigma^n, X) \)

Observe that for every subset \( X \) of \( \mathcal{N} \) the set \( X' \) is weakly closed.

It follows that for all subsets \( X \) of \( \mathcal{N} \), for all stumps \( \sigma < \tau \), if \( \sigma < \tau \), then \( \text{Der} (\tau, X) \subseteq \text{Der} (\sigma, X) \).

3.11 Lemma: Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathcal{N} \).

Consider \( Y := \{0\} \cup \bigcup_{n \in \mathbb{N}} \emptyset m \ast (1) \ast X_n \).

(i) \( 0 \) belongs to \( Y' \) if and only if there are infinitely many \( n \) such that \( X_n \) is inhabited.
(ii) If \( 0 \) belongs to \( Y' \), then \( Y' \) coincides with the closure of the set \( \{0\} \cup \bigcup_{n \in \mathbb{N}} \emptyset m \ast (1) \ast (X_n)' \).
(iii) For each stump \( \sigma \), if \( \emptyset \) belongs to \( \text{Der}(\sigma, Y) \), then \( \text{Der}(\sigma, Y) \) coincides with the closure of the set \( \{\emptyset\} \cup \bigcup_{n \in \mathbb{N}} \emptyset n \ast (1) \ast \text{Der}(\sigma, X_n) \).

**Proof:** We leave the proof to the reader. \( \Box \)

**3.12 Theorem:** For every hereditarily repetitive stump \( \sigma \), \( \text{Der}(\sigma, C B_\sigma) = \emptyset \), and for every hereditarily repetitive stump \( \tau \), if \( \sigma < \tau \), then \( \emptyset \) belongs to \( \text{Der}(\sigma, C B_\sigma) \).

**Proof:** We use induction on the set of stumps. Clearly \( \text{Der}(1, C B_1) = C B_1 = \emptyset \). Also, if \( 1 < \tau \), then \( \tau \) is non-empty and \( \emptyset \) belongs to \( C B_\tau = \text{Der}(1, C B_\tau) \).

Assume that \( \sigma \) is a non-empty hereditarily repetitive stump and that for each \( n \), \( \text{Der}(\sigma^n, C B_{\sigma^n}) = \emptyset \), and for each \( n \), for every hereditarily repetitive \( \tau \), if \( \sigma^n < \tau \), then \( \emptyset \) belongs to \( \text{Der}(\sigma^n, C B_\tau) \).

In particular \( \emptyset \) belongs to each set \( \text{Der}(\sigma^n, C B_\sigma) \).

Observe that \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_\sigma) \) coincides with the closure of the set
\[
\bigcap_{n \in \mathbb{N}} \left( \{\emptyset\} \cup \bigcup_{m \in \mathbb{N}} \emptyset mn \ast 1 \ast \text{Der}(\sigma^n, C B_{\sigma^m}) \right)
\]
and also with the closure of the set
\[
\{\emptyset\} \cup \bigcup_{m \in \mathbb{N}} \emptyset mn \ast 1 \ast \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_{\sigma^m}).
\]

As for each \( n \), \( \text{Der}(\sigma^n, C B_{\sigma^n}) = \emptyset \), we conclude that \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_\sigma) \) coincides with \( \{\emptyset\} \), and therefore \( \text{Der}(\sigma, C B_\sigma) = \emptyset \).

Now assume that \( \tau \) is a hereditarily repetitive stump and \( \sigma < \tau \).

Calculate \( m \) such that \( \sigma \leq \tau^m \) and therefore, for every \( n \), \( \sigma^n < \tau^m \). Applying the induction hypothesis, we find that \( \emptyset \) belongs to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_{\sigma^m}) \), and therefore \( \emptyset n \ast (1) \ast \emptyset \) belongs to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_\tau) \). There are infinitely many numbers \( p \) such that \( \tau^p = \tau^m \) and for each such \( p \), the sequence \( \emptyset p \ast (1) \ast \emptyset \) will belong to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, C B_\tau) \), therefore \( \emptyset \) is a limit point of this set, that is, \( \emptyset \) belongs to \( \text{Der}(\sigma, C B_\tau) \).

We conclude that the statement of the Theorem is correct. \( \Box \)

**3.13** We may conclude from Theorem 3.12 that for all hereditarily repetitive stumps \( \sigma, \tau \), \( \text{Der}(\sigma, C B_\tau) = \emptyset \) and, if \( \tau < \sigma \), then \( \text{Der}(\tau, C B_\sigma) \neq \emptyset \). In this sense, \( \sigma \) is the least stump \( \tau \) such that \( \text{Der}(\tau, C B_\sigma) = \emptyset \) and might be called the **Cantor-Bendixson-rank** of \( C B_\tau \).

We now want to prove a very non-classical counterpart to Theorem 3.12.

We introduce a binary operation Perhaps on the class of subsets of \( \mathcal{N} \). Given subsets \( X, Y \) of \( \mathcal{N} \) such that \( X \) is inhabited and \( X \) forms part of \( Y \), we let \( \text{Perhaps}(X, Y) \) be the set of all \( \alpha \) such that there exists \( \beta \) in \( X \) with the property: if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( Y \).

The sentence “\( \alpha \) belongs to Perhaps(X, Y)” might be rendered into English by the
words: “α belongs to X, perhaps (only) to Y”. We are thinking of a speaker who interrupts himself and in doing so retracts or qualifies his earlier statement and replaces it by a second, seemingly somewhat weaker one. Observe however that we also want to reckon the statement “α belongs to X, perhaps to X” as a weaker one than the unconditional statement “α belongs to X”.

Let X be an inhabited subset of \( \mathcal{N} \). X will be called perhapsive if and only if X coincides with Perhaps(X, X). Frank Waaldijk, in [25], called perhapsive subsets of \( \mathcal{N} \) weakly stable subsets of \( \mathcal{N} \).

3.14 Theorem:

(i) For all subsets X, Y of \( \mathcal{N} \), if X is inhabited and X \( \subseteq \) Y, then X \( \subseteq \) Perhaps(X, Y) \( \subseteq \) Y. 

(ii) For all subsets X, Y, Z of \( \mathcal{N} \), if X is inhabited and X \( \subseteq \) Y \( \subseteq \) Z, then Perhaps(X, Y) \( \subseteq \) Perhaps(X, Z) and Perhaps(X, Z) \( \subseteq \) Perhaps(Y, Z).

(iii) Every inhabited \( \Pi^0_3 \)-subset of \( \mathcal{N} \) is perhapsive.

(iv) For all subsets X, Y of \( \mathcal{N} \), if Y is perhapsive and X reduces to Y, then X is perhapsive.

Proof:

(i) Suppose X is inhabited and X is a subset of Y. Clearly, X is a subset of Perhaps(X, Y).

Now assume that α belongs to Perhaps(X, Y) and determine β in X such that if α≠β, then α belongs to Y. If α≠β, then α belongs to Y. If α≠β, then α coincides with β, and α belongs to X and therefore also to Y. As α=β∨(α≠β), we conclude: α≠β, that is, α belongs to Y. So Perhaps(X, Y) is a subset of Y.

(ii) obvious.

(iii) Suppose X is an inhabited subset of \( \mathcal{N} \) and X belongs to \( \Pi^0_3 \).

Let \( Y_0, Y_1, \ldots \) be a sequence of open subsets of \( \mathcal{N} \) such that X = \( \bigcap_{n \in \mathbb{N}} Y_n \). Now assume that α belongs to Perhaps(X, X) and determine β in X such that, if α≠β, then α belongs to X. Let n be a natural number. Determine m such that every γ passing through \( \exists m \) belongs to \( Y_n \) and distinguish two cases. Either \( \exists m = \exists m \), and α belongs to \( Y_n \), or \( \exists m ≠ \exists m \), therefore α≠β, and α belongs to X and therefore in particular to \( Y_n \). We conclude that Perhaps(X, X) is a subset of every set \( Y_n \), therefore, as X is also inhabited, Perhaps(X, X) coincides with X.

(iv) Suppose X, Y are subsets of \( \mathcal{N} \), and γ is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing X to Y, and Perhaps(Y, Y) is a subset of Y. Assume that α belongs to Perhaps(X, X) and determine β in X such that, if α≠β, then α belongs to X. Observe that γ|β belongs to Y. Observe also that, if γ|α≠β, then α≠β, therefore α belongs to X and γ|α belongs to Y. We conclude that γ|α belongs to Perhaps(Y, Y) and so to Y, therefore α belongs to X.
We conclude that \( \text{Perhaps}(X, X) \) is a subset of \( X \).
Therefore, if also \( X \) is inhabited, \( \text{Perhaps}(X, X) \) coincides with \( X \). ☑

3.15 We define, for every subset \( X \) of \( \mathcal{N} \) and every stump \( \sigma \), another subset of \( \mathcal{N} \), called the \( \sigma \)-th perhapsive extension of \( X \), notation \( \mathbb{P}(\sigma, X) \), as follows:

(i) \( \mathbb{P}(1, X) := X \).
(ii) For every non-empty stump \( \sigma \), \( \mathbb{P}(\sigma, X) := \text{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X)) \)

3.16 For every subset \( X \) of \( \mathcal{N} \) and every \( s \) in \( \mathbb{N} \) we let \( X|s \) be the set of all \( \alpha \) in \( X \) passing through \( s \).
3.17 Lemma:

(i) For every inhabited subset $X$ of $N$, for all stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $X \subseteq \mathbb{P}(\sigma, X) \subseteq \mathbb{P}(\tau, X) \subseteq X^{\infty}$.

(ii) For every subset $X$ of $N$, for every $s$ in $N$, for every stump $\sigma$, if $X|s$ is inhabited, then $\mathbb{P}(\sigma, X)|s = \mathbb{P}(\sigma, X|s)$.

Proof:

The proof is left to the reader.

3.18 Theorem:

(i) For every stump $\sigma$, $\mathbb{P}(\sigma, CB_\sigma)$ coincides with $\overline{CB_\sigma} = (CB_\sigma)^{\infty}$.

(ii) For all hereditarily repetitive stumps $\sigma, \tau$,

if $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\sigma)$, then $\sigma \leq \tau$.

Proof:

(i) Observe first that $\mathbb{P}(1, CB_1)$ equals $CB_1 = \emptyset$ and $\emptyset$ coincides with $\overline{\emptyset}$.

Now assume that $\sigma$ is a non-empty stump and, for each $n$, $\mathbb{P}(\sigma^n, CB_{\sigma^n})$ coincides with the closure $\overline{CB_{\sigma^n}}$ of $CB_{\sigma^n}$. We claim that the closure $\overline{CB_\sigma}$ of $CB_\sigma$ coincides with $\mathbb{P}(\sigma, CB_\sigma)$, that is, with $\mathbb{P}(\sigma, CB_\sigma)$.

For suppose $\alpha$ belongs to $\overline{CB_\sigma}$ and $\alpha$ is apart from $\overline{\emptyset}$. We determine $n, \gamma$ such that $\alpha = \overline{0n} * (1) * \gamma$ and observe that $\gamma$ belongs to $\overline{CB_{\sigma^n}}$ and therefore to $\mathbb{P}(\sigma^n, CB_{\sigma^n})$. Using Lemma 3.17(ii) we conclude that $\alpha$ belongs to $\mathbb{P}(\sigma^n, CB_\sigma)$. Therefore $\overline{CB_\sigma}$ forms part of and indeed coincides with $\mathbb{P}(\sigma, CB_\sigma)$.

(ii) For each hereditarily repetitive stump $\tau$ we define the proposition $Q(\tau)$ as follows:

\[
Q(\tau) := \begin{cases} \tau \text{ has the property } Q & \\
\text{if } \overline{CB_\sigma} \text{ forms part of } \mathbb{P}(\tau, CB_\sigma), \text{ then } \sigma \leq \tau. & 
\end{cases}
\]

We claim that every hereditarily repetitive stump $\tau$ has the property $Q$. Observe first that $1$ has the property $Q$, for, if, for some $\sigma$, $\overline{CB_\sigma}$ forms part of $\mathbb{P}(1, CB_1) = CB_\sigma$, then $CB_\sigma$ is closed and $\sigma = 1$.

Now assume that $\tau$ is a non-empty stump and that, for each $n$, $\tau^n$ has the property $Q$. We show that $\tau$ itself has the property $Q$.

Suppose that $\sigma$ is a hereditarily repetitive stump and that $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\tau)$, that is, $\overline{CB_\sigma}$ is a subset of $\mathbb{P}(\tau, CB_\tau)$. So for every $\alpha$ in $\overline{CB_\sigma}$ we may determine $\beta$ in $CB_\tau$ such that, if $\alpha$ is apart from $\beta$, then $\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\tau)$. Observe that, for every $\beta$ in $CB_\tau$, we may decide $\beta = 0$ or $\beta \neq 0$. So for every $\alpha$ in $\overline{CB_\sigma}$ there exist $i, p$ such that there exists $\beta$ with the property: either $i = 0$ and $\beta = 0$ or $i > 0$ and $\beta$ passes through $\overline{\emptyset} * (1)$, and if $\alpha \neq \beta$, then $\alpha$ belongs
to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. Observe that $CB_\sigma$ is a spread containing $0$.

We apply the Continuity Principle and distinguish two cases.

**First Case.** We find $m$ such that for every $\alpha$ in $CB_\sigma$ passing through $0_m$, if $\alpha \neq 0$, then there exists $n$ such that $\alpha$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Consider any $p \geq m$. Observe that $CB_\sigma \setminus \{(0p \cdot (1))\}$ is a spread containing $0p \cdot (1) \cdot 0$ and forming part of $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$ and apply the Continuity Principle a second time. We find $q, n$ such that every $\alpha$ in $CB_\sigma$ passing through $0p \cdot (1) \cdot 0q$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Now observe that, for every $\gamma$, $0p \cdot (1) \cdot 0q \cdot \gamma$ belongs to $CB_\sigma$, or to $CB_\sigma'$, respectively, if and only if $\gamma$ itself belongs to $CB_\sigma$, or to $CB_\sigma'$, respectively. Using Lemma 3.17(ii) we conclude that for every $\gamma$, $0p \cdot (1) \cdot 0q \cdot \gamma$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$ if and only if $\gamma$ itself belongs to $\mathbb{P}(\tau^n, CB_\sigma')$. Therefore $CB_\sigma'$ forms part of $\mathbb{P}(\tau^n, CB_\sigma')$ and thus $\sigma^0 \leq \tau^n$.

Clearly, for every $p \geq m$ there exists $n$ such that $\sigma^0 \leq \tau^n$, and so $\tau \leq \tau^n$.

**Second Case.** We find $m, p$ such that $m > p$ and for every $\alpha$ in $CB_\sigma$ passing through $0_m$ there exists $\beta$ in $CB_\sigma$ passing through $0p \cdot (1)$ with the property: if $\alpha \neq \beta$, then there exists $n$ such that $\alpha$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Observe that $CB_\sigma \setminus \{(0p \cdot (1) \cdot 0q \cdot \gamma)\}$ is a spread containing $0$ and forming part of $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. We apply the Continuity Principle a second time and find $n, q$ such that $q > m$ and every $\alpha$ in $CB_\sigma$ passing through $0q$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Reasoning as in Case (i) we obtain the conclusion that $CB_\sigma'$ forms part of $\mathbb{P}(\tau^n, CB_\sigma')$, therefore $\sigma \leq \tau^n$ and also $\sigma \leq \tau$.

3.19 **We may conclude from Theorem 3.18:** for all hereditarily repetitive stumps $\sigma, \tau$: $\sigma \leq \tau$ if and only if $CB_\sigma$ forms part of $\mathbb{P}(\tau, CB_\sigma)$. In this sense $\sigma$ is the least stump $\tau$ such that $CB_\sigma$ forms part of $\mathbb{P}(\tau, CB_\sigma)$ and might be called the *perhapsive rank* of $CB_\sigma$.

In general given an inhabited collection $C$ of (hereditarily repetitive) stumps, it is not possible to find $\sigma$ in $C$ such that, for all $\tau$ in $C$, $\sigma \leq \tau$.

3.20 **Theorem:**

(i) For all subsets $X, Y$ of $\mathcal{N}$, for every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$, if $\gamma$ maps $X$ into $Y$, then for each stump $\sigma, \gamma$ maps $\mathbb{P}(\sigma, X)$ into $\mathbb{P}(\sigma, Y)$.

(ii) For all hereditarily stumps $\sigma, \tau$, for every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$, if $\sigma < \tau$ and $\gamma$ maps $CB_\sigma$ into $CB_\tau$, then $\gamma$ does not map surjectively the closure $\overline{CB_\sigma}$ of $CB_\sigma$ onto the closure $\overline{CB_\tau}$ of $CB_\tau$.

**Proof:**

(i) The proof is by induction on the set of stumps and left to the reader.

(ii) Observe that, according to (i), $\gamma$ will map $\overline{CB_\sigma} = \mathbb{P}(\sigma, CB_\sigma)$ into $\mathbb{P}(\sigma, CB_\sigma)$ and, according to Theorem 3.18, the latter set is a proper subset of $\overline{CB_\tau}$.
4 Perhaps and Almost

We continue the study of the notion Perhaps discovered in the previous Section and introduce a further notion: Almost. We also consider countable sets that are dense in \( \mathcal{N} \).

4.1 Let \( X \) be a subset of \( \mathcal{N} \). We let Almost\((X)\) be the set of all \( \alpha \) in \( \mathcal{N} \) such that for some stump \( \sigma \), \( \alpha \) belongs to \( \mathbb{P}(\sigma, X) \). Observe that this definition involves a quantification on the set \( \text{Stp} \) of stumps. The possibility of introducing the set Almost\((X)\) depends upon our acceptance of \( \text{Stp} \) as a set and a domain of quantification.

4.2 Theorem:

(i) For all inhabited subsets \( X \) of \( \mathcal{N} \), \( X \subseteq \text{Almost}(X) \subseteq X^{\sim} \).

(ii) For all inhabited subsets \( X \) of \( \mathcal{N} \), Perhaps\((X, \text{Almost}(X))\) coincides with \( \text{Almost}(X) \).

(iii) For all subsets \( X, Y, Z \) of \( \mathcal{N} \), if \( X \) is inhabited and \( X \subseteq Y \subseteq Z \), then Perhaps\((\text{Almost}(X), Z)\) forms part of Perhaps\((X, \text{Almost}(Y), Z)\).

(iv) For all inhabited subsets \( X \) of \( \mathcal{N} \), for every stump \( \sigma \), Perhaps\((\mathbb{P}(\sigma, X), \text{Almost}(X))\) forms part of \( \text{Almost}(X) \).

(v) For all inhabited subsets \( X \) of \( \mathcal{N} \), Perhaps\((\text{Almost}(X), \text{Almost}(X))\) coincides with \( \text{Almost}(X) \), that is, \( \text{Almost}(X) \) is perhapsive.

(vi) For all inhabited subsets \( X, Y \) of \( \mathcal{N} \), if \( X \subseteq Y \) and \( Y \) is perhapsive, then \( \text{Almost}(X) \) forms part of \( Y \).

Proof:

(i) is an almost direct consequence of Lemma 3.17(i).

(ii) Let \( X \) be a subset of \( \mathcal{N} \) and suppose that \( \alpha \) belongs to Perhaps\((X, \text{Almost}(X))\). Find \( \beta \) in \( X \) such that, if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( \text{Almost}(X) \). We now build a non-empty stump \( \sigma \) by specifying successively its immediate substumps \( \sigma^0, \sigma^1, \sigma^2, \ldots \). For each \( n \), if \( \alpha(n) = \beta(n) \) or if there exists \( p < n \) such that \( \alpha(p) \neq \beta(p) \) then \( \sigma^n \) is the empty stump \( 1 \), and if \( \alpha(n) \neq \beta(n) \) and there is no \( p < n \) such that \( \alpha(n) \neq \beta(n) \), we find a stump \( \tau \) such that \( \alpha \) belongs to Perhaps\((\tau, X)\), and define \( \sigma^n := \tau \). We claim that \( \alpha \) belongs to \( \mathbb{P}(\sigma, X) \). For let \( \beta \) be the sequence we just considered and observe: \( \beta \) belongs to \( X \), and if \( \alpha \) is apart from \( \beta \), and \( n \) is the least \( p \) such that \( \alpha(p) \neq \beta(p) \), then \( \alpha \) belongs to \( \mathbb{P}(\sigma^n, X) \).

We conclude that \( \alpha \) belongs to \( \text{Almost}(X) \).

Therefore Perhaps\((X, \text{Almost}(X))\) is part of \( \text{Almost}(X) \) and, if \( X \) is inhabited, the two sets coincide.

(iii) Let \( X, Y, Z \) be subsets of \( \mathcal{N} \) such that \( X \) is inhabited and \( X \subseteq Y \subseteq Z \). Assume that \( \alpha \) belongs to Perhaps\((\text{Perhaps}(X, Y), Z)\). We intend to show that \( \alpha \) belongs to Perhaps\((X, \text{Perhaps}(Y, Z))\). First determine \( \beta \) in Perhaps\((X, Y)\) such that if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( Z \). Then determine \( \gamma \) in \( X \) such that if \( \beta \) is apart
from $\gamma$, then $\beta$ belongs to $Y$. Now assume that $\alpha$ is apart from $\gamma$ and distinguish two cases: either $\alpha$ is apart from $\beta$ and $\alpha$ belongs to $Z$, or $\gamma$ is apart from $\beta$ and $\beta$ belongs to $Y$, and therefore $\alpha$ belongs to Perhaps($Y, Z$). In both cases $\alpha$ belongs to Perhaps($Y, Z$), so if $\alpha$ is apart from $\gamma$, then $\alpha$ belongs to Perhaps($Y, Z$), therefore $\alpha$ belongs to Perhaps($X, Perhaps(Y, Z)$).

We conclude that Perhaps(Part(X, Y), Z) forms part of Perhaps($X, Perhaps(Y, Z)$).

(iv) Let $X$ be an inhabited subset of $\mathcal{N}$. We claim that for each stump $\sigma$, the set Perhaps($\mathbb{P}(\sigma, X)$, Almost($X$)) forms part of Almost($X$).

We use induction on the set of stumps.

We know from (ii) that Perhaps($\mathbb{P}(1, X)$, Almost($X$)) forms part of Almost($X$). Now assume that $\sigma$ is a non-empty stump and that, for each $n$, Perhaps($\mathbb{P}(\sigma^n, X)$, Almost($X$)) forms part of Almost($X$).

Consider Perhaps($\mathbb{P}(\sigma, X)$, Almost($X$)), observe that $\mathbb{P}(\sigma, X)$ coincides with Perhaps($X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X)$) and apply (iii) in order to conclude:

$\mathbb{P}(\sigma, X)$ forms part of Perhaps($X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X)$) and Perhaps($\mathbb{P}(\sigma^n, X)$, Almost($X$)) coincides with

$\bigcup_{n \in \mathbb{N}}$ Perhaps($\mathbb{P}(\sigma^n, X)$, Almost($X$)) and therefore with Almost($X$).

Therefore, $\mathbb{P}(\sigma, X)$ forms part of Perhaps($\mathbb{P}(\sigma, X)$, Almost($X$)).

(v) Let $X$ be an inhabited subset of $\mathcal{N}$.

Observe that Perhaps(Almost($X$), Almost($X$)) coincides with $\bigcup_{\sigma \in \text{Stp}}$ Perhaps($\mathbb{P}(\sigma, X)$, Almost($X$)), and therefore, according to (iv), with Almost($X$).

(vi) Let $X$ be inhabited subset of $\mathcal{N}$, and suppose $Y$ is a perhapsive subset of $\mathcal{N}$ containing $X$. One may prove by induction on the set of stumps that for each stump $\sigma$, $\mathbb{P}(\sigma, X)$ forms part of $Y$, and therefore Almost($X$) forms part of $Y$.

4.3 Let $X$ be an inhabited subset of $\mathcal{N}$.

In view of Theorem 4.2 we may call Almost($X$) the perhapsive closure of the set $X$: Almost($X$) is the least perhapsive set containing $X$.

4.4 Let $D$ be a subset of $\mathcal{N}$. $D$ is dense-in-itself if and only if for every $\alpha$ in $D$, for every $m$, there exists $\beta$ in $D$ such that $\beta$ is apart from $\alpha$ and $\alpha m = \beta m$. $D$ is discrete if and only if for all $\alpha$, $\beta$ in $D$ we may decide if $\alpha \# \beta$ or $\alpha = \beta$.

Let $X$ be a subset of $\mathcal{N}$ and $Y$ a subset of $X$. $Y$ is a decidable subset of $X$ if and only if we may decide, for every $\alpha$ in $X$, if $\alpha$ belongs to $Y$ or not.

4.5 Theorem:

Let $D$ be an enumerable and discrete subset of $\mathcal{N}$ that is also dense-in-itself.
(i) For each stump \( \sigma \), for every \( s \), if \( s \) contains an element of \( D \), then there exists an embedding of \( CB_\sigma \) into \( N \)\( |s \) mapping \( CB_\sigma \) itself onto a decidable subset of \( D \).

(ii) For each stump \( \sigma \), the set \( CB_\sigma \) reduces to the set \( D \).

(iii) For no stump \( \sigma \), \( D \) reduces to \( CB_\sigma \).

(iv) For each stump \( \sigma \), for each stump \( \tau \), the set \( P(\tau, CB_\sigma) \) reduces to the set \( P(\tau, D) \).

(v) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma \prec \tau \), then \( P(\sigma, D) \) is a proper subset of the set \( P(\tau, D) \).

**Proof:**

(i) We use induction on the set of stumps.

The statement is obviously true in case \( \sigma \) equals the empty stump \( \emptyset \). Now assume \( \sigma \) is a non-empty stump and for each \( n, s \) such that \( s \) contains an element of \( D \) there exists an embedding of \( CB_{\sigma^n} \) into \( N \)\( |s \) mapping \( CB_{\sigma^n} \) onto a decidable subset of \( D \).

Let \( s \) be a natural number containing an element of \( D \). Find \( a \) in \( N \) such that \( s \ast a \) belongs to \( D \). We calculate two sequences \( k_0, k_1, \ldots \) and \( p_0, p_1, \ldots \) of natural numbers such that \( k_0 < k_1 < \cdots \) and, for each \( n, p_n \) is different from \( a(k_n) \) and \( s \ast \alpha k_n \ast (p_n) \) contains an element of \( D \). For each \( n \) we construct an embedding \( \delta_n \) of \( CB_{\sigma^n} \) into \( N \)\( |s \ast \alpha k_n \ast (p_n) \) mapping \( CB_{\sigma^n} \) onto a decidable subset of \( D \). It is not difficult to define an embedding \( \gamma \) from \( CB_\sigma \) into \( N \)\( |s \) such that \( \gamma[0] \) equals \( s \ast \alpha \) and for each \( n \), for each \( \beta \) in \( CB_{\sigma^n} \), \( \gamma[n \ast (1)] \ast \beta \) equals \( s \ast \alpha k_n \ast (p_n) \ast \delta_n \ast \beta \). Observe that \( \gamma \) maps \( CB_\sigma \) itself onto a decidable subset of \( D \).

(ii) Let \( \sigma \) be a stump and, using (i), define an embedding \( \gamma \) from \( CB_\sigma \) into \( N \) mapping \( CB_\sigma \) itself onto a decidable subset of \( D \). Let \( \delta \) be an enumeration of \( D \), that is, \( D \) coincides with the set \( \{ \delta^0, \delta^1, \ldots \} \). Let \( E \) be the set of all \( s \) in \( N \) that contain an element of \( CB_\sigma \).

Observe that \( E \) is a decidable subset of \( N \). We may construct a function \( \zeta \) from \( N \) to \( N \) that for all \( \alpha \) in \( CB_\sigma \), \( \zeta[\alpha] \) coincides with \( \gamma[\alpha] \), and for each \( \alpha, n \), if \( \alpha n \) does not belong to \( E \), then for each \( i \), \( \zeta[\alpha](n + i) \) differs from \( \delta(n + i) \).

We claim that \( \zeta \) reduces \( CB_\sigma \) to \( D \).

It will be clear that for every \( \alpha \), if \( \alpha \) belongs to \( CB_\sigma \), then \( \zeta[\alpha] \) belongs to \( D \). Assume now that \( \alpha \) is an element of \( N \) and \( \zeta[\alpha] \) belongs to \( D \). Then every initial part of \( \alpha \) belongs to \( E \) and \( \alpha \) belongs to \( CB_\sigma \) and \( \zeta[\alpha] \) coincides with \( \gamma[\alpha] \). We may decide if there exists \( \beta \) in \( CB_\sigma \) such that \( \gamma[\beta] \) equals \( \gamma[\alpha] \). As \( CB_\sigma \) coincides with \( CB_\sigma \), it is excluded that \( \alpha \) does not belong to \( CB_\sigma \), so there exists \( \beta \) in \( CB_\sigma \) such that \( \gamma[\beta] \) equals \( \gamma[\alpha] \). As \( \gamma \) is an embedding, we must have \( \alpha = \beta \), that is, \( \alpha \) belongs to \( CB_\sigma \).

This completes the proof of our claim that \( \zeta \) reduces \( CB_\sigma \) to \( D \).

(iii) is an easy consequence of (ii) and Theorem 3.9(ii).

(iv) Let \( \sigma \) be a stump and let \( \zeta \) be a function from \( N \) to \( N \) embedding \( CB_\sigma \) onto a decidable subset of \( D \) and reducing \( CB_\sigma \) to \( D \). We constructed such a function in (ii). We claim that for each stump \( \tau \), \( \zeta \) reduces the set \( P(\tau, CB_\sigma) \) to the set \( P(\tau, D) \).
and prove this claim by induction on the set of stumps. The statement is obviously true if $t$ equals the empty stump $\bot$. Now assume that $t$ is a non-empty stump and, for each $n$, $\zeta$ reduces the set $\mathbb{P}(\tau^n, CB_\sigma)$ to the set $\mathbb{P}(\tau^n, D)$. Suppose that $\alpha$ belongs to $\mathbb{P}(\tau, CB_\sigma)$. Find $\beta$ in $CB_\sigma$ such that, if $\alpha$ is apart from $\beta$, then $\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. Observe that, if $\zeta|\alpha$ is apart from $\zeta|\beta$, then $\alpha$ is apart from $\beta$, and $\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$, and $\zeta|\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, D)$. As $\zeta|\beta$ belongs to $D$, this shows that $\zeta$ maps $\mathbb{P}(\tau, CB_\sigma)$ into $\mathbb{P}(\tau, D)$.

Now assume that $\alpha$ belongs to $N$ and $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$. Find $\beta$ in $D$ such that, if $\alpha$ is apart from $\beta$, then $\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, D)$. Now find out if there exists $\delta$ in $CB_\sigma$ such that $\zeta|\delta$ equals $\beta$ and distinguish two cases.

First Case. There exists $\delta$ in $CB_\sigma$ such that $\zeta|\delta$ equals $\beta$, say $\delta_0$. Observe that, if $\alpha$ is apart from $\delta_0$, then $\zeta|\alpha$ is apart from $\zeta|\delta_0 = \beta$, therefore $\zeta|\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, D)$ and $\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. As $\delta_0$ belongs to $CB_\sigma$, this shows that $\alpha$ belongs to $\mathbb{P}(\tau, CB_\sigma)$.

Second Case. There is no $\delta$ in $CB_\sigma$ such that $\zeta|\delta$ equals $\beta$. As $\beta$ belongs to $D$ and $D$ is discrete, this implies that, for every $\delta$ in $CB_\sigma$, $\beta$ is apart from $\zeta|\delta$. It follows that, for every $\delta$ in the closure $\overline{CB_\sigma}$ of $CB_\sigma$, $\beta$ is apart from $\zeta|\delta$.

Now observe that $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$ and therefore to $D^\omega$, so $\alpha$ itself belongs to $\overline{CB_\sigma} = \overline{CB_\sigma}$, therefore $\beta$ is apart from $\zeta|\alpha$, and $\zeta|\alpha$ belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, D)$ and $\alpha$ itself belongs to $\bigcup_{n\in\mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$, and therefore also to $\mathbb{P}(\tau, CB_\sigma)$.

We have shown, for every $\alpha$, if $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$, then $\alpha$ belongs to $\mathbb{P}(\tau, CB_\sigma)$ and conclude: $\zeta$ reduces $\mathbb{P}(\tau, CB_\sigma)$ to $\mathbb{P}(\tau, D)$.

(v) Let $\sigma, \tau$ be hereditarily repetitive stumps such that $\sigma < \tau$.

We have seen, in Theorem 3.18 that $\mathbb{P}(\tau, CB_\tau)$ coincides with $\overline{CB_\tau}$ and $\mathbb{P}(\sigma, CB_\tau)$ does not, and therefore, in view of Lemma 3.17, $\mathbb{P}(\sigma, CB_\tau)$ is a proper subset of $\mathbb{P}(\tau, CB_\tau)$.

Let $\zeta$ be a function from $N$ to $N$ reducing $CB_\tau$ to $D$ and also $\mathbb{P}(\sigma, CB_\tau)$ to $\mathbb{P}(\sigma, D)$ and $\mathbb{P}(\tau, CB_\tau)$ to $\mathbb{P}(\tau, D)$. (We know from our proof of (iv) that there exists such a function). Assume that $\mathbb{P}(\tau, D)$ is a subset of $\mathbb{P}(\sigma, D)$. It follows that $\mathbb{P}(\tau, CB_\tau)$ is a subset of $\mathbb{P}(\sigma, CB_\tau)$ and we obtain a contradiction.

We conclude that $\mathbb{P}(\tau, D)$ is not a subset of $\mathbb{P}(\sigma, D)$, therefore, $\mathbb{P}(\sigma, D)$ is a proper subset of $\mathbb{P}(\tau, D)$. \[\]

4.6 Let $X$ be an inhabited subset of $N$. $X$ has unbounded perhappiness if and only if for all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma < \tau$, then $\mathbb{P}(\sigma, X)$ is a proper subset of $\mathbb{P}(\tau, X)$. Theorem 4.5(v) is the statement that every enumerable and discrete subset of $N$ that is also dense-in-itself, has unbounded perhappiness. Let $X$ be a non-empty enumerable subset of $N$ and let $\delta$ be an enumeration of $X$, that is, $X = \{\delta^0, \delta^1, \ldots\}$.
We let \( \text{Almost}^*(X) \) be the set of all \( \alpha \) in \( \mathcal{N} \) such that for each \( \beta \) there exists \( n \) such that \( \overline{\alpha} (\beta(n)) = \overline{\delta^n} (\beta(n)) \). Intuitively spoken, the set \( \text{Almost}^*(X) \) is not that much different from the set \( X \). We could perhaps say that \( \text{Almost}^*(X) \) is the set of all \( \alpha \) in \( \mathcal{N} \) for which we see that every attempt to prove that \( \alpha \) is apart from every element in the sequence \( \delta^0, \delta^1, \ldots \) will fail in finitely many steps. Every \( \beta \) in \( \mathcal{N} \) might be thought of as expressing the hope that, for all \( n \), \( \overline{\alpha} (\beta(n)) \neq \overline{\delta^n} (\beta(n)) \).

4.7 Theorem:
Let \( X \) be an enumerable subset of \( \mathcal{N} \).

(i) \( X \subseteq \text{Almost}^*(X) \)
(ii) \( \text{Perhaps}(X, \text{Almost}^*(X)) = \text{Almost}^*(X) \)
(iii) \( \text{Almost}^*(X) \) is perhapsive, that is,
\[ \text{Perhaps} \ (\text{Almost}^*(X), \text{Almost}^*(X)) = \text{Almost}^*(X). \]
(iv) \( \text{Almost}(X) \subseteq \text{Almost}^*(X) \)

Proof:
(i) obvious.
(ii) Let \( \delta \) be an enumeration of \( X \), so \( X = \{\delta^0, \delta^1, \ldots\} \).
Assume that \( \alpha \) belongs to \( \text{Perhaps}(X, \text{Almost}^*(X)) \), and let \( \delta^n \) be an element of \( X \) such that, if \( \alpha \) is apart from \( \delta^n \), then \( \alpha \) belongs to \( \text{Almost}^*(X) \).
Let \( \beta \) belong to \( \mathcal{N} \) and distinguish two cases.
Either \( \overline{\alpha} (\beta(n)) \) equals \( \overline{\delta^n} (\beta(n)) \) or \( \overline{\alpha} (\beta(n)) \) is different from \( \overline{\delta^n} (\beta(n)) \). In the latter case \( \alpha \) is apart from \( \delta^n \), therefore \( \alpha \) belongs to \( \text{Almost}^*(X) \) and there exists \( m \) such that \( \overline{\alpha} (\beta(m)) = \overline{\delta^m} (\beta(m)) \).
So in both cases there exists \( m \) such that \( \overline{\alpha} (\beta(m)) = \overline{\delta^m} (\beta(m)) \).
We conclude that \( \alpha \) belongs to \( \text{Almost}^*(X) \).
(iii) Let \( \delta \) be an enumeration of \( X \), so \( X = \{\delta^0, \delta^1, \ldots\} \).
Assume that \( \alpha \) belongs to \( \text{Perhaps} \ (\text{Almost}^*(X), \text{Almost}^*(X)) \) and let \( \gamma \) be an element of \( \text{Almost}^*(X) \) such that, if \( \alpha \) is apart from \( \gamma \), then \( \alpha \) belongs to \( \text{Almost}^*(X) \).
Let \( \beta \) belong to \( \mathcal{N} \) and find \( n \) such that \( \overline{\gamma} (\beta(n)) = \overline{\delta^n} (\beta(n)) \), and distinguish two cases.
Either \( \overline{\alpha} (\beta(n)) \) equals \( \overline{\gamma} (\beta(n)) \) or \( \overline{\alpha} (\beta(n)) \) is different from \( \overline{\gamma} (\beta(n)) \). In the latter case \( \alpha \) is apart from \( \gamma \), therefore \( \alpha \) belongs to \( \text{Almost}^*(X) \), and there exists \( m \) such that \( \overline{\alpha} (\beta(m)) = \overline{\delta^m} (\beta(m)) \). So in both cases there exists \( m \) such that \( \overline{\alpha} (\beta(m)) = \overline{\delta^m} (\beta(m)) \).
We conclude that \( \alpha \) belongs to \( \text{Almost}^*(X) \).
(iv) This follows from (iii) and Theorem 4.2(vi). \( \Box \)

4.8 For every \( \delta \), we let \( D_\delta \) be the enumerable set \( \{\delta^0, \delta^1, \ldots\} \).
Let \( \sigma \) be a stump, and suppose \( \delta, \alpha \) belong to \( \mathcal{N} \).
\( \sigma \) secures \( \alpha \) with respect to \( \delta \) if and only if for every \( \beta \) there exists \( n \) such that \( \overline{\beta^n} \)
belongs to $\sigma$ and there exists $m < n$ such that $\overline{\alpha}(\beta(m)) = \overline{\delta}(\beta(m))$.

The following statement follows from Brouwer’s Thesis:

For every $\delta, \alpha$, if $\alpha$ belongs to $\text{Almost}^*(D_\delta)$,
then there exists a stump $\sigma$ that secures $\alpha$ with respect to $\delta$.

**4.9 Theorem:**

(i) For every stump $\sigma$, for every $\alpha$, for every $\delta$,
if $\sigma$ secures $\alpha$ with respect to $\delta$, then $\alpha$ belongs to $\mathbb{P}(\sigma, D_\delta)$

(ii) (Using Brouwer’s Thesis)
For every enumerable subset $X$ of $\mathcal{N}$,
$\text{Almost}^*(X) \subseteq \text{Almost}(X)$ and $\text{Almost}^*(X) \subseteq X^\sim$.

**Proof:** (i) For every stump $\sigma$ we define the proposition $P(\sigma)$ as follows:

$$P(\sigma) := \sigma \text{ has the property } P$$

$$:= \text{ For all } \alpha, \delta, \text{ if } \sigma \text{ secures } \alpha \text{ with respect to } \delta,$$
then $\alpha$ belongs to $\mathbb{P}(\sigma, D_\delta)$.

We prove, by induction on the set of stumps, that every stump has the property $P$.
Observe that the empty stump $\mathbb{1}$ has the property $P$, as there are no $\alpha, \delta$ such that $\mathbb{1}$
secures $\alpha$ with respect to $\delta$.
Now assume that $\sigma$ is a non-empty stump and that for each $n$, $\sigma^n$ has the property $P$.
We show that $\sigma$ itself has the property $P$. Suppose $\alpha, \delta$ are such that $\sigma$ secures $\alpha$
with respect to $\delta$. Assume $\alpha$ is apart from $\delta^n$ and find $n$ such that $\overline{\alpha}(n) \neq \overline{\delta}(n)$.
Define $\zeta$ in $\mathcal{N}$ such that, for each $i$, $\zeta^i = \delta^{i+1}$ and observe that $\sigma^n$ secures $\alpha$ with respect to $\zeta$.
Therefore $\alpha$ belongs to $\mathbb{P}(\sigma^n, D_\zeta)$ and, as $D_\zeta$ is a subset of $D_\delta$, also to $\mathbb{P}(\sigma^n, D_\delta)$.
Therefore, if $\alpha$ is apart from $\delta^n$, there exists $n$ such that $\alpha$ belongs to $\mathbb{P}(\sigma^n, D_\delta)$.
As $\delta^n$ belongs to $D_\delta$, $\alpha$ belongs to $\mathbb{P}(\sigma, D_\delta)$.
We conclude that $\sigma$ has the property $P$.

(ii) follows easily from (i) and the remark in Section 4.8.

**4.10** One of the results in [18] may be seen to assert that $\text{Almost}(E_2)$ does not coincide
with $E_2$.
We let $\text{Fin}$ be the set of all $\alpha$ in Cantor space $\mathcal{C}$ such that there exists $n$ such that,
for all $j > n$, $\alpha(j) = 0$. An element $\alpha$ of $\mathcal{C}$ belongs to $\text{Fin}$ if and only if $\alpha$ is the
characteristic function of a finite subset of the set $\mathbb{N}$ of natural numbers.
$\text{Fin}$ is an enumerable and discrete subset of $\mathcal{N}$ that is dense-in-itself. For each stump
$\sigma$, $C B_\sigma$ is a decidable subset of $\text{Fin}$.
According to Theorem 4.5, for every stump $\sigma$, $C B_\sigma$ reduces to $\text{Fin}$.
We considered the set $\text{Almost}^*(\text{Fin})$ in [20] en [21] and observed, that upon the additional assumption of the generalized form of Markov’s Principle, the set $\text{Almost}^*(\text{Fin})$
coincides with the set $\text{Fin}^\sim$. 

36
5 Finite unions of closed sets

We introduce a binary operation called *disjunction* on the class of subsets of $\mathcal{N}$ and study some of its properties. We show that there are uncountably many sets $X$ such that $CB_{2^\omega} \subseteq X \subseteq CB_{2^{\omega^*}}$. Thereafter, we bring in *conjunction*, and we study finite intersections of finite unions of closed sets.

5.1 We let $\text{Inf}$ be the set of all $\alpha$ in Cantor space $\mathcal{C}$ such that for each $n$ there exists $j > n$ such that $\alpha(j) = 0$.

An element $\alpha$ of $\mathcal{N}$ belongs to $\text{Inf}$ if and only if $\alpha$ is the characteristic function of an infinite subset of the set of natural numbers. $\text{Inf}$ is a countable intersection of open sets and thus belongs to the class $\Pi_2^0$.

5.2 Theorem:
Every $\Pi_2^0$-subset of $\mathcal{N}$ reduces to $\text{Inf}$.

Proof:
Let $X$ be a $\Pi_2^0$-subset of $\mathcal{N}$ and assume that $Y_0, Y_1, \ldots$ is a sequence of open subsets of $\mathcal{N}$ such that $X = \bigcap_{n \in \mathbb{N}} Y_n$. Let $C_0, C_1, \ldots$ be a sequence of decidable subsets of $\mathbb{N}$ such that for each $\alpha$, for each $n$, $\alpha$ belongs to $Y_n$ if and only if some initial part of $\alpha$ belongs to $C_n$. We define a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that for each $\alpha$, for each $n$, $(\gamma|\alpha)(n)$ belongs to $\{0, 1\}$ and $(\gamma|\alpha)(n) = 1$ if and only if the least $i < n + 1$ such that for every $j < i$ some initial part of $\alpha(n + 1)$ belongs to $C_j$ is greater than the least $i < n$ such that for every $j < i$ some initial part of $\alpha(n)$ belongs to $C_j$.

One verifies without difficulty that $\gamma$ reduces $X$ to $\text{Inf}$.

5.3 We shall see soon that the set $\text{Fin}$ is not a complete element of the class $\Sigma_2^0$, and thus thwart an expectation one might form after Theorem 5.2.

We define a binary operation $D$ on the class of subsets of Baire space $\mathcal{N}$. For all subsets $X, Y$ of $\mathcal{N}$ we let $D(X, Y)$ be the set of all $\alpha$ such that either $\alpha^0$ belongs to $X$ or $\alpha^1$ belongs to $Y$. We call the set $D(X, Y)$ the *disjunction* of the sets $X$ and $Y$.

Observe that, for all subsets $X, Y, Z$ of $\mathcal{N}$, $Z$ reduces to $D(X, Y)$ if and only if there exist subsets $Z_0, Z_1$ of $\mathcal{N}$ such that $Z = Z_0 \cup Z_1$ and $Z_0$ reduces to $X$ and $Z_1$ reduces to $Y$.

For every subset $X$ of $\mathcal{N}$ we denote $D(X, X)$ by $D^2(X)$.

We define a subset $A_1$ of $\mathcal{N}$. $A_1$ is the set of all $\alpha$ such that, for every $n$, $\alpha(n) = 0$. So the sequence $\emptyset$ is the one and only element of $A_1$.

Observe that, for every subset $X$ of $\mathcal{N}$, $X$ reduces to $A_1$ if and only if $X$ is closed and $X$ reduces to $D^2(A_1)$ if and only if there exist closed sets $X_0, X_1$ such that $X = X_0 \cup X_1$.

Observe that the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ is a spread containing $\emptyset$.

5.4 Theorem:
(i) $D^2(A_1)$ is not closed.
(ii) The closure $D^2(A_1)$ of $D^2(A_1)$ coincides with Perhaps($D^2(A_1)$, $D^2(A_1)$) and with $D^2(A_1)^{-\infty}$.
(iii) $D^2(A_1)$ is not perhapsive and does not belong to $\Pi^0_2$.
(iv) $D^2(A_1)$ belongs to $\Sigma^0_1$ but does not reduce to $\text{Fin}$.
(v) $\text{Fin}$ does not reduce to $D^2(A_1)$.

**Proof:**

(i) Suppose that $D^2(A_1)$ coincides with $D^2(A_1)$. For every $\alpha$ in the spread $D^2(A_1)$ we may decide either $\alpha^0 = 0$ or $\alpha^1 = 0$. Applying the Continuity Principle we find $m$ such that either for every $\alpha$ in $D^2(A_1)$ passing through $\bar{0}m$, $\alpha^0 = 0$ or for every $\alpha$ in $D^2(A_1)$ passing through $\bar{0}m$, $\alpha^1 = 0$.

This is absurd, as for each $m$ there exist $\alpha, \beta$ in $D^2(A_1)$ passing through $\bar{0}m$ such that $\alpha^0$ is apart from $0$ and $\beta^1$ is apart from $0$.

We conclude that $D^2(A_1)$ is not a subset of $D^2(A_1)$.

(ii) It will be clear that Perhaps($D^2(A_1), D^2(A_1)$) forms part of $D^2(A_1)^{-\infty}$ and that $D^2(A_1)^{-\infty}$ is included in $D^2(A_1)$. We now show that $D^2(A_1)$ is a subset of Perhaps($D^2(A_1), D^2(A_1)$). Assume that $\alpha$ belongs to $D^2(A_1)$. Define $\beta$ in $\mathcal{N}$ such that $\beta^0 = 0$ and for each $n$, if there exists no $p$ such that $n = (0, p)$, then $\beta(n) = \alpha(n)$. Observe that $\beta$ belongs to $D^2(A_1)$, and, if $\alpha$ is apart from $\beta$, then $\alpha^0$ is apart from $0$ and consequently $\alpha^1$ coincides with $0$, so $\alpha$ belongs to $D^2(A_1)$.

Therefore, $\alpha$ belongs to Perhaps($D^2(A_1), D^2(A_1)$).

(iii) follows from (ii) and Theorem 3.14(iii).

(iv) $D^2(A_1)$ obviously belongs to $\Sigma^0_1$. Assume now that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(A_1)$ to $\text{Fin}$. Let $B_0$ be the set of all $\alpha$ in $\mathcal{N}$ such that $\alpha^0 = 0$ and let $B_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that $\alpha^1 = 0$. Observe that $B_0, B_1$ are spreads and that $D^2(A_1) = B_0 \cup B_1$. For every $\alpha$ in $D^2(A_1)$ there exists $m$ such that, for every $i > m$, $(\gamma|\alpha)(i) = 0$. Applying the Continuity Principle two times, we find $n, m$ such that for every $\alpha$ from $B_0 \cup B_1$, if $\bar{\alpha}n = \bar{0}m$ then, for every $i > m$, $(\gamma|\alpha)(i) = 0$. Observe that now, also for every $\alpha$ from the closure $D^2(A_1)$ of $D^2(A_1)$, for every $i > m$, $(\gamma|\alpha)(i) = 0$, so $\gamma|\alpha$ belongs to $\text{Fin}$. Therefore $D^2(A_1)$ is part of $D^2(A_1)$, and this contradicts (i).

(v) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $\text{Fin}$ to $D^2(A_1)$. As, for each $m$, $\bar{\gamma}m \times 0$ belongs to $\text{Fin}$, $\gamma|1$ will belong to the closure $D^2(A_1)$ of $D^2(A_1)$ and therefore, in view of (ii), to $D^2(A_1)^{-\infty}$. So $1$ belongs to $\text{Fin}^{-\infty}$. But $1$ does not belong to $\text{Fin}$. 

5.5 Let $X$ be a subset of $\mathcal{N}$ and $n$ a positive natural number.

We define a subset of $\mathcal{N}$, the $n$-fold disjunction of $X$, notation $D^n(X)$. $D^n(X)$ is the set of all $\alpha$ in $\mathcal{N}$ such that, for some $k < n$, $\alpha^k$ belongs to $X$.

Observe that, for every subset $Z$ of $\mathcal{N}$, $Z$ reduces to $D^n(X)$ if and only if there exist subsets $Z_0, Z_1, \ldots, Z_{n-1}$ of $\mathcal{N}$, each of them reducing to $X$, such that $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_{n-1}$. It is easily seen that for each subset $X$ of $\mathcal{N}$, for every positive
n, \( D^n(X) \) reduces to \( D^{n+1}(X) \).

Observe that, for each positive \( n \), the closure \( \overline{D^n(A_1)} \) of \( D^n(A_1) \) is a spread containing \( 0 \). For every \( \alpha \), for each positive \( n \), \( \alpha \) belongs to \( \overline{D^n(A_1)} \) if and only if for each \( k \), the sequence \( 0 \) passes through one of \( \alpha^0 k, \alpha^1 k, \ldots, \alpha^{n-1} k \).

Recall that we defined a special sequence \( 0^*, 1^*, \ldots \) of stumps in Section 1.5.1. Observe that for every subset \( X \) of \( N \), for every \( n \), \( P((n+1)^*, X) \) equals \( \text{Perhaps}(X, P(n^*, X)) \), and \( P(0^*, X) = X \).

5.6 Theorem:

(i) For each \( n \), the closure \( \overline{D^{n+1}(A_1)} \) of \( D^{n+1}(A_1) \) coincides with \( P(n^*, D^{n+1}(A_1)) \).

(ii) For each \( n \), for each stump \( \sigma \), if \( \overline{D^{n+1}(A_1)} \) coincides with \( P(\sigma, D^{n+1}(A_1)) \), then \( n^* \leq \sigma \).

(iii) For each \( n \), for each function \( \gamma \) from \( N \) to \( N \), if \( \gamma \) maps \( D^n(A_1) \) into \( D^{n+1}(A_1) \), then \( \gamma \) does not map surjectively the closure \( \overline{D^n(A_1)} \) of \( D^n(A_1) \) onto the closure \( \overline{D^{n+1}(A_1)} \).

(iv) For each positive \( n \), the set \( D^{n+1}(A_1) \) does not reduce to the set \( D^n(A_1) \).

Proof:

(i) We use induction. Observe that \( P(0^*, D^1(A_1)) \) coincides with \( D^1(A_1) \), and \( D^1(A_1) \) coincides with its closure \( \overline{D^1(A_1)} \).

Let \( n \) be a natural number and assume \( D^{n+1}(A_1) \) coincides with \( P(n^*, D^{n+1}(A_1)) \). Suppose that \( \alpha \) belongs to \( \overline{D^{n+2}(A_1)} \). Define \( \beta \) such that \( \beta^{n+1} = 0 \) and for each \( p \), if there does not exist \( i \) such that \( p = (n + 1, i) \), then \( \beta(p) = \alpha(p) \). Observe that \( \beta \) belongs to \( D^{n+2}(A_1) \), and if \( \alpha \neq \beta \), then \( \alpha^{n+1} \neq 0 \) and \( \alpha \) belongs to \( \overline{D^{n+1}(A_1)} \) and so to \( P(n^*, D^{n+1}(A_1)) \) and \( P(n^*, D^{n+1}(A_1)) \) is a subset of \( P(n^*, D^{n+2}(A_1)) \).

Therefore \( \alpha \) belongs to \( P((n + 1)^*, D^{n+2}(A_1)) \).

This shows that \( D^{n+2}(A_1) \) forms part of and indeed coincides with \( P((n + 1)^*, D^{n+2}(A_1)) \).

(ii) We use induction. The statement to be proven is trivially true if \( n = 0 \). Now let \( n \) be a natural number and assume that, for each stump \( \sigma \), if \( \overline{D^{n+1}(A_1)} \) coincides with \( P(\sigma, D^{n+1}(A_1)) \), then \( n^* \leq \sigma \). Let \( \sigma \) be a stump such that \( \overline{D^{n+2}(A_1)} \) coincides with \( P(\sigma, D^{n+2}(A_1)) \). For each \( \alpha \) in \( \overline{D^{n+2}(A_1)} \) we can find \( \beta \) in \( D^{n+2}(A_1) \) such that, if \( \alpha \neq \beta \), then \( \alpha \) belongs to \( \bigcup_{k \in \mathbb{N}} P(\sigma^k, D^{n+2}(A_1)) \). For every \( \beta \) in \( D^{n+2}(A_1) \) we can find \( i < n + 2 \) such that \( \beta^i = 0 \). \( D^{n+2}(A_1) \) is a spread containing \( 0 \). We apply the Continuity Principle and find \( m, i \) such that \( i < n + 2 \) and for all \( \alpha \) in \( \overline{D^{n+2}(A_1)} \) passing through \( \overline{0 m} \) there exists \( \beta \) such that \( \beta^i = 0 \) and, if \( \alpha \neq \beta \), then \( \alpha \) belongs to \( \bigcup_{k \in \mathbb{N}} P(\sigma^k, D^{n+2}(A_1)) \). Without endangering generality, we may assume \( i = n + 1 \). Now consider the set \( B \) of all \( \alpha \) in \( \overline{D^{n+2}(A_1)} \) such that \( \overline{\alpha m} = \overline{0 m} \) and \( \alpha^{n+1} = \overline{0 m} * 1 \). Observe that \( B \) is a spread.

39
and forms part of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1)) \). As for each \( \alpha \) in \( B \), \( \alpha^{n+1} \neq \emptyset \), one may prove, for every stump \( \tau \), for each \( \alpha \) in \( B \), if \( \alpha \) belongs to \( \mathbb{P}(\tau, D^{n+2}(A_1)) \), then \( \alpha \) belongs to \( \mathbb{P}(\tau, D^{n+1}(A_1)) \). So \( B \) forms part of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). In fact, the set of all \( \alpha \) in \( D^{n+1}(A_1) \) passing through \( \emptyset \) forms part of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). (Given any such \( \alpha \), consider \( \beta \) in \( B \) such that for every \( i < n + 2 \), \( \beta^i = \alpha^i \) and observe that \( \beta \) belongs to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \) and therefore also \( \alpha \) does so.) The set \( D^{n+1}(A_1) \) is a spread containing \( \emptyset \). We apply the Continuity Principle a second time and find \( q, k \) such that every \( \alpha \) in \( D^{n+1}(A_1) \) passing through \( \emptyset \) belongs to \( \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). It is not difficult to see that now also \( D^{n+1}(A_1) \) forms part of \( \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \), therefore \( n^* \leq \sigma^k \) and \( (n + 1)^* \leq \sigma^k \).

(iii) The argument is similar to the argument for Theorem 3.20 and left to the reader.

(iv) Let \( n \) be a positive natural number and suppose \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( D^{n+1}(A_1) \) to \( D^n(A_1) \). For each \( i < n + 1 \), let \( B_i \) be the set of all \( \alpha \) such that \( \alpha^i = \emptyset \). Observe that each \( B_i \) is a spread containing \( \emptyset \) and that \( \gamma \) maps \( \bigcup_{i < n+1} B_i \) into \( \bigcup_{i < n+1} B_i \). Applying the Continuity Principle \( n + 1 \) times we find natural numbers \( p_0, p_1, \ldots, p_n \) and \( k_0, k_1, \ldots, k_n \) such that for each \( i < n + 1 \), \( k_i < n \) and for each \( \alpha \) in \( B_i \) passing through \( \emptyset \) \( p_i \), \( \gamma(\alpha) \) will belong to \( B_{k_i} \). Without loss of generality we may assume \( k_0 = k_1 = 0 \). Let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha \), \\
\( \delta(\alpha)^0 = \emptyset p_0 * \alpha^0 \) and \\
\( \delta(\alpha)^1 = \emptyset p_1 * \alpha^1 \) and for every \( i \) such that \( 1 < i < n + 1 \), \\
\( \delta(\alpha)^i = \emptyset p_i * 1 \).

Observe that \( \alpha \) belongs to \( D^2(A_1) \) if and only if \( \delta(\alpha) \) belongs to \( D^2(A_1) \) if and only if \( \gamma(\delta(\alpha)) = \emptyset \). Therefore \( D^2(A_1) \) reduces to \( A_1 \). But, as we saw in Theorem 5.4, \( D^2(A_1) \) does not reduce to \( A_1 \).

5.7 Not surprisingly, the facts reported in the last few Theorems have their counterparts in the domain of the real numbers. We leave it to the reader to define an operation Perhaps for subsets of \( \mathbb{R} \) like we did for subsets of \( \mathcal{N} \).

5.8 Theorem:

(i) The union of the closed real intervals \([0, 1]\) and \([1, 2]\) does not really-coincide with the closed real interval \([0, 2]\).

\([0, 2]\) is the least closed set containing both \([0, 1]\) and \([1, 2]\), and also the least perhapsive set containing both \([0, 1]\) and \([1, 2]\).

\([0, 2]\) coincides with \( \text{Perhaps}([0, 1] \cup [1, 2]) \).

\([0, 1] \cup [1, 2]\) is not perhapsive and does not belong to \( \mathbf{\Pi}^0_2 \).

(ii) The open real interval \((0, 1)\) does not coincide with any finite union of closed sets.

Proof: We leave the proof of (i) to the reader.
The following argument for (ii) is due to the referee of an earlier version of this paper. Let \( n \) be a natural number and suppose \( X_0, X_1, \ldots, X_{n-1} \) are really-closed sets such that \((0,1)\) coincides with \( X_0 \cup X_1 \cup \cdots \cup X_{n-1} \). Let \( Y_0, Y_1, \ldots, Y_{n-1} \) be really-open subsets of \( \mathbb{R} \) such that for each \( i < n \), \( X_i = \mathbb{R} \setminus Y_i \). Observe that 0 does not belong to any set \( Y_i \), therefore,

\[
\forall i < n \neg \exists m \left( (0, \frac{1}{m}) \text{ is a subset of } Y_i \right), \quad \text{and, by intuitionistic logic,} \\
\neg \forall i < n \exists m \left( (0, \frac{1}{m}) \text{ is a subset of } Y_i \right).
\]

Observe that if \( \forall i < n \exists m \left( (0, \frac{1}{m}) \right) \) is a subset of \( Y_i \) we may find \( m \) such that \( \forall i < n \left( (0, \frac{1}{m}) \right) \) is a subset of \( Y_i \), and the number \( \frac{1}{2m} \) will not belong to any \( X_i \). Contradiction.

Therefore \( \neg \forall i < n \exists m \left( (0, \frac{1}{m}) \right) \) is a subset of \( Y_i \) and there do not exist closed sets \( X_0, \ldots, X_{n-1} \) such that \((0,1)\) coincides with \( X_0 \cup X_1 \cup \cdots \cup X_{n-1} \).

5.9 We return to Baire space \( \mathcal{N} \). We give the new name \( T \) to the set \( CB_{2^*} \) that we introduced in Section 3.1, so \( T \) coincides with \( \{0\} \cup \{ \overline{n} \ast (1) \ast \emptyset \mid n \in \mathbb{N} \} \).

Observe that \( T \) contains all elements of Cantor space \( \mathcal{C} \) that assume the value 1 either not at all or exactly one time. Its closure \( \overline{T} \) consists of all elements of Cantor space \( \mathcal{C} \) that do not assume the value 1 two times.

The third statement of the following result improves upon conclusion (v) of Theorem 5.4.

5.10 Theorem:

(i) For all closed sets \( X, Y \), the set \( X \cup Y \) coincides with the set \((X \cup Y)^\sim\).

(ii) For each positive \( n \), for every \( n \)-sequence \( X_0, \ldots, X_{n-1} \) of closed sets, the set \( X_0 \cup \cdots \cup X_{n-1} \) coincides with the set \((X_0 \cup \cdots \cup X_{n-1})^\sim\).

(iii) For each positive \( n \), the set \( T \) does not reduce to the set \( D^n(A_1) \).

(iv) For each positive \( n \), the closure \( \overline{T} \) of \( T \) is a subset of \( D^n(A_1) \) but not of \( D^n(A_1) \).

Proof: (i) Let \( X, Y \) be closed sets. Let \( C, D \) be decidable subsets of \( \mathbb{N} \) such that every \( \alpha, \alpha \) belongs to \( X \) if and only if, for each \( n \), \( \overline{\alpha n} \) belongs to \( C \), and \( \alpha \) belongs to \( Y \) if and only if, for each \( n \), \( \overline{\alpha n} \) belongs to \( D \). Assume that \( \alpha \) belongs to \( X \cup Y \), then for each \( n \), either for every \( m \leq n \), \( \overline{\alpha m} \) belongs to \( C \), or for every \( m \leq n \), \( \overline{\alpha m} \) belongs to \( D \).

Observe that if there exists \( n \) such that \( \overline{\alpha n} \) does not belong to \( C \), then for every \( n \), \( \overline{\alpha m} \) belongs to \( D \). Therefore, also if \( \neg \neg \) (there exists \( n \) such that \( \overline{\alpha m} \) does not belong to \( C \)), then for every \( n \), \( \overline{\alpha m} \) belongs to \( D \), that is \( \alpha \) belongs to \( Y \).

So if \( \alpha \not\in X \), then \( \alpha \in Y \), and consequently \( \neg \neg (\alpha \in X \lor \alpha \in Y) \).

We thus see that \( X \cup Y \) forms part \( (X \cup Y)^\sim \).

It is obvious that \((X \cup Y)^\sim \) is a subset of \( \overline{X \cup Y} \).

(ii) The proof is left to the reader.

(iii) We use induction. We have seen, in Theorem 3.7, that the set \( T = CB_{2^*} \) does not belong to \( \Pi^0_2 \), so \( T \) is not closed and does not reduce to \( D^1(A_1) \). Let \( n \) be a

41
natural number and assume that $T$ does not reduce to $D^n(A_1)$. Suppose that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $T$ to $D^{n+1}(A_1)$. Calculate $i$ such that $i < n + 1$ and $(\gamma|0)^i = 0$. Without endangering generality we assume $i = n$, that is $(\gamma|0)^n = 0$. Using the First Axiom of Countable Choice, determine $\alpha$ in $\mathcal{N}$ such that, for each $j$, $\alpha(j) < n + 1$ and $(\gamma|0^j * \{1\} * 0^j)^{\alpha(j)} = 0$. We claim that it is possible to decide, for each $j$, if there exists $k > j$ such that $\alpha(k) = n$ or not. For suppose $j$ is a natural number and let $\beta$ be the element of $C$ such that, for each $p$, $\beta(p) = 1$ if and only if $p$ is the least $k > j$ such that $\alpha(k) = n$. Observe that $(\gamma|\beta)^n = 0$, therefore $\gamma|\beta$ belongs to $D^{n+1}(A_1)$, and $\beta$ belongs to $T$, so either $\beta = 0$ or $\beta \neq 0$. In the first case there does not exist $k > j$ such that $\alpha(k) = n$, in the second case there does.

Assume now that $j$ is a natural number and there is no $k > j$ such that $\alpha(k) = n$. Observe that the sequence $0$ belongs to the closure of the set $\{\beta | \beta \in \mathcal{N} \exists i < n ![\gamma|\beta]^i = 0\}$. As the latter set is a finite union of closed sets, we use (ii) and conclude $\neg \exists i < n ![\gamma|0]^i = 0$. If there exist $i < n$ such that $(\gamma|0)^i = 0$, we consider a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\beta$, $n$ the function $\delta$ maps the sequence $0^n * \{1\} * \beta$ onto the sequence $\gamma|0^n(n + j) * \{1\} * \beta$ and, for every $\beta$, if $\beta(0)$ differs from both 0, 1, then $\delta|\beta = \gamma|\beta$. Observe that $\delta$ reduces $T$ to $D^n(A_1)$. According to the induction hypothesis, $T$ does not reduce to $D^n(A_1)$ so there is no $i < n$ such that $(\gamma|0)^i = 0$. We conclude that we can never choose the first of the above-mentioned two alternatives, therefore, for every $j$, there exists $k > j$ such that $\alpha(k) = n$. We now define a strictly increasing sequence $\zeta$ in $\mathcal{N}$ such that for each $n$, $\alpha(\zeta(n)) = n$. We construct a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\beta$, $n$, the function $\delta$ maps the sequence $0^n * \{1\} * \beta$ onto the sequence $\gamma|0^n(n + j) * \{1\} * \beta$, and for every $\beta$, if $\beta(0)$ differs from both 0, 1, then $\delta|\beta = \gamma|\beta$. Observe that, for every $\beta$, $\beta$ belongs to $T$ if and only if $(\delta|\beta)^n = 0$.

Therefore $T$ is closed. Contradiction.

(iv) The proof is left to the reader. ☐

5.11 It follows from the fifth statement of Theorem 4.5 that for every enumerable and discrete subset $D$ of $\mathcal{N}$ that is also dense-in-itself there are uncountably many sets $X$ with the property $D \subseteq X \subseteq D^-$. This is a consequence of the fact that such a set $D$ has unbounded perhapsity. We now intend to show that there are also very many sets $X$ with the property $T \subseteq X \subseteq T^\sim$, although $T$ coincides with $\text{Perhaps}(T, T)$ and $T$ has perhapsity $1^*$. We need some preparations.

Let $f$ be a function from Cantor space $\mathcal{C}$ to itself such that for every $\alpha$, for every $n$, $(f|\alpha)(n) = 1$ if and only if there exists $m \leq n$ such that $n = \alpha m$. Observe that for all $\alpha$ in $\mathcal{C}$ there exist infinitely many $j$ such that $(f|\alpha)(j) = 1$ and that for all $\alpha, \beta$ in $\mathcal{C}$, if $\alpha \# \beta$, then there are only finitely many $j$ such that $\alpha(j) = \beta(j) = 1$.

Let $\tilde{\cdot}$ be the binary operation on $\mathcal{C}$ that is defined by:

\[
(\alpha - \beta)(n) := \alpha(n) - \beta(n).
\]

Let Max be the binary operation on $\mathcal{C}$ that is defined by: for all $\alpha, \beta$ in $\mathcal{C}$, for all $n$, $(\alpha \text{ Max} \beta)(n) := \alpha(n) - \beta(n)$.
\((\text{Max}(\alpha, \beta))(n) = \text{Max}(\alpha(n), \beta(n))\).

For every \(\alpha\) in \(\mathcal{C}\), we define subsets \(\mathbb{T}(\alpha)\) and \(\mathbb{U}(\alpha)\) of \(\mathcal{C}\), as follows:

\[
\mathbb{T}(\alpha) := \{0\} \cup \{ \eta n \ast \langle 1 \rangle \ast \eta | n \in \mathbb{N} \text{ and } \alpha(n) = 1 \} \text{ and } \mathbb{U}(\alpha) := \overline{\mathbb{T}(f|\alpha)} \cup \overline{\mathbb{T}(1 - (f|\alpha))}.
\]

Observe that, for every \(\alpha\) in \(\mathcal{C}\) the closure \(\overline{\mathbb{T}(\alpha)}\) of \(\mathbb{T}(\alpha)\) is a spread containing 0.

5.12 Theorem:

(i) For all \(\alpha, \beta, \gamma\) in \(\mathcal{C}\), if \(\overline{\mathbb{T}(\alpha)}\) is a subset of \(\overline{\mathbb{T}(\beta)} \cup \overline{\mathbb{T}(\gamma)}\), then either there exists \(n\) such that for every \(j > n\), if \(\alpha(j) = 1\), then \(\beta(j) = 1\) or there exists \(n\) such that for every \(j > n\), if \(\alpha(j) = 1\) then \(\gamma(j) = 1\).

(ii) For all \(\alpha, \beta\) in \(\mathcal{C}\), \(\mathbb{T}(\alpha) \cup \mathbb{T}(\beta) = \mathbb{T}(\text{Max}(\alpha, \beta))\).

(iii) For all \(\alpha, \beta\) in \(\mathcal{C}\), if \(\mathbb{U}(\alpha)\) is a subset of \(\mathbb{U}(\beta)\), then \(\alpha = \beta\).

(iv) For each \(\alpha\) in \(\mathcal{C}\), \(\mathbb{T}(1 - (f|\alpha)) \subseteq \mathbb{T}(\infty)\), and the set \(D^2(A_1)\) reduces to \(\mathbb{U}(\alpha)\) and \(\mathbb{U}(\alpha)\) does not coincide with either \(\mathbb{T}\) or \(\mathbb{T}^-\).

(v) There is no set \(C\) such that \(\mathbb{T} \subseteq C \subseteq \mathbb{T}^-\) and the set \(D^3(A_1)\) reduces to the set \(C\).

(vi) For each positive \(n\) there exists a set \(C\) such that \(\mathbb{T} \subseteq C \subseteq \mathbb{T}^-\) and \(C\) is a union of \(n + 1\) closed sets not coinciding with any union of \(n\) closed sets.

(vii) There is a set \(C\) such that \(\mathbb{T} \subseteq C \subseteq \mathbb{T}^-\) and \(C\) is a countable union of closed sets not coinciding with any finite union of closed sets.

Proof: (i) Suppose that \(\overline{\mathbb{T}(\alpha)}\) is a subset of \(\overline{\mathbb{T}(\beta)} \cup \overline{\mathbb{T}(\gamma)}\). Using the Continuity Principle we find \(n\) such that either every \(\delta\) in \(\overline{\mathbb{T}(\alpha)}\) passing through \(\eta n\) belongs to \(\overline{\mathbb{T}(\beta)}\) or every \(\delta\) in \(\overline{\mathbb{T}(\alpha)}\) passing through \(\eta n\) belongs to \(\overline{\mathbb{T}(\gamma)}\). If the first alternative obtains, then for all \(j > n\), \(\alpha(j) = 1\) entails \(\beta(j) = 1\), and if the second one does, then for all \(j > n\), \(\alpha(j) = 1\) entails \(\gamma(j) = 1\).

(ii) easily follows from (i).

(iii) Suppose that that \(\mathbb{U}(\alpha)\) is a subset of \(\mathbb{U}(\beta)\). In particular, \(\mathbb{T}(1 - (f|\alpha))\) is a subset of \(\overline{\mathbb{T}(f|\beta)} \cup \overline{\mathbb{T}(1 - (f|\beta))}\). Observe that there is no \(n\) such for every \(j > n\), \((f|\alpha)(j) = 0\) entails \((f|\beta)(j) = 1\), therefore, by (i), there exists \(n\) such that for every \(j > n\), \((f|\alpha)(j) = 0\) entails \((f|\beta)(j) = 0\), and thus \(\alpha = \beta\).

(iv) Let \(\alpha\) belong to \(\mathcal{C}\). We may construct a function \(\gamma\) from \(\mathcal{N}\) to \(\mathcal{N}\) reducing \(D^2(A_1)\) to \(\mathbb{U}(\alpha) = \mathbb{T}(f|\alpha) \cup \mathbb{T}(1 - (f|\alpha))\). It suffices to ensure that for every \(\beta\), for every \(n\), firstly, if both \(\beta^0\) and \(\beta^1\) pass through \(\eta n\), then \(\gamma|\beta\) passes through \(\eta n\), and secondly, if \(n\) is the least \(k\) such that exactly one of \(\beta^0, \beta^1\) passes through \(\eta k\), then \(\gamma|\beta\) must pass through some finite sequence \(\eta\ell \ast \langle 1 \rangle\), where \((f|\alpha)(\ell) = 1\) if \(\beta^0\) passes through \(\eta n\) and \((f|\alpha)(\ell) = 0\) if \(\beta^1\) passes through \(\eta n\), and thirdly, as long as at least one of \(\beta^0, \beta^1\) passes through \(\eta n\), then \((\gamma|\beta)(n)\) assumes at most one value different from 0, but if neither one of \(\beta^0, \beta^1\) passes through \(\eta n\), then \((\gamma|\beta)(n)\) assumes two times a value different from 0 (and so does not belong to \(T^-\)). If \(\gamma\) satisfies these conditions it will be clear that, for all \(\beta\) in \(\mathcal{C}\), \(\beta^0 = 0\) if and only if \(\gamma|\beta\) belongs to \(\overline{\mathbb{T}(f|\alpha)}\), and \(\beta^1 = 0\) if and only if \(\gamma|\beta\) belongs to \(\overline{\mathbb{T}(1 - (f|\alpha))}\). So \(\gamma\) reduces \(D^2(A_1)\) to \(\mathbb{U}(\alpha)\).

43
Observe that $D^2(A_1)$ does not reduce to either $T$ or $T^-$, so $U(\alpha)$ is different from both $T$ and $T^-$. 

(v) Suppose $C$ is a set such that $T \subseteq C \subseteq T^-$ and $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^3(A_1)$ to $C$. We claim that $\gamma$ maps every $\alpha$ in $(D^3(A_1))^-$ onto $0$. We prove this claim as follows:

Assume $\alpha$ belongs to $(D^3(A_1))^-$ and $\gamma(\alpha) \neq 0$. Let $m$ be the least $p$ such that $\gamma(\alpha)(p) \neq 0$ and calculate $n$ such that, for each $\beta$ passing through $\alpha n$, $\gamma(\alpha)(m) = \gamma(\beta)(m)$. Observe that we must have $\gamma(\alpha)(m + 1) = \alpha m * \langle 1 \rangle$, otherwise $\gamma(\alpha)$ would not belong to $T$, and $\alpha$ itself would not belong to $D^3(A_1)$. There are two elements $i$ of $\{0, 1, 2\}$ such that $\alpha^i n = \alpha^i n$, and without loss of generality we may assume $\alpha^0 n = \alpha^0 n = \alpha^0 n$. Define a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\beta$, $(\delta(\beta))^0 = \alpha^0 n * \beta^0$, and $(\delta(\beta))^1 = \alpha^1 n * \beta^1$, and $(\delta(\beta))^2 = \alpha^2 n * \beta^2$. Observe that for every $\beta$, $\beta$ belongs to $D^2(A_1)$ if and only if $\gamma(\delta(\beta))$ equals $\alpha^i m * \langle 1 \rangle$, therefore $D^2(A_1)$ is closed. Contradiction. Therefore $\gamma$ indeed maps $(D^3(A_1))^-$ onto $\{0\}$ and $(D^3(A_1))^-$ is a subset of $D^3(A_1)$. Observe that $(D^3(A_1))^-$ coincides with $D^3(A_1)$ and $D^3(A_1)$ is not closed, see Theorems 5.4 and 5.10. Contradiction.

(vi) For each positive $n$, for each $i < n + 1$, we let $E^i_n$ be the closure of the set $\{0\} \cup \{0(k(n + 1) + i) * \langle 1 \rangle * 0|k \in \mathbb{N}\}$, and we define $C_n := \bigcup_{i < n+1} E^i_n$. It will be clear that $C_n$ is a union of $n + 1$ closed sets and that $T \subseteq C_n \subseteq T^-$. Suppose that we find closed sets $F_0, F_1, \ldots, F_{n-1}$ such that $C_n$ coincides with $\bigcup F_i$. Applying the Continuity Principle $n + 1$ times we find for each $i < n$ numbers $m_i, n_i$ such that every $\alpha$ in $E^i_n$ passing through $\alpha m_i$ belongs to $F_{n_i}$.

Without endangering generality we may assume $n_0 = n_1 = 0$. It easily follows that the closure of $E^0_n \cup E^1_n$ forms part of $\bigcup F_i$ and therefore of $C_n$. Using once more the Continuity Principle, we obtain a contradiction.

(vii) For each $i$, we let $E^i$ be the closure of the set $\{0\} \cup \{0(2k \cdot (2i + 1) - 1) * \langle 1 \rangle * 0|k \in \mathbb{N}\}$ and we define $C := \bigcup_{i \in \mathbb{N}} E^i$. It is not difficult to see that $C$ satisfies the requirements.

5.13 For each positive $n$, there exists a subset of $\mathbb{R}$ that is a union of $n + 1$ closed sets and does not coincide with any union of $n$ closed sets. In order to find such sets, one may start from the set $T^*$ consisting of the real numbers really-coinciding with one of the rational numbers $0, 1, 2, 3, 4, \ldots$ and reconsider the sixth statement of Theorem 5.12.

5.14 Let $X, Y$ be subsets of $\mathcal{N}$. We let the conjunction of $X$ and $Y$, notation: $C(X, Y)$, be the set of all $\alpha$ such that $\alpha^0$ belongs to $X$ and $\alpha^1$ belongs to $Y$.

More generally, let $n$ be a positive natural number and let $X_0, \ldots, X_{n-1}$ be an $n$-sequence of subsets of $\mathcal{N}$. We let the conjunction of $X_0, \ldots, X_{n-1}$, notation $C(X_0, \ldots, X_{n-1})$ or $C_{i=0}^{n-1}(X_i)$, be the set of all $\alpha$ such that, for all $j < n$, $\alpha^j$ belongs to $X_j$. 

44
Observe that for every positive \( n \), for all subsets \( Z, X_0, \ldots, X_{n-1} \) of \( \mathcal{N} \), \( Z \) reduces to \( C(X_0, \ldots, X_{n-1}) \) if and only if there exists an \( n \)-sequence \( Z_0, \ldots, Z_{n-1} \) of subsets of \( \mathcal{N} \) such that \( Z = Z_0 \cap \cdots \cap Z_{n-1} \) and for each \( j < n \), \( Z_j \) reduces to \( X_j \).

Let \( X \) be a subset of \( \mathcal{N} \) and \( n \) a positive natural number. We let the \( n \)-fold conjunction of \( X \), notation: \( C^n(X) \), be the set of all \( \alpha \) such that for all \( j < n \), \( \alpha^j \) belongs to \( X \).

Observe that for every positive \( n \), for all subsets \( Z, X \) of \( \mathcal{N} \), \( Z \) reduces to \( C^n(X) \) if and only if there exists an \( n \)-sequence \( Z_0, \ldots, Z_{n-1} \) of subsets of \( \mathcal{N} \), each of them reducing to \( X \), such that \( Z = Z_0 \cap \cdots \cap Z_{n-1} \).

Recall that for every natural number \( m \) there exists a natural number \( k = \text{length}(m) \) and natural numbers \( (m)_0, \ldots, (m)_{k-1} \) such that \( m = ((m)_0, \ldots, (m)_{k-1}) \).

For all natural numbers \( m, n \) we define: \( m \text{ bows to } n \), notation: \( m \prec n \), if and only if \( \text{length}(m) = \text{length}(n) \) and for all \( i < \text{length}(m), (m)_i < (n)_i \).

For all natural numbers \( m \) we let \( P_m \) be the set of all \( \alpha \) such that for every \( j < \text{length}(m) \), \( \alpha^j(m)_j = 0 \).

Observe that, for every \( m, P_m \) is a spread containing \( 0 \).

For all natural numbers \( n \), we let \( Q_n \) be the (finite) union of all sets \( P_m \) such that \( m \) bows to \( n \). Observe that, for each \( n, k \), if \( k = \text{length}(n) \), then \( Q_n \) coincides with \( C(D^{(n)_0}(A_1), \ldots, D^{(n)_{k-1}}(A_1)) \).

Observe that \( Q_0 = Q_\emptyset = \mathcal{N} \) and, for each \( n, \) if there exists \( i < \text{length}(n) \) such that \( (n)_i = 0 \), then \( Q_n = 0 \).

For all \( n, j, p \) such that \( j < \text{length}(n) \) and \( p > 0 \) we let \( c = c(n, j, p) \) be the natural number satisfying the following conditions: \( \text{length}(c) = \text{length}(n) \) and for all \( i < \text{length}(n) \), if \( i \neq j \), then \( (c)_i = (n)_i \), and \( (c)_j \) is the greatest natural number \( q \) such that \( p \cdot q < (n)_j \).

5.15 Theorem:

(i) For all positive natural numbers, \( p, q \), the set \( C(D^p(A_1), D^q(A_1)) \) reduces to the set \( D^{pq}(A_1) \).

(ii) For all positive natural numbers \( p, m, n \), the set \( Q_p \ast m \) reduces to the set \( Q_n \) if and only if there exists \( j < \text{length}(n) \) such that the set \( Q_m \) reduces to the set \( Q_{c(n,j,p)} \).

Proof: (i) Let \( p, q \) be positive natural numbers. Let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that for all \( i < p, j < q, \) for all \( n, \) the number \( (\gamma_i(n)^{pq}) \) equals the number \( \text{Min}(\alpha^0(n), \alpha^1(n)) \). The function \( \gamma \) reduces the set \( C(D^p(A_1), D^q(A_1)) \) to the set \( D^{pq}(A_1) \).

(ii) Let \( p, m, n \) be positive natural numbers and assume \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( Q_{(p) \ast m} \) to \( Q_n \). Using the Continuity Principle a finite number of times we find \( s \) in \( \mathbb{N} \) and a function \( F \) from the set of numbers bowing to \( (p) \ast m \) to the set of numbers bowing to \( n \) such that, for every \( t < (p) \ast m \), the function \( \gamma \) maps every \( \alpha \) in \( P_t \), passing through \( 0 \)'s into the set \( P_{F(t)} \).
We claim that there exists $j < \text{length}(n)$ such that for all $t, u$ bowing to $(p) \ast m$, if $t_0 \neq u_0$, then $(F(t))_j \neq (F(u))_j$.

For suppose there is no such $j$. Let $X$ be the set of all $\alpha$ in Cantor space $\mathcal{C}$ such that $\alpha^0$ assumes at most one time the value 1 and for each $i$, if there is no $q$ such that $i = (0, q)$, then $\alpha(i) = 0$. Observe that $X$ is a spread containing $0$. Remark also that for every $\alpha$ in $X$, for every $j < \text{length}(n)$, the sequence $\gamma | \alpha$ belongs to $D^{(n)j} (A_1)$. Indeed, let $j$ be a natural number and let $t, u$ be numbers bowing to $(p) \ast m$ such that $t_0 \neq u_0$ and $k := (F(t))_j = (F(u))_j$. Observe that for every $\alpha$, if $\alpha$ belongs to $P_t \cup P_u$, then $(\gamma | \alpha)_j = 0$. As $X$ forms part of $(P_t \cup P_u)^\sim$, also for every $\alpha$, if $\alpha$ belongs to $X$, then $(\gamma | \alpha)_j = 0$.

We conclude that $\gamma$ maps $X$ into $Q_n$, therefore $X$ forms part of $Q(p) \ast m$, and, in particular, for every $\alpha$ in $X$ there exists $k$ such that $\alpha^{0,k} = 0$. Therefore the spread consisting of all $\alpha$ in $\mathcal{C}$ that assume the value 1 at most one time forms part of $D^0 (A_1)$. Using the Continuity Principle we find $s, k$ such that every such $\alpha$ passing through $0s$ has the property $\alpha^k = 0$. This is false, and our claim holds true.

Now choose $j < \text{length}(n)$ such that for all $t, u$ bowing to $(p) \ast m$, if $t_0 \neq u_0$, then $(F(t))_j \neq (F(u))_j$.

For each $k < p$ we let $C_k$ be the set of all numbers $(F(t))_j$ where $t$ is some number bowing to $(p) \ast m$ such that $t_0 = k$.

Observe that for all $k, \ell < p$, if $k \neq \ell$, then $C_k$ and $C_\ell$ are mutually disjoint subsets of the set $\{0, 1, \ldots, (n)j - 1\}$. We now determine $k < p$ such that for every $\ell < p$, the number of elements of $C_k$ does not exceed the number of elements of $C_\ell$. Observe that the number of elements of $C_k$ does not exceed the greatest natural number $q$ such that $p \cdot q \leq (n)j$. In order to see that $Q_m$ reduces to $Q_c(n,j,p)$ we define a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$, $(\delta | \alpha)^{0,k} = 0$ and for all $j < \text{length}(m)$, $(\delta | \alpha)^{j+1} = \alpha^j$.

We then remark that for every $\alpha$, $\alpha$ belongs to $Q_m$ if and only if $\gamma | (\delta | \alpha)$ belongs to $Q_n$ and, in addition, there is some $i$ in $C_k$ such that $(\gamma | (\delta | \alpha))^{2,i} = 0$. We conclude that, if $Q(p) \ast m$ reduces to $Q_n$, then there exists $j > n$ such that $Q_m$ reduces to $Q_c(n,j,p)$.

Now assume that $p, m, n$ are natural numbers such that for some $j < \text{length}(n)$, the set $Q_m$ reduces to the set $Q_c(n,j,p)$. Using (i) one may prove that $Q(p) \ast m$ reduces to $Q_n$.

5.16 Theorem 5.15 enables us, given any $m, n$, to decide in finitely many steps if the set $Q_m$ reduces to the set $Q_n$ or not.

We may prove, for instance, that the sets $C^4 (D^2(A_1))$ and $C^2 (D^3(A_1))$ do not reduce to each other.

6 Forming limits and finding more hierarchies

We consider various upper bounds for a given sequence of subsets of $\mathcal{N}$. If upper bounds are taken repeatedly, hierarchies arise similar to the Cantor-Bendixson-
hierarchy discussed in Section 3.

6.1 For all subsets $X, Y$ of $\mathcal{N}$ we let the (disjoint) sum of $X$ and $Y$, notation $X \oplus Y$, be the set $(0) \ast X \cup (1) \ast Y$.

For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ we let the (countable) (disjoint) sum of the sequence $X_0, X_1, \ldots$, notation $\sum_{n \in \mathbb{N}} X_n$, be the set $\bigcup_{n \in \mathbb{N}} (n) \ast X_n$.

The following Theorem establishes that (the reducibility degrees of) the subsets of $\mathcal{N}$ behave as the elements of a countably complete upper semi-lattice.

6.2 Theorem:

(i) For all subsets $X, Y, Z$ of $\mathcal{N}$, the set $X \oplus Y$ reduces to $Z$ if and only if both $X$ and $Y$ reduce to $Z$.

(ii) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, for every subset $Z$ of $\mathcal{N}$, the set $\sum_{n \in \mathbb{N}} X_n$ reduces to $Z$ if and only if, for each $n$, $X_n$ reduces to $Z$.

Proof: The proof is straightforward and left to the reader. One has to apply the Second Axiom of Countable Choice when proving (ii). □

6.3 Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$. We consider a sample of four from the many sets $Y$ there are with the property that each set $X_n$ reduces to $Y$. We define:

$$0 - \text{Lim}(X_n) := \bigcup_{n \in \mathbb{N}} (0) \ast n \ast (1) \ast X_n,$$

$$1 - \text{Lim}(X_n) := \{ \alpha | \alpha \in \mathcal{N} | \alpha = 0 \text{ or } \alpha \text{ belongs to } 0 - \text{Lim}(X_n) \},$$

$$2 - \text{Lim}(X_n) := \{ \alpha | \alpha \in \mathcal{N} | \text{If } \alpha \neq 0, \text{ then } \alpha \text{ belongs to } 0 - \text{Lim}(X_n) \},$$

and

$$3 - \text{Lim}(X_n) := \bigcup_{n \in \mathbb{N}} (0) \ast n \ast (1) \ast X_{2n} \cup \bigcup_{n \in \mathbb{N}} (2n) \ast (3) \ast X_{2n+1} + 1.$$}

Suppose that, for each $n$, the set $X_n$ coincides with the set $\{0\}$. Observe that now, for all $i, j < 3$, if $i - \text{Lim}(X_n)$ reduces to $j - \text{Lim}(X_n)$, then $i = j$. Moreover, the set $1 - \text{Lim}(X_n)$ coincides with $CB_{2^*}$ and the set $2 - \text{Lim}(X_n)$ coincides with the closure $CB_{2^*}$ of $CB_{2^*}$. Next assume that, for each $n$, the set $X_{2n}$ coincides with $C^n(D^3(A_1))$ and the set $X_{2n+1}$ coincides with $C^1(D^4(A_1), C^n(D^2(A_1)))$. Then the set $3 - \text{Lim}(X_n)$ reduces to the set $0 - \text{Lim}(X_n)$ but $0 - \text{Lim}(X_n)$ does not reduce to $3 - \text{Lim}(X_n)$.

We leave it to the reader to verify these statements.

It will be clear that every infinite sequence of subsets of $\mathcal{N}$ admits of a wide variety of upper bounds.

6.4 Let $n$ be a positive natural number and let $X_0, \ldots, X_{n-1}$ be an $n$-sequence of subsets of $\mathcal{N}$. We let the disjunction of $X_0, \ldots, X_{n-1}$, notation $D(X_0, \ldots, X_{n-1})$ or
$D_{i=0}^{n-1}(X_i)$, be the set of all $\alpha$ such that, for some $i < n$, $\alpha^i$ belongs to $X_i$.

We introduce, for every stump $\sigma$, a subset $DCD_\sigma$ of $\mathcal{N}$, as follows, by transfinite induction:

1. $DCB_1 := \{0\}$
2. For every non-empty stump $\sigma$, $DCB_\sigma := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{m \ast \langle 1 \rangle \ast D_{i=0}^{n-1}(DCB_{\sigma^i})\}$.

We have added the letter $D$ of Disjunction to the letters $C$, $B$ of Cantor and Bendixson. We need a principle of induction in order to prove nice properties of this collection of sets. Some other variants of induction on the set of stumps are mentioned in [23].

Let $(\sigma_0, \ldots, \sigma_{m-1})$ and $(\tau_0, \ldots, \tau_{n-1})$ be finite sequences of stumps. We call the sequence $(\tau_0, \ldots, \tau_{n-1})$ a simplification of the sequence $(\sigma_0, \ldots, \sigma_{m-1})$ if and only if there exists $i < m$ and a positive natural number $k$ such that $\sigma_i$ is non-empty and $n = m + (k - 1)$ and for each $j < i$, $\sigma_j = \tau_j$, and for each $j < k$, $\tau_{i+j} = (\sigma_i)^j$ and, for each $j < m - i - 1$, $\tau_{i+k+j} = \tau_{i+j+1}$. So the sequence $(\tau_0, \ldots, \tau_{n-1})$ results from the sequence $(\sigma_0, \ldots, \sigma_{m-1})$ if one replaces $(\sigma_i)$ by $((\sigma_i)^0, \ldots, (\sigma_i)^{k-1})$.

We denote the set of finite sequences of stumps by $\text{Stp}^*$.

**6.5 Theorem:** *(A principle of Induction on $\text{Stp}^*$)*

Let $P$ be a subset of $\text{Stp}^*$ such that every finite sequence of stumps belongs to $\text{Stp}^*$ as soon as each one of its simplifications belongs to $\text{Stp}^*$.

Then every finite sequence of stumps belongs to $\text{Stp}^*$.

**6.6** The proof of Theorem 6.5 requires some preparations and will be given only in Section 6.11.

Let $\sigma$ be a stump. We let $\mathbb{B}(\sigma)$ be the set of all natural numbers belonging to $\sigma$, that is such that $\sigma(n) = 0$, see Section 1.5.3.

It is wise to think of $\mathbb{B}(\sigma)$ as a set of (code numbers of) finite sequences of natural numbers.

Let $A$ be a countable set and $<_0$ be a binary relation on $A$. Let $P$ be a subset of $A$. $P$ is called $<_0$-hereditary if and only if, for every $a$ in $A$, $a$ belongs to $P$ as soon as every $b$ in $A$ such that $b <_0 a$ belongs to $P$. $<_0$ is called an inductive relation on $A$ if and only if every $<_0$-hereditary subset of $A$ coincides with $A$. $<_0$ is called a stumpy relation on $A$ if and only if there exists a stump $\sigma$ such that $(A, <_0)$ embeds isomorphically into $(\mathbb{B}(\sigma), <^\#)$ where, for all $m, n$ in $\mathbb{N}$, $m <^\# n$ if and only if there exists $p$ such that $m = n \ast \langle p \rangle$, that is, $m$ is, as a finite sequence, an immediate successor of $n$. So $(A, <_0)$ is stumpy if and only if there exists an injective mapping $f$ from $A$ into some $\mathbb{B}(\sigma)$ such that for every $a_0, a_1$ in $A$, $a_0 <_0 a_1$ if and only if $f(a_0) <^\# f(a_1)$.

Let $A, B$ be countable sets and $<_0 <_1$ binary relation on $A, B$, respectively. We define a relation $<_2$ on the set $A \times B$ as follows. For all $a_0, a_1$ in $A$, $b_0, b_1$ in $B$, $(a_0, b_0) <_2 (a_1, b_1)$ if and only if either $a_0 <_0 a_1$ and $b_0 = b_1$ or $a_0 = a_1$ and
We call the relation \(<_2\) the interweaving of the relations \(<_0,<_1\) and denote the structure \((A \times B, <_2)\) by \((A,<_0)\otimes(B,<_1)\).

6.7 Lemma: (Principle of Double Induction on the set \(\text{Stp}\) of stumps)

Let \(P\) be a subset of \(\text{Stp} \times \text{Stp}\) such that every pair \((\sigma,1)\) and every pair \((1,\tau)\) belong to \(P\), and every pair \((\sigma,\tau)\) of non-empty stumps belongs to \(P\) as soon as every pair \((\sigma^n,\tau)\) and every pair \((\sigma,\tau^n)\) belong to \(P\).

Then every pair of stumps belongs to \(P\).

**Proof:** Let \(Q\) be the set of all stumps \(\sigma\) that for every stump \(\tau\), the pair \((\sigma,\tau)\) belongs to \(P\). Now use 1.5.2, the First Principle of Induction on the set \(\text{Stp}\) of stumps. \(\Box\)

6.8 Lemma:

(i) Let \(A\) be a countable set and \(<_0\) a binary relation on \(A\).

If \(A\) is stumpy, then \(<_0\) is inductive.

(ii) For all stumps \(\sigma, \tau\), the interweaving \((B(\sigma),<_#)\otimes(B(\tau),<_#)\) is stumpy.

**Proof:** (i) Let \(\sigma\) be a stump and let \(f\) be an isomorphic embedding from \((A,<_0)\) into \((B(\sigma),<_#)\). Let \(P\) be a \(<_0\)-hereditary subset of \(Q\). Let \(Q\) be the set of all \(m\) in \(B(\sigma)\) such that, for every \(a\) in \(A\), if \(f(a) = m\), then \(a\) belongs to \(P\). Observe that \(Q\) is a \(<_#\)-hereditary subset of \(B(\sigma)\), therefore, by the principle of Stump Induction, Theorem 1.5.4, \(Q\) coincides with \(B(\sigma)\) and \(P\) coincides with \(A\).

(ii) We use the just-mentioned Principle of Double Induction on the set of stumps, 6.7. Observe that the statement holds if either \(\sigma\) or \(\tau\) is the empty stump. So assume that both \(\sigma\) and \(\tau\) are non-empty and that for each \(n\), there exist stumps \(\rho_{n,0}\) and \(\rho_{n,1}\) such that the interweaving \((B(\sigma),<_#)\otimes(B(\tau^n),<_#)\) embeds into \((B(\rho_{n,0}),<_#)\) and the interweaving \((B(\sigma^n),<_#)\otimes(B(\tau),<_#)\) embeds into \((B(\rho_{n,1}),<_#)\). Now form a stump \(\varphi\) such that for each \(n\), \(\varphi^{2n} = \rho_{n,0}\) and \(\varphi^{2n+1} = \rho_{n,1}\) and observe that the interweaving \((B(\sigma),<_#)\otimes(B(\tau),<_#)\) embeds into \((B(\varphi),<_#)\). \(\Box\)

6.9 Let \(A\) be a set and \(<_0\) a binary relation on \(A\).

We define a binary relation \(<_1\) on the set \(A^*\) of finite sequences of elements of \(A\). For all \((a_0,\ldots,a_{m-1})\), \((b_0,\ldots,b_{n-1})\) in \(A^*\) we define: \((b_0,\ldots,b_{n-1})\) \(<_0^*\) \((a_0,\ldots,a_{m-1})\), or: \((b_0,\ldots,b_{n-1})\) is a \(<_0\)-simplification of \((a_0,\ldots,a_{m-1})\), if and only if there exist \(i < m\) and a positive natural number \(k\) such that \(n = m + (k - 1)\), and for each \(j < i\), \(a_j = b_j\), and for each \(j < k\), \(b_i+j < a_i\), and for each \(j < m-i-1\), \(b_{i+k+j} = a_{i+j+1}\).

So the sequence \((b_0,\ldots,b_{n-1})\) is obtained from the sequence \((a_0,\ldots,a_{m-1})\) if one replaces \((a_i)\) by \((b_i,\ldots,b_{i+k-1})\) where, for each \(j < k\), \(b_{i+j} < a_i\).

We call the relation \(<_0^*\) the sequencing of the relation \(<_0\).
6.10 Lemma:
For every stump \( \sigma \), the sequencing \( (\#)^{\ast} \) of the relation \( \# \) is a stumpy relation on \( (B(\sigma))^{\ast} \).

Proof: We use induction on the set of stumps. The statement of the Lemma is trivially true if \( \sigma \) is the empty stump. Now assume that \( \sigma \) is non-empty and that, for each \( n \), the relation \( (\#)^{\ast} \) is stumpy on \( (B(\sigma^{n}))^{\ast} \).

For all elements \( a = (a_{0}, \ldots, a_{n-1}) \) and \( b = (b_{0}, \ldots, b_{n-1}) \) of \( (B(\sigma))^{\ast} \) we define: \( b \prec^{\ast} a \) if and only if there exists a finite sequence \( c_{0}, \ldots, c_{k-1} \) of elements of \( (B(\sigma))^{\ast} \) such that \( b = c_{0} \) and \( a = c_{k-1} \) and, for each \( j < k \), \( c_{j} \) is a \( \# \)-simplification of \( c_{j+1} \).

So the relation \( \prec^{\ast} \) is the transitive closure of the relation \( (\#)^{\ast} \).

For each \( a \) in \( (B(\sigma))^{\ast} \) we let \( (B(\sigma))^{\ast} \restriction a \) be the set of all \( b \) in \( (B(\sigma))^{\ast} \) such that \( b \prec^{\ast} a \).

Observe that it suffices to show that, for each non-empty sequence \( a \) in \( (B(\sigma))^{\ast} \), the structure \( (B(\sigma))^{\ast} \restriction a, (\#)^{\ast} \) is stumpy.

We prove this by induction on \( |a| \).

If \( a \) has length 1, we determine \( t \) in \( \mathbb{N} \) such that \( a = (t) \). We may assume that \( t \) is positive and determine \( k > 0 \) such that \( t = (t(0), \ldots, t(k-1)) \).

We observe that \( a \prec^{\ast} (t(0)) \) and use the fact that \( (\#)^{\ast} \) is stumpy on \( (B(\sigma^{t(0)}))^{\ast} \).

If \( a \) has length greater than 1, we determine \( b \) in \( (B(\sigma))^{\ast} \) and \( t \) in \( \mathbb{N} \) such that \( a = b^{\ast}(t) \).

We may assume that \( t \) is coding a non-empty finite sequence \( t = (t(0), \ldots, t(k-1)) \).

We also may assume that \( (\#)^{\ast} \) is stumpy on \( (B(\sigma))^{\ast} \restriction b \). By the induction hypothesis \( (\#)^{\ast} \) is stumpy on \( (B(\sigma))^{\ast} \restriction (t) \) as \( (B(\sigma))^{\ast} \restriction (t) \) forms part of \( (B(\sigma^{t(0)}))^{\ast} \).

Now observe that \( (B(\sigma))^{\ast} \restriction a, (\#)^{\ast} \) may be seen as the result of interweaving \( (B(\sigma))^{\ast} \restriction b, (\#)^{\ast} \) and \( (B(\sigma))^{\ast} \restriction (t), (\#)^{\ast} \) and conclude by Lemma 6.10 that \( (\#)^{\ast} \) is stumpy on \( (B(\sigma))^{\ast} \restriction a \). \( \Box \)

6.11 Proof of Theorem 6.5.
Let \( P \) be a subset of \( \text{Stp}^{\ast} \) such that every finite sequence of stumps belongs to \( P \) as soon as each one of its simplifications belongs to \( P \).

Let \( (\sigma_{0}, \ldots, \sigma_{n-1}) \) be a finite sequence of stumps. Let \( \tau \) be some stump such that for each \( i \leq n \), \( \tau^{i} = \sigma_{i} \). For each \( t \) in \( B(\tau) \) we define a stump \( \tau^{t} \) as follows. The definition is by induction on \( |t| \). We define \( 0_{\tau} := \tau \) and for each \( t, i \), if \( t \ast (i) \) belongs to \( B(\tau) \), then \( 1_{t}^{(i)} := \tau^{t} \).

Observe that for all elements \( a = (a(0), \ldots, a(m-1)) \) and \( b = (b_{0}, \ldots, b_{n-1}) \) of \( B(\tau)^{\ast} \), \( b(0), \ldots, b_{n-1} \) is a \( \# \)-simplification of \( (a(0), \ldots, a_{m-1}) \) if and only if the sequence \( (a(0)_{\tau}, \ldots, (m-1)_{\tau}) \) is a simplification of the sequence \( (a(0)_{\tau}, \ldots, a(m-1)_{\tau}) \). Now let \( Q \) be the set of all elements \( (a(0), \ldots, a(m-1)) \) of \( B(\tau)^{\ast} \) such that \( (a(0)_{\tau}, \ldots, a(m-1)_{\tau}) \) belongs to \( P \). Conclude by Lemma 6.10 that \( Q \) coincides with \( B(\tau)^{\ast} \) in particular \( (0), (1), \ldots, (n-1) \) belongs to \( Q \), therefore \( (0)_{\tau}, \ldots, (n-1)_{\tau} \) belongs to \( P \), that is \( (\sigma_{0}, \ldots, \sigma_{n-1}) \) belongs to \( P \). \( \Box \)
6.12 Theorem:

(i) For each stump σ, the set DCBσ belongs to the class Σ^0_2, and its closure DCB^σ coincides with its double complement (DCBσ)^¬¬.

(ii) For all hereditarily repetitive stumps σ, τ, if σ ≤ τ then the set DCBσ reduces to the set DCBτ.

(iii) For every stump σ, for every n, the set DCBσ reduces to the set DCBσ | 0n.

(iv) For every finite sequence (σ_0, σ_1, ..., σ_{n-1}) of stumps, the set D(σ_0, σ_1, ..., σ_{n-1}) does not reduce to the set D^n(σ_0, σ_1, ..., σ_{n-1}).

(v) For all hereditarily repetitive stumps σ, τ, if σ < τ, then the set DCBσ does not reduce to the set DCBτ.

(vi) For each stump σ, for each n, the set D^{n+1}(DCBσ) does not reduce to the set D^n(DCBσ).

Proof: (i) We leave the proof to the reader as it is similar to the proof of Theorem 3.5(ii).

(ii) We use induction on the set of hereditarily repetitive stumps. It is obvious that for each stump τ, the set DCB_τ reduces to the set DCB_τ. Now assume that σ, τ are hereditarily repetitive stumps, σ is non-empty and σ ≤ τ. Using our Axioms of Countable Choice we find a strictly increasing α such that, for each m, σ^m ≤ τ_α(m) and also γ such that, for each m, γ^m is a function from N to N reducing the set DCB^m to the set DCB^m_α(m).

We leave it to the reader to define S in such a way that, for each m, is a function from N to N reducing the set D^{n+1}(DCB^m) to the set D^n(DCB^m). Finally we construct a function Z from N to N such that for every m, for every α, the sequence Z(m) equals the sequence Oα(m) * (1) * (Sm|e).

It will be clear that Z maps O onto O and reduces DCBσ to DCBτ.

(iii) We leave the proof to the reader observing only that for each stump σ, for each n, the set D^n(DCBσ) reduces to the set D^{n+1}(DCBσ).

(iv) We intend to use the Principle of Induction on the set Stp of finite sequences of stumps expressed in Theorem 6.5. Let (σ_0, σ_1, ..., σ_{n-1}) be a finite sequence of stumps and assume that the statement has been proved for every finite sequence of stumps that is a simplification of the sequence (σ_0, σ_1, ..., σ_{n-1}). (Observe that, if the sequence (σ_0, σ_1, ..., σ_{n-1}) has no simplifications, then the statement to be proved is equivalent to the statement: D^{n+1}(A_1) does not reduce the set D^n(A_1) and therefore true by Theorem 5.6(iv). But the argument we are about to explain furnishes another proof of this special case). Let us assume that γ is a function from N to N reducing the set D(A_1, D^{n+1}_i(DCB_σ_i)) to the set D^n_{i=0}(DCB_σ_i). Consider the sets B_0, B_1, ..., B_n which are defined as follows: B_0 := {α|α ∈ N | α^0 = 0} and, for each i < n, B_{i+1} := {α|α ∈ N | α^{i+1} = 0}. Observe that every one of these sets is a spread containing 0 and forming part of D(A_1, D^{n+1}_i(DCB_σ_i)) and applying the Continuity Principle n + 1 times we find for each i ≤ n natural numbers m_i and k_i such that, for
every α in B, passing through 0m, the sequence (γ|α)i belongs to DCB. Without
loss of generality we assume k₀ = k₁ = 0.
Applying the Continuity Principle two more times we find p₀, p₁, such that for every
i < 2 either ti = 0 and for every α in B passing through 0p, the sequence (γ|α)i
coincides with 0, or ti = 1 and for every α in B passing through 0p, there exists β
such that (γ|α)i equals 0p + (1) * β and β belongs to Dn(1)T. Let us first
assume t₀ = t₁ = 0. Observe that now for every α in B, passing through 0p the
sequence (γ|α)0 coincides with 0, and also for every α in B passing through 0p, the
sequence (γ|α)0 coincides with 0. We now see that the set D²(A₁) reduces to the
set A₁ as follows. We construct a function δ from N to N such that, for every α,
(δ|α)0 = 0p0 * α0 and if α1 = 0, then (δ|α)1,0 = 0, but if α1 ≠ 0, then (δ|α)1,0 does
not belong to DCB. And for each i, if 0 < i ≤ n − 1, then (δ|α)i does not belong
to DCB, and (δ|α)p = 0p where p = max(p₀, p₁). It is not difficult to verify that
for each α, the sequence (γ|α)0 belongs to D²(A₁) if and only if δα belongs to D(A₁, Dn⁻¹(DCB))
if and only if (γ|(δ|α))0 = 0. We know, however, from Theorem 5.4(i) that the set
D²(A₁) is not closed. So we conclude that either t₀ or t₁ differs from 0, and, without
loss of generality we may assume that t₀ = 1. We now calculate q such that for every
α, if α passes through 0q, then (γ|α)0 passes through 0n0 * (1). Let ε be a function
from N to N such that, for every α, the sequence (γ|0q * α) equals 0n0 * (1) * (ε|α).
Observe that, for every α, 0q * α belongs to D(A₁, Dn⁻¹(DCB)) if and only if
either ε|α belongs to Dn⁻¹(DCB) or for some positive i < n, the sequence
(γ|0q * α)i belongs to DCB. Using (iii) we conclude that D(A₁, Dn⁻¹(DCB)) reduces to D(Dn⁻¹(DCB), Dn⁻²(DCB)). Observe that Dn⁻¹(DCB) reduces to DCB, and therefore D(A₁, D(Dn⁻¹(DCB), Dn⁻²(DCB))) reduces to D(A₁, Dn⁻¹(DCB)) and therefore also to
D(Dn⁻¹(DCB), Dn⁻²(DCB)). We now obtain a contradiction by our induction hypothesis, as the finite sequence ⟨(σ₀)⁰, ..., (σ₀)n⁻¹, σ₁, ..., σn⁻¹⟩ is a simplification of the finite sequence ⟨σ₀, σ₁, ..., σn⁻¹⟩.
(v) This easily follows from (iv). Let σ, τ be hereditarily repetitive stumps such that
σ < τ. Calculate m such that σ ≤ τm. Observe that DCB reduces to DCBm. On the other hand Dm⁻¹(DCB) reduces to DCB but not to DCBm. Therefore
DCB does not reduce to DCBm.
(vi) This follows immediately from (iv).

6.13 We introduce, for every stump σ, a subset CCB of N, as follows, by transfinite
induction:
(i) CCB₁ := D²(A₁)
(ii) For every non-empty stump σ, CCBσ := {0} ∪ ∪n∈N 0n * (1) * Cn⁻¹(CCB)σ₁.
We have added the letter C of Conjunction to the letters C, B of Cantor and Bendixson.
6.14 Theorem:

(i) For each stump \( \sigma \), the set \( CCB_{\sigma} \) belongs to the class \( \Sigma_0^0 \) and its closure \( CCB_{\sigma} \) coincides with its double complement \( (CCB_{\sigma})^\bot \).

(ii) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma \leq \tau \), then the set \( CCB_{\sigma} \) reduces to the set \( CCB_{\tau} \).

(iii) For every stump \( \sigma \), for every \( n \), the set \( CCB_{\sigma} \) reduces to the set \( CCB_{\sigma} \cap \exists n \).

(iv) For every finite sequence \( (\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) \) of stumps, the set \( C[D^2(A_1), C^{n-1}_{i=0}(CCB_{\sigma_i})] \) does not reduce to the set \( C^{n-1}_{i=0}(CCB_{\sigma_i}) \).

(v) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma \leq \tau \), then the set \( CCB_{\sigma} \) does not reduce to the set \( CCB_{\tau} \).

(vi) For each stump \( \sigma \), for each \( n \), the set \( C^{n+1}(CCB_{\sigma}) \) does not reduce to the set \( C^n(CCB_{\sigma}) \).

Proof: We only prove (iv) and leave it to the reader to prove the remaining statements of the Theorem.

We again want to use the Principle of Induction on the set of finite sequences of stumps expressed in Theorem 6.5. For every finite sequence \( (\sigma_0, \ldots, \sigma_{n-1}) \) of stumps we define: \( (\sigma_0, \ldots, \sigma_{n-1}) \) has the property \( P := \) either there exists \( i < n \) such that \( \sigma_i = 1 \) or for every \( k \), the set \( C[C^{k+1}(D^2(A_1)), C^{n-1}_{i=0}(CCB_{\sigma_i})] \) does not reduce to the set \( C[C^k(D^2(A_1)), C^{n-1}_{i=0}(CCB_{\sigma_i})] \).

It is easy to see that it suffices to show that every finite sequence of stumps has the property \( P \).

Let us assume that \( (\sigma_0, \ldots, \sigma_{n-1}) \) is a finite sequence of stumps such that every simplification of the finite sequence \( (\sigma_0, \ldots, \sigma_{n-1}) \) has the property \( P \). We may assume that none of the stumps \( \sigma_0, \ldots, \sigma_{n-1} \) is the empty stump. Suppose we find \( k \) in \( N \) and a function \( \gamma \) from \( N \) to \( N \) reducing the set \( C[C^{k+1}(D^2(A_1)), C^{n-1}_{i=0}(CCB_{\sigma_i})] \) to the set \( C[C^k(D^2(A_1)), C^{n-1}_{i=0}(CCB_{\sigma_i})] \). For every \( k+1 \)-sequence \( i = (i_0, \ldots, i_k) \) from \( \{0,1\}^{k+1} \) we let \( B(i_0, \ldots, i_k) \) be the set of all \( \alpha \) in \( N \) such that, for every \( j \leq k \), the sequence \( \alpha^{i_0,j, \alpha_j} \) coincides with \( \emptyset \) and, for every \( j < n - 1 \), the sequence \( \alpha^{i_0,j, \alpha_j} \) coincides with \( \emptyset \). Observe that each set \( B(i_0, \ldots, i_k) \) is a spread containing \( \emptyset \) and forming part of \( C[C^{k+1}(D^2(A_1)), C^{n-1}_{i=0}(CCB_{\sigma_i})] \).

Repeatedly applying the Continuity Principle we find, for each finite sequence \( (i_0, \ldots, i_k) \) from \( \{0,1\}^* \), for each \( j \leq n - 1 \), natural numbers \( n = n(i_0, \ldots, i_k, j) \) and \( t = t(i_0, \ldots, i_k, j) \) such that \( \text{either } t = 0 \) and for every \( \alpha \) in \( B(i_0, \ldots, i_k) \) passing through \( \emptyset_n \), the sequence \( \gamma|_\alpha^{1,j} \) coincides with \( \emptyset \), or \( t = 1 \) and for every \( \alpha \) in \( B(i_0, \ldots, i_k) \), the sequence \( \gamma|_\alpha^{1,j} \) is apart from \( \emptyset \).

We claim that there must exist a finite sequence \( (i_0, \ldots, i_k) \) in \( \{0,1\} \) and \( j \leq n - 1 \) such that \( t = t(i_0, \ldots, i_k, j) = 1 \). For suppose not. We then calculate \( N = \max\{n(i_0, \ldots, i_k, j)|(i_0, \ldots, i_k) \in \{0,1\}^{k+1}, j \leq n - 1\} \) and construct a function \( \delta \) from \( N \) to \( N \) such that for every \( j < k + 1 \), the sequence \( (\delta|_\alpha^{0,j}) \) coincides with \( \emptyset N * \alpha_j \).
and for every $j \leq n - 1$, the sequence $(\delta|\alpha)^{1,j}$ coincides with 0, and $(\delta|\alpha)(0) = 0$.
Observe that, for every $\alpha$, $\alpha$ belongs to $C^{k+1}(D^2(A_1))$ if and only if the sequence 
$(\gamma|\alpha)^{0}$ belongs to $C^k(D^2(A_1))$. Therefore the set $C^{k+1}(D^2(A_1))$ reduces to the 
set $C^k(D^2(A_1))$, and we have a contradiction, according to Theorem 5.15.
Without loss of generality we may assume that $t = t(0, \ldots, 0) = 1$. We now determine 
p, q such that for every $\alpha$ in $B(0, \ldots, 0)$ passing through $\gamma\alpha$ the sequence ($\gamma|\alpha)^{1,0}$ 
passes through $\delta q * 1$. As in the proof of Theorem 6.12 we may conclude that the set 
$C(C^{k+1}(D^2(A_1)), C_{i=0}^{n-2}(CCB_{\sigma_i}))$ reduces to the set $C(C^k(D^2(A_1)), C_{i=0}^{n-2}(CCB_{\sigma_i}))$.
But then also the set $C((C^{k+1}(D^2(A_1))))$ will reduce to the set $C(C^k(D^2(A_1)), C_{i=0}^{n-2}(CCB_{\sigma_i})))$, and we ob-
tain a contradiction, as the finite sequence 
$((\sigma_0)^{0}, \ldots, (\sigma_0)^{q}, \sigma_1, \ldots, \sigma_{n-1})$ is a simplification of the finite sequence 
$(\sigma_0, \ldots, \sigma_{n-1})$.

7 The Borel Hierarchy Theorem

We introduce positively Borel sets and canonical classes of positively Borel sets. We 
discuss the custom of calling some given positively Borel sets $X, Y$ each other’s comple-
ment and remark that almost every positively Borel set has very many comple-
ments. We show which conclusion one may draw from the argument given by Borel 
and Lebesgue.

We then prove the Hierarchy Theorem, first for the finite levels only, and then, rea-
soning more shrewdly, the general case.

7.1 The class Borel of positively Borel subsets of $\mathcal{N}$ is given by the following inductive 
definition.
(i) Every subset of $\mathcal{N}$ belonging to either $\Pi^0_1$ or $\Sigma^0_1$ is positively Borel.
(ii) For any given sequence $X_0, X_1, \ldots$ of positively Borel subsets of $\mathcal{N}$, the sets 
$\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ are themselves positively Borel.
(iii) Clauses (i) and (ii) produce all positively Borel subsets of $\mathcal{N}$.

7.2 We define the class of the non-zero stumps by the following inductive definition: 
a stump $\sigma$ is non-zero if either $\sigma$ coincides with $\{\langle \rangle\} = 1^*$ or $\sigma$ is non-empty and for 
each $n$, $\sigma^n$ is a non-zero stump.
Every non-zero stump is non-empty but the converse is false.
Observe that we may decide, for every non-zero stump $\sigma$, if $\sigma$ equals $1^*$ or not.
For every non-zero stump $\sigma$ we define classes $\Sigma^0\sigma$ and $\Pi^0\sigma$ of subsets of $\mathcal{N}$, by the 
following inductive definition:
(i) $\Sigma_1^0$, coincides with the class $\Sigma_1^0$ of the open subsets of $\mathcal{N}$ and $\Pi_1^0$, coincides with the class $\Pi_1^0$ of the closed subsets of $\mathcal{N}$.

(ii) For every non-zero stump $\sigma$ different from $1^*$, for every subset $X$ of $\mathcal{N}$:

$X$ belongs to $\Sigma_0^\sigma$ if and only if there exist a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, and a sequence $p_0, p_1, \ldots$ of natural numbers such that for each $n$, $X_n$ belongs to $\Pi_{p_n}^{\sigma^2}$ and $X$ coincides with $\bigcup_{n \in \mathbb{N}} X_n$.

$X$ belongs to $\Pi_0^\sigma$ if and only if there exist a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ and a sequence $p_0, p_1, \ldots$ of natural numbers such that, for each $n$, $X_n$ belongs to $\Sigma_{p_n}^{\sigma^2}$ and $X$ coincides with $\bigcap_{n \in \mathbb{N}} X_n$.

The classes $\Sigma_0^\sigma, \Pi_0^\sigma$ are called the canonical classes of positively Borel sets.

Observe that a subset $X$ of $\mathcal{N}$ is positively Borel if and only if, for some non-zero stump $\sigma$, $X$ belongs to $\Sigma_0^\sigma$.

7.3 We define the class of complementary pairs (of positively Borel sets) by the following definition.

(i) For every open subset $X$ of $\mathcal{N}$, the two-element collection $\{X, N \setminus X\}$ is a complementary pair.

(ii) For every sequence $X_0, Y_0, X_1, Y_1, \ldots$ of positively Borel sets, if, for each $n$, $\{X_n, Y_n\}$ is a complementary pair, then the two-element collection $\{\bigcup_{n \in \mathbb{N}} X_n, \bigcap_{n \in \mathbb{N}} Y_n\}$ is a complementary pair.

(iii) Clauses (i) and (ii) produce all complementary pairs.

7.3.1 Theorem:

(i) For every positively Borel set $X$ there exists a positively Borel set $Y$ such that $\{X, Y\}$ is a complementary pair.

(ii) For every complementary pair $\{X, Y\}$, every element of $X$ is apart from every element of $Y$.

(iii) $\operatorname{Inf}$ is the set of all elements in $\mathcal{C}$ apart from every element of $\operatorname{Fin}$. Almost-$^\ast(\operatorname{Fin})$ is the set of all elements of $\mathcal{C}$ apart from every element of $\operatorname{Inf}$. For every stump $\sigma$, the two-element collection $\{P(\sigma, \operatorname{Fin}), \operatorname{Inf}\}$ is a complementary pair of positively Borel sets.

(iv) The statement: for every closed set $X$, for all open sets $Y, Z$, if both $\{X, Y\}$ and $\{X, Z\}$ are complementary pairs, then $Y = Z$, is not provable intuitionistically.

Proof: The proofs of (i) and (ii) are straightforward and left to the reader. (iii) is a consequence of earlier results, see Sections 4.6-10. It shows clearly that in the realm of positively Borel sets we do not have unicity of complements. (iv) claims that we
cannot prove unicity of complements for sets from the first level of the hierarchy. The statement mentioned is equivalent to the so-called generalized Markov Principle. This principle claims that, for every \( \alpha \), if \( \neg \forall n [\alpha(n) = 0] \), then \( \exists n [\alpha(n) = 0] \). We do not see why this principle should be true, and do not want to use it.

7.3.2 Complementary pairs of positively Borel sets are considered by Brouwer, see [6], page 89, line 21-27, (he is studying \( \Pi^0_1 \) and \( \Sigma^0_1 \)), and also, more generally by Martin-Löf in [14], page 80, and by Bishop and Bridges in [1], pages 73-75.

7.4 We define a function \( \langle \rangle \) from \( \mathcal{N} \times \mathcal{N} \) to \( \mathcal{N} \) such that for all \( \alpha, \beta \), \( \langle \alpha, \beta \rangle^0 = \alpha \) and for each \( n > 0 \), \( \langle \alpha, \beta \rangle^n = \beta \).

For every subset \( X \) of \( \mathcal{N} \), for every \( \alpha \), we let \( X_\alpha \) be the set of all \( \beta \) in \( \mathcal{N} \) such that \( \langle \alpha, \beta \rangle \) belongs to \( X \).

We introduce, for each non-zero stump \( \sigma \), subsets \( US_\sigma \) and \( UP_\sigma \) of \( \mathcal{N} \) by means of the following definition.

(i) \( US_1^* \) is the set of all \( \alpha \) such that for some \( m, n \), \( \alpha^0(m) = \alpha^1(n) + 1 \).

\( UP_1^* \) is the set of all \( \alpha \) that do not belong to \( US_1^* \).

(ii) For every non-zero stump \( \sigma \) different from \( 1^* \), \( US_\sigma \) is the set of all \( \alpha \) such that for some \( n \), \( \langle \alpha^0, n, \alpha^1 \rangle \) belongs to \( UP_\sigma \), and \( UP_\sigma \) is the set of all \( \alpha \) such that for all \( n \), \( \langle \alpha^0, n, \alpha^1 \rangle \) belongs to \( US_\sigma \).

We call \( US_\sigma \) and \( UP_\sigma \) the universal or cataloguing sets of level \( \sigma \).

We also introduce, for each non-zero stump \( \sigma \), subsets \( E_\sigma \) and \( A_\sigma \) of \( \mathcal{N} \) by means of the following inductive definition:

(i) \( E^*_1 \) is the set of all \( \alpha \) such that for some \( n \), \( \alpha(n) \neq 0 \).

\( A^*_1 \) is the set of all \( \alpha \) such that for all \( n \), \( \alpha(n) = 0 \).

(ii) For every non-zero stump \( \sigma \) different from \( 1^* \), \( E_\sigma \) is the set of all \( \alpha \) such that for some \( n \), \( \alpha^n \) belongs to \( A_\sigma \), and \( A_\sigma \) is the set of all \( \alpha \) such that for all \( n \), \( \alpha^n \) belongs to \( E_\sigma \).

We call \( E_\sigma \) and \( A_\sigma \) the canonical complete sets of level \( \sigma \), and also the leading sets of \( \Sigma^0_\sigma \), \( \Pi^0_\sigma \), respectively.

7.4.1 Theorem:

(i) For every non-zero stump \( \sigma \), for all subsets \( X, Y \) of \( \mathcal{N} \), if both \( X \) and \( Y \) belong to \( \Sigma^0_\sigma \), then \( X \cap Y \) belongs to \( \Sigma^0_\sigma \), and: if both \( X \) and \( Y \) belong to \( \Pi^0_\sigma \), then \( X \cap Y \) belongs to \( \Pi^0_\sigma \).

(ii) For every non-zero stump \( \sigma \), for every sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), if, for each \( n \), \( X_n \) belongs to \( \Sigma^0_\sigma \), then \( \bigcup_{n \in \mathbb{N}} X_n \) belongs to \( \Sigma^0_\sigma \), and: if, for each \( n \), \( X_n \) belongs to \( \Pi^0_\sigma \), then \( \bigcap_{n \in \mathbb{N}} X_n \) belongs to \( \Pi^0_\sigma \).
(iii) For every non-zero stump $\sigma$, for all subsets $X, Y$ of $\mathcal{N}$, if $Y$ belongs to $\Sigma^0_\sigma$, and $X$ reduces to $Y$, then $X$ belongs to $\Sigma^0_\sigma$, and: if $Y$ belongs to $\Pi^0_\sigma$ and $X$ reduces to $Y$, then $X$ belongs to $\Pi^0_\sigma$.

(iv) For every non-zero hereditarily repetitive stump $\sigma$ different from $1^*$, for every subset $X$ of $\mathcal{N}$:

$X$ belongs to $\Sigma^0_\sigma$ if and only if there exists a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that for every $n$, $X_n$ belongs to $\Pi^0_{\sigma^n}$, and $X = \bigcup X_n$.

and: $X$ belongs to $\Pi^0_\sigma$ if and only if there exists a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that for every $n$, $X_n$ belongs to $\Sigma^0_{\sigma^n}$, and $X = \bigcap X_n$.

(v) For every non-zero hereditarily repetitive stump $\sigma$, for every subset $X$ of $\mathcal{N}$, $X$ belongs to $\Sigma^0_\sigma$ if and only if, for some $\alpha$, $X$ coincides with $U \Sigma_\sigma \{\alpha\}$, and: $X$ belongs to $\Pi^0_\sigma$ if and only if, for some $\alpha$, $X$ coincides with $U \Pi_\sigma \{\alpha\}$. In addition, \{\$U \Sigma_\sigma, U \Pi_\sigma\$\} is a complementary pair of positively Borel sets.

(vi) For every non-zero hereditarily repetitive stump $\sigma$, for every subset $X$ of $\mathcal{N}$, $X$ belongs to $\Sigma^0_\sigma$ if and only if $X$ reduces to $A_\sigma$, and: $X$ belongs to $\Pi^0_\sigma$ if and only if $X$ reduces to $A_\sigma$.

In addition, \{\$E_\sigma, A_\sigma\$\} is a complementary pair of positively Borel sets.

**Proof:** The proofs are straightforward and left to the reader. One sometimes has to use the Second Axiom of Countable Choice.

\[ \square \]

7.4.2 As a further remark we may add that for all non-zero stumps $\sigma, \tau$, $\Sigma^0_\sigma$ forms part of $\Sigma^0_\tau$ if and only if $\Pi^0_\sigma$ forms part of $\Pi^0_\tau$ if and only if $\sigma \leq \tau$. Also, for every non-zero stump $\sigma$ we may find a hereditarily repetitive non-zero stump $\tau$ such that both $\sigma \leq \tau$ and $\tau \leq \sigma$, and therefore $\Sigma^0_\sigma = \Sigma^0_\tau$ and $\Pi^0_\sigma = \Pi^0_\tau$. So it does no harm, in this context, to restrict attention to hereditarily repetitive stumps. Observe that, for each positive $n$, the stump $n^*$ is hereditarily repetitive.

7.5 For every $\alpha$, we define an infinite sequence $^a \alpha$, as follows, by induction on length($a$): $^1 \alpha := \alpha$ and for all $a, n : ^{a+n} \alpha := (^a \alpha)^n$. So for every $\alpha, a, m$ one has $^a \alpha(m) = \alpha(a \ast m)$. For all $a, b$ we define: $A$ does not compare with $b$, or $a, b$ are incompatible, notation $a \perp b$, if and only if there is no $m$ such that either $a \ast m = b$ or $b \ast m = a$.

Observe that for every non-zero stump $\sigma$, for every $a$, one may decide if there exists $n$ such that $a \ast \langle n \rangle$ belongs to $\sigma$ or not. If there is no such $n$, we say that $a$ is a final position in $\sigma$.

For every stump $\sigma$, for every $a$, we say that $a$ is just outside $\sigma$ if and only if there exists $b, n$ such that $a = b \ast \langle n \rangle$ and $b$ belongs to $\sigma$ while $a$ does not. For every stump $\sigma$, for every $a$, one may decide if $a$ is just outside $\sigma$ or not.

7.5.1 Theorem: (The Classical Borel Hierarchy Theorem)
For every non-zero hereditarily repetitive stump \( \sigma \), if either \( \Pi^0_0 \) forms part of \( \Sigma^0_0 \) or \( \Sigma^0_0 \) forms part of \( \Pi^0_0 \), then there exists \( \gamma \) belonging to neither one of \( US_\sigma \), \( UP_\sigma \).

(ii) For every function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \) there exists \( \alpha \) in \( \mathcal{N} \) such that \( \alpha \) belongs to \( E_1 \) if and only if \( f(\alpha) \) belongs to \( E_1 \). For every decidable subset \( A \) of \( \mathbb{N} \) consisting of mutually incompatible numbers, for every function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \) there exists \( \alpha \) such that for each \( a \) in \( A \), \( \alpha \) belongs to \( E_1 \) if and only if \( \alpha(f(\alpha)) \) belongs to \( E_1 \).

(iii) For every non-zero stump \( \sigma \), for every function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \), there exists \( \alpha \) such that \( \alpha \) belongs to \( E_\sigma \) if and only if \( f |\alpha \) belongs to \( E_\sigma \), and: \( \alpha \) belongs to \( A_\sigma \) if and only if \( \alpha |\alpha \) belongs to \( A_\sigma \).

Proof: (i) For every non-zero hereditarily repetitive stump \( \sigma \), we let \( DS_\sigma \) be the set of all \( \alpha \) such that \( \langle \alpha, \alpha \rangle \) belongs to \( US_\sigma \), and we let \( DP_\sigma \) be the set of all \( \alpha \) such that \( \langle \alpha, \alpha \rangle \) belongs to \( UP_\sigma \). We call \( DS_\sigma \) and \( DP_\sigma \) the diagonal sets of level \( \sigma \).

Observe that \( DS_\sigma \), \( DP_\sigma \) belong to \( \Sigma^0_0 \), \( \Pi^0_0 \), respectively. Suppose, for instance, that \( \Pi^0_0 \) forms part of \( \Sigma^0_0 \), then in particular \( DP_\sigma \) belongs to \( \Sigma^0_0 \), and we may find \( \beta \) such that \( DP_\sigma \) coincides with \( US_\sigma |\beta \), and therefore for every \( \alpha \), \( \langle \alpha, \alpha \rangle \) belongs to \( UP_\sigma \) if and only if \( \langle \beta, \alpha \rangle \) belongs to \( US_\sigma \). Define \( \gamma := \langle \beta, \beta \rangle \) and observe that \( \gamma \) cannot belong to either \( UP_\sigma \) or \( US_\sigma \).

(ii) Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \). We define an infinite sequence \( \alpha \), by induction, as follows. Let \( n \) be a natural number and suppose we decided already on \( \alpha(0), \ldots, \alpha(n-1) \). We now consider if there exist \( i, j < n \) such that \( f^{i}(\alpha) \) > 1, whereas, for each every \( q < j \), \( f^{q}(\alpha) = 0 \). If so, we define \( \alpha(n) := 1 \); if not, we define \( \alpha(n) := 0 \). It is not difficult to see that \( \alpha \) belongs to \( E_1 \) if and only if \( f |\alpha \) belongs to \( E_1 \). Now let \( A \) be a decidable subset of \( \mathbb{N} \) consisting of mutually incompatible natural numbers and let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \). We define an infinite sequence \( \alpha \), by induction, as follows. Let \( n \) be a natural number and suppose we decided already on \( \alpha(0), \ldots, \alpha(n-1) \). We now consider if there exist \( a \) in \( A \), \( k \) in \( \mathbb{N} \) such that \( n = a * k \) and for some \( i, j < n \), \( f^{a+i}(\alpha) \) > 1, whereas, for every \( q < j \), \( f^{a+q}(\alpha) = 0 \). If so, we define \( \alpha(n) := 1 \); if not, we define \( \alpha(n) := 0 \). Observe that for each \( a \) in \( A \), there exists \( k \) such that \( \alpha(a * k) = 1 \) if and only if there exists \( i \) such that \( (f|\alpha)(a + i) \neq 0 \), that is \( \alpha \) belongs to \( E_1 \) if and only if \( \alpha |\alpha \) belongs to \( E_1 \).

(iii) Let \( \sigma \) be a non-zero stump and let \( A \) be the set of all final positions in \( \sigma \). Let \( F \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \). Using (ii), construct \( \alpha \) such that for every \( a \) in \( A \), \( \alpha |\alpha \) belongs to \( E_1 \) if and only if \( \alpha |\alpha \) belongs to \( E_1 \). Using the Principle of Stump Induction, see Section 1.5.4, prove that for every \( a \) belonging to \( \sigma \), both \( \alpha |\alpha \) belongs to \( E_1 \) and only if \( a(f|\alpha) \) belongs to \( E_1 \) and: \( \alpha \) belongs to \( A_\sigma \) if and only if \( a(f|\alpha) \) belongs to \( A_\sigma \). In particular: \( \alpha \) belongs to \( E_\sigma \) if and only if \( f |\alpha \) belongs to \( E_\sigma \).
and: $\alpha$ belongs to $A_\sigma$ if and only if $f|\alpha$ belongs to $A_\sigma$.

(iv) Let $\sigma$ be a non-zero hereditarily repetitive stump and let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $A_\sigma$ to $E_\sigma$, that is, for every $\alpha$, $\alpha$ belongs to $A_\sigma$ if and only if $f|\alpha$ belongs to $E_\sigma$. Using (iii), construct $\alpha$ such that $\alpha$ belongs to $A_\sigma$ if and only if $f|\alpha$ belongs to $E_\sigma$ and: $\alpha$ belongs to $E_\sigma$ if and only if $f|\alpha$ belongs to $E_\sigma$. If $\alpha$ should belong to $A_\sigma$, $f|\alpha$ would belong to both $E_\sigma$ and $A_\sigma$, contradiction. If $\alpha$ should belong to $E_\sigma$, $f|\alpha$ would belong to $E_\sigma$ and therefore $\alpha$ would belong both to $A_\sigma$ and $E_\sigma$, contradiction. Therefore $\alpha$ belongs to neither one of $A_\sigma$, $E_\sigma$.

7.5.2 The second statement of Theorem 7.5.1 is a kind of fixed point theorem and the argument given may be compared to the argument developed by S.C. Kleene for his First Recursion Theorem.

The reader will perhaps be surprised by the careful formulation of the first and fourth statement of Theorem 7.5.1. Does not the assumption that, for some non-zero stump $\sigma$, some $\alpha$ does not belong to either one of $A_\sigma$, $E_\sigma$, lead to a contradiction?

In fact, we are only entitled to draw this conclusion in case $\sigma = 1^*$. It thus follows from Theorem 7.5.1 that $\Pi_1^0$ is not included in $\Sigma_1^0$ and that $\Sigma_1^0$ is not included in $\Pi_1^0$ but that is all.

One might hope for the conclusion that the classes $\Pi_2^0$ and $\Sigma_2^0$ are not included in each other, but this hope realizes only if one makes some unfounded assumption like the generalized Markov Principle: for every $\alpha$, if $\forall \exists n[\alpha(n) = 0]$, then $\exists n[\alpha(n) = 0]$, or S. Kuroda’s axiom: for every subset $P$ of $\mathbb{N}$, if $\forall n[\neg P(n)]$, then $\forall \forall n \in \mathbb{N}[P(n)]$.

It is not true, although stated in [17] and [19], that assuming Markov’s Principle enables one to climb all further steps of the hierarchy: already the third level is still out of reach.

7.5.3 We add an example showing that in intuitionistic mathematics it is possible that statements $\neg \exists x \forall y \exists z[P(x, y, z)]$ and $\forall x \exists y \forall z[\neg P(x, y, z)]$ and $\forall x \forall y \forall z[P(x, y, z) \lor \neg P(x, y, z)]$ are simultaneously true. We claim that (i) $\exists x \forall n \exists m[\alpha(n) = 0 \land \alpha(m) \neq 0]$ and (ii) $\neg \exists x \forall n \exists m[\alpha(n) \neq 0 \lor \alpha(m) = 0]$ and (iii) $\forall x \exists n \forall m[\alpha(n) = 0 \land \alpha(m) \neq 0] \lor (\alpha(n) \neq 0 \lor \alpha(m) = 0)]$.

We only prove (ii). Assume $\forall x \exists n \forall m[\alpha(n) \neq 0 \lor \alpha(m) = 0]$.

Using the Continuity Principle, we find $n, p$ such that for every $\alpha$ passing through $\tilde{0}^p$ either $\alpha(n) \neq 0$ or $\alpha = 0$. Now consider $\alpha = \tilde{0}^q * \underline{1}$ where $q$ is greater than both $n, p$. Contradiction.

This example shows that it is impossible to obtain from Theorem 7.5.1 the conclusion that $\Pi_2^0$ is not included in $\Sigma_3^0$, if one only uses the rules of intuitionistic logic and no further mathematical assumptions.

7.6 We now want to prove the finite case of the Intuitionistic Borel Hierarchy Theorem.
Given any subset $X$ of $\mathcal{N}$ we let the infinite disjunction of $X$, notation $D^n(X)$, be the set of all $\alpha$ such that, for some $n$, $\alpha^n$ belongs to $X$, and we let the infinite conjunction of $X$, notation $C^n(X)$, be the set of all $\alpha$ such that, for all $n$, $\alpha^n$ belongs to $X$.

A subset $Y$ of $\mathcal{N}$ reduces to $D^n(X)$, $C^n(X)$ respectively, if and only if there exists a sequence $Y_0, Y_1, \ldots$ of subsets of $\mathcal{N}$, each of them reducing to $X$ with the property that the set $Y$ coincides with the set $\bigcup_{n \in \mathbb{N}} Y_n$, $\bigcap_{n \in \mathbb{N}} Y_n$, respectively.

Observe that, for each positive $n$, the set $D^n(A_n)$ coincides with the set $E_{n+1}$ and the set $C^n(E_n)$ coincides with the set $A_{n+1}$.

Let $(X, Y)$ be a pair of sets such that every element of $X$ is apart from every element of $Y$. We say that $X$ is strongly irreducible to $Y$ if and only if for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $X$ into $Y$ there exists also $\alpha$ in $Y$ such that $f|\alpha$ belongs to $Y$. We say that $X$ is very strongly irreducible to $Y$ if and only if for every decidable subset $A$ of $\mathbb{N}$ consisting of mutually incompatible numbers, (that is, for all $a, b$ in $A$, if $a \neq b$, then $a \perp b$), for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $a$ in $A$, for every $\alpha$, if $^a\alpha$ belongs to $X$, then $^a(f|\alpha)$ belongs to $Y$, there exists $\alpha$ such that for every $a$ in $A$ both $^a\alpha$ and $^a(f|\alpha)$ belong to $Y$. Observe that, if $X$ is very strongly irreducible to $Y$, then it is also true that for every decidable subset $A$ of $\mathbb{N}$ consisting of mutually incompatible numbers, for every $\rho$ in $\mathcal{N}$, for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ with the property that for all $a$ in $A$, for all $\alpha$, if $^a\alpha$ belongs to $X$, then $^{(\rho)(a)}(f|\alpha)$ belongs to $Y$, there exists $\alpha$ such that for all $a$ in $A$, both $^a\alpha$ and $^{(\rho)(a)}(f|\alpha)$ belong to $Y$.

Let $X$ be a subset of $\mathcal{N}$. $X$ is called strictly analytic if and only if there exists a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $\gamma$, so for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, $\beta$ coincides with $\gamma(\alpha)$.

**7.6.1 Theorem:**

The set $A_1$ is very strongly irreducible to the set $E_1$ and the set $E_1$ is very strongly irreducible to the set $A_1$.

**Proof:** Let $A$ be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all $a$ in $A$, for all $\alpha$, if $^a\alpha$ belongs to $A_1$, then $^a(f|\alpha)$ belongs to $E_1$. Using Theorem 7.5.1(ii), find $\alpha$ such that for each $a$ in $A$, $^a\alpha$ belongs to $E_1$ if and only if $^a(f|\alpha)$ belongs to $E_1$. We claim that for each $a$ in $A$ both $^a\alpha$ and $^a(f|\alpha)$ belong to $E_1$. Let $a$ belong to $A$ and let $\beta$ be such that $^a\beta = 0$ and for each $i$, if there is no $j$ such that $i = a \ast j$, then $\beta(i) = \alpha(i)$. Calculate $q$ such that $(^a(f|\beta))(q) \neq 0$, and find $p$ such that $f^{a+i}(\beta p) > 1$ and, for every $j < p$, $f^{a+i}(\beta j) = 0$. Now distinguish two cases. Either $\overline{\alpha p} = \overline{\beta p}$, therefore $^a(f|\alpha)$ and also $^a\alpha$ belong to $E_1$, or $\overline{\alpha p} \neq \overline{\beta p}$ and therefore $^a\alpha$ and also $^a(f|\alpha)$ belong to $E_1$.

The proof of the second statement is similar.

**7.6.2 Lemma:** (First Continuity Lemma)
Let $X$ be a strictly analytic subset of $\mathcal{N}$.

Let $R$ be a subset of $\mathcal{N} \times \mathcal{N}$ and $a$ a natural number such that for every $\alpha$, if $^{a}\alpha$ belongs to $C^{\omega}(X)$, then there exists $m$ such that $aRm$. Then:

(i) For every $\alpha$ such that $^{a}\alpha$ belongs to $C^{\omega}(X)$, there exists $n, m$ with the property that for every $\beta$ such that $^{a}\beta$ belongs to $C^{\omega}(X)$, if $\overline{\alpha}n = \overline{\beta}n$ and for all $j < n$, $^{a*}(j)\alpha = {^{a*}(j)}\beta$, then $\beta Rm$.

(ii) For every $\gamma$ in $C^{\omega}(X)$ there exist functions $\mu, \nu$ from $\mathcal{N}$ to $\mathbb{N}$ such that for every $\alpha$ such that $^{a}\alpha$ equals $\gamma$, for every $\beta$ such that $^{a}\beta$ belongs to $C^{\omega}(X)$, if both $\overline{\alpha}\nu(\alpha) = \overline{\beta}\nu(\alpha)$ and for all $j < \nu(\alpha)$, $^{a*}(j)\beta = \gamma j$, then $\beta R\mu(\alpha)$.

**Proof:** Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $f$. Now let $g$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$, $(g|\alpha)(0) = \alpha(0)$ and for each $n$, $(g|\alpha)^n = f((\alpha)^n)$. Observe that $C^{\omega}(X)$ coincides with the range of $g$. Let $h$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$, $^{a}(h|\alpha) = g((\alpha)^1)$ and for each $n$, if there is no $j$ such that $n = a + j$, then $(h|\alpha)(n) = \alpha^0(n)$. Observe that the set of all $\alpha$ such that $^{a}\alpha$ belongs to $C^{\omega}(X)$ coincides with the range of $h$.

Assume that for every $\alpha$ such that $^{a}\alpha$ belongs to $C^{\omega}(X)$ there exists $m$ such that $\alpha Rm$. Then for every $\beta$ there exists $m$ such that $(h|\beta)Rm$.

(i) Assume that we have some $\alpha$ such that $^{a}\alpha$ belongs to $C^{\omega}(X)$.

Find $\gamma$ such that $\alpha = h|\gamma$, and using the Continuity Principle, find $m, n$ such that for every $\delta$, if $\overline{\gamma}n = \overline{\delta}n$, then $(h|\delta)Rm$.

Observe that for every $\beta$ such that $^{a}\beta$ belongs to $C^{\omega}(X)$ and $\overline{\alpha}n = \overline{\beta}n$ and for all $j < n$, $^{a*}(j)\beta$ there exists $\delta$ such that $\beta = h|\delta$ and $\overline{\delta}n = \overline{\beta}n$.

(ii) Using the First Axiom of Continuous Choice, determine functions $\pi, \rho$ from $\mathcal{N}$ to $\mathbb{N}$ such that for every $\beta$, $(h|\beta)R\pi(\beta)$ and $\rho(\beta) := \mu[|\pi(\beta) \neq 0] + 1$. Observe that for every $\beta$, for every $\alpha$, if $^{a}\alpha$ belongs to $C^{\omega}(X)$ and both $\overline{\alpha}\rho(\beta) = \overline{\beta}\rho(\beta)$ and for each $j < \rho(\beta)$, $^{a*}(j)\alpha = {^{a*}(j)}(h|\beta)$, there will exist $\delta$ passing through $\overline{\beta}\rho(\beta)$ such that $\alpha = h|\beta$, and therefore $aR\pi(\beta)$. Let $\gamma$ belong to $C^{\omega}(X)$. Construct a function $\eta$ from $\mathcal{N}$ to $\mathcal{N}$ such that for each $\alpha$ such that $^{a}\alpha = \gamma$, the sequence $h|(\eta|\alpha)$ coincides with $\alpha$.

Define functions $\mu, \nu$ from $\mathcal{N}$ to $\mathbb{N}$ by: for all $\alpha$, $\mu(x) := \pi[(\eta|\alpha)$ and $\nu(\alpha) := \rho[(\eta|\alpha)$.

One easily verifies that $\mu, \nu$ satisfy the requirements.

**7.6.3 Theorem:** Let $X, Y$ be strictly analytic subsets of $\mathcal{N}$ such that every element of $X$ is apart from every element of $Y$.

(i) If $X$ is very strongly irreducible to $Y$, then $D^{\omega}(X)$ is very strongly irreducible to $C^{\omega}(Y)$.

(ii) If $X$ is very strongly irreducible to $Y$, then $C^{\omega}(X)$ is very strongly irreducible to $D^{\omega}(Y)$.

**Proof:** (i) Let $A$ be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every $a$ in $A$, for every
\(\alpha\), if \(\alpha\) belongs to \(D^\omega(X)\), then \(\alpha(f|\alpha)\) belongs to \(C^\omega(Y)\). It follows that for every \(a\) in \(A\), for every \(n\), for every \(\alpha\), if \(\alpha^{(n)}(\alpha)\) belongs to \(X\), then \(\alpha^{(n)}(f|\alpha)\) belongs to \(Y\). Observe that the set of all numbers \(a * \langle n \rangle\), where \(a\) belongs to \(A\) and \(n\) to \(\mathbb{N}\) is a decidable set of mutually incompatible numbers. We use the fact that \(X\) is very strongly irreducible to \(Y\) and find some \(\alpha\) such that for all \(a\) in \(A\), for all \(n\), both \(\alpha^{(n)}(\alpha)\) and \(\alpha^{(n)}(f|\alpha)\) belong to \(Y\), and therefore, for all \(a\) in \(A\), both \(\alpha\) and \(f|\alpha\) belong to \(C^\omega(Y)\).

(ii) Let \(A\) be a decidable subset of \(\mathbb{N}\) consisting of mutually incompatible numbers and let \(f\) be a function from \(\mathcal{N}\) to \(\mathcal{N}\) such that for every \(a\) in \(A\), for every \(\alpha\), if \(\alpha\) belongs to \(C^\omega(X)\), then \(\alpha(f|\alpha)\) belongs to \(D^\omega(Y)\). Let \(\gamma\) be an element of \(C^\omega(X)\). Using Lemma 7.6.2(ii) we find for each \(a\) in \(A\) functions \(\mu_\alpha\) and \(\nu_\alpha\) from \(\mathcal{N}\) to \(\mathbb{N}\) such that for every \(\alpha\) such that \(\alpha\) equals \(\gamma\), for every \(\beta\) such that \(\alpha\) belongs to \(C^\omega(X)\), if both \(\bar{\nu}_\alpha(\alpha) = \bar{\beta}\nu_\alpha(\alpha)\) and for all \(j < \nu(\alpha)\), \(\alpha^{(j)}(\beta)\) equals \(\gamma^j\), then \(\alpha^{(\mu(\alpha))(\beta)}(f|\beta)\) belongs to \(Y\).

We now define a sequence \(g_0, g_1, \ldots\) of functions from \(\mathcal{N}\) to \(\mathcal{N}\) such that, for each \(\beta\), the sequence \(g_0|\beta, g_1|\beta, \ldots\) is a convergent sequence of elements of \(\mathcal{N}\).

We let \(a_0\) be an element of \(\mathcal{N}\) with the property that for each \(a\) in \(A\), \(\alpha(\alpha_0)\) equals \(\gamma\), and define, for each \(\beta\), \(g_0|\beta := a_0\). Now assume that \(a\) belongs to \(\mathcal{N}\) and that \(g_a\) has been defined. If \(a\) does not belong to \(A\), we define: \(g_{a+1} := g_a\).

If \(a\) belongs to \(A\) we let \(p_a\) be the largest of all numbers \(\nu_\alpha(g_a|\beta)\), \(b\) in \(A\), \(b \leq a\).

We then define the function \(g_{a+1}\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that for every \(\beta\), the sequence \(\alpha^{(p_a)}(g_{a+1}|\beta)\) equals \(\beta\) and for all \(j\), if there is no \(i\) such that \(j = a * \langle p_a \rangle * i\), then \((g_{a+1}|\beta)(j) = (g_a|\beta)(j)\).

It will be clear that, for each \(\beta\), the sequence \(g_0|\beta, g_1|\beta, \ldots\) converges. We define the function \(g\) from \(\mathcal{N}\) to \(\mathcal{N}\) by: for each \(\beta\), \(g|\beta := \lim_{a \to \infty} g_a|\beta\).

Observe that for each \(\beta\), for each \(a\) in \(A\), if \(\alpha\) belongs to \(X\), then \(\alpha^{(p_a)}(g|\beta)\) belongs to \(X\), and \(\alpha(g|\beta)\) belongs to \(C^\omega(X)\), and \(\alpha^{(\mu(\mu(\alpha))(\beta)}(f|\beta)\) belongs to \(Y\). So we may determine \(\beta\) such that for all \(a\) in \(A\), both \(\alpha^{(p_a)}(g|\beta)\) and \(\alpha^{(\mu(\mu(\alpha))(\beta)}(f|\beta)\) belong to \(Y\), therefore both \(\alpha(g|\beta)\) and \(\alpha(f|\beta)\) belong to \(D^\omega(Y)\).

7.6.4 Theorem: (Finite Borel Hierarchy Theorem)

(i) For each \(n\), for every function \(f\) from \(\mathcal{N}\) to \(\mathcal{N}\) mapping \(A_n\) into \(E_n\), there exists \(\alpha\) such that both \(\alpha\) and \(f|\alpha\) belong to \(E_n\).

(ii) For each \(n\), for every function \(f\) from \(\mathcal{N}\) to \(\mathcal{N}\) mapping \(E_n\) into \(A_n\), there exists \(\alpha\) such that both \(\alpha\) and \(f|\alpha\) belong to \(A_n\).

Proof: Observe that, for each \(n\), \((A_n, E_n)\) is a complementary pair of positively Borel and strictly analytic sets. Using Theorems 7.6.1 and 7.6.3 conclude that, for each \(n\), the set \(A_n\) is very strongly irreducible to the set \(E_n\) and the set \(E_n\) is very strongly irreducible to the set \(A_n\).
7.7 We intend to prove the Intuitionistic Borel Hierarchy Theorem itself.

7.7.1 For every non-zero stump \( \sigma \), for every \( \alpha \), we introduce a game \( G(\sigma, \alpha) \) for players I, II. It is a game of perfect information. Player I starts and chooses a natural number \( n_0 \), then player II chooses a natural number \( n_1 \), and so they continue choosing alternately a natural number. The play ends as soon as the number \( \langle n_0, \ldots, n_{k-1} \rangle \) is just outside \( \sigma \). Player I is the winner if and only if either \( k \) is odd and \( \alpha(\langle n_0, \ldots, n_{k-1} \rangle) \) differs from 0, or \( k \) is even and \( \alpha(\langle n_0, \ldots, n_{k-1} \rangle) \) equals 0. We then say that the number \( \langle n_0, \ldots, n_{k-1} \rangle \) is a *win for player I in the game \( G(\sigma, \alpha) \).* Player II is the winner if and only if player I is not. In that case the number \( \langle n_0, \ldots, n_{k-1} \rangle \) is called a *win for player II in the game \( G(\sigma, \alpha) \).*

An element \( \gamma \) of \( \mathcal{N} \) may be thought of as a strategy for either player I or player II, as follows. For every \( s = \langle (s)_0, \ldots, (s)_{k-1} \rangle \) where \( k = \text{length}(s) \), for every \( \gamma \), we define: \( s \) *I-obeyes* \( \gamma \), or \( \gamma \) *I-governs* \( s \), if and only if, for every \( i \), if \( 2i < k \), then \( (s)_{2i} = \gamma(\langle (s)_1, (s)_3, \ldots, (s)_{2i-1} \rangle) \) and: \( s \) *II-obeyes* \( \gamma \), or \( \gamma \) *II-governs* \( s \), if and only if, for every \( i \), if \( 2i + 1 < k \), then \( (s)_{2i+1} = \gamma(\langle (s)_0, (s)_2, \ldots, (s)_{2i} \rangle) \).

Suppose that for some non-zero stump \( \sigma \), for some \( \gamma, \alpha \), every position just outside \( \sigma \) I-obeying to \( \gamma \) is a win for player I in the game \( G(\sigma, \alpha) \). We then say that \( \gamma \) is a *winning strategy for player I in the game \( G(\sigma, \alpha) \).* Also, if every position just outside \( \sigma \) II-obeying to \( \gamma \) is a win for player II in the game \( G(\sigma, \alpha) \), we say that \( \gamma \) is a *winning strategy for player II in the game \( G(\sigma, \alpha) \).*

For every \( \gamma \), \( \alpha \), for every non-zero stump \( \sigma \), we define two elements of \( \mathcal{N} \), \( \text{Corr}_I^\gamma(\gamma, \alpha) \) and \( \text{Corr}_II^\gamma(\gamma, \alpha) \), as follows. For every \( s \), if \( s \) is not a position just outside \( \sigma \) I-obeying \( \gamma \), then \( \text{Corr}_I^\gamma(\gamma, \alpha)(s) = \alpha(s) \), but if \( s \) is a position just outside \( \sigma \) I-obeying \( \gamma \), then, if \( \text{length}(s) \) is odd, \( \text{Corr}_I^\gamma(\gamma, \alpha)(s) = \text{the larger one of the numbers } 1, \alpha(s) \), and if \( \text{length}(s) \) is even, \( \text{Corr}_I^\gamma(\gamma, \alpha)(s) = 0 \). Also, for every \( s \), if \( s \) is not a position just outside \( \sigma \) II-obeying \( \gamma \), then \( \text{Corr}_II^\gamma(\gamma, \alpha)(s) = \alpha(s) \), but if \( s \) is a position just outside \( \sigma \) II-obeying \( \gamma \), then, if \( \text{length}(s) \) is odd, \( \text{Corr}_II^\gamma(\gamma, \alpha)(s) = 0 \), and if \( \text{length}(s) \) is even, \( \text{Corr}_II^\gamma(\gamma, \alpha)(s) = \text{the larger one of the numbers } 1, \alpha(s) \).

We might pronounce “\( \text{Corr}_I^\gamma(\gamma, \alpha) \)” as: “\( \alpha \)-as-corrected-according-to-\( \gamma \)-as a strategy for player I in the game \( G(\sigma, \alpha) \).”

7.7.2 Theorem:

(i) For every non-zero stump \( \sigma \), for every \( \alpha \), \( E_\sigma(\alpha) \) if and only if there exists \( \gamma \) such that \( \gamma \) is a winning strategy for player I in the game \( G(\sigma, \alpha) \) and \( \alpha \) equals \( \text{Corr}_I^\gamma(\gamma, \alpha) \).

(ii) For every non-zero stump \( \sigma \), for every \( \alpha \), \( A_\sigma(\alpha) \) if and only if there exists \( \gamma \) such that \( \gamma \) is a winning strategy for player II in the game \( G(\sigma, \alpha) \) and \( \alpha \) equals \( \text{Corr}_II^\gamma(\gamma, \alpha) \).

Proof: The proof is straightforward and uses the Second Axiom of Countable Choice.
7.7.3 We want to show that for every non-zero hereditarily repetitive stump $\sigma$, for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $A_2$ into $E_{\sigma}$, there exists $\alpha$ such that both $\alpha$ itself and its image $f|\alpha$ belong to $E_{\sigma}$, that is, there exist $\gamma, \delta$ such that $\alpha = \text{Corr}^T_1(\gamma, \alpha)$ and $f|\alpha = \text{Corr}^T_1(\delta, f|\alpha)$. With this aim in mind we introduce some further notions.

Let $C$ be a spread. We say that $C$ is a value-dictating spread if there exist a decidable subset $A$ of $\mathbb{N}$ and an element $\alpha$ of $\mathcal{N}$ such that for every $\beta$, $\beta$ belongs to $C$ if and only if for each $n$ in $A$, $\beta(n) = \alpha(n)$.

For every spread $C$, for all $m, n$, we say that $C$ dictates the value $m$ in $n$ if and only if, for all $\alpha$ in $C$, $\alpha(m) = n$. We say that $m$ is without choice in $C$ if, for some $n$, $C$ dictates the value $n$ in $C$. We say that $m$ is completely free in $C$ if and only if, for all $\alpha$ in $C$, for all $\beta$, if for every $j \neq m$, $\alpha(j) = \beta(j)$, then also $\beta$ belongs to $C$. We say that $m$ is almost completely free in $C$ if and only if, for all $p$, the number $m + p$ is free in $C$. We say that $m$ is completely free in $C$ if and only if, for all $p$, the number $m + p$ is free in $C$, and for all $p$, either $m + p$ is free in $C$, or $C$ dictates the value $0$ in $m + p$.

Let $C$ be a spread. We define the minimal element of $C$, notation $\text{Min}(C)$, as follows: for each $n$, $\text{Min}(C)(n)$ := the least $k$ such that some element of $C$ passes through $\text{Min}(C)n * (k)$.

7.7.4 Lemma: (First Basic Lemma)

Let $C$ be a value-dictating spread and $s$ a natural number such that $s$ is almost completely free in $C$.

Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$.

Suppose that for all $\alpha$ in $C$, if $s\alpha$ belongs to $A_1$, then $s(f|\alpha)$ belongs to $E_1$.

There exists a value-dictating subspread $D$ of $C$ such that for all $\alpha$ in $D$, both $s\alpha$ and $s(f|\alpha)$ belong to $E_1$, and for all $t$ such that $t \perp s$, if $t$ is almost completely free in $C$ then $t$ is almost completely free in $D$.

Proof: Let $\text{Min}(C)$ be the minimal element of $C$. Observe that $s\text{Min}(C)$ belongs to $A_1$ and find $n$ such that $s(f|\text{Min}(C))(\langle n \rangle) \neq 0$. Calculate $m$ such that for every $\alpha$ in $C$, if $\bar{\alpha}m = \text{Min}(C)m$, then $s(f|\alpha)(\langle n \rangle) = s(f|\text{Min}(C))(\langle n \rangle)$. Let $D$ be the set of all $\alpha$ in $C$ such that $\bar{\alpha}m = \text{Min}(C)m$ and $s\alpha = \emptyset m + 1$.

It is easy to see that $D$ satisfies the requirements. 

7.7.5 Lemma: (Second Basic Lemma)

Let $C$ be a value-dictating spread and $s$ a natural number such that $s$ is almost completely free in $C$.
Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$.

Suppose that for all $\alpha$ in $C$, if $s\alpha$ belongs to $E_1$, then $s(f|\alpha)$ belongs to $A_1$.

There exists a value-dictating subspread $D$ of $C$ such that for all $\alpha$ in $D$, both $s\alpha$ and $s(f|\alpha)$ belong to $A_1$, and for all $t$ such that $t \perp s$, if $t$ is almost completely free in $C$, then $t$ is almost completely free in $D$.

**Proof:** Let $D$ be the set of all $\alpha$ in $C$ such that $s\alpha = 0$. It is easy to see that $D$ satisfies the requirements.

### 7.7.6 Lemma: (Second Continuity Lemma)

Let $C$ be a value-dictating spread and $s$ a natural number such that $s$ is almost completely free in $C$.

Let $\tau$ be a non-zero stump different from $1^*$ and let $R$ be a subset of $\mathcal{N} \times \mathcal{N}$, such that for every $\alpha$ in $C$, if $s\alpha$ belongs to $A_\tau$, then there exists $m$ such that $\alpha R m$.

Then, given any $\alpha$ in $C$ such that $s\alpha$ belongs to $A_\tau$, we may calculate $m, n$ such that for every $t$ in $C$ such that $s t$ belongs to $A_\tau$, if $s t = a n$ and for every $j < n$, $s^{(j)} \alpha = s^{(j)} \beta$, then $t R m$.

**Proof:** Assume $a$ belongs to $C$ and $s\alpha$ belongs to $A_\tau$. We calculate $\gamma$ such that $s\alpha$ coincides with $\text{Corr}_{\tau}(\gamma, s\alpha)$, and $p$ such that for every $q > p$, $s * q$ is free in $C$. We let $X$ be the set of all $\delta$ such that for every $q \leq p$, $\delta q = \gamma q$. We define a function $h$ from $X \times C$ to $C$, as follows. For every $\delta$ in $X$, every $\beta$ in $C$ we require that $s(h(\delta, \beta))$ equals $\text{Corr}_{\tau}(\delta, s\beta)$, and for every $t$, if there is no $j$ such that $t = s * j$, then $(h(\delta, \beta))(t) = \beta(t)$.

Observe that both $X$ and $C$ are spreads.

Applying the Continuity Principle we find $m, n$ such that $n > p$ and for all $\delta$ in $X$, for all $\beta$ in $C$, if $\beta n = \alpha n$ and $\delta n = \gamma n$, then $(h(\delta, \beta)) R m$.

Now assume that $\beta$ belongs to $C$, and $s\beta$ belongs to $A_\tau$, and $\beta n = \alpha n$, and for all $j < n$, $s^{(j)} \beta$ coincides with $s^{(j)} \alpha$. Calculate $\delta$ in $X$ such that $\beta = h(\delta, \beta)$ and $\beta n = \gamma n$, and conclude $\beta R m$.

### 7.7.7 Lemma: (Main Lemma)

Let $C$ be a value-dictating spread and $s$ a natural number such that $s$ is almost completely free in $C$.

Let $\tau$ be a non-zero hereditarily repetitive stump different from $1^*$, and let $g$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all $\alpha$ in $C$, if $s\alpha$ belongs to $A_\tau$, then $g(\alpha)$ belongs to $E_\tau$.

There exist $m, p$ and a value-dictating subspread $D$ of $C$ such that $\tau^m = \tau^p$, and $s * \langle m \rangle$ is completely free in $D$, and for all $\alpha$ in $D$, if $s^{(m)} \alpha$ belongs
to $E_{r_m}$, then $^s \alpha$ belongs to $A^r$ and $(g|\alpha)^p$ belong to $A_{r^p} = A_{r_m}$, and for all $t$ such that $t \perp s$, if $t$ is almost completely free in $C$, then $t$ is almost completely free in $D$.

**Proof:** Let $\alpha$ be some element of $C$ such that $^s \alpha$ belongs to $A_r$.

Applying the previous Lemma, we find $p, n$ such that for every $\beta$ in $C$, if $^s \beta$ belongs to $A_r$ and $\overline{\beta}n = \overline{\alpha}n$ and for all $j < n$, $^{s^{(j)}} \beta$ coincides with $^{s^{(j)}} \alpha$, then $(g|\beta)^p$ belongs to $A_{r^p}$. We now determine $m$ such that $m > n$ and $r^p = r^m$ and $s \ast (m)$ is completely free in $C$ and let $D$ be the set of all $\beta$ in $C$ such that $\overline{\beta}m = \overline{\alpha}m$ and for all $j$, if $j \neq m$, then $^{s^{(j)}} \beta = ^{s^{(j)}} \alpha$. $D$ is easily seen to be a spread satisfying all our requirements.

\[ \blacksquare \]

7.7.8 Let $s, c$ be natural numbers. $s$ I-obeys $c$ if and only if for each $i$, if $2i < \text{length}(s)$, then $m := ((s)_{1}, (s)_{3}, \ldots, (s)_{2i-1})$ is smaller than $\text{length}(c)$ and $(s)_{2i}$ equals $(c)_{m}$.

Let $s, t$ be natural numbers of equal length. We say that $s$ is II-similar to $t$ if and only if for each odd $i < \text{length}(s)$, $(s)_{i} = (t)_{i}$. If $s$ is II-similar to $t$, then player II made the same moves in the course of reaching the position $s$ as he made in the course of reaching the position $t$.

7.7.9 Theorem: (*Intuitionistic Borel Hierarchy Theorem, First Part*)

Let $\sigma$ be a non-zero hereditarily repetitive stump and let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $A_{\sigma}$ into $E_{\sigma}$.

There exists $\alpha$ in $\mathcal{N}$ such that both $\alpha$ itself and $f|\alpha$ belong to $E_{\sigma}$.

**Proof:** We intend to build $\alpha, \gamma, \delta$ such that $\alpha$ coincides with $\text{Corr}_{\sigma}^\gamma(\gamma, \alpha)$ and $f|\alpha$ coincides with $\text{Corr}_{\sigma}^\gamma(\delta, f|\alpha)$.

We define the sequences $\gamma, \delta$ step by step, first $\gamma(0), \delta(0)$, then $\gamma(1), \delta(1), \ldots$, and at the same time we define a sequence $C_0, C_1, \ldots$ of value-dictating spreads such that the following conditions are satisfied:

(1) for each $n$, for each pair $(s, t)$ of II-similar positions of equal length, if $s$ belongs to $B(\sigma)$ and $s$ I-obeys $\gamma_n$ and $t$ I-obeys $\delta_n$, then also $t$ belongs to $B(\sigma)$ and $^s \sigma = ^\tau \sigma$ and one of the four following cases obtains:

   (i) $s$ is a final position in $\sigma$ of even length, and for all $\alpha$ in $C_n$, both $^s \alpha$ and $^{t(f|\alpha)}$ belong to $A_{1}$.

   (ii) $s$ is a final position in $\sigma$ of odd length, and for all $\alpha$ in $C_n$, both $^s \alpha$ and $^{t(f|\alpha)}$ belong to $E_{1}$.

   (iii) $s$ is a non-final position in $\sigma$ of even length, and for all $\alpha$ in $C_n$, if $^s \alpha$ belongs to $A_{(\sigma)}$, then $^{t(f|\alpha)}$ belongs to $A_{(\tau)} = A_{(\tau_{\sigma})}$.

   (iv) $s$ is a non-final position in $\sigma$ of odd length, and for all $\alpha$ in $C_n$, if $^s \alpha$ belongs to $E_{(\tau_{\sigma})}$, then $^{t(f|\alpha)}$ belongs to $E_{(\tau_{\sigma})} = E_{(\tau_{\sigma})}$.

66
(2) For each \( n \), each non-final position \( s \) of \( \sigma \) obeying \( 7\sigma n \) is almost free in \( C_n \).

(3) For each \( n \), \( C_{n+1} \) is a subspread of \( C_n \) and each \( i < n \) is without choice in \( C_n \).

We define \( C_0 = \mathcal{N} \). Observe that the empty position \( 0 = \langle \rangle \) is the only position \( I \)-obeying \( 0 = 70 = \delta 0 \). Remark that for all \( \alpha \) in \( C_0 \), if \( 0\alpha \) belongs to \( A_\sigma \), then \( 0(f|\alpha) \) belongs to \( E_\sigma \).

Now assume that, for some \( n \), \( C_n, 7\sigma n, \delta n \) have been defined and the conditions (1)-(3) are satisfied so far.

We determine \( k = \text{length}(n) \) and consider the finite sequence coded by \( n \), \( n = \langle (n)_0, \ldots, (n)_{k-1} \rangle \). The elements of this sequence have to be thought of as moves by player II. We consider

\[
s := \langle (\gamma(0), (n)_0), \gamma((n)_0), \ldots, \gamma((n)_0, \ldots, (n)_{k-2}), (n)_{k-1} \rangle.
\]

Observe that length(\( s \)) = 2\( k \). We also consider

\[
t := \langle (\delta(0), (n)_0), \delta((n)_0), \ldots, \delta((n)_0, \ldots, (n)_{k-2}), (n)_{k-1} \rangle.
\]

Observe that \( s \) is \( I \)-similar to \( t \), and \( s \) obeys \( 7\sigma n \), and \( t \) obeys \( 5\sigma n \).

We now distinguish several cases.

(*) \( s \) does not belong to \( A \).

We now define: \( \gamma(n) := 0 \) and \( \delta(n) := 0 \) and \( C_{n+1} \) is the set of all \( \alpha \) in \( C_n \) passing through \( \text{Min}(C_n)(n + 1) \).

(**) \( s \) is a final position in \( \sigma \).

We then know: for all \( \alpha \) in \( C_n \), if \( s\alpha \) belongs to \( A_1 \), then \( t(f|\alpha) \) belongs to \( E_1 \). We apply our first Basic Lemma 7.7.4 and determine a value-dictating subspread \( D \) of \( C_n \), such that for all \( \alpha \) in \( D \) both \( s\alpha \) and \( t(f|\alpha) \) belong to \( E_1 \) and for all \( u \), if \( u \perp s \) and \( u \) is almost completely free in \( C_n \), then \( u \) is almost completely free in \( D \). We then define: \( \gamma(n) := 0 \) and \( \delta(n) := 0 \) and let \( C_{n+1} \) be the set of all \( \alpha \) in \( D \) passing through \( \text{Min}(D)(n + 1) \).

(***) \( s \) is a non-final position in \( \sigma \).

We then know: for all \( \alpha \) in \( C_n \), if \( s\alpha \) belongs to \( A_{(s, \sigma)} \), then \( t(f|\alpha) \) belongs to \( E_{(s, \sigma)} = E_{(s, \sigma)} \), and \( \sigma \) is a non-zero hereditarily repetitive stump different from \( 1^* \). We apply our Main Lemma 7.7.7 and determine \( m, p \) and a value dictating subspread \( D \) of \( C_n \) such that \( s^*(m)\sigma = t^*(p)\sigma \) and \( s * (m) \) is completely free in \( D \), and for all \( \alpha \) in \( D \), if \( s^*(m)\alpha \) belongs to \( E_{(s^*(m), \sigma)} \), then \( t^*(p)(f|\alpha) \) belongs to \( A_{(s^*(m), \sigma)} \) and for every \( u \), if \( u \perp s \) and \( u \) is almost completely free in \( C_n \), then \( u \) is almost completely free in \( D \). We now distinguish two further subcases.

(***)' \( s^*(m) \) and therefore also \( t^*(p) \) are final positions in \( \sigma \). We now define \( \gamma(n) := m \) and \( \delta(n) := p \) and we let \( D' \) be the set of all \( \alpha \) in \( D \) such that \( s^*(m)\alpha \) belongs to \( A_1 \). We let \( C_{n+1} \) be the set of all \( \alpha \) in \( D' \) passing through \( \text{Min}(D')(n + 1) \). Observe that according to the Second Basic Lemma 7.7.5 for all \( \alpha \) in \( C_{n+1} \), both \( s^*(m)\alpha \) and \( t^*(p)(f|\alpha) \) belong to \( A_1 \).

(***)'' \( s * (m) \) and therefore also \( t * (p) \) are non-final positions in \( \sigma \). We now define
\( \gamma(n) := m \) and \( \delta(n) := p \) and we let \( C_{n+1} \) be the set of all \( \alpha \) in \( D \) passing through \( \text{Min}(D)(n + 1) \).

Observe that for all \( \alpha \) in \( C_{n+1} \), for all \( j \), if \( \ast^{(m,j)}\alpha \) belongs to \( A_{(\ast^{(m,j)},\sigma)} \) then \( \ast^{(p,j)}(f|\alpha) \) belongs to \( E_{(\ast^{(p,j)},\sigma)} \).

This completes the description of the construction. Let \( \alpha \) be the unique element of \( N \) that belongs to every \( C_n \). Observe that for every final position \( s \) in \( \sigma \)-obeying \( \gamma \), if \( \text{length}(s) \) is even, then \( \ast\alpha \) belongs to \( E_1 \) and if \( \text{length}(s) \) is odd, then \( \ast\alpha \) belongs to \( A_1 \). Therefore \( \alpha \) belongs to \( E_\sigma \). Observe that for every final position \( t \) in \( \sigma \)-obeying \( \delta \), if \( \text{length}(t) \) is even, then \( \ast(t)|\alpha \) belongs to \( E_1 \) and if \( \text{length}(t) \) is odd, then \( \ast(t)|\alpha \) belongs to \( A_1 \). Therefore \( f|\alpha \) belongs to \( E_\sigma \).

7.7.10 Theorem: (Intuitionistic Borel Hierarchy Theorem, Second Part)

Let \( \sigma \) be a non-zero hereditarily repetitive stump and let \( f \) be a function from \( N \) to \( N \) mapping \( E_\sigma \) into \( A_\sigma \).

There exists \( \alpha \) in \( N \) such that both \( \alpha \) itself and \( f|\alpha \) belong to \( A_\sigma \).

**Proof:** We know from Theorem 7.4.1(ii) that the class \( \Pi_0^0 \) is closed under the operation of countable union and construct a function \( h \) from \( N \) to \( N \) such that for every \( \alpha \), \( h|\alpha \) belongs to \( A_\sigma \) if and only if, for each \( n \), \( \alpha^n \) belongs to \( A_\sigma \). Let \( \tau \) be the non-empty stump such that, for each \( n \), \( \tau^n \) equals \( \sigma \). \( \tau \) is sometimes called the successor of \( \sigma \). We let \( g \) be a function from \( N \) to \( N \) such that, for each \( \alpha \), for each \( n \), \( (g|\alpha)^n = f|(\alpha^n) \). Finally, we let \( k \) be a function from \( N \) to \( N \) such that, for each \( \alpha \), for each \( n \), \( (k|\alpha)^n = h|(g|\alpha) \). Observe that, for each \( \alpha \), if \( \alpha \) belongs to \( A_\tau \), then for each \( n \), \( \alpha^n \) belongs to \( E_\sigma \), and \( (g|\alpha)^n \) belongs to \( A_\sigma \), therefore \( h|(g|\alpha) \) belongs to \( A_\sigma \) and \( k|\alpha \) belongs to \( E_\tau \). Using Theorem 7.7.9 we find \( \beta \) such that both \( \beta \) itself and \( k|\beta \) belong to \( E_\tau \). Now find \( n \) such that \( \beta^n \) belongs to \( A_\sigma \), and observe: \( h|(g|\beta) \) belongs to \( A_\sigma \), therefore \( (g|\beta)^n = f|(\beta^n) \) belongs to \( A_\sigma \).

8 The never-ending productivity of disjunction

We show that, for every non-zero hereditarily repetitive stump \( \sigma \), the set \( D(A_1, A_\sigma) \) does not reduce to the set \( A_{S(\sigma)} \) and that, for each \( n \), the set \( D^{n+1}(A_\sigma) \) does not reduce to the set \( D^n(A_\sigma) \). We thus answer a question asked but not answered in [17]. A special case of this result has been shown in [24].

8.1 Let \( T \) be the set \( \{0\} \cup \{0n \ast (1) * 0 | n \in N \} \). We studied this countable set in Section 5.9. We now prove a simple fact that we want to use in establishing the main result of this Section.
8.2 Lemma: For all subsets \( A, B \) of the closure \( \overline{T} \) of \( T \), if \( \overline{T} \) forms part of \( A \cup B \), then there exists \( n \) such that either for all \( p \), if \( p > n \), then \( \overline{0}_p \ast (1) \ast 0 \) belongs to \( A \), or for all \( p \), if \( p > n \), then \( \overline{0}_p \ast (1) \ast 0 \) belongs to \( B \).

Proof: Applying the Continuity Principle we find \( n \) such that either every \( \alpha \) in \( \overline{T} \) passing through \( \overline{0}_n \) belongs to \( A \), or every \( \alpha \) in \( \overline{T} \) passing through \( \overline{0}_n \) belongs to \( B \). The conclusion follows easily.

8.3 Lemma: Let \( \sigma \) be a non-zero hereditarily repetitive stump.

(i) Let \( X \) be a subset of \( N \) reducing to \( E_\sigma \), and let \( f \) be a function from \( N \) to \( N \) mapping \( A_\sigma \) into \( X \).

There exists \( \alpha \) in \( N \) such that \( \alpha \) belongs to \( E_\sigma \) and \( f|\alpha \) belongs to \( X \).

(ii) For every \( n \), for every \( n \)-sequence \( \tau_0, \ldots, \tau_{n-1} \) of non-zero hereditarily repetitive stumps, if for each \( i < n \), \( \tau_i < \sigma \), then \( C_i \) reduces to \( E_\sigma \).

Proof: (i) Let \( g \) be a function from \( N \) to \( N \) reducing \( X \) to \( E_\sigma \). Let \( h \) be the function from \( N \) to \( N \) such that, for every \( a \), \( h|\alpha = g|(f|\alpha) \). Observe that \( h \) maps \( A_\sigma \) into \( X_\sigma \), and using Theorem 7.7.9, find \( \alpha \) such that both \( \alpha \) and \( h|\alpha \) belong to \( E_\sigma \), therefore \( f|\alpha \) belongs to \( X \).

(ii) Remark that each set \( A_\sigma \) reduces to \( E_\sigma \) and that the set \( C_\sigma \) reduces to \( E_\sigma \) as the class \( \Sigma^0_1 \) is closed under the operation of intersection of sets.

8.4 Lemma:

Let \( \sigma \) be a non-zero hereditarily repetitive stump.

Let \( \alpha \) be an element of \( A_\sigma \) and let \( c \) be a natural number such that for every \( t \) \( \Pi \)-obeying \( c \), if \( t \) belongs to \( \sigma \) and \( \text{length}(t) \) is even, then \( t|\sigma \) belongs to \( A(t_\sigma) \).

Then there exists \( \gamma \) passing through \( c \) such that \( \alpha \) coincides with \( \text{Corr}_{II}(\gamma, \alpha) \).

Proof: Let \( H \) be the set of all numbers \( t \) of even length that belong to \( \sigma \) and \( \Pi \)-obey \( c \). Observe that \( H \) is a finite set and calculate \( k \) such that \( 2k = \max\{\text{length}(t)|t \in H\} \).

We wish to prove, for every \( i \leq k \), for every \( t \) in \( H \), if \( \text{length}(t) = 2k - 2i \), then there exists \( \gamma \) passing through \( c \) such that \( t|\sigma \) coincides with \( t(\text{Corr}_{II}(\gamma, \alpha)) \). We use induction. Assume that \( t \) belongs to \( H \) and, for every \( s \) in \( H \) such that \( \text{length}(s) > \text{length}(t) \), there exists \( \gamma \) such that \( s|\gamma \) coincides with \( s(\text{Corr}_{II}(\gamma, \alpha)) \). Calculate \( j \) such that \( \text{length}(t) = j \) and find \( m \) such that \( t = (m)_{0}, c((m)_{0}), (m)_{1}, c((m)_{0}, (m)_{1}), \ldots, (m)_{j-1}, c(m) \).

We list the finitely many elements of \( H \) such that \( \text{length}(u) = j + 2 \), calling them \( u_{0}, \ldots, u_{\ell-1} \). We may assume that for each \( p < \ell \), \( u_{p} = t \ast (p, c(m \ast (p))) \). For each \( p < \ell \) we determine \( \gamma_{p} \) such that \( \text{Corr}_{II}(\gamma_{p}, \alpha) = \text{Corr}_{II}(\gamma_{p}, \alpha) \). We also determine \( \delta \) such that \( \text{Corr}_{II}(\delta, \alpha) = \text{Corr}_{II}(\delta, \alpha) \). We then define \( \gamma \) passing through \( c \) such that, for each \( p \), if \( p < \ell \), then \( m_{p}(\gamma) = m_{p}(\gamma) \) and, if \( p \geq \ell \), then \( m_{p}(\gamma) \) coincides with \( m_{p}(\gamma) \), and observe that \( \gamma(\text{Corr}_{II}(\gamma, \alpha)) \) coincides with \( \gamma(\alpha) \).
After $k$ steps we obtain the conclusion that there exists $\gamma$ passing through $c$ such that $\alpha$ coincides with $\text{Corr}^\gamma_H(\gamma, \alpha)$.

8.5 Lemma:

Let $\sigma$ be a non-zero hereditarily repetitive stump.

Let $g$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $g|\alpha$ belongs to $A_\sigma$.

Then for every $\alpha$ there exists $\gamma, \varepsilon$ such that $g|\alpha$ coincides with $\text{Corr}^\gamma_H(\gamma, g|\alpha)$ and for every $n$, for every $\beta$, if $\beta$ passes through $\alpha(\varepsilon(n))$, then there exists $\delta$ passing through $\gamma n$ such that $g|\beta$ coincides with $\text{Corr}^\delta_H(\delta, g|\beta)$.

Proof: The conclusion of this Lemma follows easily if one applies the Second Axiom of Continuous Choice, but we want to make it clear that one needs only the Continuity Principle and the First Axiom of Dependent Choices. Observe that for every $n$, for every $\alpha$, we may determine $c$ such that $\text{length}(c) = n$ and there exists $\gamma$ passing through $c$ with the property that $g|\alpha$ coincides with $\text{Corr}^\gamma_H(\gamma, g|\alpha)$. Using the Continuity Principle, we see that for every $n$, for every $\alpha$, there exists $m, c$ such that $\text{length}(c) = n$ and for all $\beta$ passing through $\alpha m$ there exists $\delta$ passing through $c$ with the property that $g|\beta$ coincides with $\text{Corr}^\delta_H(\delta, g|\beta)$.

Now let $\alpha$ belong to $\mathcal{N}$. We consider the set $X$ of all pairs $(m, c)$ such that for every $\beta$ passing through $\alpha m$ there exists $\delta$ passing through $c$ with the property that $g|\beta$ coincides with $\text{Corr}^\delta_H(\delta, g|\beta)$.

We have just seen that $X$ contains at least one member.

Another application of the Continuity Principle ensures that to any $(m, c)$ in $X$ one may find $(p, d)$ in $X$ such that $\text{length}(d) = \text{length}(c) + 1$ and $c$ is an initial part of $d$. Applying the First Axiom of Dependent Choices we find $\beta, \zeta$ in $\mathcal{N}$ such that for each $p$, $\text{length}(\zeta(p)) = n$, and $\zeta(n)$ is an initial part of $\zeta(n + 1)$, and for each $\beta$, if $\beta$ passes through $\alpha(\varepsilon(n))$, then there exists $\delta$ passing through $\zeta(n)$ with the property that $g|\beta$ coincides with $\text{Corr}^\delta_H(\delta, g|\beta)$. We now let $\gamma$ be the element of $\mathcal{N}$ such that for each $n$, $\gamma n = \zeta(n)$. Observe that for every $s$, if $s$ II-obey $\gamma$, then there exists $n$ such that for every $p$, if $p > n$, then $s$ II-obey $\zeta(p)$. It follows that $g|\alpha$ coincides with $\text{Corr}^\gamma_H(\gamma, g|\alpha)$.

8.6 Theorem:

Let $\sigma$ be a non-zero hereditarily repetitive stump. The set $D(A_1, A_\sigma)$ does not reduce to the set $A_S(\sigma)$.

Proof: Assume that $f$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D(A_1, A_\sigma)$ to $A_S(\sigma)$. We want to obtain a contradiction.

We define a function $h$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $(h|\alpha)^1$ coincides with $\text{Corr}^\gamma_H(\alpha^1, \alpha^0)$ and, for every $t$, if there is no $j$ such that $t = (1) \ast j$, then $(h|\alpha)(t) := \alpha^0(t)$.
Observe that, for each \( a \), the sequence \((h|a)^1\) belongs to \( A_{\mathcal{S}(\sigma)} \) and the sequence \( f|(h|a) \) belongs to \( A_{\mathcal{S}(\sigma)} \). Observe that \((h|0)^0\) belongs to \( A_1 \) and that \((h|0)^1\) belongs to \( A_\sigma \).

We apply Lemma 8.5 and determine \( \gamma, \varepsilon \) such that the sequence \( f|(h|0) \) coincides with \( \text{Corr}_{\mathcal{S}(\sigma)}(\gamma, f|(h|0)) \) and for every \( n \), for every \( \alpha \), if \( \alpha \) passes through \( \overline{0}(\varepsilon(n)) \), then there exists \( \delta \) passing through \( \overline{\gamma}n \) such that \( f|(h|\alpha) \) coincides with \( \text{Corr}_{\mathcal{S}(\sigma)}(\delta, f|\alpha) \).

We assume that \( \varepsilon \) is strictly increasing, that is, for each \( n \), \( \varepsilon(n) < \varepsilon(n+1) \). We now want to define elements \( \beta, \delta \) of \( \mathcal{N} \) with the following properties.

(i) for each \( n \), \( \beta^{2n-1} \) belongs to \( E_\sigma \) and \( \beta^{2n+1,0} \) belongs to \( E_1 \).

(ii) for each \( n \), \( \beta^n \) passes through \( (h|0)(\varepsilon(n)) \) and the sequence \( \delta^n \) passes through \( \overline{\gamma}n \) and \( f|\beta^n \) coincides with \( \text{Corr}_{\mathcal{S}(\sigma)}(\delta^n, f|\beta^n) \).

Let \( n \) belong to \( \mathbb{N} \). We show how to define \( \beta^{2n} \) and \( \beta^{2n+1} \), and also \( \delta^{2n} \) and \( \delta^{2n+1} \).

We let \( C \) be the set of all \( \alpha \) such that \( \alpha \) passes through \( (h|0)\varepsilon(2n) \) and, for all \( j \), if \( j < \varepsilon(2n) \) then \( \alpha^{-1,j} = (h|0)^{-1,j} \). Let \( X \) be the set of \( \alpha \) in \( C \) such that for each \( j \), \( \varepsilon(2n) \), \( \alpha^{-1,j} \) belongs to \( (1) \). Observe that for each \( \alpha \), if \( \alpha \) belongs to \( X \), then there exists \( \gamma \) passing through \( \overline{0}(2n) \) such that \( h|\gamma = \alpha \), therefore there exists \( \delta \) passing through \( \overline{\gamma}(2n) \) such that \( f|\alpha \) coincides with \( \text{Corr}_{\mathcal{S}(\sigma)}(\delta, f|\alpha) \), in particular, for every \( t \) of even length belonging to \( S(\sigma) \) and II-obeying \( \overline{\gamma}(2n) \) the sequence \( f|\alpha \) belongs to \( A_{\mathcal{S}(\sigma)} \). We now define a function \( g \) from \( \mathcal{N} \) to \( \mathcal{N} \) mapping \( A_\sigma \) into \( X \).

We first construct \( \mu \) in \( \mathbb{N} \) such that \( \mu \) is strictly increasing, \( \varepsilon(2n) \leq \mu(0) \) and for each \( i \), \( \sigma^i \) equals \( \sigma^{\mu(i)} \). We then define: for every \( \alpha \), for every \( i \), \( (g|\alpha)^{\mu(i)} := \alpha^{i} \) and for each \( p \), if there do not exist \( i, s \) such that \( p = (1, \mu(i)) * s \), then \( (g|\alpha)(p) := (h|0)(p) \). We now consider the function \( k \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \), \( k|\alpha = f|(g|\alpha) \). Observe that for every \( \alpha \), if \( \alpha \) belongs to \( A_\sigma \), then \( (g|\alpha)^{1} \) belongs to \( A_\sigma \) and for every \( t \) of positive even length belonging to \( \sigma \) and II-obeying \( \overline{\gamma}(2n) \), the sequence \( f|k|\alpha \) belongs to \( A_{\mathcal{S}(\sigma)} \) and for every \( t \) of even length \( \sigma \) and II-obeying \( \overline{\gamma}(2n) \), the sequence \( f|k|\alpha \) belongs to \( A_{\mathcal{S}(\sigma)} \) and for every \( t \) of such \( t \), \( \overline{\sigma}(S(\sigma)) < \sigma \).

The set \( Y \) consisting of all \( \xi \) such that for every such \( t \), \( \overline{\xi} \) belongs to \( A_{\mathcal{S}(\sigma)} \) reduces to \( E_\sigma \). We apply Lemma 8.3 and find \( \alpha \) in \( E_\sigma \) such that \( k|\alpha \) belongs to \( Y \).

We define \( \beta^{2n} := g|\alpha \). Observe that \( \beta^{2n+1} \) belongs to \( E_\sigma \), as \( \alpha \) does so. Observe that \( \beta^{2n,0} \) belongs to \( A_1 \), therefore \( f|\beta^{2n,0} \) belongs to \( A_\sigma \). Also \( \alpha \) belongs to \( Y \), that is, for every \( t \) of positive even length belonging to \( \sigma \) and II-obeying \( \overline{\gamma}(2n) \), the sequence \( f|k|\alpha \) belongs to \( A_{\mathcal{S}(\sigma)} \) and for every \( t \) of such \( t \), \( \overline{\sigma}(S(\sigma)) < \sigma \).

We also define: \( \beta^{2n+1} := h|(\overline{0}(\varepsilon(2n+1)) * 1) \). Observe that \( \beta^{2n+1,0} \) belongs to \( E_1 \) and \( \beta^{2n+1,1} \) belongs to \( A_\sigma \), and \( f|\beta^{2n+1,1} \) belongs to \( A_{\mathcal{S}(\sigma)} \) and we may define \( \delta^{2n+1} \) passing through \( \overline{\gamma}(2n+1) \) such that \( f|\beta^{2n+1,1} \) coincides with \( \text{Corr}_{\mathcal{S}(\sigma)}(\delta^{2n+1} + f|\beta^{2n+1,1}) \).

Let \( T \) be the set \( \{0\} \cup \{0^n * 1 \} * 0 \) \( n \in \mathbb{N} \).

We build functions \( b \) and \( d \) from the closure \( \overline{T} \) of \( T \) to \( \mathcal{N} \) such that for every \( n \),
b(|\bar{0}n \ast (1) \ast 0) equals \beta^n and d(|\bar{0}n \ast (1) \ast 0) equals \delta^n. Observe that b|0 coincides with h|0 and d|0 coincides with \gamma, therefore f|(b|0) coincides with Corr_{II}^{S(\sigma)}(d|0, f|(b|0)). We claim that for every \alpha in T, the sequence f|(b|\alpha) coincides with the sequence Corr_{II}^{S(\sigma)}(d|\alpha, f|(b|\alpha)).

For suppose that for some \alpha, p, the value of the sequence f|(b|\alpha) at p differs from the value of Corr_{II}^{S(\sigma)}(d|\alpha, f|(b|\alpha)) at p. Then the sequence \alpha must be different from every sequence \bar{0}n \ast (1) \ast 0, therefore \alpha = 0. But the sequence b|0 coincides with the sequence Corr_{II}^{S(\sigma)}(d|0, f|(b|0)). contradiction.

We conclude that for every \alpha in T, f|(b|\alpha) belongs to \text{AS}(\text{S}(\sigma)), therefore b|\alpha belongs to D(A_1, A_\sigma).

Applying Lemma 8.1 we find n such that either for every p > n if p > n, then b(|\bar{0}p \ast (1) \ast 0) = \beta_p^{p,0} belongs to A_1, or for every p, if p > n, then \beta_p^{p,1} belongs to A_\sigma. Contradiction.

8.7 Theorem: Let \sigma be a non-zero hereditarily repetitive stump. For each positive n, the set D(A_1, D^n(A_\sigma)) does not reduce to the set D^n(\text{AS}(\text{S}(\sigma))).

Proof: Assume that n is a positive natural number and f is a function from \mathcal{N} to \mathcal{N} reducing D(A_1, D^n(A_\sigma)) to D^n(\text{AS}(\text{S}(\sigma))). We let X be the set of all \alpha such that \alpha^0 = 0 belongs to A_1, and for each i < n, we let h_i be the function from \mathcal{N} to \mathcal{N} such that for every \alpha, the sequence (h_i|\alpha)^{1,i} coincides with the sequence Corr_{II}^{S(\sigma)}(\alpha^1, \alpha^{0,1,i}) and for every t, if there is no j such that t = (1, i) \ast j, then (h|\alpha)(t) = \alpha^0(t). Applying the Continuity Principle we first calculate p, q such that q < n and, for every \alpha in X, if \alpha passes through \bar{0}p, then (f|\alpha)^q belongs to \text{AS}(\text{S}(\sigma)). Applying the Continuity Principle again, we find for each i < n numbers p_i, q_i such that for every \alpha, if \alpha passes through \bar{0}p_i then (f|(h_i|\alpha))^q_i belongs to A. We now distinguish two cases. First Case. We find i such that q_i = q. We then consider the set Z consisting of all \alpha such that \alpha passes through \bar{0}p_i and through \bar{0}p_i and for each j < p_i, \alpha^{1,i,j} coincides with (h_i|0)^{1,i,j}. Observe that for every \alpha in Z, if \alpha^{1,i} belongs to A_\sigma, there exists \beta passing through \bar{0}p_i such that \alpha = h_i|\beta and therefore (f|\alpha)^q belongs to \text{AS}(\text{S}(\sigma)). Also if \alpha^0 belongs to A_1, (f|\alpha)^q belongs to \text{AS}(\text{S}(\sigma)). We leave it to the reader to conclude from this that the set D(A_1, A_\sigma) reduces to the set \text{AS}(\text{S}(\sigma)). We now have a contradiction, according to Theorem 8.6. Second Case. We find i, j < n such that i \neq j and q_i = q_j. We now let p be the greatest of p_i, p_j, and let Z be the set of all \alpha passing through \bar{0}p such that for every k < p, both \alpha^{1,i,k} coincides with (h_i|0)^{1,i,k} and \alpha^{1,j,k} coincides with (h_j|0)^{1,j,k}. Observe that for every \alpha in Z, if \alpha^{1,i} belongs to A_\sigma or \alpha^{1,j} belongs to A_\sigma, (f|\alpha)^q belongs to A. We may conclude from this that the set D(A_\sigma, D^n(A_\sigma)) reduces to the set \text{AS}(\text{S}(\sigma)), therefore D(A_1, A_\sigma) reduces to \text{AS}(\sigma), and this contradicts Theorem 8.6.

8.8 Corollary: Let \sigma be a non-zero hereditarily repetitive stump. For each positive n, the set D^{n+1}(A_\sigma) does not reduce to the set D^n(A_\sigma).
9 On (strictly) analytic, co-analytic and projective sets.

Using Brouwer’s Thesis, we recover some famous results on strictly analytic sets. We introduce the wider class of analytic sets and its counterpart, the class of co-analytic sets. We show the collapse of the projective hierarchy.

9.1 Let $X$ be a subset of $\mathcal{N}$. $X$ is called strictly analytic if and only if there exists a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $\gamma$, that is, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, $\alpha$ coincides with $\gamma|\beta$. Every strictly analytic subset $X$ of $\mathcal{N}$ is inhabited, that is, there exists $\alpha$ such that $\alpha$ belongs to $X$. Every one of the leading sets of the Borel hierarchy, that is, every set $A_\sigma$ and every set $E_\delta$, is strictly analytic.

9.2 Theorem: (Lusin Separation Theorem)

Let $\gamma, \delta$ be functions from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha, \beta$, $\gamma|\alpha$ is apart from $\delta|\beta$. There exist positively Borel sets $C, D$, such that for every $\alpha, \beta$, $\gamma|\alpha$ belongs to $C$ and $\delta|\beta$ belongs to $D$, and every element of $C$ is apart from every element of $D$.

Proof: We use the Principle Induction on Monotone Bars, 1.6.4. We define subsets $P, Q$ of $\mathbb{N} \times \mathbb{N}$, as follows: for every $a, b$ in $\mathbb{N}$,

$P(a, b) :=$ For every $\alpha$ passing through $a$, for every $\beta$ passing through $b$, the sequence $\gamma|\alpha$ is apart from the sequence $\delta|\beta$.

$Q(a, b) :=$ There exist positively Borel sets $C, D$ such that for every $\alpha$ passing through $a$, for every $\beta$ passing through $b$, $\gamma|\alpha$ belongs to $C$ and $\delta|\beta$ belongs to $D$, and every element of $C$ is apart from every element of $D$.

It will be clear that $P$ is a monotone bar in $\mathcal{N} \times \mathcal{N}$ and that $P$ is a subset of $Q$. Now assume $a, b$ belong to $\mathbb{N}$ and for every $m$, for every $n$, the pair $(a \star (m), b \star (n))$ belongs to $Q$. Using the Second Axiom of Countable Choice, we determine, for every $m, n$, positively Borel sets $C_{m,n}$ and $D_{m,n}$ such that for every $\alpha$ passing through $a \star (m)$, for every $\beta$ passing through $b \star (n)$, $\gamma|\alpha$ belongs to $C_{m,n}$ and $\delta|\beta$ belongs to $D_{m,n}$ and every element of $C_{m,n}$ is apart from every element of $D_{m,n}$. Define $C := \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} C_{m,n}$ and $D := \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} D_{m,n}$ and remark that for every $\alpha$ passing through $a$, for every $\beta$
passing through $b$, $\gamma|\alpha$ belongs to $C$ and $\delta|\beta$ belongs to $D$, and every element of $C$ is apart from every element of $D$.

The Principle of Double Induction on Monotone Bars allows us to conclude that the pair $\langle (\cdot, \cdot) \rangle$ belongs to $Q$, and the conclusion of the Theorem follows.

9.3 Theorem: (One half of Lusin's Regular Representation Theorem)

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$, such that for all $\alpha, \beta$, if $\alpha$ is apart from $\beta$, then $\gamma|\alpha$ is apart from $\gamma|\beta$.

Then the range of $\gamma$, that is, the set of all $\alpha$ coinciding with some $\gamma|\beta$, is positively Borel.

Proof: Assume that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ and that for all $\alpha, \beta$, if $\alpha$ is apart from $\beta$, then $\gamma|\alpha$ is apart from $\gamma|\beta$.

We intend to define a sequence $H_0, H_1, \ldots$ of positively Borel sets such that the range of $\gamma$ coincides with $\bigcap_{n \in \mathbb{N}} H_n$.

To this end, we first define a sequence $B_0, B_1, \ldots$ of decidable subsets of $\mathbb{N}$. Each of these will be a bar in $\mathcal{N}$.

We let $B_0$ be the set $\{\langle \cdot, \cdot \rangle\}$ consisting of the empty sequence only. For each $n$, we let $B_{n+1}$ be the set of all $a$ of minimal length such that some proper initial segment of $a$ belongs to $B_n$, and for some initial part $b$ of $a$, $\gamma^n(b) \neq \emptyset$. Observe that for each $n$, for all $a$ in $B_n$, for all $\alpha, \beta$ passing through $a$, $\langle \gamma|\alpha \rangle_n = \langle \gamma|\beta \rangle_n$. We let $\varepsilon$ be an element of $\mathcal{N}$ such that for each $n$, for each $a$ in $B_n$, for each $\alpha$ passing through $a$, $\langle \gamma|\alpha \rangle_n$ equals $\varepsilon^n(a)$. Using Theorem 9.2 and the Second Axiom of Countable Choice we find for each $n$, for all $a, b$ in $B_n$ such that $a \neq b$, positively Borel sets $C_{a,b}, D_{a,b}$ such that for all $\alpha$, if $\alpha$ passes through $a$, then $\gamma|\alpha$ belongs to $C_{a,b}$ and, if $\alpha$ passes through $b$, then $\gamma|\alpha$ belongs to $D_{a,b}$, and every element of $C_{a,b}$ is apart from every element of $D_{a,b}$. For each $n$, for each $a$ in $B_n$ we let $K_a$ be the set of all $\beta$ passing through $\varepsilon^n(a)$ and belonging to every set $C_{a,b} \cap D_{b,a}$, where $b$ belongs to $B_n$ but differs from $a$. Remark that for all $n$, for all $a, b$ in $B_n$, if $a \neq b$, then every element of $K_a$ is apart from every element of $K_b$. For each $n$, for each $a$ in $B_n$ we let $L_a$ be the set of all $\beta$ belonging to every set $K_b$, where $b$ is an initial segment of $a$, and for some $i \leq n$, $b$ belongs to $B_i$. We define, for each $n$, $H_n := \bigcup_{a \in B_n} L_a$.

It will be clear that the range of $\gamma$ forms part of every set $H_n$. Now assume that $\beta$ belongs to every set $H_n$. For each $n$, we determine $a_n$ in $B_n$ such that $\beta$ belongs $L_{(a_n)}$. Observe that for each $n$, $a_n$ must be a proper initial segment of $a_{n+1}$ and consider the sequence $\alpha$ that passes through every $a_n$. Observe that for each $n$, $\beta$ belongs to $K_{(a_n)}$ and therefore passes through $\langle \gamma|\alpha \rangle_n$, so $\beta$ coincides with $\gamma|\alpha$.

9.4 Recall from Section 1.3.3, that a subset $X$ of $\mathcal{N}$ is weakly closed if and only if every $\alpha$ such that, for each $n$, $\alpha^n$ contains an element of $X$ belongs to $X$. If in
addition, one may decide for each $a$, if $a$ contains an element of $X$ or not, then $X$ is a spread. As subset $X$ of $\mathcal{N}$ is closed if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for each $n$, $\overline{\alpha}n$ belongs to $C$.

9.5 Theorem:

(i) Every spread is a strictly analytic subset of $\mathcal{N}$.

(ii) For every weakly-closed subset of $X$ of $\mathcal{N}$, $X$ is strictly analytic if and only if there exists $\delta$ such that for every $a$, $a$ contains a member of $X$ if and only if, for some $n$, $\delta(n)$ equals $a$.

(iii) Not every strictly analytic and closed subset of $X$ is a spread.

(iv) Not every inhabited and closed subset of $X$ is strictly analytic.

Proof: (i) For every inhabited spread $X$ may define a so-called retraction from $\mathcal{N}$ to $X$, that is a function $\gamma$ from $\mathcal{N}$ to $X$ such that for every $\alpha$ in $X$, $\gamma(\alpha) = \alpha$. We have done so in Section 1.3.5. We conclude that every spread is a strictly analytic subset of $\mathcal{N}$.

(ii) Let us first assume that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$. Observe that for each $a$, $a$ contains a member of the range of $\gamma$ if and only if there exists $b$ such that for each $i < \text{length}(a)$ there exists an initial segment $c$ of $b$ such that $\gamma^i(c) = (a)_i + 1$ and for all initial segments $d$ of $c$, $\gamma^i(d) = 0$.

It is easy to construct $\delta$ enumerating all numbers $a$ with this property. Now assume that we have a $\delta$ enumerating all numbers $a$ containing an element of $X$. Observe that for every $n$ there exist $m,p$ such that $\delta(m) = \delta(n) * (p)$. We construct $\varepsilon$ such that for each $a$, $\text{length}(a) = \text{length}(\varepsilon(a))$, as follows. We define $\varepsilon(\langle \rangle) := \langle \rangle$ and for each $a,n$, $\varepsilon(a * \langle n \rangle) := \delta(n)$ if $\delta(n)$ is an immediate successor of $\varepsilon(a)$, and $\varepsilon(a * (n)) = \varepsilon(a) * (p_0)$ where $p_0$ is such that for some $k$, $\delta(k) = \varepsilon(a) * (p_0)$ and for no $i < k$ there exists $p$ such that $\delta(k) = \varepsilon(a) * (p)$. We then let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every $n$, for every $\alpha$, $\gamma(\alpha)$ passes through $\varepsilon(\overline{\alpha}n)$. $X$ will coincide with the range of $\gamma$.

(iii) For every $\alpha$, we define a decidable subset $C_\alpha$ of $\mathbb{N}$, as follows. For each $\alpha$, $\alpha$ belongs to $C_\alpha$ if and only if either $\langle \rangle$ passes through $a$, or $\text{length}(a) > 0$ and $(a)_0 = 1$ and, if $\text{length}(a) > 1$, there exists $n < (a)_1$ such that $\alpha(n) \neq 0$. We let $X_\alpha$ be the set of all $\beta$ such that, for each $n$, $\overline{\beta}n$ belongs to $X_\alpha$. Observe that $X_\alpha$ is closed and inhabited. We claim that $X_\alpha$ is also strictly analytical, as, for every $\alpha$, we may construct $\delta$ enumerating the natural numbers containing an element of $X_\alpha$ as follows. For each $a$, if either $\langle \rangle$ passes through $a$ or $\text{length}(a) > 1$ and $(a)_0 = 1$ and there exists $n < (a)_1$ such that $\alpha(n) \neq 0$, then $\delta(a) := (a)$, and if $\text{length}(a) = 1$ and $(a)_0$ is the least $n$ such that $\alpha(n) \neq 0$, then $\delta(a) := \langle 1 \rangle$, and if these conditions do not apply then $\delta(a) := \langle \rangle$. Assume that for each $\alpha$, $X_\alpha$ is a spread, therefore $\langle 1 \rangle$ contains an element of $X_\alpha$ or not, therefore either: for some $n$, $\alpha(n) \neq 0$, or: $\alpha = \langle \rangle$. We easily obtain a contradiction by the Continuity Principle.

(iv) For each $\alpha$, we let $Y_\alpha$ be the set of all $\beta$ such that, for each $n$, either $\overline{\beta}n = \overline{\langle 0 \rangle}n$. 

75
or both $\overline{b} n = \overline{1} n$ and $\overline{a} n = \overline{0} n$. Observe that, for each $\alpha$, $Y_\alpha$ is closed. Assume that, for each $\alpha$, $Y_\alpha$ is strictly analytical, that is, there exists $\delta$ enumerating the set of all numbers $a$ containing an element of $Y_\alpha$. Applying the Second Axiom of Continuous Choice we find a function $e$ from $\mathcal{N}$ to $\mathcal{N}$ such that for all $\alpha$, $e|\alpha$ enumerates the numbers containing an element of $Y_\alpha$. Calculate $m$ such that $(e|0)(m) = 1$. Calculate $p$ such that for every $\alpha$, if $\overline{e} p = \overline{0} p$, then $(e|\alpha)(m) = (e|0)(m) = 1$, therefore $\overline{1}$ belongs to $Y_\alpha$ and $a = 0$. Contradiction.

9.6 Suppose that $\gamma$ belongs to $\text{Fun}$, that is, for every $\alpha$ there exists $n$ such that $\gamma(\overline{\alpha} n) \neq 0$. Let $a$ be a natural number. We say that $a$ belongs to $\gamma$ if and only if for every initial segment $b$ of $a$, $\gamma(b) = 0$. Observe that $\text{Stp}$ is a subset of $\text{Fun}$, and that our terminology accords with the one introduced in Section 1.5.3. Suppose that $\gamma, \delta$ belong to $\text{Fun}$. Let $\varepsilon$ be an element of $\mathcal{N}$. We say that $\varepsilon$ embeds $\gamma$ into $\delta$ if and only if $\varepsilon$ maps every number belonging to $\gamma$ onto a number belonging to $\delta$ in such a way that for every $a, b$ belonging to $\gamma$, if $a$ is a proper initial segment of $b$, then $\varepsilon(a)$ is a proper initial segment of $\varepsilon(b)$. We say that $\gamma$ embeds into $\delta$, notation: $\gamma \leq^* \delta$, if and only if there exists $\varepsilon$ embedding $\gamma$ into $\delta$. We say that $\gamma$ properly embeds into $\delta$, notation: $\gamma <^* \delta$, if and only if, for some $n$, $\gamma$ embeds into $\delta^n$.

Let $\gamma$ belong to $\text{Fun}$. We define an element of $\mathcal{N}$, called the successor of $\gamma$, notation: $S(\gamma)$, as follows. $(S(\gamma))(\langle \rangle) := 0$ and, for each $n$, $(S(\gamma))^n := \gamma$.

9.7 Theorem:

(i) For all $\alpha$ in $\text{Fun}$, $\alpha \leq^* \alpha$ and not: $\alpha <^* \alpha$.

For all $\alpha, \beta$ in $\text{Fun}$, if $\alpha <^* \beta$, then $\alpha \leq^* \beta$.

For all $\alpha, \beta, \gamma$ in $\text{Fun}$, if $\alpha \leq^* \beta$ and $\beta <^* \gamma$, then $\alpha <^* \gamma$, and: if $\alpha <^* \beta$ and $\beta \leq^* \gamma$, then $\alpha <^* \gamma$, and if $\alpha \leq^* \beta$ and $\beta <^* \gamma$, then $\alpha \leq^* \gamma$.

For all $\alpha, \beta$ in $\text{Fun}$, $\alpha \leq^* \beta$ if and only if $\alpha <^* S(\beta)$.

(ii) (Brouwer's Thesis) For every $\gamma$ in $\text{Fun}$ there exists a stump $\sigma$ such that the set of all natural numbers belonging to $\gamma$ coincides with the set of all numbers belonging to $\sigma$.

(iii) (Cantor's Argument) For every function $\gamma$ from $\mathcal{N}$ to $\text{Fun}$ there exists $\alpha$ in $\text{Fun}$ such that, for every $\beta$, $\gamma|\beta$ is apart from $\alpha$.

(iv) (Boundedness Theorem) For every function $\gamma$ from $\mathcal{N}$ to $\text{Fun}$ there exists $\alpha$ in $\text{Fun}$ such that, for every $\beta$, $\gamma|\beta$ embeds into $\alpha$.

Proof: We leave the proofs of (i) and (ii) to the reader. As to (iii), suppose $\gamma$ is a function from $\mathcal{N}$ to $\text{Fun}$, and let $\alpha$ be an element of $\text{Fun}$ such that, for every $\beta$, $\alpha(\beta) = (\gamma(\beta))(\beta) + 1$. Clearly, $\alpha$ is apart from every $\gamma|\beta$. We now prove (iv). Again suppose $\gamma$ is a function from $\mathcal{N}$ to $\text{Fun}$. Observe that for every $\beta$, for every $\overline{\delta}$, there exists $n$ such that $(\gamma|\beta)(\overline{\delta} m) \neq 0$ and there exists $m$ such that $\gamma^m(\overline{\delta} m) = (\gamma|\beta)(\overline{\delta} m) + 1$. 76
We define \( \alpha \) in \( \mathcal{C} \) such that for every \( a, n \), if \( n = \text{length}(a) \) and \( a = \langle (a)_0, \ldots, (a)_{n-1} \rangle \), then \( \alpha(a) = 1 \) if and only if there exist \( i, j \) such that \( 2i < n \) and \( 2j + 1 < n \) and \( \gamma(\langle (a)_0, \ldots, (a)_{2i} \rangle, \langle (a)_1, (a)_3, \ldots, (a)_{2j+1} \rangle) \) differs from 0. Clearly, \( \alpha \) belongs to \( \text{Fun} \).

Now consider any \( \beta \in \mathcal{N} \). Define an element \( \varepsilon \) of \( \mathcal{N} \), as follows. For every \( d, a \), if \( n = \text{length}(d) \) and \( d = \langle (d)_0, \ldots, (d)_{n-1} \rangle \), then \( \varepsilon(d) := \langle (d)_0, \beta(0), \ldots, (d)_{n-1}, \beta(n - 1) \rangle \).

One verifies easily that \( \varepsilon \) embeds \( \gamma \beta \) into \( \alpha \).

9.8 We introduce two projection operations on the class of subsets of \( \mathcal{N} \). For every subset \( X \) of \( \mathcal{N} \), we let the \( \text{(existential) projection} \) of \( X \), notation \( Ex(X) \), be the set of all \( \alpha \) such that, for some \( \beta \), \( \langle \alpha, \beta \rangle \) belongs to \( X \).

For every subset \( X \) of \( \mathcal{N} \), we let the \( \text{universal projection} \) of \( X \), notation \( Un(X) \), be the set of all \( \alpha \) such that for every \( \beta \), \( \langle \alpha, \beta \rangle \) belongs to \( X \).

A subset \( \mathcal{N} \) is called \( \text{analytic} \) if and only if there exists a closed subset \( Y \) of \( \mathcal{N} \) such that \( X \) coincides with \( Ex(Y) \). The class of the analytic subsets of \( \mathcal{N} \) is denoted by \( \Sigma_1^1 \).

A subset \( X \) of \( \mathcal{N} \) is called \( \text{co-analytic} \) if and only if there exists an open subset \( Y \) of \( \mathcal{N} \) such that \( X \) coincides with \( Un(Y) \). The class of the co-analytic subsets of \( \mathcal{N} \) is denoted by \( \Pi_1^1 \).

We use \( S^* \) to denote the successor function from \( \mathbb{N} \) to \( \mathbb{N} \), and \( o \) to denote the operation of composition on \( \mathbb{N} \). So, for each \( \alpha, \beta, n, \alpha \circ \beta(n) = \alpha(\beta(n)) \).

9.9 Theorem: \( \text{(Closure Properties of the classes } \Sigma_1^1 \text{ and } \Pi_1^1 \text{)} \)

(i) For every sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), if, for each \( n, X_n \) belongs to \( \Sigma_1^1 \), then both \( \bigcup_{n \in \mathbb{N}} X_n \) and \( \bigcap_{n \in \mathbb{N}} X_n \) belong to \( \Sigma_1^1 \).

(ii) For every subset \( X \) of \( \mathcal{N} \), if \( X \) is positively Borel, then \( X \) is analytic.

(iii) Every strictly analytic subset of \( \mathcal{N} \) is inhabited and analytic, but not every inhabited and closed subset of \( \mathcal{N} \) is strictly analytic.

(iv) Every co-analytic subset of \( \mathcal{N} \) is perhapsive, and the set \( D^2(A_1) \) is not co-analytic.

(v) For every sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), if, for each \( n, X_n \) belongs to \( \Pi_1^1 \), then \( \bigcap_{n \in \mathbb{N}} X_n \) belongs to \( \Pi_1^1 \).

(vi) Every \( \Pi_2^1 \)-subset of \( \mathcal{N} \) is co-analytic.

(vii) For every subset \( X \) of \( \mathcal{N} \), if \( X \) belongs to \( \Sigma_1^1 \), then \( Ex(X) \) belongs to \( \Sigma_1^1 \) and:

if \( X \) belongs to \( \Pi_1^1 \), then \( Un(X) \) belongs to \( \Pi_1^1 \).

(viii) For all subsets \( X, Y \) of \( \mathcal{N} \), if \( X \) reduces to \( Y \) and \( Y \) is analytic, then \( X \) is analytic, and: if \( X \) reduces to \( Y \) and \( Y \) is co-analytic, then \( X \) is co-analytic.

(ix) For every subset \( X \) of \( \mathcal{N} \), if \( X \) is strictly analytic, then \( Ex(X) \) is strictly analytic.

In particular, if \( X \) is a spread, or a weakly closed strictly analytic subset of \( \mathcal{N} \), then \( Ex(X) \) is strictly analytic.
Proof: (i) We use the Second Axiom of Countable Choice. Let \( Y_0, Y_1, \ldots \) be a sequence of closed subsets of \( \mathcal{N} \) such that, for each \( n \), \( X_n = \text{Ex}(Y_n) \). We now define subsets \( Z_0, Z_1 \) of \( \mathcal{N} \), as follows.

\( Z_0 \) is the set of all \( \alpha \) such that \( \langle \alpha^0, \alpha^1 \circ S^* \rangle \) belongs to \( Y_{\alpha(0)} \) and \( Z_1 \) is the set of all \( \alpha \) such that for each \( n \), \( \langle \alpha^0, \alpha^{1,n} \rangle \) belongs to \( Y_n \). Observe that \( Z_0, Z_1 \) are closed, and \( \bigcup X_n \) coincides with \( \text{Ex}(Z_0) \) and \( \bigcap X_n \) coincides with \( \text{Ex}(Z_1) \).

(ii) follows easily from (i).

(iii) Let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \). Let \( Z \) be the set of all \( \alpha \) such that for each \( n, m \), if \( \gamma^n(\alpha^{1,m}) \neq 0 \) and for every \( i < m \), \( \gamma^n(\alpha^{1,i}) = 0 \), then \( \gamma^n(\alpha^{1,m}) = \alpha^n(n) + 1 \). Observe that \( Z \) is a closed subset of \( \mathcal{N} \) and that the range of \( \gamma \) coincides with \( \text{Ex}(Z) \).

(iv) Let \( X \) be a co-analytical subset of \( \mathcal{N} \) and let \( Y \) be an open subset of \( \mathcal{N} \) such that \( X = \bigcup Y \). Now suppose we are given \( \alpha, \gamma \) such that \( \gamma \) belongs to \( X \), and: if \( \alpha \) is apart from \( \gamma \), then \( \alpha \) belongs to \( X \). We claim that \( \alpha \) itself belongs to \( X \). In order to see this, find a decidable subset \( C \) of \( \mathbb{N} \) such that for every \( \delta, \delta \) belongs to \( Y \) if and only if, for some \( n \), \( \delta n \) belongs to \( C \).

Let \( \beta \) be an element of \( \mathbb{N} \). Find \( n \) such that \( \langle \gamma, \beta \rangle n \) belongs to \( C \), and distinguish two cases. Either \( \langle \alpha, \beta \rangle n = \langle \gamma, \beta \rangle n \), and \( \langle \alpha, \beta \rangle n \) belongs to \( C \), and therefore \( \langle \alpha, \beta \rangle \) belongs to \( Y \) or \( \langle \alpha, \beta \rangle \neq \langle \gamma, \beta \rangle n \), therefore \( \alpha \) is apart from \( \gamma \), and \( \alpha \) belongs to \( X \) and \( \langle \alpha, \beta \rangle \) belongs to \( Y \). In any case \( \langle \alpha, \beta \rangle \) belongs to \( Y \). Therefore, for every \( \beta, \langle \alpha, \beta \rangle \) belongs to \( Y \), and \( \alpha \) itself belongs to \( X \). This shows that every co-analytical subset of \( \mathcal{N} \) is perhapsive.

The set \( D^2(A_1) \) is not perhapsive, see Theorem 5.4, and therefore does not belong to \( \Pi_1^1 \).

(v) We use the Second Axiom of Countable Choice. Let \( Y_0, Y_1, \ldots \) be a sequence of open subsets of \( \mathcal{N} \) such that, for each \( n \), \( X_n = \bigcup Y_n \). We define a subset \( Z \) of \( \mathcal{N} \) as follows. \( Z \) is the set of all \( \alpha \) such that \( \langle \alpha^0, \alpha^1 \circ S^* \rangle \) belongs to \( Y_{\alpha(0)} \). Observe that \( Z \) is open and \( \bigcap X_n \) coincides with \( \bigcup Y_n(Z) \).

(vi) Use (v) and the simple fact that every open subset of \( \mathcal{N} \) is co-analytical.

(vii) Suppose that \( X \) belongs to \( \Sigma_1^1 \). Let \( Y \) be a closed subset of \( \mathcal{N} \) such that \( X = \text{Ex}(Y) \). Observe that for every \( \alpha, \alpha \) belongs to \( \text{Ex}(X) \) if and only if for some \( \beta, \langle \alpha, \beta \rangle \) belongs to \( X \) if and only if for some \( \beta, \gamma \), the sequence \( \langle \langle \alpha, \beta \rangle, \gamma \rangle \) belongs to \( Y \). Let \( Z \) be the set of all \( \alpha \) such that \( \langle \langle \alpha^0, \alpha^{1,0}, \alpha^{1,1} \rangle \rangle \) belongs to \( Y \). Observe that \( Z \) is closed and \( \text{Ex}(X) \) coincides with \( \text{Ex}(Z) \), therefore \( \text{Ex}(X) \) belongs to \( \Sigma_1^1 \). The proof of the second statement is similar.

(viii) Let \( X, Y \) be subsets of \( \mathcal{N} \) and let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( X \) to \( Y \). Assume that \( Y \) is analytical and let \( Z \) be a closed subset of \( \mathcal{N} \) such that \( Y \) coincides with \( \text{Ex}(Z) \). Observe that for every \( \alpha, \alpha \) belongs to \( X \) if and only if \( \gamma \alpha \) belongs to \( Y \) if and only if for some \( \beta, \langle \beta, \alpha \rangle \) belongs to \( Z \).

Let \( V \) be the set of all \( \alpha \) such that \( \langle \gamma, \alpha^0, \alpha^1 \rangle \) belongs to \( Z \). Observe that \( V \) is closed and \( X \) coincides with \( \text{Ex}(V) \), therefore \( X \) is analytical. The proof of the second
9.10 We have proved, in Theorem 4.7, that for every enumerable subset $X$ of $\mathcal{N}$, the set $\text{Almost}^*(X)$ is perhapsive. Observe that for such sets $X$, the set $\text{Almost}^*(X)$ is co-analytical, so Theorem 4.7(iii) follows from Theorem 9.9(iv).

We define subsets $US_1^1$ and $UP_1^1$ of $\mathcal{N}$ as follows.

$US_1^1$ is the set of all $\alpha$ such that, for some $\beta$, the sequence $\langle \alpha^0, (\alpha^1, \beta) \rangle$ belongs to $UP_1^*$. $UP_1^*$ is the set of all $\alpha$ such that, for all $\beta$, the sequence $\langle \alpha^0, (\alpha^1, \beta) \rangle$ belongs to $US_1^*$. We call $US_1^1$, $UP_1^1$ the cataloguing sets of $\Sigma_1^1$, $\Pi_1^1$, respectively.

We define subsets $E_1^1$ and $A_1^1$ of $\mathcal{N}$ as follows.

$E_1^1$ is the set of all $\alpha$ such that for some $\beta$, for all $n$, $\alpha(\beta n) = 0$. $A_1^1$ coincides with $\text{Fun}$, so $A_1^1$ consists of all $\alpha$ such that for every $\beta$ there exists $n$ such that $\alpha(\beta n) \neq 0$. We call $E_1^1$, $A_1^1$ the leading sets of $\Sigma_1^1$, $\Pi_1^1$, respectively.

9.11 Theorem: (Classical remarks on cataloguing and leading sets in $\Sigma_1^1$ and $\Pi_1^1$)

(i) For every subset $X$ of $\mathcal{N}$, $X$ belongs to $\Sigma_1^1$ if and only if, for some $\alpha$, $X$ coincides with $US_1^1|\alpha$, and: $X$ belongs to $\Pi_1^1$ if and only if, for some $\alpha$, $X$ coincides with $UP_1^1|\alpha$.

Moreover, $UP_1^1$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $US_1^1$.

(ii) For every subset $X$ of $\mathcal{N}$, $X$ belongs to $\Sigma_1^1$ if and only if $X$ reduces to $E_1^1$, and: $X$ belongs to $\Pi_1^1$ if and only if $X$ reduces to $A_1^1$.

Moreover, $A_1^1$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $E_1^1$.

(iii) If $UP_1^1$ belongs to $\Sigma_1^1$, and also if $US_1^1$ belongs to $\Pi_1^1$, there exists $\alpha$ not belonging to either one of $UP_1^1$, $US_1^1$.

(iv) For every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ there exists $\alpha$ such that $\alpha$ belongs to $A_1^1$ if and only if $f|\alpha$ belongs to $E_1^1$ and also: $\alpha$ belongs to $E_1^1$ if and only if $f|\alpha$ belongs to $E_1^1$.

(v) If $A_1^1$ reduces to $E_1^1$ and also if $E_1^1$ reduces to $A_1^1$, there exists $\alpha$ not belonging to either one of $A_1^1$, $E_1^1$.

Proof: (i) Let $X$ belong to $\Sigma_1^1$ and let $Y$ be a closed subset of $\mathcal{N}$ such that $X = Ex(Y)$. Find $\alpha$ such that $Y$ coincides with $UP_1^1|\alpha$ and observe that, for every $\beta$, $\beta$ belongs to $X$ if and only if, for some $\gamma$, $\langle \beta, \gamma \rangle$ belongs to $Y$ if and only if, for some $\gamma$, $\langle \alpha, (\beta, \gamma) \rangle$ belongs to $UP_1^*$ if and only if, for some $\alpha$, $\langle (\alpha, \beta) \rangle$ belongs to $US_1^*$ if and only if $\alpha(\beta n) = 0$. Therefore $X$ coincides with $US_1^1|\alpha$. The proof of the second statement is similar. We leave the proof of the third statement to the reader.

(ii) Let $X$ belong to $\Sigma_1^1$ and let $Y$ be a closed subset of $\mathcal{N}$ such that $X = Ex(Y)$ and let $C$ be a decidable subset of $\mathcal{N}$ such that, for every $\beta$, $\beta$ belongs to $Y$ if and only if, for each $n$, $\beta n$ belongs to $C$. Observe that for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, for every $n$, $\langle \alpha, \beta \rangle n$ belongs to $C$. Define a function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha, \beta, n$, $f|\alpha|(\beta n) = 0$ if and only if $\langle \alpha, \beta \rangle n$ belongs to $C$ and observe
that $f$ reduces $X$ to $E_1^1$. The proof of the second statement is similar. We leave the proof of the third statement to the reader.

(iii) Suppose that $UP_1^1$ belongs to $\Sigma_1^1$ and consider the diagonal set $DP_1^1$ consisting of all $\alpha$ such that $(\alpha, \alpha)$ belongs to $UP_1^1$. Also $DP_1^1$ belongs to $\Sigma_1^1$. Find $\beta$ such that $DP_1^1$ coincides with $US_1^1|\beta$ and observe: $(\beta, \beta)$ belongs to $UP_1^1$ if and only if $(\beta, \beta)$ belongs to $US_1^1$. Therefore $(\beta, \beta)$ can not belong to either $UP_1^1$ or $US_1^1$.

(iv) Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$. We define $\alpha$ in $\mathcal{C}$ as follows. For each $a$, $\alpha(a) = 1$ if and only if there exist initial segments $b, c$ of $a$ such that $f^b(c) > 1$ and for every proper initial segment $d$ of $c$, $f^b(d) = 0$. Observe that for every $\beta$, for some $n$, $\alpha(\beta n) = 1$ if and only if, for some $n$, $(f|\alpha)(\beta n) > 0$. It follows that $\alpha$ belongs to $A_1^1$ if and only if $f|\alpha$ belongs to $A_1^1$ and: $\alpha$ belongs to $E_1^1$ if and only if if $f|\alpha$ belongs to $E_1^1$.

(v) Suppose that $f$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing to $A_1^1$ to $E_1^1$. Find $\alpha$ such that $\alpha$ belongs to $A_1^1$ if and only if $(f|\alpha)$ belongs to $A_1^1$ and: $\alpha$ belongs to $E_1^1$ if and only if $f|\alpha$ belongs to $E_1^1$. We also know that $\alpha$ belongs to $A_1^1$ if and only if $f|\alpha$ belongs to $E_1^1$. It follows that $\alpha$ cannot belong to either one of $A_1^1$, $E_1^1$. The proof of the second statement is similar.

9.12 Theorem: (Intuitionistic Observations on the leading sets in $\Sigma_1^1$ and $\Pi_1^1$)

(i) For every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1^1$ into $A_1^1$ there exists $\alpha$ such that both $\alpha$ and $f|\alpha$ belong to $A_1^1$.
(ii) The set $E_1^1$ is strictly analytical but not co-analytical, and the set $A_1^1$ is co-analytical but not strictly analytical.
(iii) For every non-zero stump $\sigma$ there exist functions $\gamma, \delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that $\gamma$ reduces $A_1^1$ to $E_1^1$ and maps $E_1^1$ to $A_1^1$ and $\delta$ reduces $E_1^1$ to $A_1^1$ and maps $A_1^1$ to $E_1^1$.
(iv) For every non-zero stump $\sigma$, for every positively Borel subset $X$ of $\mathcal{N}$, for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1^1$ into $X$ there exists $\alpha$ in $A_1^1$ such that $f|\alpha$ belongs to $X$.
(v) For every non-zero stump $\sigma$, for every positively Borel subset $X$ of $\mathcal{N}$ for every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $A_1^1$ into $X$ there exists $\alpha$ in $E_1^1$ such that $f|\alpha$ belongs to $X$.
(vi) The sets $E_1^1, A_1^1$ are not positively Borel.

Proof: (i) Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1^1$ into $A_1^1$. We define $\alpha$ in $\mathcal{C}$ as follows. For each $a$, $\alpha(a) = 1$ if and only if there exist initial segments $b, c$ of $a$ such that $f^b(c) > 1$ and for every proper initial segment $d$ of $c$, $f^b(d) = 0$. Observe that for every $\beta$, for some $n$, $\alpha(\beta n) = 1$ if and only if, for some $n$, $(f|\alpha)(\beta n) \neq 0$. We claim that $\alpha$ belongs to $A_1^1$. Let $\beta$ belong to $\mathcal{N}$ and consider the sequence $\gamma$ such that for each $n$, $\gamma(\beta n) = 0$ and for each $a$, if $a$ is not an initial segment of $\beta$, then $\gamma(a) = \alpha(a)$. Observe that $\gamma$ belongs to $E_1^1$, therefore $f|\gamma$ belongs to $A_1^1$ and find $n$. 

80
such that \((f|\gamma)(\beta n) \neq 0\). Also find \(m\) such that \(f^m(\gamma m) \neq 0\), and distinguish two cases. Either \(\beta m = \gamma m\), and therefore \((f|\alpha)(\beta m) \neq 0\), and also for some \(k\), \(\alpha(\beta k) = 1\), or \(\beta m \neq \gamma m\) and therefore, for some \(k\), \(\alpha(\beta k) \neq \gamma(\beta k)\) and \(\alpha(\beta k) \neq 0\).

(ii) We define a function \(f\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that, for each \(\alpha\), for each \(n\), \((f|\alpha)(\alpha^1 n) = 0\), and for each \(a\), if \(a\) is not an initial segment of \(\alpha^1\), then \((f|\alpha)(a) = \alpha^0(a)\). It is easy to see that \(E_1^1\) coincides with the range of \(f\). Therefore \(E_1^1\) is strictly analytic. It follows from (i) that \(E_1^1\) does not reduce to \(A_1^1\) and so is not co-analytic. One may draw the same conclusion from Theorem 9.9(iv). It follows from Theorem 9.7(iv), the Boundedness Theorem, that the set \(A_1^1\) is not strictly analytic.

(iii) We use induction on the set of stumps. We first define a function \(\gamma\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that for each \(\alpha\), for each \(n\), \((\gamma|\alpha)(\emptyset n) = 0\) if and only if for each \(j \leq n\), \(\alpha(j) = 0\), and, for each \(a\), if \(a\) is not an initial segment of \(\emptyset n\), then \((\gamma|\alpha)(a) \neq 0\). Observe that \(\gamma\) simultaneously reduces \(A_1^*\) to \(E_1^1\) and \(E_1^1\) to \(A_1^1\). We then define a function \(\delta\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that for each \(\alpha\), for each \(n\), \((\delta|\alpha)(\emptyset n) \neq 0\) if and only if \(\alpha(\emptyset n) = 0\), and, for each \(a\), if \(\alpha(a) \neq 1\), then \((\delta|\alpha)(a) = 0\). Observe that \(\delta\) simultaneously reduces \(E_1^1\) to \(E_1^1\) and \(A_1^*\) to \(A_1^1\). Now assume that \(\sigma\) is a non-zero stump different from \(1^*\) and that for each \(n\) there exist functions \(\gamma, \delta\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that \(\gamma\) reduces \(A_\sigma\) to \(E_1^1\) and maps \(E_\sigma\) into \(A_1^1\) and \(\delta\) reduces \(E_\sigma\) to \(E_1^1\) and maps \(A_\sigma\) into \(A_1^1\). Using the Second Axiom of Countable Choice we find \(\gamma, \delta\) such that, for each \(n\), \(\gamma^n, \delta^n\) are functions from \(\mathcal{N}\) to \(\mathcal{N}\), and \(\gamma^n\) reduces \(A_{(\sigma^n)}\) to \(E_1^1\) and maps \(E_{(\sigma^n)}\) into \(A_1^1\) and \(\delta^n\) reduces \(E_{(\sigma^n)}\) to \(E_1^1\) and maps \(A_{(\sigma^n)}\) into \(A_1^1\). We define a function \(f\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that for all \(\alpha\), for all \(n\), for all \(p\), \((f|\alpha)(\beta p) = 0\) if and only if for all \(n, j\), if \(n \leq p\), then \((\delta^n|\alpha^n)(\beta^p j) = 0\). Observe that \(f|\alpha\) belongs to \(E_1^1\) if and only if, for each \(n\), \(\delta^n|\alpha^n\) belongs to \(E_1^1\), therefore \(f\) reduces \(A_\sigma\) to \(E_1^1\). Observe also that for all \(\alpha\), if \(\alpha\) belongs to \(E_\sigma\), then for some \(n\), \(\delta^n|\alpha^n\) belongs to \(A_1^1\), and therefore \(f|\alpha\) belongs to \(A_1^1\). We also define a function \(g\) from \(\mathcal{N}\) to \(\mathcal{N}\) such that, for all \(\alpha\), \((g|\alpha)(\langle\rangle) = 0\) and for all \(n, a\), \((g|\alpha)(\emptyset n + a) = (\gamma^n|\alpha)(a)\). Observe that \(g|\alpha\) belongs to \(E_1^1\) if and only if, for some \(n\), \(\gamma^n|\alpha^n\) belongs to \(E_1^1\), therefore \(g\) reduces \(E_\sigma\) to \(E_1^1\). Observe also that for each \(\alpha\), if \(\alpha\) belongs to \(A_\sigma\), then for each \(n\), \(\gamma^n|\alpha^n\) belongs to \(A_1^1\) and therefore \(g|\alpha\) belongs to \(A_1^1\).

(iv) Let \(X\) be a positively Borel set and let \(f\) be a function from \(\mathcal{N}\) to \(\mathcal{N}\) mapping \(E_1^1\) into \(X\). Find a non-zero stump \(\sigma\) and a function \(g\) from \(\mathcal{N}\) to \(\mathcal{N}\) reducing \(X\) to \(A_{\sigma^n}\). Also find a function \(h\) from \(\mathcal{N}\) to \(\mathcal{N}\) reducing \(E_\sigma\) to \(E_1^1\) and mapping \(A_\sigma\) into \(A_1^1\). Observe that the composition of \(g, f\) and \(h\) maps \(E_\sigma\) into \(A_{\sigma^n}\) and, using the Borel Hierarchy Theorem, find \(\alpha\) such that both \(\alpha\) and \(g((f|h|\alpha))\) belong to \(A_{\sigma^n}\). Remark that \(h|\alpha\) belongs to \(A_1^1\) and \(f((h|\alpha))\) belongs to \(X\).

(v) The proof is similar to the proof of (iv) and left to the reader.

(vi) is an easy consequence of (iv) and (v). In [17] the question how to prove that \(A_1^1\) is not positively Borel was asked but not answered.

9.13 Theorem: (One Half of Souslin’s Theorem)
(i) For every stump $\sigma$, the set of all $\alpha$ such that $\alpha$ embeds into $\sigma$ is a positively Borel subset of $\mathcal{N}$.

(ii) For every subset $X$ of $\mathcal{N}$, if $X$ is both strictly analytic and co-analytic, then $X$ is positively Borel.

(iii) The set $A_1$ is not strictly analytic.

**Proof:** (i) This fact follows easily from the observation that for every non-zero stump $\sigma$, for every $\alpha$, $\alpha$ embeds into $\sigma$ if and only if for each $m$ there exists $n$ such that $\alpha^m$ embeds into $\sigma^n$.

(ii) Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $f$. Let $g$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $A_1^n$. Let $h$ be the composition of $g$ and $f$. Using the Boundedness Theorem, Theorem 9.7(iv), find $\beta$ in $A_1$ such that, for every $\alpha$, $h|\alpha$ embeds into $\beta$. Using Brouwer’s Thesis, see Theorem 9.7(ii), we may assume that $\beta$ is a stump. Observe that, for each $\alpha$, $\alpha$ is positively Borel if and only if $g|\alpha$ embeds into $\beta$, therefore, by (i), $X$ is positively Borel.

(iii) This follows from (ii) and Theorem 9.12(v).

**9.14** Theorem 9.13 is of limited application as every co-analytic subset of $\mathcal{N}$ is perhaps and “most” positively Borel sets are not.

The distinction we have to make between analytic and strictly analytic is sometimes annoying. For instance, the set $A_1$ is not strictly analytic, but we would like to prove that the set $A_1$ is not analytic.

One may argue that $A_1$, if analytic, must also be co-analytic, as follows. Suppose that there exists a closed subset $X$ of $\mathcal{N}$ such that $A_1 = Ex(X)$. It is reasonable to assume that the set $X$ is given by a finite description. Therefore, for each $\alpha$ we may consider the question if $a$ contains an element of $X$ as a well-circumscribed proposition not involving incomplete objects. Using a so-called Brouwer-Kripke axiom we may form $\beta$ in $\mathcal{N}$ such that $a$ contains an element of $X$ if and only if for some $n$, $\beta(n) = 1$. Applying also the Second Axiom of Countable Choice we may form $\delta$ enumerating all numbers $a$ such that $a$ contains an element of $X$. But then, according to Theorem 9.5(ii), $X$ is strictly analytic, and therefore, by Theorem 9.9(ix), also $A_1 = Ex(X)$ is strictly analytic. We then must conclude that $A_1$ is not analytic.

John Burgess, in [7], following [8], uses a term of Brouwer’s and calls strictly analytic subsets of $\mathcal{N}$ “dressed spreads”. Applying an unrestricted Brouwer-Kripke-axiom he proves that every analytic subset of $\mathcal{N}$ is strictly analytic. The argument just given is essentially his.

**9.15** Let $X$ be a subset of $\mathcal{N}$. $X$ is called an (existential) projection of a co-analytic set if and only if there exists a co-analytic subset $Y$ of $\mathcal{N}$ such that $X = Ex(Y)$. The class of these sets is denoted by $\Sigma^1_2$. A subset $X$ of $\mathcal{N}$ is called a universal projection of an analytic set if and only if there exists an analytic subset $Y$ of $\mathcal{N}$ such that
\( X = Un(Y) \). The class of these sets of denoted by \( \Pi^1_2 \).

We define subsets \( US^1_2 \) and \( UP^1_2 \) of \( \mathcal{N} \), as follows. \( US^1_2 \) is the set of all \( \alpha \) such that, for some \( \beta \), the sequence \( \langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle \) belongs to \( UP^1_1 \), and \( UP^1_2 \) is the set of all \( \alpha \) such that, for all \( \beta \), the sequence \( \langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle \) belongs to \( US^1_1 \). \( US^1_1, UP^1_1 \) are called the cataloguing sets of \( \Sigma^1_2, \Pi^1_1 \), respectively.

We define subsets \( E^1_2 \) and \( A^1_2 \) of \( \mathcal{N} \), as follows. \( E^1_2 \) is the set of all \( \alpha \) such that for some \( \beta \), for all \( \gamma \) there exists \( n \), such that \( \alpha((\beta n, \gamma n)) \neq 0 \), and \( A^1_2 \) is the set of all \( \alpha \) such that for all \( \beta \) there exists \( \gamma \) such that, for all \( n \), \( \alpha((\beta n, \gamma n)) = 0 \). \( E^1_2, A^1_2 \) are called the leading sets of \( \Sigma^1_2, \Pi^1_1 \), respectively.

9.16 Theorem: (Properties of the classes \( \Sigma^1_2 \) and \( \Pi^1_1 \)).

(i) For every subset \( X \) of \( \mathcal{N} \), \( X \) belongs to \( \Sigma^1_2 \) if and only if, for some \( \alpha \), \( X \) coincides with \( US^1_2 \lfloor \alpha \) if and only if \( X \) reduces to \( E^1_2 \).

(ii) For every subset \( X \) of \( \mathcal{N} \), \( X \) belongs to \( \Pi^1_1 \) if and only if, for some \( \alpha \), \( X \) coincides with \( UP^1_1 \lfloor \alpha \) if and only if \( X \) reduces to \( A^1_2 \).

(iii) Both \( \Sigma^1_1 \) and \( \Pi^1_1 \) are subclasses of \( \Sigma^1_2 \) as well as of \( \Pi^1_2 \).

(iv) For every sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), if, for each \( n \), \( X_n \) belongs to \( \Sigma^1_2 \), then both \( \bigcup_{n \in \mathbb{N}} X_n \) and \( \bigcap_{n \in \mathbb{N}} X_n \) belong to \( \Sigma^1_2 \).

(v) \((\text{Collapse of the Projective Hierarchy})\)

For every subset \( X \) of \( \mathcal{N} \), if \( X \) belongs to \( \Sigma^1_2 \), then both \( Ex(X) \) and \( Un(X) \) belong to \( \Sigma^1_2 \).

(vi) \( \Pi^1_2 \) is a subclass of \( \Sigma^1_2 \).

(vii) For every sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), if, for each \( n \), \( X_n \) belongs to \( \Pi^1_2 \), then \( \bigcap_{n \in \mathbb{N}} X_n \) belongs to \( \Pi^1_2 \).

(viii) For every subset \( X \) of \( \mathcal{N} \), if \( X \) belongs to \( \Pi^1_2 \), then \( Un(X) \) belongs to \( \Pi^1_2 \).

Proof: We only show part of (v) and leave the rest of the proof to the reader. Let \( X \) be a subset of \( \mathcal{N} \) belonging to \( \Sigma^1_2 \). We want to prove that \( Un(X) \) belongs to \( \Sigma^1_2 \). Let \( Y \) be a subset of \( \mathcal{N} \) such that \( X \) coincides with \( Ex(Y) \) and \( Y \) belongs to \( \Pi^1_1 \). Let \( Z \) be an open subset of \( \mathcal{N} \) such that \( Y = Un(Z) \).

Let \( C \) be a decidable subset of \( \mathbb{N} \) such that for every \( \alpha, \alpha \) belongs to \( Z \) if and only if, for some \( n, \alpha n \) belongs to \( C \). Apply the Second Axiom of Continuous Choice and observe that for every \( \alpha, \alpha \) belongs to \( Un(X) \) if and only if, for every \( \beta, \langle \alpha, \beta \rangle \) belongs to \( X \), if and only if for every \( \beta \) there exists \( \gamma \) such that \( \langle \langle \alpha, \beta, \gamma \rangle \rangle \) belongs to \( Y \), if and only if there exists \( \varepsilon \) in \( \text{Fun} \) such that \( \varepsilon(0) = 0 \) and for each \( \beta, \langle \langle \alpha, \beta, \varepsilon \rangle \rangle \) belongs to \( Y \). Then remark that for all \( \varepsilon, \alpha, \alpha \) belongs to \( \text{Fun} \) and \( \varepsilon(0) = 0 \) and for each \( \beta, \langle \langle \alpha, \beta, \varepsilon \rangle \rangle \) belongs to \( Y \) if and only if \( \varepsilon \) belongs to \( \text{Fun} \) and \( \varepsilon(0) = 0 \) and for all \( \beta, \delta \) there exists \( n \) such that \( \langle \langle \langle \alpha, \beta, \varepsilon \rangle \rangle, \delta \rangle \) belongs to \( C \), if and only if \( \varepsilon \) belongs to \( \text{Fun} \) and \( \varepsilon(0) = 0 \) and for all \( \beta, \delta, \zeta, \mu \) there exists \( n \) such that either \( \langle \langle \langle \alpha, \beta, \zeta \rangle \rangle, \delta \rangle \) belongs to \( C \)
belongs to $C$ or for some $i < n$, for some $j < \mu(n)$, $\varepsilon^{i}(\overline{\beta}j) \neq 0$, or, for some $i < n$, $\varepsilon^{i}(\overline{\beta}\mu(i)) \neq \zeta(i) + 1$. This shows that the set all pairs $\langle \varepsilon, \alpha \rangle$ such that $\varepsilon$ belongs to $\text{Fun}$ and $\varepsilon(0) = 0$ and for each $\beta$, $\langle \langle \alpha, \beta \rangle, \varepsilon|\beta \rangle$ belongs to $Y$, is a member of the class $\Pi_{1}^{1}$, and that the set $U \cap (X)$ itself is a member of the class $\Sigma_{2}^{1}$.

9.17 The previous Theorem shows that, in intuitionistic mathematics, $\Sigma_{2}^{1}$ is the class of all positive projective sets. One may ask if $\Pi_{1}^{1}$ is a proper subclass of $\Sigma_{2}^{1}$ and if the class $\Pi_{2}^{1}$ is closed under the operation of disjunction. We were unable to answer these questions.

9.18 Theorem: (A Remnant of the Classical Projective Hierarchy Theorem)

(i) There exists $\gamma$ belonging to neither one of $US_{2}^{1}$, $UP_{2}^{1}$.

(ii) For every function $f$ from $\mathcal{N}$ to $\mathcal{N}$ there exists $\alpha$ such that $\alpha$ belongs to $E_{2}^{1}$ if and only if $f|\alpha$ belongs to $E_{2}^{1}$ and also: $\alpha$ belongs to $A_{1}^{1}$ if and only if $f|\alpha$ belongs to $A_{1}^{1}$.

(iii) There exists $\alpha$ belonging to neither one of $E_{2}^{1}$, $A_{1}^{1}$.

Proof: (i) We let $DP_{1}^{1}$ be the set of all $\alpha$ such that $\langle \alpha, \alpha \rangle$ belongs to $UP_{1}^{1}$. According to Theorem 9.15, $DP_{2}^{1}$ is a member of $\Sigma_{2}^{1}$ and we may find $\beta$ such that $DP_{1}^{1}$ coincides with $US_{1}^{1}|\beta$, therefore, for every $\alpha$, $\langle \alpha, \alpha \rangle$ belongs to $UP_{1}^{1}$ if and only if $\langle \beta, \alpha \rangle$ belongs to $US_{2}^{1}$. Define $\gamma := \langle \beta, \beta \rangle$ and observe that $\gamma$ does not belong to either $US_{2}^{1}$ or $UP_{2}^{1}$, as every element of $US_{2}^{1}$ is apart from every element of $UP_{1}^{1}$.

(ii) Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$. We construct $\alpha$ in $C$ in such a way that for all $\beta, \gamma$, there exists $n$ such that $\alpha(\langle \beta n, \gamma n \rangle) = 1$ if and only if there exists $n$ such that $(f|\alpha)(\langle \beta n, \gamma n \rangle) \neq 0$.

We define $\alpha$ by induction, as follows. Let $n$ be a natural number and suppose we already decided about $\alpha(0), \ldots, \alpha(n - 1)$. We now say: $\alpha(n) = 1$ if and only if there exists $b, c, p, q, j$ such that $\text{length}(b) = \text{length}(c)$ and $n = \langle b, c \rangle$ and $\text{length}(p) = \text{length}(q)$ and $p, q$ are initial parts of $b, c$ respectively and $j < n$ and $f^{\langle p, q \rangle}(\overline{\alpha j}) > 1$ and, for all $k < j$, $f^{\langle p, q \rangle}((\overline{\alpha k})) = 0$.

It is easy to see that $\alpha$ satisfies the requirements, and that $\alpha$ belongs to $E_{2}^{1}$ if and only if $f|\alpha$ belongs to $E_{2}^{1}$ and: $\alpha$ belongs to $A_{1}^{1}$ if and only if $f|\alpha$ belongs to $A_{1}^{1}$.

(iii) Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $A_{1}^{1}$ to $E_{2}^{1}$ and apply (ii), keeping in mind that every element of $E_{2}^{1}$ is apart from every element of $A_{1}^{1}$.

9.19 The reader may compare Theorem 9.18 to our Remark 7.5.3.
References


Subfaculteit Wiskunde
Katholieke Universiteit Nijmegen
Postbus 9010, 6500 GL Nijmegen, the Netherlands
e-mail: veldman@sci.kun.nl