ON A PROBLEM OF ADJAMAGBO

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Abstract

In this note we construct an unramified, finite and birational polynomial map from \( \mathbb{C}^2 \) to an irreducible affine variety \( V \) of dimension two which is injective on each line through the origin but which is not injective (hence no isomorphism). More precisely, every point of the singular locus of \( V \) has exactly two preimages.

1 Introduction

The Jacobian Conjecture asserts that every unramified polynomial map \( F : \mathbb{C}^n \to \mathbb{C}^n \) (i.e. a map whose Jacobian matrix has maximal rank at each point of \( \mathbb{C}^n \)) is an isomorphism.

Several attempts have been made to generalise this conjecture by allowing one of the \( \mathbb{C}^n \)'s to be replaced by an irreducible affine variety of dimension \( n \). One such attempt was made by Bass in \([4]\). He conjectured that any étale map (i.e. flat and unramified) from a complex irreducible affine and unirational variety of dimension \( n \) whose invertible regular functions are all constants to \( \mathbb{C}^n \) is an isomorphism. However, based on a result of Kulikov in \([8]\), Adjamagbo showed in \([1]\) that this conjecture is false for any \( n \geq 2 \) and is true if \( n = 1 \).

In this paper we consider the following problem, due to Adjamagbo, which arose from an attempt of his to generalise the Jacobian Conjecture.

Adjamagbo’s Problem

Let \( V \subset \mathbb{C}^m \) (\( m \geq 1 \)) be an irreducible affine variety of dimension \( n \) and \( F : \mathbb{C}^n \to V \) an unramified polynomial map. Assume that \( F \) satisfies the following conditions

a) \( F \) is birational
b) \( F \) is finite
c) \( F \) is injective on each line through the origin of \( \mathbb{C}^n \).

Does it follow that \( F \) is an isomorphism?

Various special cases of this problem have been investigated and seem to indicate that the answer to Adjamagbo’s Problem is affirmative: namely if \( n = 1 \) it is well-known that every unramified injective polynomial map from \( \mathbb{C} \) to \( \mathbb{C}^m \) is an embedding i.e. \( F : \mathbb{C} \to \mathbb{C}^m \) is an isomorphism. Furthermore in case \( V = \mathbb{C}^n \) each of the two conditions a) and b) imply that \( F \) is an isomorphism: if condition a) is satisfied this is Keller’s theorem (see \([7]\) or \([3]\) or \([5]\)). If condition b) is satisfied the result is
"classical" (see [3] or [5]). Furthermore if $n = 2$ it was shown by Gwoździewicz in [6]
that already injectivity of $F$ on one line is sufficient to imply that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is an
isomorphism. So in case $V = \mathbb{C}^2$ each of the three conditions a), b), c) above suffices
to imply that $F$ is an isomorphism. However the main result of this paper, Proposition
2.1, shows that there exists an irreducible affine variety $V$ of dimension two in $\mathbb{C}^3$ and
an unramified polynomial map $F : \mathbb{C}^2 \to V$ which satisfies all three conditions a), b) and c) and is nevertheless not injective (hence no isomorphism). By extending this example in an obvious way we obtain a negative answer to Adjamagbo’s Problem for all $n \geq 2$.

2 The example

Throughout this section we have the following notations: by $f = (f_1, f_2, f_3) : \mathbb{C}^2 \to \mathbb{C}^3$
we denote the polynomial map given by the following polynomials

$$f_1 = X_1 + X_2^2 X_3^2, f_2 = X_2 - X_1 \text{ and } f_3 = X_1^2$$

By $V \subset \mathbb{C}^3$ we denote the Zariski closure of $f(\mathbb{C}^2)$ in $\mathbb{C}^3$. So $f$ induces a morphism
$F : \mathbb{C}^2 \to V$. Finally $S$ denotes the singular locus of $V$.

**Proposition 2.1** $V$ is an irreducible affine variety of dimension two such that the
invertible elements of its coordinate ring are constants. Furthermore, the map $F$ has
the following properties.
1) $F$ is unramified.
2) $F$ is injective on each line through the origin of $\mathbb{C}^2$.
3) $F$ is finite.
4) $F$ is birational.
5) $F$ is not injective (hence no isomorphism). More precisely, if $y \in S$ then $\# F^{-1}(y) = 2$ and if $y \in V \setminus S$ then $\# F^{-1}(y) = 1$.

**Proof.** The ring homomorphism $f^* : \mathbb{C}[Y] := \mathbb{C}[Y_1, Y_2, Y_3] \to \mathbb{C}[f] := \mathbb{C}[f_1, f_2, f_3]$ defined by $f^*(Y_i) = f_i$ for all $i$ is surjective. Hence, since $\mathbb{C}[f]$ has dimension two
the kernel of $f^*$ is generated by one irreducible polynomial in $\mathbb{C}[Y]$. Consequently
the Zariski closure of $f(\mathbb{C}^2)$ in $\mathbb{C}^3$, which equals the zero-set of this polynomial in
$\mathbb{C}^3$, is an irreducible affine variety of dimension two contained in $\mathbb{C}^3$. The statement
concerning the units of the coordinate ring of $V$ is obvious since $\mathbb{C}[f] \subset \mathbb{C}[X_1, X_3]$.

1) To see that $F$ is unramified we need to show that the three $2 \times 2$ minors of the
Jacobian matrix of $F$ have no common zero in $\mathbb{C}^2$. However from the equalities
$\det J(f_1, f_2) = 1 + 2X_1^2X_3^2 + 2X_1X_2^3$ and $\det J(f_1, f_3) = -4X_2X_3^3$ one readily verifies
that already these two minors have no common zero in $\mathbb{C}^3$.

2) To see that $F$ is injective on each line through $0 \in \mathbb{C}^2$ we first consider the lines
spanned by a vector of the form $(1, v_2)$, with $v_2 \in \mathbb{C}$ and assume that $F(t, tv_2) = F(s, sv_2)$ for some $t, s \in \mathbb{C}$. Then in particular, looking at the third component of $F$,
we get $t^2 = s^2$, so $t = s$ or $t = -s$. If $t = -s$ then $f_1(t, tv_2) = f_1(s, sv_2)$ implies that
$-s + s^2v_2^2 = s + s^2v_2^2$, whence $s = 0$ and hence $t = -s = 0$, so $s = t$. Consequently
\( t = s \) in any case, so \( F \) is injective on the lines spanned by the vectors of the form \((1, v_2)\). Finally, looking at the second component of \( F \), one readily verifies that \( F \) is also injective on the \( Y \)-axis.

3) To show the last three points of the proposition we compute the reduced Gröbner basis of the ideal

\[
I := (Y_1 - f_1, Y_2 - f_2, Y_3 - f_3) \subset \mathbb{C}[X_1, X_2, Y_1, Y_2, Y_3]
\]

with respect to the pure lexicographical ordering with \( X_1 > X_2 > Y_1 > Y_2 > Y_3 \). The result is that \( B = \{b_1, \ldots, b_5\} \), where

\[
\begin{align*}
  b_1 &= X_1 - X_2 + Y_2 \\
  b_2 &= X_2^2 - 2Y_2X_2 + Y_2^2 - Y_3 \\
  b_3 &= c_1(Y)X_2 + d_1(Y) \\
  b_4 &= c_2(Y)X_2 + d_2(Y) \\
  b_5 &= -2Y_1Y_3^2 - 4Y_2^2Y_2 + Y_3^4 - 2Y_1Y_2Y_2^2 - Y_3^3 - 2Y_1Y_2Y_2^2 + Y_2^4 \\
  c_1 &= 2Y_1 + Y_2 - 2Y_3 \\
  d_1 &= -Y_2^2Y_3 + Y_2Y_3 - 3Y_1Y_2 - Y_2^2 - 2Y_3 \\
  c_2 &= 1 + 2Y_3Y_2 \\
  d_2 &= -(Y_1 + Y_3Y_2^2 + Y_2 - Y_3^2).
\end{align*}
\]

From \( b_1 \) and \( b_2 \) we see that \( X_1 - X_2 + f_2 = 0 \) and \( X_2^2 - 2f_2X_2 + (f_2^2 - f_3) = 0 \). So \( \mathbb{C}[X_1, X_2] \) is finite over \( \mathbb{C}[f] \) i.e. \( F \) is finite.

4) From \( b_4 \) we get \( X_2 = -d_2(f)/c_2(f) \in \mathbb{C}(f) \) (\( c_2(f) \) is non-zero). Hence also \( X_1 = X_2 - f_2 \in \mathbb{C}(f) \), whence \( \mathbb{C}(X_1, X_2) = \mathbb{C}(f) \) i.e. \( F \) is birational.

5) Finally we show that \( F \) is not injective. First observe that since \( b := b_5 \) is the only polynomial in \( \mathbb{C}[Y] \) contained in \( B \), it follows from the relation algorithm (see for example Proposition C.2.2 in [5]) that \( \ker f^* = \langle b \rangle \). So \( S \) is given by the ideal \( \langle b, b_{y_1}, b_{y_2}, b_{y_3} \rangle \), where \( b_{y_k} \) denotes the partial derivative of \( b \) with respect to \( Y_k \). Computing the reduced Gröbner basis of this ideal with respect to the pure lexicographical ordering with \( Y_1 > Y_2 > Y_3 \) we find \( \{c_1, c_2\} \) ! So \( S = V(c_1, c_2) \). In particular, looking at \( c_2 \) we see that if \( y \in S \) then \( y_{2y_3} \neq 0 \). Since \( F^* : \mathbb{C}^2 \to V \) is finite, it is surjective, so \#\( F^{-1}(y) \) \( \geq 1 \) for all \( y \in V \). Assume first that \( y \in V \setminus S \). So either \( c_1(y) \) or \( c_2(y) \) is non-zero. Let \( x \in F^{-1}(y) \). If \( c_1(y) \neq 0 \) it follows from \( b_3 \) that \( x_2 = -d_1(y)/c_1(y) \) and using \( b_1 \) we get that \( x_1 = -y_2 - d_1(y)/c_1(y) \). So \#\( F^{-1}(y) \) \( = 1 \). A similar argument, using \( b_4 \), shows that \#\( F^{-1}(y) \) \( = 1 \) if \( c_2(y) \neq 0 \). Finally suppose that \( y \in S \). So, as observed above \( y_3 \neq 0 \). Choose \( z_1 \in \mathbb{C} \) with \( z_1^2 = y_3 \).

Claim: \( F^{-1}(y) = \{(z_1, z_1 + y_2), (-z_1, -z_1 + y_2)\} \).

The inclusion "\( \subset \)" follows immediately from \( x_2 - x_1 = y_2 \) and \( x_1^2 = y_3 \). To see the converse inclusion it remains to see that \( z_1 + z_1^2(z_1 + y_2) = y_1 \) and \( -z_1 + (-z_1)^2(-z_1 + y_2)^2 = y_1 \) or equivalently that \( z_1(1 + 2y_3y_2) + y_3y_2^2 + y_2^2 - y_1 = 0 \) and \( -z_1(1 + 2y_3y_2) + (y_3y_2^2 + y_2^2 - y_1) = 0 \). However, \( y \in S \) i.e. \( c_1(y) = 0 \) and \( c_2(y) = 0 \). So \( 1 + 2y_3y_2 = c_1(y) = 0 \) and \( y_3y_2^2 + y_2^2 - y_1 = (-1/2)c_1(y) + (1/2)y_2c_2(y) = 0 \), which completes the proof. \( \square \)
Corollary 2.2  Adjamagbo’s Problem has a negative answer for all $n \geq 2$.

Proof. The case $n = 2$ is 2.1. So let $n \geq 3$ and define $f_i = X_{i-1}$ for all $4 \leq i \leq n+1$ and $f_1, f_2, f_3$ as in 2.1. Let $f : \mathbb{C}^n \to \mathbb{C}^{n+1}$ be given by $f(x) = (f_1(x), \ldots, f_{n+1}(x))$ for all $x \in \mathbb{C}^n$ and denote by $V$ the Zariski closure of $f(\mathbb{C}^n)$ in $\mathbb{C}^{n+1}$. Then it is left to the reader to verify that $V$ is an irreducible affine variety of dimension $n$ and that the induced map $F : \mathbb{C}^n \to V$ satisfies all five properties of 2.1, thereby supplying a negative answer to Adjamagbo’s Problem.

Final remark. The map $F$ constructed above is not étale: namely if $F$ is étale then (by the preservation of normality under étale maps (see [2]),(12))) $V$ is normal i.e. $\mathbb{C}[f]$ is integrally closed. Since $\mathbb{C}(f) = \mathbb{C}(X)$ and $\mathbb{C}[X]$ is integral over $\mathbb{C}[f]$ this implies that $\mathbb{C}[f] = \mathbb{C}[X]$ i.e. $F$ is an isomorphism, a contradiction.

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References


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