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THE CANCELLATION PROBLEM
IN DIMENSION FOUR

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The Cancellation Problem in Dimension Four

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Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let $k$ be a field of characteristic zero and let $V$ be an algebraic variety over $k$. The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \ldots, X_n]$ implies that $A \cong k[X_1, \ldots, X_{n-1}]$ for an affine $k$-domain $A$. One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \ldots, X_n]$ with a slice is isomorphic to $k[X_1, \ldots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing $\mathbb{Q}$, this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing $\mathbb{Q}$ for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let $A$ be a ring. A derivation on $A$ is a map $D: A \to A$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $R$ is a ring and $A$ is an $R$-algebra via $f: R \to A$, then $A$ is called an $R$-derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation $D$ is called locally nilpotent if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of $D$ is denoted by $A^D$. If $s \in A$ is such that $D(s) = 1$, then $s$ is called a slice of $D$.

The following proposition (see [Wri81]) is well-known.

**Proposition 2.1.** Let $A$ be a $\mathbb{Q}$-algebra and let $D$ be a locally nilpotent derivation on $A$. Assume that $s \in A$ is a slice of $D$. Then $A = A^D[s]$ and $s$ is algebraically independent over $A^D$. Furthermore, $D = d/ds$. □

In the applications, $A$ will invariably be a polynomial ring $R[X] := R[X_1, \ldots, X_n]$ over a ring $R$. An $R$-derivation $D$ on such a ring is called triangular if $D(X_i) \in R[X_{i+1}, \ldots, X_n]$ for all $i$. Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a coordinate if there is a polynomial automorphism $F$ of $R[X]$ with $s$ as one of its components. More generally, a sequence $(s_1, \ldots, s_k)$ of elements of $R[X]$ with $1 \leq k \leq n$ is called a partial coordinate system if there are polynomials $f_{k+1}, \ldots, f_n \in R[X]$ such that $(s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

**Corollary 2.2.** Let $R$ be a ring and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and let $s \in R[X]$ be a slice of $D$. Then $s$ is a coordinate if and only if $R[X]^D \cong R^{(n-1)}$. □

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

**Problem 2.3 (Cancellation Problem).** Let $k$ be a field of characteristic zero and let $n \geq 2$. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{(n-1)}$, i.e., is $s$ a coordinate in $k[X]$?

More generally, one can ask the following question.

**Problem 2.4 (Generalized Cancellation Problem).** Let $k$ be a field of characteristic zero, let $R$ be an affine $k$-domain, and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{(n-1)}$, i.e., is $s$ a coordinate in $R[X]$?

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

**Theorem 2.5.** Let $k$ be a field of characteristic zero. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$. □
Nowadays even stronger results have been obtained by Bhadwadel and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field $k$ in the theorem can in fact be replaced by an arbitrary $\mathbb{Q}$-algebra $R$.

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that $D$ has a slice. Then $k[X]^D \cong k^{[2]}$.

This paper now proves the Generalized Cancellation Problem for $n = 3$ in case $R$ is a Dedekind domain over $\mathbb{Q}$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where $\partial_i$ denotes $\partial/\partial X_i$. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely

$$D := (2X_1^2 - 3)\partial_1 + (4X_1^2 - 8X_4)\partial_2 + (5X_1^2 - 10)\partial_3 + X_5\partial_4.$$

### 3 Local Coordinates

Let $R$ be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables over $R$. This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_m[X]$, for all maximal ideals $m$ of $R$, provided that $R$ is Hermite, and similarly for partial coordinate systems. Recall that $R$ is called Hermite if every unimodular row $(r_1, \ldots, r_k)$ can be extended to an invertible square matrix over $R$.

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define Loc($R$) := \{ $R_r \mid r \in R \setminus \{0\}$ \}.

**Proposition 3.2 (Quillen Induction).** Let $P \subseteq$ Loc($R$). Write $P(L)$ instead of $L \in P$ for $L \in$ Loc($R$). In that case, $L$ is said to have property $P$. Assume that

(a) for all $m \in$ Max($R$): there exists an $r \in R \setminus m$ such that $P(R_r)$;

(b) for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.

Then $P(L)$ for all $L \in$ Loc($R$). In particular $P(R)$.
Proof. Let \( S \) be the collection of all \( r \in R \setminus \{0\} \) such that \( P(R_r) \) together with 0. This is an ideal of \( R \). It is not empty because 0 \( \in S \), closed under addition because of (b) (for \( r, s \in S \) take \( t := r + s \)), and closed under multiplication with elements of \( R \) also because of (b) (for \( r \in R \) and \( \xi \in S \), take \( r := \xi, s := \xi, \) and \( t := r\xi \)).

Suppose that \( S \neq R \). Then \( S \) is contained in some maximal ideal of \( R \), say \( m \). By (a) there is an \( r \in R \setminus m \) such that \( P(R_r) \). But then \( r \in S \subseteq m \), which contradicts \( r \notin m \). So \( S = R \) and therefore \( P(L) \) for all \( L \in \text{Loc}(R) \).

**Definition 3.3.** An element \( H \) of \( \text{End}_R[R[X]] \) is called **nice** if it is of the form \( H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.}) \). Here \( \text{h.o.t.} \) stands for higher order terms, i.e., terms of degree 2 or greater, and \( \text{End}_R[R[X]] \) has been identified with \( R[X]^n \). A coordinate \( h \in R[X] \) is called nice if there is a nice \( H \in \text{Aut}_R(R[X]) \) which has \( h \) as its first component. Similarly, a partial coordinate system \( (h_1, \ldots, h_k) \in R[X]^k \) is called nice if there is a nice \( H \in \text{Aut}_R(R[X]) \) which has \( (h_1, \ldots, h_k) \) as its first \( k \) components.

**Lemma 3.4.** A partial coordinate system \( (h_1, \ldots, h_k) \in R[X]^k \) is nice if and only if it is of the form \( (X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.}) \). In particular, a coordinate \( h \in R[X] \) is nice if and only if it is of the form \( X_1 + \text{h.o.t.} \).

**Proof.** By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with \( h_1, \ldots, h_k \) as its first \( k \) components.

**Definition 3.5.** Let \( H \in \text{End}_R[R[X]] \) be nice. Then \( ^TH \in \text{End}_R[R[T][X]] \) is defined by

\[
^TH := T^{-1}H[X_1 := TX_1, \ldots, X_n := TX_n].
\]

(This is defined over \( R[T] \) and not just over \( R[T, T^{-1}] \) because \( H \) is nice.) If \( r \in R \), then \(^TH[T := r] \in \text{End}_R(R[X]) \) is denoted by \(^rH\).

One can easily see that \((\det JH)[X := TX] = \det J^rH \) and that \( H \) is invertible if and only if \(^rH \) is. Here \( JH \) denotes the Jacobian matrix \((\partial H_i/\partial X_j)_{ij} \) of \( H \). Even better, if \( r \in R \setminus \{0\} \), then \( \det J^rH \in R^* \) if and only if \( \det JH \in R^* \) and \( ^rH \) is invertible if and only if \( H \) is.

The map \(^TH \) is called the **clearing map** because of the following: if \( K \) is the quotient field of \( R \) and \( H \in \text{End}_K(K[X]) \) is of the form \( H = X + \text{h.o.t.} \), then there is an \( r \in R \setminus \{0\} \) such that \( ^rH \in \text{End}_R(R[X]) \). So, the denominators of \( H \) are cleared. See Chapter 1 of [Ess00].

**Lemma 3.6.** Let \( r, s \in R \setminus \{0\} \) be such that \( rR + sR = R \) and let \( H \in \text{Aut}_{R_r}(R_{rs}[X]) \) be nice. Then there are nice \( H_1 \in \text{Aut}_{R_r}(R_r[X]) \) and \( H_2 \in \text{Aut}_{R_s}(R_{rs}[X]) \) such that \( H = H_1H_2 \).

**Proof.** Note that

\[
^TH = H_{(1)} + TH_{(2)} + T^2H_{(3)} + \cdots + T^{d-1}H_{(d)}
\]
where each $H(i)$ is the homogeneous part of degree $i$ of $H$ and $d$ is the degree of $H$. Hence
\[
1-TH = H(1) + (1-T)H(2) + (1-T)^2H(3) + \cdots + (1-T)^{d-1}H(d)
\]
\[
= H(1) + H(2) + H(3) + \cdots + H(d) + T(h.o.t.)
\]
where, as before, h.o.t. stands for some terms of $X$-degree at least two. As a consequence
\[
H^{-1} \circ 1-TH = H^{-1} \circ (H + T(h.o.t.))
\]
\[
= X + T(h.o.t.).
\]

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR + sR = R$ it follows that $r^kR+s^kR = R$. Take $v, w \in R$ with $r^k v + s^k w = 1$. If $k$ is sufficiently large, then $s^k wH$ and $s^k w(H^{-1})$ are elements of $\text{End}_{R_i}(R_v[X])$. They are also each others inverse and hence they are in fact elements of $\text{Aut}_{R_i}(R_v[X])$. Take $H_1 := s^k wH$ and compute $H^{-1}H_1$. This gives
\[
H^{-1}H_1 = H^{-1} \circ T[H := s^k w]
\]
\[
= H^{-1} \circ 1-T[H := r^k v]
\]
\[
= (X + T(h.o.t.))[T := r^k v]
\]
\[
= X + r^k v(h.o.t.)
\]
and similarly
\[
H_1^{-1}H = X + r^k v(h.o.t.).
\]

For $k$ sufficiently large, $H_2 := H_1^{-1}H$ and its inverse apparently are elements of $\text{Aut}_{R_i}(R_v[X])$. So now $H = H_1 H_2$ with $H_1$ and $H_2$ are both of the required form. \[\square\]

**Lemma 3.7.** Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.

**Proof.** Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^kR + s^lR = R$. Write $r^k x + s^l y = 1$ for some $x, y \in R$. Then $t = (r^k x + s^l y)t = vx + wy \in R$. \[\square\]

**Lemma 3.8 (Patching Lemma).** Let $r, s \in R$ with $rR + sR = R$. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that there is a nice $F \in \text{Aut}_{R_i}(R_v[X])$ with first $k$ components equal to $h_1, \ldots, h_k$ and that there is a nice $G \in \text{Aut}_{R_r}(R_v[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. Then there is a nice $H \in \text{Aut}_R(R[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. 5
Proof. Consider the polynomial map \( F^{-1}G \in \text{Aut}_{R_r}(R_{rs}[X]) \) and note that it is fact an \( R_{rs}[X_1, \ldots, X_k] \)-automorphism of \( R_{rs}[X] = R_{rs}[X_1, \ldots, X_k][X_{k+1}, \ldots, X_n] \). Now apply Lemma 3.6 to the ring \( R[X_1, \ldots, X_k] \) and write \( F^{-1}G = H_2H_1 \) with \( H_1 \in \text{Aut}_{R_r}[X_1, \ldots, X_k](R_r[X]) \) and \( H_2 \in \text{Aut}_{R_r}[X_1, \ldots, X_k](R_r[X]) \), where both \( H_i \) are of the form \( X + \text{h.o.t.} \). Considered as automorphisms over respectively \( R_r \) and \( R_x \), the first \( k \) components of \( H_1 \) and \( H_2 \) of course equal \( X_1, \ldots, X_k \). Hence \( H := FH_1 = GH_2^{-1} \) is a nice polynomial automorphism (over \( R_{rs} \), a priori) whose first \( k \) components equal \( h_1, \ldots, h_k \). It is defined over \( R_r \) (because \( H = FH_1 \) and \( F \) and \( H_1 \) are defined over \( R_r \)) and it is defined over \( R_x \) (because \( H = GH_2^{-1} \) and \( G \) and \( H_2 \) are defined over \( R_x \)). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over \( R \).

Theorem 3.9. Let \( k \in \{1, \ldots, n\} \) and let \( h_1, \ldots, h_k \in R[X] \) be polynomials of the form \( h_i = X_i + \text{h.o.t.} \). Assume that for every maximal ideal \( m \) of \( R \), \( (h_1, \ldots, h_k) \) is a nice partial coordinate system when considered as an element of \( R_m[X]^k \). Then \( (h_1, \ldots, h_k) \) is a nice partial coordinate system.

Proof. Let \( P \subseteq \text{Loc}(R) \) be the collection of all \( R_r \), \( r \in R \setminus \{0\} \), such that \( (h_1, \ldots, h_k) \) is a nice partial coordinate system over \( R_r \). Now check the two conditions for Quillen Induction.

(a) Let \( m \) be a maximal ideal of \( R \). It is assumed that \( (h_1, \ldots, h_k) \) is a nice partial coordinate system over \( R_m \). Using Lemma 3.4, choose \( F \in \text{Aut}_{R_m}(R_m[X]) \) nice with first \( k \) components equal to \( h_1, \ldots, h_k \). There are only finitely many elements of \( R \) appearing in the denominator of a coefficient of a component of \( F \) and its inverse. Denote the product of these denominators by \( r \). None of these denominators is an element of \( m \) and, because \( m \) is prime, \( r \) is not an element of \( m \) either. Furthermore, obviously, \( P(R_r) \).

(b) Let \( r, s, t \in R \setminus \{0\} \) be such that \( rR_t + sR_t = R_t \) and assume \( P(R_r) \) and \( P(R_s) \). Then \( P(R_t) \) follows by applying the Patching Lemma (Lemma 3.8) to the ring \( R_t \).

So, using Quillen Induction (Proposition 3.2), \( P(R) \), which means that \( (h_1, \ldots, h_k) \) is a nice partial coordinate system over \( R \).

Corollary 3.10. Assume that \( R \) is Hermite. Let \( k \in \{1, \ldots, n\} \) and \( h_1, \ldots, h_k \in R[X] \). Assume that \( (h_1, \ldots, h_k) \) is a partial coordinate system when considered as an element of \( R_m[X]^k \), for every maximal ideal \( m \) of \( R \). Then \( (h_1, \ldots, h_k) \) is a partial coordinate system.

Proof. First of all note that it is possible to assume that the \( h_i \) have no constant part. Write \( h_i = r_{i1}X_1 + \cdots + r_{in}X_n + \text{h.o.t.} \) for all \( i \), with \( r_{ij} \in R \).

Consider a maximal ideal \( m \) of \( R \). Then \( (h_1, \ldots, h_k) \) is a partial coordinate system over \( R_m \), which means that there are \( f_{k+1}, \ldots, f_n \in R_m[X] \) such that \( F := (h_1, \ldots, h_k, f_{k+1}, \ldots, f_n) \in \text{Aut}_{R_m}(R_m[X]) \). The \( f_i \) can be chosen in such a way
that they have no constant part. Then \( \det JF \in R_m[X]^* \) and hence substituting 
\( X_1 := 0, \ldots, X_n := 0 \) gives

\[
\begin{bmatrix}
  r_{11} & \cdots & r_{1n} \\
  \vdots & & \vdots \\
  r_{k1} & \cdots & r_{kn}
\end{bmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_m^n.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \( R_m \)-module homomorphism 
from \( R_m^n \) to \( R_k^n \).

Because this holds for every maximal ideal of \( R \), it follows that the matrix \((r_{ij})_{ij}\) 
represents a surjective \( R \)-module homomorphism from \( R^n \) to \( R^k \). Now \( R \) is Hermite, 
which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix 
\( M \) over \( R \) (see [Lam78], Corollary 4.5). Viewing this matrix \( M \) as a polynomial 
automorphism of \( R\[X\] \) and applying its inverse to the polynomials \( h_i \), it follows 
that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})\).

By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \( R_m[X] \), for every 
\( m \in \text{Max}(R) \). Now apply Theorem 3.9. \( \square \)

The condition that \( R \) be Hermite in the previous corollary is necessary. For let \( R \) be 
a non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \( R \) that cannot be 
extended to an invertible square matrix. Then \( h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n] \) 
is not a coordinate (if it were, the coefficients of the linear part of an automorphism 
with \( h \) as its first component would form an invertible square matrix over \( R \) 
extending \((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \( m \) of \( R \), \((a_1, \ldots, a_n)\) is extendible 
to an invertible square matrix over \( R_m \) (since \( R_m \) is local) and so \( h \) is a coordinate 
over \( R_m \).

\section{Main Result}

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two 
variables over an discrete valuation ring containing \( \mathbb{Q} \).

\begin{theorem}
Let \( R \) be a discrete valuation ring containing \( \mathbb{Q} \). Denote the unique 
maximal ideal of \( R \) by \( m \), write \( K \) for the quotient field \( \mathbb{Q}(R) \) of \( R \), and write \( k \) 
for the residue field \( R/m \) of \( R \). Let \( A \) be a finitely generated affine \( R \)-domain and assume 
that \( K \otimes R A \cong K[2] \) and that \( k \otimes_R A \cong k[2] \). Then \( A \cong R[2] \). \( \square \)
\end{theorem}

In order to use this result, a lemma is needed on the behaviour of the kernel of a 
locally nilpotent derivation with a slice under tensoring.

\begin{lemma}
Let \( s \in R[X] := R[X_1, \ldots, X_n] \) and let \( A \) be an \( R \)-algebra via the map 
\( \varphi: R \to A \). Denote the induced map \( R[X] \to A[X] \) by \( \varphi# \). Then

\[ A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi#(s)A[X]) \]
\end{lemma}
In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then

$$A \otimes_R R[X]^D \cong A[X]^{\tilde{D}},$$

where $\tilde{D}$ denotes the extension of $D$ to $A[X]$.

**Proof.** The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

\[
\begin{array}{ccccccccc}
 sR[X] & \longrightarrow & R[X] & \longrightarrow & R[X]/sR[X] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A \otimes_R sR[X] & \longrightarrow & A \otimes_R A[X] & \longrightarrow & A \otimes_R R[X]/sR[X] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \varphi_#(s)A[X] & \longrightarrow & A[X] & \longrightarrow & A[X]/(\varphi_#(s)A[X]) & \longrightarrow & 0 \\
\end{array}
\]

The map $A \otimes_R sR[X] \to \varphi_#(s)A[X]$ is surjective: take an element $\varphi_#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_#(s)f$ is the image of $\sum c_\alpha \otimes_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_#(s)A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1.

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to $0$. Let $D$ be the locally nilpotent derivation $Y \partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\tilde{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\tilde{D} = A[X].$

**Lemma 4.3.** Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.

**Proof.** Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K'[X,Y,Z]$ by $\tilde{D}$. By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X,Y,Z]^D \cong K[X,Y,Z]^{\tilde{D}} \cong K[2].$$

In exactly the same way it follows that

$$k \otimes_R R[X,Y,Z]^D \cong k[2].$$

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. 

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Theorem 4.4. Let $R$ be a Dedekind domain containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice. Then $R[X,Y,Z]^D \cong R[2]$.

Proof. Let $s \in R[X,Y,Z]$ be a slice of $D$. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, $R$ is Hermite. By Corollary 3.10 it is enough to show that $s$ is a coordinate in $R_m[X,Y,Z]$ for every maximal ideal $m$ of $R$.

So let $m$ be a maximal ideal of $R$. Then $R_m$ is a discrete valuation ring. Because $R$ contains $\mathbb{Q}$, $R_m$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_m[X,Y,Z]^D \cong R_m[2]$. In other words, $s$ is a coordinate in $R_m[X,Y,Z]$.

Corollary 4.5. Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X,Y,Z,W]$ of the form

$$D := a(X,Y,Z,W)\partial_X + b(X,Y,Z,W)\partial_Y + c(X,Y,Z,W)\partial_Z + d(W)\partial_W.$$ 


Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since $D$ is locally nilpotent. So $d^{-1}W$ is a slice of $D$. This slice is also a coordinate and hence $k[X]^D \cong k[3]$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$.

References


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