ON A PROBLEM OF ADJAMAGBO

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Abstract

In this note we construct an unramified, finite and birational polynomial map from $\mathbb{C}^2$ to an irreducible affine variety $V$ of dimension two which is injective on each line through the origin but which is not injective (hence no isomorphism). More precisely, every point of the singular locus of $V$ has exactly two preimages.

1 Introduction

The Jacobian Conjecture asserts that every unramified polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ (i.e. a map whose Jacobian matrix has maximal rank at each point of $\mathbb{C}^n$) is an isomorphism.

Several attempts have been made to generalise this conjecture by allowing one of the $\mathbb{C}^n$’s to be replaced by an irreducible affine variety of dimension $n$. One such attempt was made by Bass in [4]. He conjectured that any étale map (i.e. flat and unramified) from a complex irreducible affine and unirational variety of dimension $n$ whose invertible regular functions are all constants to $\mathbb{C}^n$ is an isomorphism. However, based on a result of Kulikov in [8], Adjamagbo showed in [1] that this conjecture is false for any $n \geq 2$ and is true if $n = 1$.

In this paper we consider the following problem, due to Adjamagbo, which arose from an attempt of his to generalise the Jacobian Conjecture.

Adjamagbo’s Problem

Let $V \subset \mathbb{C}^m$ ($m \geq 1$) be an irreducible affine variety of dimension $n$ and $F : \mathbb{C}^n \rightarrow V$ an unramified polynomial map. Assume that $F$ satisfies the following conditions

a) $F$ is birational
b) $F$ is finite
c) $F$ is injective on each line through the origin of $\mathbb{C}^n$.

Does it follow that $F$ is an isomorphism?

Various special cases of this problem have been investigated and seem to indicate that the answer to Adjamagbo’s Problem is affirmative: namely if $n = 1$ it is well-known that every unramified injective polynomial map from $\mathbb{C}$ to $\mathbb{C}^m$ is an embedding i.e. $F : \mathbb{C} \rightarrow \mathbb{C}^m$ is an isomorphism. Furthermore in case $V = \mathbb{C}^n$ each of the two conditions a) and b) imply that $F$ is an isomorphism: if condition a) is satisfied this is Keller’s theorem (see [7] or [3] or [5]). If condition b) is satisfied the result is
"classical" (see [3] or [5]). Furthermore if \( n = 2 \) it was shown by Gwoździewicz in [6] that already injectivity of \( F \) on one line is sufficient to imply that \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is an isomorphism. So in case \( V = \mathbb{C}^2 \) each of the three conditions \( a), b), c) \) above suffices to imply that \( F \) is an isomorphism. However the main result of this paper, Proposition 2.1, shows that there exists an irreducible affine variety \( V \) of dimension two in \( \mathbb{C}^3 \) and an unramified polynomial map \( F : \mathbb{C}^2 \to V \) which satisfies all three conditions \( a), b) \) and \( c) \) and is nevertheless not injective (hence no isomorphism). By extending this example in an obvious way we obtain a negative answer to Adjamagbo's Problem for all \( n \geq 2 \).

2 The example

Throughout this section we have the following notations: by \( f = (f_1, f_2, f_3) : \mathbb{C}^2 \to \mathbb{C}^3 \) we denote the polynomial map given by the following polynomials

\[
\begin{align*}
f_1 &= X_1 + X_2^2 X_3^2, \\
f_2 &= X_2 - X_1 \\
f_3 &= X_1^2
\end{align*}
\]

By \( V \subset \mathbb{C}^3 \) we denote the Zariski closure of \( f(\mathbb{C}^2) \) in \( \mathbb{C}^3 \). So \( f \) induces a morphism \( F : \mathbb{C}^2 \to V \). Finally \( S \) denotes the singular locus of \( V \).

**Proposition 2.1** \( V \) is an irreducible affine variety of dimension two such that the invertible elements of its coordinate ring are constants. Furthermore, the map \( F \) has the following properties.

1) \( F \) is unramified.
2) \( F \) is injective on each line through the origin of \( \mathbb{C}^2 \).
3) \( F \) is finite.
4) \( F \) is birational.
5) \( F \) is not injective (hence no isomorphism). More precisely, if \( y \in S \) then \(#F^{-1}(y) = 2 \) and if \( y \in V \setminus S \) then \(#F^{-1}(y) = 1 \).

**Proof.** The ringhomomorphism \( f^* : \mathbb{C}[Y] := \mathbb{C}[Y_1, Y_2, Y_3] \to \mathbb{C}[f] := \mathbb{C}[f_1, f_2, f_3] \) defined by \( f^*(Y_i) = f_i \) for all \( i \) is surjective. Hence, since \( \mathbb{C}[f] \) has dimension two the kernel of \( f^* \) is generated by one irreducible polynomial in \( \mathbb{C}[Y] \). Consequently the Zariski closure of \( f(\mathbb{C}^2) \) in \( \mathbb{C}^3 \), which equals the zero-set of this polynomial in \( \mathbb{C}^3 \), is an irreducible affine variety of dimension two contained in \( \mathbb{C}^3 \). The statement concerning the units of the coordinate ring of \( V \) is obvious since \( \mathbb{C}[f] \subset \mathbb{C}[X_1, X_3] \).

1) To see that \( F \) is unramified we need to show that the three \( 2 \times 2 \) minors of the Jacobian matrix of \( F \) have no common zero in \( \mathbb{C}^2 \). However from the equalities

\[
\begin{align*}
\det J(f_1, f_2) &= 1 + 2X_1 X_2^2 + 2X_1 X_2^2 \\
\det J(f_1, f_3) &= -4X_2 X_3^2
\end{align*}
\]

one readily verifies that already these two minors have no common zero in \( \mathbb{C}^3 \).

2) To see that \( F \) is injective on each line through \( 0 \in \mathbb{C}^2 \) we first consider the lines spanned by a vector of the form \((1, v_2)\), with \( v_2 \in \mathbb{C} \) and assume that \( F(t, tv_2) = F(s, sv_2) \) for some \( t, s \in \mathbb{C} \). Then in particular, looking at the third component of \( F \), we get \( t^2 = s^2 \), so \( t = s \) or \( t = -s \). If \( t = -s \) then \( f_1(t, tv_2) = f_1(s, sv_2) \) implies that \(-s + s^4 v_2^2 = s + s^4 v_2^2 \), whence \( s = 0 \) and hence \( t = -s = 0 \), so \( s = t \). Consequently
\( t = s \) in any case, so \( F \) is injective on the lines spanned by the vectors of the form \((1, w_2)\). Finally, looking at the second component of \( F \), one readily verifies that \( F \) is also injective on the \( Y \)-axis.

3) To show the last three points of the proposition we compute the reduced Gröbner basis of the ideal

\[ I := (Y_1 - f_1, Y_2 - f_2, Y_3 - f_3) \subset \mathbb{C}[X_1, X_2, Y_1, Y_2, Y_3] \]

with respect to the pure lexicographical ordering with \( X_1 > X_2 > Y_1 > Y_2 > Y_3 \). The result is that \( B = \{b_1, \ldots, b_5\} \), where

\[
\begin{align*}
    b_1 &= X_1 - X_2 + Y_2 \\
    b_2 &= X_2^2 - 2Y_2X_2 + Y_2^2 - Y_3 \\
    b_3 &= c_1(Y)X_2 + d_1(Y) \\
    b_4 &= c_2(Y)X_2 + d_2(Y) \\
    b_5 &= -2Y_1Y_3^2 - 4Y_2Y_3 + Y_3^4 - 2Y_3^3Y_2^2 - Y_3 + Y_1^2 - 2Y_1Y_3Y_2^2 + Y_2^2Y_3^2 \\
    c_1 &= 2Y_1 + Y_2 - 2Y_2^2 \\
    c_2 &= 1 + 2Y_3Y_2^2 \\
    d_1 &= -Y_2Y_3 + Y_2^3Y_3 - 3Y_1Y_2 - 2Y_2 - 2Y_3 \\
    d_2 &= -(Y_1 + Y_3Y_2^2 + Y_2 - Y_3^2) 
\end{align*}
\]

From \( b_1 \) and \( b_2 \) we see that \( X_1 - X_2 + f_2 = 0 \) and \( X_2^2 - 2f_2X_2 + (f_2^2 - f_3) = 0 \). So \( \mathbb{C}[X_1, X_2] \) is finite over \( \mathbb{C}[f] \) i.e. \( F \) is finite.

4) From \( b_4 \) we get \( X_2 = -d_2(f)/c_2(f) \in \mathbb{C}(f) \) \( (c_2(f) \text{ is non-zero}) \). Hence also \( X_1 = X_2 - f_2 \in \mathbb{C}(f) \), whence \( \mathbb{C}[X_1, X_2] = \mathbb{C}(f) \) i.e. \( F \) is birational.

5) Finally we show that \( F \) is not injective. First observe that since \( b := b_5 \) is the only polynomial in \( \mathbb{C}[Y] \) contained in \( B \), it follows from the relation algorithm (see for example Proposition C.2.2 in [5]) that \( \ker f^* = (b) \). So \( S \) is given by the ideal \((b, b_1, b_2, b_3)\), where \( b_1, b_2, b_3 \) denote the partial derivative of \( b \) with respect to \( Y_1 \). Computing the reduced Gröbner basis of this ideal with respect to the pure lexicographical ordering with \( Y_1 > Y_2 > Y_3 \) we find \( \{c_1, c_2\} \)! So \( S = V(c_1, c_2) \). In particular, looking at \( c_2 \) we see that if \( y \in S \) then \( y_2y_3 \neq 0 \). Since \( F : \mathbb{C}^2 \to V \) is finite, it is surjective, so \( \#F^{-1}(y) \geq 1 \) for all \( y \in V \). Assume first that \( y \in V \setminus S \). So either \( c_1(y) \) or \( c_2(y) \) is non-zero. Let \( x \in F^{-1}(y) \). If \( c_1(y) \neq 0 \) it follows from \( b_3 \) that \( x_2 = -d_1(y)/c_1(y) \) and using \( b_1 \) we get that \( x_1 = -y_2 - d_1(y)/c_1(y) \). So \( \#F^{-1}(y) = 1 \). A similar argument, using \( b_4 \), shows that \( \#F^{-1}(y) = 1 \) if \( c_2(y) \neq 0 \). Finally suppose that \( y \in S \). So, as observed above \( y_3 \neq 0 \). Choose \( z_1 \in \mathbb{C} \) with \( z_1^2 = y_3 \).

Claim: \( F^{-1}(y) = \{(z_1, z_1 + y_2), (-z_1, -z_1 + y_2)\} \).

The inclusion "\( \subset \)" follows immediately from \( x_2 - x_1 = y_2 \) and \( x_1^2 = y_3 \). To see the converse inclusion it remains to see that \( z_1 + z_1^2(z_1 + y_2)^2 = y_1 \) and \(-z_1 + (-z_1)^2(-z_1 + y_2)^2 = y_1 \) or equivalently that \( z_1(1 + 2y_3y_2) + y_3y_2^2 + y_3^2 - y_1 = 0 \) and \(-z_1(1 + 2y_3y_2) + (y_3y_2^2 + y_3^2 - y_1) = 0 \). However, \( y \in S \) i.e. \( c_1(y) = 0 \) and \( c_2(y) = 0 \). So \( 1 + 2y_3y_2 = c_1(y) = 0 \) and \( y_3y_2^2 + y_3^2 - y_1 = (-1/2)c_1(y) + (1/2)y_2c_2(y) = 0 \), which completes the proof. □

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Corollary 2.2 Adjamagbo’s Problem has a negative answer for all $n \geq 2$.

Proof. The case $n = 2$ is 2.1. So let $n \geq 3$ and define $f_i = X_{i-1}$ for all $4 \leq i \leq n + 1$ and $f_1, f_2, f_3$ as in 2.1. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$ be given by $f(x) = (f_1(x), \ldots, f_{n+1}(x))$ for all $x \in \mathbb{C}^n$ and denote by $V$ the Zariski closure of $f(\mathbb{C}^n)$ in $\mathbb{C}^{n+1}$. Then it is left to the reader to verify that $V$ is an irreducible affine variety of dimension $n$ and that the induced map $F : \mathbb{C}^n \rightarrow V$ satisfies all five properties of 2.1, thereby supplying a negative answer to Adjamagbo’s Problem.

Final remark. The map $F$ constructed above is not étale: namely if $F$ is étale then (by the preservation of normality under étale maps (see [2],(12)i)) $V$ is normal i.e. $\mathbb{C}[f]$ is integrally closed. Since $\mathbb{C}(f) = \mathbb{C}(X)$ and $\mathbb{C}[X]$ is integral over $\mathbb{C}[f]$ this implies that $\mathbb{C}[f] = \mathbb{C}[X]$ i.e. $F$ is an isomorphism, a contradiction.

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References


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