THE CANCELLATION PROBLEM
IN DIMENSION FOUR

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Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let $k$ be a field of characteristic zero and let $V$ be an algebraic variety over $k$. The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \ldots, X_n]$ implies that $A \cong k[X_1, \ldots, X_{n-1}]$ for an affine $k$-domain $A$. One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \ldots, X_n]$ with a slice is isomorphic to $k[X_1, \ldots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing $\mathbb{Q}$, this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing $\mathbb{Q}$ for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let \( A \) be a ring. A derivation on \( A \) is a map \( D : A \to A \) satisfying \( D(a + b) = D(a) + D(b) \) and \( D(ab) = aD(b) + D(a)b \) for all \( a, b \in A \). If \( R \) is a ring and \( A \) is an \( R \)-algebra via \( f : R \to A \), then \( A \) is called an \( R \)-derivation if \( D(f(r)) = 0 \) for all \( r \in R \). A derivation \( D \) is called locally nilpotent if for all \( a \in A \) there is an \( n \in \mathbb{N} \) such that \( D^n(a) = 0 \). The kernel of \( D \) is denoted by \( A^D \). If \( s \in A \) is such that \( D(s) = 1 \), then \( s \) is called a slice of \( D \).

The following proposition (see [Wri81]) is well-known.

\[ \text{Proposition 2.1.} \text{ Let } A \text{ be a } \mathbb{Q} \text{-algebra and let } D \text{ be a locally nilpotent derivation on } A. \text{ Assume that } s \in A \text{ is a slice of } D. \text{ Then } A = A^D[s] \text{ and } s \text{ is algebraically independent over } A^D. \text{ Furthermore, } D = d/ds. \]

In the applications, \( A \) will invariably be a polynomial ring \( R[X] := R[X_1, \ldots, X_n] \) over a ring \( R \). An \( R \)-derivation \( D \) on \( R \) is called triangular if \( D(X_i) \in R[X_{i+1}, \ldots, X_n] \) for all \( i \). Such a derivation is automatically locally nilpotent. An element \( s \in R[X] \) is called a coordinate if there is a polynomial automorphism \( F \) of \( R[X] \) with \( s \) as one of its components. More generally, a sequence \( (s_1, \ldots, s_k) \) of elements of \( R[X] \) with \( 1 \leq k \leq n \) is called a partial coordinate system if there are polynomials \( f_{k+1}, \ldots, f_n \in R[X] \) such that \( (s_1, \ldots, s_k, f_{k+1}, \ldots, f_n) \) is a polynomial automorphism of \( R[X] \).

Proposition 2.1 implies the following.

\[ \text{Corollary 2.2.} \text{ Let } R \text{ be a ring and let } n \geq 2. \text{ Let } D \text{ be a locally nilpotent } R\text{-derivation on } R[X] := R[X_1, \ldots, X_n] \text{ and let } s \in R[X] \text{ be a slice of } D. \text{ Then } s \text{ is a coordinate if and only if } R[X]^D \cong R^{[n-1]}. \]

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

\[ \text{Problem 2.3 (Cancellation Problem).} \text{ Let } k \text{ be a field of characteristic zero and let } n \geq 2. \text{ Let } D \text{ be a locally nilpotent } k\text{-derivation on } k[X] := k[X_1, \ldots, X_n] \text{ and assume that } D \text{ has a slice } s \in k[X]. \text{ Is then } k[X]^D \cong k^{[n-1]}, \text{ i.e., is } s \text{ a coordinate in } k[X] ? \]

More generally, one can ask the following question.

\[ \text{Problem 2.4 (Generalized Cancellation Problem).} \text{ Let } k \text{ be a field of characteristic zero, let } R \text{ be an affine } k\text{-domain, and let } n \geq 2. \text{ Let } D \text{ be a locally nilpotent } R\text{-derivation on } R[X] := R[X_1, \ldots, X_n] \text{ and assume that } D \text{ has a slice } s \in R[X]. \text{ Is then } R[X]^D \cong R^{[n-1]}, \text{ i.e., is } s \text{ a coordinate in } R[X] ? \]

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

\[ \text{Theorem 2.5.} \text{ Let } k \text{ be a field of characteristic zero. Let } D \text{ be a locally nilpotent } k\text{-derivation on } k[X] := k[X_1, X_2]. \text{ Then } k[X]^D \cong k^{[1]}. \]

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Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field $k$ in the theorem can in fact be replaced by an arbitrary $\mathbb{Q}$-algebra $R$.

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that $D$ has a slice. Then $k[X]^D \cong k^{[2]}$.

This paper now proves the Generalized Cancellation Problem for $n=3$ in case $R$ is a Dedekind domain over $\mathbb{Q}$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where $\partial_i$ denotes $\partial/\partial X_i$. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely $D := (2X_1^2 - 3)\partial_1 + (4X_2^4 - 8X_4)\partial_2 + (5X_3^4 - 10)\partial_3 + X_5\partial_4$.

### 3 Local Coordinates

Let $R$ be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables over $R$. This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_m[X]$, for all maximal ideals $m$ of $R$, provided that $R$ is Hermite, and similarly for partial coordinate systems. Recall that $R$ is called **Hermite** if every unimodular row $(r_1, \ldots, r_k)$ can be extended to an invertible square matrix over $R$.

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define $\text{Loc}(R) := \{R_r \mid r \in R \setminus \{0\}\}$.

**Proposition 3.2 (Quillen Induction).** Let $P \subseteq \text{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \text{Loc}(R)$. In that case, $L$ is said to have property $P$. Assume that

(a) for all $m \in \text{Max}(R)$: there exists an $r \in R \setminus m$ such that $P(R_r)$;

(b) for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.

Then $P(L)$ for all $L \in \text{Loc}(R)$. In particular $P(R)$.
Proof. Let $S$ be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of $R$. It is not empty because 0 is in $S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of $R$ also because of (b) (for $\tilde{r} \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}, s := \tilde{s}$, and $t := \tilde{r}\tilde{s}$).

Suppose that $S \neq R$. Then $S$ is contained in some maximal ideal of $R$, say $m$. By (a) there is an $r \in R \setminus m$ such that $P(R_r)$. But then $r \in S \subseteq m$, which contradicts $r \not\in m$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$.

Definition 3.3. An element $H$ of $\text{End}_R(R[X])$ is called nice if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater, and $\text{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $h$ as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first $k$ components.

Lemma 3.4. A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first $k$ components.

Definition 3.5. Let $H \in \text{End}_R(R[X])$ be nice. Then $T^H \in \text{End}_{R[T]}(R[T][X])$ is defined by

$$T^H := T^{-1} H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because $H$ is nice.) If $r \in R$, then $T^H[T := r] \in \text{End}_R(R[X])$ is denoted by $^rH$.

One can easily see that $(\det JH)[X := TX] = \det J^T H$ and that $H$ is invertible if and only if $^T H$ is. Here $JH$ denotes the Jacobian matrix $(\partial H_i/\partial X_j),_{ij}$ of $H$. Even better, if $r \in R \setminus \{0\}$, then $\det J^r H \in R^\ast$ if and only if $\det JH \in R^\ast$ and $^rH$ is invertible if and only if $H$ is.

The map $T^H$ is called the clearing map because of the following: if $K$ is the quotient field of $R$ and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that $^rH \in \text{End}_R(R[X])$. So, the denominators of $H$ are cleared. See Chapter 1 of [Ess00].

Lemma 3.6. Let $r, s \in R \setminus \{0\}$ be such that $r R + s R = R$ and let $H \in \text{Aut}_{R_r}(R_{r_s}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_r}(R_r[X])$ and $H_2 \in \text{Aut}_{R_s}(R_s[X])$ such that $H = H_1 H_2$.

Proof. Note that

$$T^H = H_{(1)} + T^H_{(2)} + T^2 H_{(3)} + \cdots + T^{d-1} H_{(d)}$$

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where each \( H_{(i)} \) is the homogeneous part of degree \( i \) of \( H \) and \( d \) is the degree of \( H \). Hence

\[
1-TH = H_{(1)} + (1-T)H_{(2)} + (1-T)^2H_{(3)} + \cdots + (1-T)^{d-1}H_{(d)} \\
= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(h.o.t.) \\
= H + T(h.o.t.),
\]

where, as before, h.o.t. stands for some terms of \( X \)-degree at least two. As a consequence

\[
H^{-1} \circ 1-TH = H^{-1} \circ (H + T(h.o.t.)) \\
= X + T(h.o.t).
\]

Now let \( k \in \mathbb{N} \) be sufficiently large. From \( rR + sR = R \) it follows that \( r^kR + s^kR = R \). Take \( v, w \in R \) with \( r^kv + s^kw = 1 \). If \( k \) is sufficiently large, then \( s^k wH \) and \( s^k w(H^{-1}) \) are elements of \( \text{End}_{R_r}(R_r[X]) \). They are also each others inverse and hence they are in fact elements of \( \text{Aut}_{R_r}(R_r[X]) \).

Take \( H_1 := s^k wH \) and compute \( H^{-1} H_1 \). This gives

\[
H^{-1} H_1 = H^{-1} \circ T [T := s^k w] \\
= H^{-1} \circ 1-T [T := r^k v] \\
= (X + T(h.o.t.))[T := r^k v] \\
= X + r^k v(h.o.t.)
\]

and similarly

\[
H_1^{-1} H = X + r^k v(h.o.t.).
\]

For \( k \) sufficiently large, \( H_2 := H_1^{-1} H \) and its inverse apparently are elements of \( \text{Aut}_{R_r}(R_r[X]) \). So now \( H = H_1 H_2 \) with \( H_1 \) and \( H_2 \) are both of the required form.

**Lemma 3.7.** Let \( r, s \in R \) be such that \( rR + sR = R \). Take \( t \in R_{rs} \) such that \( t \in R_r \cap R_s \). Then \( t \in R \).

**Proof.** Write \( t = v/r^k = w/s^l \) with \( v, w \in R \) and \( k, l \in \mathbb{N} \). Because \( rR + sR = R \), also \( r^kR + s^l R = R \). Write \( r^k x + s^l y = 1 \) for some \( x, y \in R \). Then \( t = (r^k x + s^l y)t = vx + wy \in R \).

**Lemma 3.8 (Patching Lemma).** Let \( r, s \in R \) with \( rR + sR = R \). Let \( k \in \{1, \ldots, n\} \) and let \( h_1, \ldots, h_k \in R[X] \) be polynomials of the form \( h_i = X_i + h.o.t. \). Assume that there is a nice \( F \in \text{Aut}_{R_r}(R_r[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \) and that there is a nice \( G \in \text{Aut}_{R_r}(R_r[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \). Then there is a nice \( H \in \text{Aut}_R(R[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \).
Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R} \{R_{e}[X]\}$ and note that it is fact an $R_{er}[X_{1},\ldots,X_{k}]-\text{automorphism}$ of $R_{er}[X] = R_{er}[X_{1},\ldots,X_{k}][X_{k+1},\ldots,X_{n}]$. Now apply Lemma 3.6 to the ring $R[X_{1},\ldots,X_{k}]$ and write $F^{-1}G = H_{1}H_{2}$ with $H_{1} \in \text{Aut}_{R_{e}[X_{1},\ldots,X_{k}]}(R_{e}[X])$ and $H_{2} \in \text{Aut}_{R_{e}[X_{1},\ldots,X_{k}]}(R_{e}[X])$, where both $H_{i}$ are of the form $X + \text{h.o.t.}$. Considered as automorphisms over respectively $R_{e}$ and $R_{a}$, the first $k$ components of $H_{1}$ and $H_{2}$ of course equal $X_{1},\ldots,X_{k}$. Hence $H := KHH_{2}^{-1}$ is a nice polynomial automorphism (over $R_{es}$, a priori) whose first $k$ components equal $h_{1},\ldots,h_{k}$. It is defined over $R_{r}$ (because $H = KH_{1}$ and $F$ and $H_{1}$ are defined over $R_{e}$) and it is defined over $R_{e}$ (because $H = GH_{2}^{-1}$ and $G$ and $H_{2}$ are defined over $R_{e}$). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over $R_{r}$.

Theorem 3.9. Let $k \in \{1,\ldots,n\}$ and let $h_{1},\ldots,h_{k} \in R[X]$ be polynomials of the form $h_{i} = X_{i} + \text{h.o.t.}$. Assume that for every maximal ideal $m$ of $R$, $(h_{1},\ldots,h_{k})$ is a nice partial coordinate system when considered as an element of $R_{m}[X]^{k}$. Then $(h_{1},\ldots,h_{k})$ is a nice partial coordinate system.

Proof. Let $P \subseteq \text{Loc}(R)$ be the collection of all $R_{r}, r \in R \setminus \{0\}$, such that $(h_{1},\ldots,h_{k})$ is a nice partial coordinate system over $R_{r}$. Now check the two conditions for Quillen Induction.

(a) Let $m$ be a maximal ideal of $R$. It is assumed that $(h_{1},\ldots,h_{k})$ is a nice partial coordinate system over $R_{m}$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_{e}}(R_{m}[X])$ nice with first $k$ components equal to $h_{1},\ldots,h_{k}$. There are only finitely many elements of $R$ appearing in the denominator of a coefficient of a component of $F$ and its inverse. Denote the product of these denominators by $r$. None of these denominators is an element of $m$ and, because $m$ is prime, $r$ is not an element of $m$ either. Furthermore, obviously, $P(R_{r})$.

(b) Let $r,s,t \in R \setminus \{0\}$ be such that $rR_{t} + sR_{t} = R_{t}$ and assume $P(R_{r})$ and $P(R_{s})$. Then $P(R_{t})$ follows by applying the Patching Lemma (Lemma 3.8) to the ring $R_{t}$.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_{1},\ldots,h_{k})$ is a nice partial coordinate system over $R$.\)

Corollary 3.10. Assume that $R$ is Hermite. Let $k \in \{1,\ldots,n\}$ and $h_{1},\ldots,h_{k} \in R[X]$. Assume that $(h_{1},\ldots,h_{k})$ is a partial coordinate system when considered as an element of $R_{m}[X]^{k}$, for every maximal ideal $m$ of $R$. Then $(h_{1},\ldots,h_{k})$ is a partial coordinate system.

Proof. First of all note that it is possible to assume that the $h_{i}$ have no constant part. Write $h_{i} = r_{i1}X_{1} + \cdots + r_{in}X_{n} + \text{h.o.t.}$ for all $i$, with $r_{ij} \in R$.

Consider a maximal ideal $m$ of $R$. Then $(h_{1},\ldots,h_{k})$ is a partial coordinate system over $R_{m}$, which means that there are $f_{k+1},\ldots,f_{n} \in R_{m}[X]$ such that $F := (h_{1},\ldots,h_{k},f_{k+1},\ldots,f_{n}) \in \text{Aut}_{R_{m}}(R_{m}[X])$. The $f_{i}$ can be chosen in such a way
that they have no constant part. Then det \( JF \in R_m[X]^* \) and hence substituting \( X_1 := 0, \ldots, X_n := 0 \) gives

\[
\begin{vmatrix}
  r_{11} & \cdots & r_{1n} \\
  \vdots & \ddots & \vdots \\
  r_{k1} & \cdots & r_{kn}
\end{vmatrix}
= \det J(F[X] := 0) = (\det JF)[X := 0] \in R_m^*.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \(R_m\)-module homomorphism from \(R^n_m\) to \(R^k_m\).

Because this holds for every maximal ideal of \(R\), it follows that the matrix \((r_{ij})_{ij}\) represents a surjective \(R\)-module homomorphism from \(R^n\) to \(R^k\). Now \(R\) is Hermite, which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix \(M\) over \(R\) (see [Lam78], Corollary 4.5). Viewing this matrix \(M\) as a polynomial automorphism of \(R[X]\) and applying its inverse to the polynomials \(h_i\), it follows that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})\). By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \(R_m[X]\), for every \(m \in \text{Max}(R)\). Now apply Theorem 3.9.

The condition that \(R\) be Hermite in the previous corollary is necessary. For let \(R\) be any non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \(R\) that cannot be extended to an invertible square matrix. Then \(h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n]\) is not a coordinate (if it were, the coefficients of the linear part of an automorphism with \(h\) as its first component would form an invertible square matrix over \(R\) extending \((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \(m\) of \(R\), \((a_1, \ldots, a_n)\) is extendible to an invertible square matrix over \(R_m\) (since \(R_m\) is local) and so \(h\) is a coordinate over \(R_m\).

\[\text{4 Main Result}\]

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing \(\mathbb{Q}\).

**Theorem 4.1.** Let \(R\) be a discrete valuation ring containing \(\mathbb{Q}\). Denote the unique maximal ideal of \(R\) by \(m\), write \(K\) for the quotient field \(Q(R)\) of \(R\), and write \(k\) for the residue field \(R/m\) of \(R\). Let \(A\) be a finitely generated affine \(R\)-domain and assume that \(K \otimes_R A \cong K^2]\) and that \(k \otimes_R A \cong k^2\). Then \(A \cong R^2\).

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

**Lemma 4.2.** Let \(s \in R[X] := R[X_1, \ldots, X_n]\) and let \(A\) be an \(R\)-algebra via the map \(\varphi: R \to A\). Denote the induced map \(R[X] \to A[X]\) by \(\varphi\). Then

\[
A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi(s)A[X])
\]
In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then

$$A \otimes_R R[X]^D \cong A[X]^D,$$

where $\tilde{D}$ denotes the extension of $D$ to $A[X]$.

**Proof.** The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

\[
\begin{array}{cccccccc}
\quad & sR[X] & \longrightarrow & R[X] & \longrightarrow & R[X]/sR[X] & \longrightarrow & 0 \\
\downarrow & & & & & & & \\
A \otimes_R sR[X] & \longrightarrow & A \otimes_R A[X] & \longrightarrow & A \otimes_R R[X]/sR[X] & \longrightarrow & 0 \\
\downarrow & & & & & & & \\
\quad & \varphi_\#(s)A[X] & \longrightarrow & A[X] & \longrightarrow & A[X]/(\varphi_\#(s)A[X]) & \longrightarrow & 0 \\
\end{array}
\]

The map $A \otimes_R sR[X] \to \varphi_\#(s)A[X]$ is surjective: take an element $\varphi_\#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_\#(s)f$ is the image of $\sum c_\alpha \otimes sX_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_\#(s)A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1.

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to 0. Let $D$ be the locally nilpotent derivation $Y \partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\tilde{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\tilde{D} = A[X]$.

**Lemma 4.3.** Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.

**Proof.** Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\tilde{D}$. By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X,Y,Z]^D \cong K[X,Y,Z]^\tilde{D} \cong K^{[2]}.$$

In exactly the same way it follows that

$$k \otimes_R R[X,Y,Z]^D \cong k^{[2]}.$$

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. 

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Theorem 4.4. Let $R$ be a Dedekind domain containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice. Then $R[X,Y,Z]^D \cong R^{[2]}$.

Proof. Let $s \in R[X,Y,Z]$ be a slice of $D$. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, $R$ is Hermite. By Corollary 3.10 it is enough to show that $s$ is a coordinate in $R_m[X,Y,Z]$ for every maximal ideal $m$ of $R$.

So let $m$ be a maximal ideal of $R$. Then $R_m$ is a discrete valuation ring. Because $R$ contains $\mathbb{Q}$, $R_m$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_m[X,Y,Z]^D \cong R_m^{[2]}$. In other words, $s$ is a coordinate in $R_m[X,Y,Z]$.

Corollary 4.5. Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X,Y,Z,W]$ of the form

$$D := a(X,Y,Z,W)\partial_X + b(X,Y,Z,W)\partial_Y + c(X,Y,Z,W)\partial_Z + d(W)\partial_W.$$ 

Assume that $D$ has a slice. Then $k[X,Y,Z,W]^D \cong k^{[3]}$.

Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since $D$ is locally nilpotent. So $d^{-1}W$ is a slice of $D$. This slice is also a coordinate and hence $k[X]^D \cong k^{[3]}$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$.

References


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