THE CANCELLATION PROBLEM
IN DIMENSION FOUR

Harm Derksen, Arno van den Essen, Peter van Rossum

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Harm Derksen∗ Arno van den Essen Peter van Rossum

Abstract
This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let \( k \) be a field of characteristic zero and let \( V \) be an algebraic variety over \( k \). The Cancellation Problem asks if \( V \times k \cong k^n \) implies that \( V \cong k^{n-1} \). This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if \( A[T] \cong k[X_1, \ldots, X_n] \) implies that \( A \cong k[X_1, \ldots, X_{n-1}] \) for an affine \( k \)-domain \( A \). One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on \( k[X_1, \ldots, X_n] \) with a slice is isomorphic to \( k[X_1, \ldots, X_{n-1}] \). For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing \( \mathbb{Q} \), this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing \( \mathbb{Q} \) for \( n = 3 \). As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for \( n = 4 \) over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let $A$ be a ring. A derivation on $A$ is a map $D: A \to A$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $R$ is a ring and $A$ is an $R$-algebra via $f: R \to A$, then $A$ is called an $R$-derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation $D$ is called locally nilpotent if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of $D$ is denoted by $A^D$. If $s \in A$ is such that $D(s) = 1$, then $s$ is called a slice of $D$.

The following proposition (see [Wri81]) is well-known.

**Proposition 2.1.** Let $A$ be a $\mathbb{Q}$-algebra and let $D$ be a locally nilpotent derivation on $A$. Assume that $s \in A$ is a slice of $D$. Then $A = A^D[s]$ and $s$ is algebraically independent over $A^D$. Furthermore, $D = d/ds$. \hfill \Box

In the applications, $A$ will invariably be a polynomial ring $R[X] := R[X_1, \ldots, X_n]$ over a ring $R$. An $R$-derivation $D$ on such a ring is called triangular if $D(X_i) \in R[X_{i+1}, \ldots, X_n]$ for all $i$. Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a coordinate if there is a polynomial automorphism $F$ of $R[X]$ with $s$ as one of its components. More generally, a sequence $(s_1, \ldots, s_k)$ of elements of $R[X]$ with $1 \leq k \leq n$ is called a partial coordinate system if there are polynomials $f_{k+1}, \ldots, f_n \in R[X]$ such that $(s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

**Corollary 2.2.** Let $R$ be a ring and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and let $s \in R[X]$ be a slice of $D$. Then $s$ is a coordinate if and only if $R[X]^D \cong R^{[n-1]}$. \hfill \Box

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

**Problem 2.3 (Cancellation Problem).** Let $k$ be a field of characteristic zero and let $n \geq 2$. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{[n-1]}$, i.e., is $s$ a coordinate in $k[X]$?

More generally, one can ask the following question.

**Problem 2.4 (Generalized Cancellation Problem).** Let $k$ be a field of characteristic zero, let $R$ be an affine $k$-domain, and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{[n-1]}$, i.e., is $s$ a coordinate in $R[X]$?

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

**Theorem 2.5.** Let $k$ be a field of characteristic zero. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$. \hfill \Box
Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field \( k \) in the theorem can in fact be replaced by an arbitrary \( \mathbb{Q} \)-algebra \( R \).

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let \( k \) be a field of characteristic zero and let \( D \) be a locally nilpotent \( k \)-derivation on \( k[X] := k[X_1, X_2, X_3] \). Assume that \( D \) has a slice. Then \( k[X]^D \cong k^{[2]} \).

This paper now proves the Generalized Cancellation Problem for \( n = 3 \) in case \( R \) is a Dedekind domain over \( \mathbb{Q} \). As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

\[
D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4
\]

for \( n = 4 \), where \( \partial_i \) denotes \( \partial/\partial X_i \). In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for \( n = 4 \). This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for \( n = 5 \) which is triangular, namely

\[
D := (2X_1^2 - 3)\partial_1 + (4X_2^3 - 8X_4)\partial_2 + (5X_4^3 - 10)\partial_3 + X_5\partial_4.
\]

## 3 Local Coordinates

Let \( R \) be a domain, \( n \in \mathbb{N} \), and \( R[X] := R[X_1, \ldots, X_n] \) the polynomial ring in \( n \) variables over \( R \). This section shows that a polynomial in \( R[X] \) is a coordinate if and only if it is a coordinate when considered as an element of \( R_m[X] \), for all maximal ideals \( m \) of \( R \), provided that \( R \) is Hermite, and similarly for partial coordinate systems.

Recall that \( R \) is called Hermite if every unimodular row \((r_1, \ldots, r_k)\) can be extended to an invertible square matrix over \( R \).

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define \( \text{Loc}(R) := \{ R_r \mid r \in R \setminus \{0\} \} \).

**Proposition 3.2 (Quillen Induction).** Let \( P \subseteq \text{Loc}(R) \). Write \( P(L) \) instead of \( L \in P \) for \( L \in \text{Loc}(R) \). In that case, \( L \) is said to have property \( P \). Assume that

(a) for all \( m \in \text{Max}(R) \): there exists an \( r \in R \setminus m \) such that \( P(R_r) \);

(b) for all \( r, s, t \in R \setminus \{0\} \): if \( rR_t + sR_t = R_t \), \( P(R_t) \) and \( P(R_s) \), then \( P(R_t) \).

Then \( P(L) \) for all \( L \in \text{Loc}(R) \). In particular \( P(R) \).
Proof. Let $S$ be the collection of all $r \in R \setminus \{0\}$ such that $P(R_e)$ together with 0. This is an ideal of $R$. It is not empty because $0 \in S$, closed under addition because of (b), (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of $R$ also because of (b) (for $r \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}$, $s := \tilde{s}$, and $t := \tilde{r}s$).

Suppose that $S \neq R$. Then $S$ is contained in some maximal ideal of $R$, say $m$. By (a) there is an $r \in R \setminus m$ such that $P(R_e)$. But then $r \in S \subseteq m$, which contradicts $r \not\in m$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$.

Definition 3.3. An element $H$ of $\text{End}_R(R[X])$ is called nice if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater, and $\text{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $h$ as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first $k$ components.

Lemma 3.4. A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first $k$ components.

Definition 3.5. Let $H \in \text{End}_R(R[X])$ be nice. Then $^TH \in \text{End}_{R[T]}(R[T][X])$ is defined by

$$^TH := T^{-1}H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because $H$ is nice.) If $r \in R$, then $^TH[T := r] \in \text{End}_R(R[X])$ is denoted by $^rH$.

One can easily see that $(\det JH)[X := TX] = \det J^rH$ and that $H$ is invertible if and only if $^rH$ is. Here $JH$ denotes the Jacobian matrix $(\partial H_i/\partial X_j)_{ij}$ of $H$. Even better, if $r \in R \setminus \{0\}$, then $\det J^rH \in R^*$ if and only if $\det JH \in R^*$ and $^rH$ is invertible if and only if $H$ is.

The map $^TH$ is called the clearing map because of the following: if $K$ is the quotient field of $R$ and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that $^rH \in \text{End}_R(R[X])$. So, the denominators of $H$ are cleared.

Lemma 3.6. Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \text{Aut}_{R_e}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_e}(R_e[X])$ and $H_2 \in \text{Aut}_{R_e}(R_e[X])$ such that $H = H_1H_2$.

Proof. Note that

$$^TH = H_{(1)} + TH_{(2)} + T^2H_{(3)} + \cdots + T^{d-1}H_{(d)}$$

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where each \( H_{(i)} \) is the homogeneous part of degree \( i \) of \( H \) and \( d \) is the degree of \( H \). Hence

\[
1-TH = H_{(1)} + \left(1 - T\right)H_{(2)} + \left(1 - T\right)^2 H_{(3)} + \cdots + \left(1 - T\right)^{d-1} H_{(d)} \\
= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(h.o.t.) \\
= H + T(h.o.t.),
\]

where, as before, \( h.o.t. \) stands for some terms of \( X \)-degree at least two. As a consequence

\[
H^{-1} \circ 1-TH = H^{-1} \circ (H + T(h.o.t.)) \\
= X + T(h.o.t.).
\]

Now let \( k \in \mathbb{N} \) be sufficiently large. From \( rR + sR = R \) it follows that \( r^kR + s^kR = R \). Take \( v, w \in R \) with \( r^kv + s^kw = 1 \). If \( k \) is sufficiently large, then \( s^wH \) and \( s^w(H^{-1}) \) are elements of \( \text{End}_{R_r}(R_r[X]) \). They are also each others inverse and hence they are in fact elements of \( \text{Aut}_{R_r}(R_r[X]) \).

Take \( H_1 := s^wH \) and compute \( H^{-1}H_1 \). This gives

\[
H^{-1}H_1 = H^{-1} \circ T[H := s^w] \\
= H^{-1} \circ 1-T[H := r^k] \\
= (X + T(h.o.t.))[H := r^k] \\
= X + r^k(h.o.t.)
\]

and similarly

\[
H_1^{-1}H = X + r^k(h.o.t.).
\]

For \( k \) sufficiently large, \( H_2 := H_1^{-1}H \) and its inverse apparently are elements of \( \text{Aut}_{R_r}(R_r[X]) \). So now \( H = H_1H_2 \) with \( H_1 \) and \( H_2 \) are both of the required form. \( \square \)

**Lemma 3.7.** Let \( r, s \in R \) be such that \( rR + sR = R \). Take \( t \in R_{rs} \) such that \( t \in R_r \cap R_s \). Then \( t \in R \).

**Proof.** Write \( t = v/\ell^k = w/\ell^l \) with \( v, w \in R \) and \( k, l \in \mathbb{N} \). Because \( rR + sR = R \), also \( r^kR + s^lR = R \). Write \( r^kx + s^ly = 1 \) for some \( x, y \in R \). Then \( t = (r^kx + s^ly)t = vx + wy \in R \). \( \square \)

**Lemma 3.8** (Patching Lemma). Let \( r, s \in R \) with \( rR + sR = R \). Let \( k \in \{1, \ldots , n\} \) and let \( h_1, \ldots , h_k \in R[X] \) be polynomials of the form \( h_i = X_i + h.o.t. \). Assume that there is a nice \( F \in \text{Aut}_{R_r}(R_r[X]) \) with first \( k \) components equal to \( h_1, \ldots , h_k \) and that there is a nice \( G \in \text{Aut}_{R_r}(R_r[X]) \) with first \( k \) components equal to \( h_1, \ldots , h_k \). Then there is a nice \( H \in \text{Aut}_{R}(R[X]) \) with first \( k \) components equal to \( h_1, \ldots , h_k \).
Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R_r}(R_{rs}[X])$ and note that it is fact an $R_{rs}[X_1,\ldots,X_k]$-automorphism of $R_{rs}[X] = R_{rs}[X_1,\ldots,X_k][X_{k+1},\ldots,X_n]$. Now apply Lemma 3.6 to the ring $R[X_1,\ldots,X_k]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \text{Aut}_{R_r}[X_1,\ldots,X_k](R_r[X])$ and $H_2 \in \text{Aut}_{R_r}[X_1,\ldots,X_k](R_r[X])$, where both $H_i$ are of the form $X + \text{h.o.t.}$. Considered as automorphisms over respectively $R_r$ and $R_s$, the first $k$ components of $H_1$ and $H_2$ are of course equal $X_1,\ldots,X_k$. Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over $R_{rs}$, a priori) whose first $k$ components equal $h_1,\ldots,h_k$. It is defined over $R_r$ (because $H = FH_1$ and $F$ and $H_1$ are defined over $R_r$) and it is defined over $R_s$ (because $H = GH_2^{-1}$ and $G$ and $H_2$ are defined over $R_s$). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over $R_r$.

\[\text{Corollary 3.10.}\] Assume that $R$ is Hermite. Let $k \in \{1,\ldots,n\}$ and $h_1,\ldots,h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that for every maximal ideal $\mathfrak{m}$ of $R$, $(h_1,\ldots,h_k)$ is a nice partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$. Then $(h_1,\ldots,h_k)$ is a nice partial coordinate system.

Proof. Let $P \subseteq \text{Loc}(R)$ be the collection of all $R_r$, $r \in R \setminus \{0\}$, such that $(h_1,\ldots,h_k)$ is a nice partial coordinate system over $R_r$. Now check the two conditions for Quillen Induction.

(a) Let $\mathfrak{m}$ be a maximal ideal of $R$. It is assumed that $(h_1,\ldots,h_k)$ is a nice partial coordinate system over $R_{\mathfrak{m}}$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$ nice with first $k$ components equal to $h_1,\ldots,h_k$. There are only finitely many elements of $R$ appearing in the denominator of a coefficient of a component of $F$ and its inverse. Denote the product of these denominators by $\mathfrak{m}$. None of these denominators is an element of $\mathfrak{m}$ and, because $\mathfrak{m}$ is prime, $r$ is not an element of $\mathfrak{m}$ either. Furthermore, obviously, $P(R_r)$.

(b) Let $r,s,t \in R \setminus \{0\}$ be such that $rR_r + sR_r = R_r$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring $R_t$.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_1,\ldots,h_k)$ is a nice partial coordinate system over $R$. 

\[\text{Corollary 3.10.}\] Assume that $R$ is Hermite. Let $k \in \{1,\ldots,n\}$ and $h_1,\ldots,h_k \in R[X]$. Assume that $(h_1,\ldots,h_k)$ is a partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$, for every maximal ideal $\mathfrak{m}$ of $R$. Then $(h_1,\ldots,h_k)$ is a partial coordinate system.

Proof. First of all note that it is possible to assume that the $h_i$ have no constant part. Write $h_i = r_{ij}X_1 + \cdots + r_{in}X_n + \text{h.o.t.}$ for all $i$, with $r_{ij} \in R$.

Consider a maximal ideal $\mathfrak{m}$ of $R$. Then $(h_1,\ldots,h_k)$ is a partial coordinate system over $R_{\mathfrak{m}}$, which means that there are $f_{k+1},\ldots,f_n \in R_{\mathfrak{m}}[X]$ such that $F := (h_1,\ldots,h_k,f_{k+1},\ldots,f_n) \in \text{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$. The $f_i$ can be chosen in such a way
that they have no constant part. Then det \( JF \in R_m[X]^* \) and hence substituting \( X_1 := 0, \ldots, X_n := 0 \) gives

\[
\begin{vmatrix}
  r_{11} & \cdots & r_{1n} \\
  \vdots & & \vdots \\
  r_{k1} & \cdots & r_{kn} \\
  * & \cdots & * \\
  \vdots & & \vdots \\
  * & \cdots & *
\end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_m^*.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \(R_m\)-module homomorphism from \(R^n_m\) to \(R^k_m\).

Because this holds for every maximal ideal of \(R\), it follows that the matrix \((r_{ij})_{ij}\) represents a surjective \(R\)-module homomorphism from \(R^n\) to \(R^k\). Now \(R\) is Hermite, which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix \(M\) over \(R\) (see [Lam78], Corollary 4.5). Viewing this matrix \(M\) as a polynomial automorphism of \(R[X]\) and applying its inverse to the polynomials \(h_i\), it follows that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + h.o.t., \ldots, X_k + h.o.t.)\).

By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \(R_m[X]\), for every \(m \in \text{Max}(R)\). Now apply Theorem 3.9.

The condition that \(R\) be Hermite in the previous corollary is necessary. For let \(R\) be any non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \(R\) that cannot be extended to an invertible square matrix. Then \(h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n]\) is not a coordinate (if it were, the coefficients of the linear part of an automorphism with \(h\) as its first component would form an invertible square matrix over \(R\) extending \((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \(m\) of \(R\), \((a_1, \ldots, a_n)\) is extendible to an invertible square matrix over \(R_m\) (since \(R_m\) is local) and so \(h\) is a coordinate over \(R_m\).

4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing \(\mathbb{Q}\).

**Theorem 4.1.** Let \(R\) be a discrete valuation ring containing \(\mathbb{Q}\). Denote the unique maximal ideal of \(R\) by \(m\), write \(K\) for the quotient field \(Q(R)\) of \(R\), and write \(k\) for the residue field \(R/m\) of \(R\). Let \(A\) be a finitely generated affine \(R\)-domain and assume that \(K \otimes_R A \cong K[2]\) and that \(k \otimes_R A \cong k[2]\). Then \(A \cong R[2]\). □

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

**Lemma 4.2.** Let \(s \in R[X] := R[X_1, \ldots, X_n]\) and let \(A\) be an \(R\)-algebra via the map \(\varphi : R \to A\). Denote the induced map \(R[X] \to A[X]\) by \(\varphi_\#\). Then

\[
A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi_\#(s)A[X])
\]

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In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then
\[ A \otimes_R R[X]^D \cong A[X]^\tilde{D}, \]
where $\tilde{D}$ denotes the extension of $D$ to $A[X]$.

**Proof.** The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

![Diagram](image)

The map $A \otimes_R sR[X] \to \varphi_\#(s)A[X]$ is surjective: take an element $\varphi_\#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum c_\alpha X_1^{a_1} \ldots X_n^{a_n}$ with each $c_\alpha \in A$. Then $\varphi_\#(s)f$ is the image of $\sum c_\alpha \otimes sX_1^{a_1} \ldots X_n^{a_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_\#A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1. \qed

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to 0. Let $D$ be the locally nilpotent derivation $Y \partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\tilde{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\tilde{D} = A[X]$.

**Lemma 4.3.** Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.

**Proof.** Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\tilde{D}$. By Lemma 4.2 and Theorem 2.6 it follows that
\[ K \otimes_R R[X,Y,Z]^D \cong K[X,Y,Z]^\tilde{D} \cong K^{[2]} \]
In exactly the same way it follows that
\[ k \otimes_R R[X,Y,Z]^D \cong k^{[2]} \]
Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. \qed

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Theorem 4.4. Let \( R \) be a Dedekind domain containing \( \mathbb{Q} \) and let \( D \) be a locally nilpotent \( R \)-derivation on \( R[X, Y, Z] \) with a slice. Then \( R[X, Y, Z]^D \cong R[^2] \).

Proof. Let \( s \in R[X, Y, Z] \) be a slice of \( D \). Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, \( R \) is Hermite. By Corollary 3.10 it is enough to show that \( s \) is a coordinate in \( R_m[X, Y, Z] \) for every maximal ideal \( m \) of \( R \).

So let \( m \) be a maximal ideal of \( R \). Then \( R_m \) is a discrete valuation ring. Because \( R \) contains \( \mathbb{Q} \), \( R_m \) contains \( \mathbb{Q} \) as well. Now Lemma 4.3 implies that \( R_m[X, Y, Z]^D \cong R_m[^2] \). In other words, \( s \) is a coordinate in \( R_m[X, Y, Z] \).

Corollary 4.5. Let \( k \) be a field of characteristic zero and let \( D \) be a locally nilpotent \( k \)-derivation on \( k[X, Y, Z, W] \) of the form

\[
D := a(X, Y, Z, W)\partial_X + b(X, Y, Z, W)\partial_Y + c(X, Y, Z, W)\partial_Z + d(W)\partial_W.
\]

Assume that \( D \) has a slice. Then \( k[X, Y, Z, W]^D \cong k[^3] \).

Proof. If \( d(W) \neq 0 \), then \( d(W) \in k^* \), since \( D \) is locally nilpotent. So \( d^{-1}W \) is a slice of \( D \). This slice is also a coordinate and hence \( k[X]^D \cong k[^3] \). Otherwise, if \( d(W) = 0 \), apply Theorem 4.4 with \( R = k[W] \).

References


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Harm Derksen <hderksen@math.lsa.umich.edu>
Department of Mathematics
University of Michigan
525 East University
Ann Arbor, MI 48109-1109
U.S.A.

Arno van den Essen <essen@sci.kun.nl>
Peter van Rossum <petervr@sci.kun.nl>
Department of Mathematics
University of Nijmegen
Toernooiveld 1
6525 VW Nijmegen
The Netherlands