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THE CANCELLATION PROBLEM
IN DIMENSION FOUR

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The Cancellation Problem in Dimension Four

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Abstract
This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let $k$ be a field of characteristic zero and let $V$ be an algebraic variety over $k$. The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \ldots, X_n]$ implies that $A \cong k[X_1, \ldots, X_{n-1}]$ for an affine $k$-domain $A$. One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \ldots, X_n]$ with a slice is isomorphic to $k[X_1, \ldots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing $\mathbb{Q}$, this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing $\mathbb{Q}$ for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let $A$ be a ring. A derivation on $A$ is a map $D: A \rightarrow A$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $R$ is a ring and $A$ is an $R$-algebra via $f: R \rightarrow A$, then $A$ is called an $R$-derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation $D$ is called locally nilpotent if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of $D$ is denoted by $A^D$. If $s \in A$ is such that $D(s) = 1$, then $s$ is called a slice of $D$.

The following proposition (see [Wri81]) is well-known.

Proposition 2.1. Let $A$ be a $\mathbb{Q}$-algebra and let $D$ be a locally nilpotent derivation on $A$. Assume that $s \in A$ is a slice of $D$. Then $A = A^D[s]$ and $s$ is algebraically independent over $A^D$. Furthermore, $D = d/ds$. □

In the applications, $A$ will invariably be a polynomial ring $R[X] := R[X_1, \ldots, X_n]$ over a ring $R$. An $R$-derivation $D$ on such a ring is called triangular if $D(X_i) \in R[X_{i+1}, \ldots, X_n]$ for all $i$. Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a coordinate if there is a polynomial automorphism $F$ of $R[X]$ with $s$ as one of its components. More generally, a sequence $(s_1, \ldots, s_k)$ of elements of $R[X]$ with $1 \leq k \leq n$ is called a partial coordinate system if there are polynomials $f_{k+1}, \ldots, f_n \in R[X]$ such that $(s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

Corollary 2.2. Let $R$ be a ring and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and let $s \in R[X]$ be a slice of $D$. Then $s$ is a coordinate if and only if $R[X]^D \cong R^{[n-1]}$. □

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

Problem 2.3 (Cancellation Problem). Let $k$ be a field of characteristic zero and let $n \geq 2$. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{[n-1]}$, i.e., is $s$ a coordinate in $k[X]$?

More generally, one can ask the following question.

Problem 2.4 (Generalized Cancellation Problem). Let $k$ be a field of characteristic zero, let $R$ be an affine $k$-domain, and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{[n-1]}$, i.e., is $s$ a coordinate in $R[X]$?

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

Theorem 2.5. Let $k$ be a field of characteristic zero. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$. □
Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field $k$ in the theorem can in fact be replaced by an arbitrary $\mathbb{Q}$-algebra $R$.

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that $D$ has a slice. Then $k[X]^D \cong k[2]$.

This paper now proves the Generalized Cancellation Problem for $n = 3$ in case $R$ is a Dedekind domain over $\mathbb{Q}$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where $\partial_i$ denotes $\partial/\partial X_i$. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely $D := (2X_1^2 - 3)\partial_1 + (4X_2^4 - 8X_4)\partial_2 + (5X_4^4 - 10)\partial_3 + X_5\partial_4$.

3 Local Coordinates

Let $R$ be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables over $R$. This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_m[X]$, for all maximal ideals $m$ of $R$, provided that $R$ is Hermite, and similarly for partial coordinate systems.

Recall that $R$ is called Hermite if every unimodular row $(r_1, \ldots, r_k)$ can be extended to an invertible square matrix over $R$.

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define $\text{Loc}(R) := \{R_r \mid r \in R \setminus \{0\}\}$.

**Proposition 3.2 (Quillen Induction).** Let $P \subseteq \text{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \text{Loc}(R)$. In that case, $L$ is said to have property $P$. Assume that

(a) for all $m \in \text{Max}(R)$: there exists an $r \in R \setminus m$ such that $P(R_r)$;

(b) for all $r, s, t \in R \setminus \{0\}$: if $rR_t = sR_t$, then $P(R_r)$.

Then $P(L)$ for all $L \in \text{Loc}(R)$. In particular $P(R)$.
Proof. Let $S$ be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of $R$. It is not empty because 0 is in $S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of $R$ also because of (b) (for $r \in R$ and $s \in S$, take $r := \hat{s}$, $s := \hat{s}$, and $t := \hat{rs}$).

Suppose that $S \neq R$. Then $S$ is contained in some maximal ideal of $R$, say $m$. By (a) there is an $r \in R \setminus m$ such that $P(R_r)$. But then $r \in S \subseteq m$, which contradicts $r \not\in m$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$.

**Definition 3.3.** An element $H$ of $\text{End}_R(R[X])$ is called nice if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater, and $\text{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $h$ as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first $k$ components.

**Lemma 3.4.** A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first $k$ components.

**Definition 3.5.** Let $H \in \text{End}_R(R[X])$ be nice. Then $^TH \in \text{End}_R(R[T][X])$ is defined by

$$^TH := T^{-1}H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because $H$ is nice.) If $r \in R$, then $^TH[T := r] \in \text{End}_R(R[X])$ is denoted by $^rH$.

One can easily see that $(\det JH)[X := TX] = \det J^TH$ and that $H$ is invertible if and only if $^TH$ is. Here $JH$ denotes the Jacobian matrix $(\partial h_i/\partial X_j)_H$ of $H$. Even better, if $r \in R \setminus \{0\}$, then $\det J^rH \in R^*$ if and only if $\det JH \in R^*$ and $^rH$ is invertible if and only if $H$ is.

The map $^TH$ is called the clearing map because of the following: if $K$ is the quotient field of $R$ and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that $^rH \in \text{End}_R(R[X])$. So, the denominators of $H$ are cleared. See Chapter 1 of [Ess00].

**Lemma 3.6.** Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \text{Aut}_{R_r}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_r}(R_r[X])$ and $H_2 \in \text{Aut}_{R_s}(R_s[X])$ such that $H = H_1H_2$.

Proof. Note that

$$^TH = H_{(1)} + ^TH_{(2)} + ^T^2H_{(3)} + \cdots + ^T^{d-1}H_{(d)}$$

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where each $H_{(i)}$ is the homogeneous part of degree $i$ of $H$ and $d$ is the degree of $H$. Hence

\[
1^{-TH} = H_{(1)} + (1 - T)H_{(2)} + (1 - T)^2H_{(3)} + \cdots + (1 - T)^{d-1}H_{(d)} \\
= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(h.o.t.) \\
= H + T(h.o.t.),
\]

where, as before, h.o.t. stands for some terms of $X$-degree at least two. As a consequence

\[
H^{-1} \circ 1^{-TH} = H^{-1} \circ (H + T(h.o.t.)) = X + T(h.o.t.).
\]

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR + sR = R$ it follows that $r^kR + s^kR = R$. Take $v, w \in R$ with $r^k v + s^k w = 1$. If $k$ is sufficiently large, then $s^k w H$ and $s^k w (H^{-1})$ are elements of $\text{End}_{R_t}(R_r[X])$. They are also each others inverse and hence they are in fact elements of $\text{Aut}_{R_t}(R_r[X])$.

Take $H_1 := s^k w H$ and compute $H^{-1} H_1$. This gives

\[
H^{-1} H_1 = H^{-1} \circ TH [T := s^k w] = H^{-1} \circ 1^{-TH} [T := r^k v] = (X + T(h.o.t.)) [T := r^k v]) = X + r^k v(h.o.t.)
\]

and similarly

\[
H_1^{-1} H = X + r^k v(h.o.t.).
\]

For $k$ sufficiently large, $H_2 := H_1^{-1} H$ and its inverse apparently are elements of $\text{Aut}_{R_t}(R_s[X])$. So now $H = H_1 H_2$ with $H_1$ and $H_2$ are both of the required form.

\[\square\]

Lemma 3.7. Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.

Proof. Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^k R + s^l R = R$. Write $r^k x + s^l y = 1$ for some $x, y \in R$. Then $t = (r^k x + s^l y)t = vx + wy \in R$. \[\square\]

Lemma 3.8 (Patching Lemma). Let $r, s \in R$ with $rR + sR = R$. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that there is a nice $F \in \text{Aut}_{R_t}(R_r[X])$ with first $k$ components equal to $h_1, \ldots, h_k$ and that there is a nice $G \in \text{Aut}_{R_t}(R_s[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. Then there is a nice $H \in \text{Aut}_{R_t}(R[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. Then
Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R_\infty}(R_\infty[X])$ and note that it is fact an $R_\infty[X_1, \ldots, X_k]$-automorphism of $R_\infty[X] = R_\infty[X_1, \ldots, X_k][X_{k+1}, \ldots, X_n]$. Now apply Lemma 3.6 to the ring $R[X_1, \ldots, X_k]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \text{Aut}_{R_\infty}[X_1, \ldots, X_k](R_\infty[X])$ and $H_2 \in \text{Aut}_{R_\infty}[X_1, \ldots, X_k](R_\infty[X])$, where both $H_i$ are of the \textit{form} $X + h.o.t..$ Considered as automorphisms over respectively $R_i$ and $R_\infty$, the first $k$ components of $H_1$ and $H_2$ ofcourse equal $X_1, \ldots, X_k$. Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over $R_\infty$, a priori) whose first $k$ components equal $h_1, \ldots, h_k$. It is defined over $R_r$ (because $H = FH_1$ and $F$ and $H_1$ are defined over $R_r$) and it is defined over $R_s$ (because $H = GH_2^{-1}$ and $G$ and $H_2$ are defined over $R_s$). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over $R_r$. \hfill \square

\textbf{Theorem 3.9.} Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + h.o.t.$ Assume that for every maximal ideal $\mathfrak{m}$ of $R$, $(h_1, \ldots, h_k)$ is a nice partial coordinate system when considered as an element of $R_\mathfrak{m}[X]^k$. Then $(h_1, \ldots, h_k)$ is a nice partial coordinate system.

\textbf{Proof.} Let $P \subseteq \text{Loc}(R)$ be the collection of all $R_r$, $r \in R \setminus \{0\}$, such that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_r$. Now check the two conditions for Quillen Induction.

(a) Let $\mathfrak{m}$ be a maximal ideal of $R$. It is assumed that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_\mathfrak{m}$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_\infty}(R_\mathfrak{m}[X])$ nice with first $k$ components equal to $h_1, \ldots, h_k$. There are only finitely many elements of $R$ appearing in the denominator of a coefficient of a component of $F$ and its inverse. Denote the product of these denominators by $r$. None of these denominators is an element of $\mathfrak{m}$ and, because $\mathfrak{m}$ is prime, $r$ is not an element of $\mathfrak{m}$ either. Furthermore, obviously, $P(R_r)$.

(b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring $R_t$.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R$. \hfill \square

\textbf{Corollary 3.10.} Assume that $R$ is Hermite. Let $k \in \{1, \ldots, n\}$ and $h_1, \ldots, h_k \in R[X]$. Assume that $(h_1, \ldots, h_k)$ is a partial coordinate system when considered as an element of $R_\mathfrak{m}[X]^k$, for every maximal ideal $\mathfrak{m}$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system.

\textbf{Proof.} First of all note that it is possible to assume that the $h_i$ have no constant part. Write $h_i = r_{i1}X_1 + \cdots + r_{in}X_n + h.o.t.$ for all $i$, with $r_{ij} \in R$.

Consider a maximal ideal $\mathfrak{m}$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system over $R_\mathfrak{m}$, which means that there are $f_{k+1}, \ldots, f_n \in R_\mathfrak{m}[X]$ such that $F := (h_1, \ldots, h_k, f_{k+1}, \ldots, f_n) \in \text{Aut}_{R_\mathfrak{m}}(R_\mathfrak{m}[X])$. The $f_i$ can be chosen in such a way
that they have no constant part. Then \( \det JF \in R_m[X]^* \) and hence substituting \( X_1 := 0, \ldots, X_n := 0 \) gives

\[
\begin{bmatrix}
    r_{11} & \cdots & r_{1n} \\
    \vdots & & \vdots \\
    r_{k1} & \cdots & r_{kn} \\
    * & \cdots & * \\
    \vdots & & \vdots \\
    * & \cdots & * 
\end{bmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_m^*.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \( R_m \)-module homomorphism from \( R^n_m \) to \( R^k_m \).

Because this holds for every maximal ideal of \( R \), it follows that the matrix \((r_{ij})_{ij}\) represents a surjective \( R \)-module homomorphism from \( R^n \) to \( R^k \). Now \( R \) is Hermite, which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix \( M \) over \( R \) (see [Lam78], Corollary 4.5). Viewing this matrix \( M \) as a polynomial automorphism of \( R[X] \) and applying its inverse to the polynomials \( h \), it follows that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})\).

By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \( R_m[X] \), for every \( m \in \text{Max}(R) \). Now apply Theorem 3.9.

The condition that \( R \) be Hermite in the previous corollary is necessary. For let \( R \) be any non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \( R \) that cannot be extended to an invertible square matrix. Then \( h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n] \) is not a coordinate (if it were, the coefficients of the linear part of an automorphism with \( h \) as its first component would form an invertible square matrix over \( R \) extending \((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \( m \) of \( R \), \((a_1, \ldots, a_n)\) is extendible to an invertible square matrix over \( R_m \) (since \( R_m \) is local) and so \( h \) is a coordinate over \( R_m \).

## 4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing \( \mathbb{Q} \).

**Theorem 4.1.** Let \( R \) be a discrete valuation ring containing \( \mathbb{Q} \). Denote the unique maximal ideal of \( R \) by \( \mathfrak{m} \), write \( K \) for the quotient field \( \mathbb{Q}(R) \) of \( R \), and write \( k \) for the residue field \( R/\mathfrak{m} \) of \( R \). Let \( A \) be a finitely generated affine \( R \)-domain and assume that \( K \otimes_R A \cong K[2] \) and that \( k \otimes_R A \cong k[2] \). Then \( A \cong R[2] \).

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

**Lemma 4.2.** Let \( s \in R[X] := R[X_1, \ldots, X_n] \) and let \( A \) be an \( R \)-algebra via the map \( \varphi: R \to A \). Denote the induced map \( R[X] \to A[X] \) by \( \varphi_\# \). Then

\[
A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi_\#(s)A[X])
\]

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In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then
\[ A \otimes_R R[X]^D \cong A[X]^\hat{D}, \]
where $\hat{D}$ denotes the extension of $D$ to $A[X]$.

Proof. The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

```
sR[X] \longrightarrow R[X] \longrightarrow R[X]/sR[X] \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
A \otimes_R sR[X] \longrightarrow A \otimes_R A[X] \longrightarrow A \otimes_R R[X]/sR[X] \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\varphi_#(s)A[X] \longrightarrow A[X] \longrightarrow A[X]/(\varphi_#(s)A[X]) \longrightarrow 0
```

The map $A \otimes_R sR[X] \to \varphi_#(s)A[X]$ is surjective: take an element $\varphi_#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum \alpha c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_#(s)f$ is the image of $\sum \alpha c_\alpha \otimes sX_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism.

Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_#(s)A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1. \hfill \Box

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to 0. Let $D$ be the locally nilpotent derivation $Y \partial_Y$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\hat{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\hat{D} = A[X]$.

Lemma 4.3. Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.

Proof. Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\hat{D}$. By Lemma 4.2 and Theorem 2.6 it follows that
\[ K \otimes_R R[X,Y,Z]^\hat{D} \cong K[X,Y,Z]^\hat{D} \cong K^{[2]} . \]

In exactly the same way it follows that
\[ k \otimes_R R[X,Y,Z]^D \cong k^{[2]} . \]

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. \hfill \Box

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Theorem 4.4. Let $R$ be a Dedekind domain containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice. Then $R[X,Y,Z]^D \cong R^2$.

Proof. Let $s \in R[X,Y,Z]$ be a slice of $D$. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, $R$ is Hermite. By Corollary 3.10 it is enough to show that $s$ is a coordinate in $R_m[X,Y,Z]$ for every maximal ideal $m$ of $R$.

So let $m$ be a maximal ideal of $R$. Then $R_m$ is a discrete valuation ring. Because $R$ contains $\mathbb{Q}$, $R_m$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_m[X,Y,Z]^D \cong R_m^2$. In other words, $s$ is a coordinate in $R_m[X,Y,Z]$.

Corollary 4.5. Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X,Y,Z,W]$ of the form

$$D := a(X,Y,Z,W)\partial_X + b(X,Y,Z,W)\partial_Y + c(X,Y,Z,W)\partial_Z + d(W)\partial_W.$$ 

Assume that $D$ has a slice. Then $k[X,Y,Z,W]^D \cong k^3$.

Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since $D$ is locally nilpotent. So $d^{-1}W$ is a slice of $D$. This slice is also a coordinate and hence $k[X]^D \cong k^3$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$.

References


[ER00] Arno van den Essen and Peter van Rossum. Triangular derivations related to problems on affine $n$-space. Report 0005, Department of Mathematics, University of Nijmegen, March 2000.


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