Sets in the Plane with Many Conyclic Subsets

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Abstract

We study sets of points in the Euclidian plane having property \( R(t,s) \): every \( t \)-tuple of its points contains a concyclic \( s \)-tuple. Typical examples of the kind of theorems we prove are: 1) a set with \( R(19,10) \) must have all its points on two circles or all its points, with the exception of at most 9, are on one circle and 2) of a set with \( R(8,5) \) and \( N > 27 \) points at least \( N - 3 \) points lie on one circle.

1. Introduction

If all points, or all points but one, of a set \( V \) of points in the Euclidean plane are on a circle, then clearly every 5-subset of \( V \) contains a concyclic 4-subset. In [2] it was proved that the converse also holds, unless \( |V| = 6 \). In [1] other proofs were given and also the following was proved. If every 6-subset of a set \( V, |V| > 77 \), of points in the Euclidean plane contains a concyclic 4-subset, then all points of \( V \) with the exception of at most two are on a circle. The same then must hold if the condition is strengthened to: every 7-subset contains a concyclic 5-subset. We shall see below that then the condition \( |V| > 77 \) can be omitted. More generally we investigate sets satisfying the condition one gets by replacing the pair \((7,5)\) by \((t,s)\), \( t \geq s > 3 \).

It may be noteworthy that the essential point of the proofs in [1] and [2] is that the 2-(7,4,2) design (the complementary design of the 2-(7,3,1) design) has no realisation in the plane with concyclic quadruples as blocks. This means that there is no configuration of 7 points and 7 circles such that every circle contains 4 of the points and every pair of circles intersect in 2 of the points.

2. Preliminaries and Examples

Above, where we wrote “concyclic” and “circle” one may read “concyclic or collinear” and “circle or line”, respectively. The reason is that the only property of the set of circles that plays a role is that its elements are determined by three of their points (in [1], but not below, it is also used that two pairs of points on a circle do or do not separate each other). The same holds for the set of circles and lines and for any subset of that set. So in what follows, to avoid lengthy expressions, we shall silently assume that there is a prescribed set \( S \) of circles and/or lines, and call a set \( \mathcal{S} \) if it has all its points on a circle or line of \( S \). Its support will be that line or that circle. (The reader may still prefer to think of \( S \) as the set of all circles and accordingly read “round” as “concyclic”.)

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By set we will always mean a set of points in the Euclidian plane; \( V \) will denote such a set and \( N \) its cardinality. We say that \( V \) has property \( R(t, s) \) if each of its \( t \)-subsets contains a round \( s \)-subset. It will be understood throughout that \( 3 < s \leq t \leq N \).

(Sets with \( R(s, s), s > 3 \), are easily seen to be round and of course \( N = t \) is also a trivial case, but admitting these cases allows easier formulations.)

A set has property \( C(r) \) if there is a round set containing all its points with the exception of at most \( r \).

Trivial examples of sets with \( R(t, s) \) are sets with \( C(t - s) \). So we can state the theorems mentioned in the introduction in a condensed form as follows. If \( R(5, 4) \) and \( N > 6 \) then \( C(1) \) and if \( R(6, 4) \) and \( N > 77 \) then \( C(2) \). (The validity for infinite \( N \) easily follows from that for finite \( N \). This being also the case for all theorems below the reader may find it comfortable to think of finite sets only.)

**Lemma 1.** A set has property \( R(t, s) \) if it is the union of \( k \) round sets, \( k > 0 \), and a set of \( p \) points, where \( p \leq t - 1 - k(s - 1) \).

**Proof.** A \( t \)-subset has at least \( t - p \) of its points in the union of the \( k \) round sets. Since \( t - p \geq 1 + k(s - 1) \), at least one of the round sets must contain at least \( s \) of these \( t - p \) points. (Note: \( k = 0 \) would cover the case \( N < t \).)

\( C(t - s) \) is the case \( k = 1 \) in the lemma. Another special case is that with \( p = 0 \): all points are on \( k \) round sets, where \( k(s - 1) \leq t - 1 \) (equivalently: \( k < \frac{t - 1}{s - 1} \)).

In Figure 1 we see examples (with \( S \) the set of all circles and lines) of a 12-set with \( R(6, 4) \), a 10-set with \( R(8, 5) \), a 9-set with \( R(8, 5) \) and an 11-set with \( R(6, 4) \), all escaping the condition in the lemma.

![Figure 1](image-url)

A set satisfying the condition in the lemma, thus with \( R(t, s) \) in a rather trivial way, will be said to have property \( RR(t, s) \). If \( 2s \geq t + 2 \) we can only have \( RR(t, s) \) with \( k = 1 \), so if we have \( C(t - s) \).

Clearly \( R(t, s) \) entails \( R(t + 1, s) \) and, if \( s > 4 \), \( R(t, s - 1) \) and \( R(t - 1, s - 1) \). The same holds with \( RR \) instead of \( R \).

The first example above has \( R(6, 4) \) so it has \( R(7, 4) \). It has not \( RR(6, 4) \) but it has \( RR(7, 4) \). The third example has \( R(8, 5) \) but not \( RR(8, 5) \); it has \( RR(8, 4) \), however.

It is easy to construct examples for large \( t \):
Lemma 2. If $V_1$ is a set with $R(t_1, s)$ and $V_2$ is a set with $R(t_2, s)$, then $V_1 \cup V_2$ has $R(t_1 + t_2 - 1, s)$. The same holds with $RR$ instead of $R$.

Proof. The $R$-case: a $(t_1 + t_2 - 1)$-subset of $V_1 \cup V_2$ contains a $t_1$-subset of $V_1$ or it contains a $t_2$-subset of $V_2$, so it contains a round $s$-subset of $V_1$ or a round $s$-subset of $V_2$. The $RR$-case: if $V_i, i = 1, 2$, consists of $k_i$ round sets and a $p_i$-set with $p_i \leq t_i - 1 - k_i(s-1)$, then $V_1 \cup V_2$ consists of $k_1 + k_2$ round sets and a $(p_1 + p_2)$-set with $p_1 + p_2 \leq (t_1 + t_2 - 1) - 1 - (k_1 + k_2)(s-1)$.

Corollary 3. If $V$ is a set with $R(t_1, s_1)$ and $W$ is a set with $R(t_2, s_2)$, then $V \cup W$ has $R(t_1 + t_2 - 1, \min(s_1, s_2))$. The same holds with $RR$ instead of $R$.

In the $RR$ cases a certain reverse also holds:

Lemma 4. If $Q$ is a set with $RR(t, s)$, $t > 2s$, then i) $Q$ has $C(t - s)$ or ii) $Q$ is the union of at least 2 and at most $\left\lfloor \frac{t+1}{s+1} \right\rfloor$ round sets or iii) $Q$ is the union of a set $V$ with $RR(s+1, s)$ and a set $W$ with $RR(t-s, s)$.

Proof. $Q$ consists of $k$ round sets and a set $P$ with $p$ points, where $p \leq t - 1 - k(s-1)$ (and $k > 0$). If one of these round sets has less than $s$ points (and thus $k > 1$) we add its points to $P$ to get $k-1$ round sets and a set with at most $p + s - 1 \leq t - 1 - (k-1)(s-1)$ points. So we may suppose the round sets to have cardinality $\geq s$. If now $k = 1$ we have $C(t-s)$. Otherwise $Q$ is the union of at least 2 and at most $\left\lfloor \frac{t+1}{s+1} \right\rfloor$ round sets or we have $p > 0$. In the latter case let $V$ consist of one of the round sets and one point from $P$. Then $V$ has $C(1)$, i.e. $RR(s+1, s)$.

$W := Q - V$ consists of $k - 1$ ($\geq 1$) round sets and a set with $p - 1$ points. Now $p - 1 \leq t - 1 - k(s-1) - 1 = (t-s)-1-(k-1)(s-1)$ so if $|W| \geq t-s$ then $W$ has $RR(t-s, s)$; else $Q$ has $C(t-s)$.

(Case ii) in the lemma is not a subcase of iii): as a consequence of our convention $t \leq N$ the union of 4 round 6-sets with disjoint supports has $RR(21, 6)$ but is not the union of a set with $RR(7, 6)$ and a set with $RR(15, 6)$.

3. Trivial Cases

Proposition 5. Let $2s \geq t + 4$. Then every set $V$ with $|V| \geq t$ that has property $R(t, s)$ has property $C(t-s)$.

Proof. $V$ has a round $s$-subset $W$, say $W = \{1, 2, \ldots, s\}$. If $C(t-s)$ does not hold, $V$ has a $(t-s+1)$-subset $U = \{s+1, s+2, \ldots, t+1\}$ of points not on the support of $W$. The $t$-subset $T = (W - \{1\}) \cup U$ contains a round $s$-subset that can not contain 3 points of $W$, so it contains $\geq s-2$ points of $U$. Then $s-2 \leq |U| = t-s+1$, so $2s \leq t+3$, contradiction.

Proposition 6. Let $2s = t + 3$ and $s \geq 5$. Then every set $V$ with $|V| \geq t$ that has property $R(t, s)$ has property $C(t-s)$.
**Proof.** Suppose \( C(t - s) = C(s - 3) \) does not hold and take \( W, U \) and \( T \) as in the previous proof; \(|U| = s - 2 \geq 3\). \( T \) contains a round \( s \)-set \( S \) containing \( U \) and exactly 2 points of \( W \), say \( S = \{2, 3\} \cup U \). Likewise \( T' = (W \setminus \{2\}) \cup U \) contains a round \( s \)-set \( S' = \{v, w\} \cup U, \{v, w\} \neq \{2, 3\} \). So the support of \( U \) contains \( \geq 3 \) points of \( W \), so \( U \) on the support of \( W \), contradiction. \( \square \)

This deals with the case \( R(7, 5) \), mentioned in the Introduction.

**Proposition 7.** Let \( 2s = t + 2 \) and \( s \geq 6 \). Then every set \( V \) with \(|V| \geq t \) that has property \( R(t, s) \) has property \( C(t - s) \).

**Proof.** Suppose that \( C(t - s) = C(s - 2) \) does not hold and take \( W, U \) and \( T \) as in the previous proofs; \(|U| = s - 1 \geq 5\).

If \( U \) is not round, then we may suppose that the round \( s \)-subset in \( T \) is \( \{2, 3\} \cup (U \setminus \{t + 1\}) \). In \( T' = (W \setminus \{2\}) \cup U \) there is a round \( s \)-set of the form \( \{x, y\} \cup (U \setminus \{z\}) \) with \( \{x, y\} \neq \{2, 3\} \). Since \( U \setminus \{t + 1\} \) and \( U \setminus \{z\} \) are round but \( U \) is not, we have \( z = t + 1 \). The support of \( U \setminus \{t + 1\} \) thus contains \( \geq 3 \) points of that of \( W \). Contradiction as in the previous proof.

If \( U \) is round, then we see from \( T \) that at least one point of \( W \) is on the support of \( U \), say \( 2 \). With \( T' \) it then follows that there is a second point, say \( 1 \). If a point \( q \in V \setminus (W \cup U) \) would be on the support of \( W \), then \( R(t, s) \) would not hold, see \( \{q, 3, 4, \ldots, s\} \cup U \). If all points of \( V \setminus (W \cup U) \) would be on the support of \( U \), then \( C(s - 2) \) would hold. So there is a point \( q \in V \) not on the support of \( W \) or on the support of \( U \). But then we can, instead of \( U \), take the not round \( (U \setminus \{t + 1\}) \cup \{g\} \) and finish the proof as before. \( \square \)

The third example above has \( R(8, 5) \) and shows that the condition \( s \geq 6 \) can not be omitted.

Note that the propositions cover all cases with \( 2s \geq t + 2 \) (i.e. the cases where the properties \( RR(t, s) \) and \( C(t - s) \) coincide) except for \( (t, s) = (5, 4), (6, 4) \) or \( (8, 5) \). For the first two cases see the Introduction. The third case will be dealt with in Section 5.

**Proposition 8.** Let \( s \geq 7 \) and \( 2s - 1 \leq t \leq 3s - 8 \). Then every set with \( R(t, s) \) has \( RR(t, s) \).

**Proof.** Let \( V \) be such a set. Take a round \( s \)-set \( W = \{1, 2, \ldots, s\} \) in \( V \) and let \( C \) be its support. Let \( Q = V \setminus (V \cap C) \). If \(|Q| \leq t - s \) we have \( C(t - s) \), so let \(|Q| \geq t - s + 1 \).

1) We first prove that \( Q \) has property \( R(t - s + 1, s - 1) \). Let \( U = \{s + 1, s + 2, \ldots, t + 1\} \) be an arbitrary \( (t - s + 1) \)-subset of \( Q \). Suppose \( U \) contains no round \( (s - 1) \)-set. Let \( T = (W \cup U) \setminus \{1\} \) so \(|T| = t \). \( T \) contains a round \( s \)-set \( S \) for which we must have \(|S \cap W| \leq 2 \) and \(|S \cap U| \leq s - 2 \), so we can suppose that \( S = \{s - 1, s, s + 1, \ldots, s - 2\} \). Now let \( T' = (W \cup U) \setminus \{s\} \). For a round \( s \)-set \( S' \) in \( T' \) we again have \(|S' \cap W| = 2 \) but also \(|S' \cap \{s + 1, s + 2, \ldots, s - 2\}| \leq 2 \), otherwise the support of the \( (s - 2) \)-set \( \{s + 1, s + 2, \ldots, s - 2\} \) would contain another point of \( U \) or 3 points of \( W \). So \( S' \) must contain at least \( s - 4 \) points from \( \{2s - 1, 2s, \ldots, t + 1\} \). Since \( s - 4 > t - 2s + 3 \) this is impossible.
Now \(2(s+1) > (t-s+1)+5\) so \(Q\) has \(C(t-2s+2)\) by Proposition 5. So \(V\) is the union of a round set \(R_1\) with support \(C_1\) and with \(|R_1| \geq s\), a second round set \(R_2\) disjoint from \(C_1\) with support \(C_2\) and with \(|R_2| \geq s-1\) and a set \(R_3\) of at most \(t-2s+2\) points disjoint from \(C_1 \cup C_2\). If \(|R_3| \leq t-2s+1\) we have \(RR(t,s)\), since \(t-2s+1 = t-1-2(s-1)\), so now we assume that \(|R_3| = t-2s+2\).

3) Let \(X_i\) be an \((s-1)\)-set in \(R_i, i = 1, 2\). A round \(s\)-set in \(X\) can contain at most 2 points from \(X_i\) so it contains at least \(s-2-|1| = 3s-t-4 > 3\) points of \(X_2\). So its support is \(C_2\) and it thus contains at least one point of \(C_2 \cap X_1\). Let \(x\) be such a point. Likewise, after replacing \(x\) in \(X_1\) by a point of \(R_1 \setminus X_1\), we find a second point \(y\) in \(C_2 \cap X_1\).

4) If \(|R_2| > s-2\) we can make a \(t\)-set by taking \(s-1\) points from each of \(R_1\) and \(R_2\) and all points from \(R_3\) without using the (at most 2) points of \(R_1 \cap C_2\). This \(t\)-set does not contain a cyclic \(s\)-set, since \(2+2+(t-2s+2) = t-2s+6 < s\). So \(|R_2| = s-2\) and we have \(C(t-s)\) since \(s-2 + t-2s+2 = t-s\).

\[\Box\]

4. Large Round Sets

The theorems mentioned in the Introduction could suggest that a set with \(R(t,s)\), if sufficiently large, is trivial in the sense that it has \(RR(t,s)\). This is true for certain pairs \((t,s)\) as we shall show in the next section. On the other hand one can take a round set of arbitrary cardinality together with the 6 vertices of two squares that share a side to get a set with \(R(8,4)\) and not \(RR(8,4)\).

That at least large round subsets can not be avoided when we increase \(|V|\) can be shown using the Ramsey numbers \(Ram(p,q;4)\). Indeed, if \(|V| \geq Ram(n,t-s+4;4)\) for a set \(V\) with \(R(t,s)\), then \(V\) has an \(n\)-subset in which all 4-tuples are round, so that is itself round, or a \((t-s+4)\)-subset \(U\) in which no 4-tuple is round. The latter is impossible: adding \(s-4\) points to \(U\) would give a \(t\)-tuple which would contain a round \(s\)-tuple with at least 4 points in \(U\). We want a more concrete bound, however.

**Theorem 9.** Let \(V\) be a set with property \(R(t,s)\), let \(N = |V|\) and let \(d\) and \(q\) be integers with \(3 < d < q < s\). Then \(V\) contains a round \(n\)-set if

\[
(N-d)(N-d-1) \cdots (N-q+1) \binom{s}{q} > \left( \binom{n-d}{q-d} - 1 \right) \binom{t}{q} (q-d)!.
\]

**Proof.** We shall prove that there is a \(d\)-subset of \(V\) that is contained in \(\binom{n-d}{q-d}\) round \(q\)-subsets of \(V\); their union then is a round set of cardinality \(\geq n\). Let \(r\) be the number of round \(q\)-subsets in \(V\). Suppose every \(d\)-subset of \(V\) is contained in at most \(m = \binom{n-d}{q-d} - 1\) of these subsets. Counting in two ways pairs \(\langle Q,T \rangle\) with \(Q\) a round \(q\)-set, \(T\) a \(t\)-set and \(Q \subset T \subset V\), we find, since every \(t\)-subset contains a round \(s\)-set,

\[
r \binom{N-q}{t-q} \geq \binom{N}{t} \binom{s}{q}.
\]

Counting in two ways pairs \(\langle D,Q \rangle\) with \(D\) a \(d\)-set, \(Q\) a round \(q\)-set and \(D \subset Q \subset V\),
we find
\[ r \left( \frac{q}{d} \right) \leq \binom{N}{d}. \] (2)

From (1) and (2):
\[ \binom{N}{t} \binom{s}{q} \frac{q}{d} \leq \binom{N}{d} \binom{N - q}{t - q}, \]
and so:
\[ (N - d)(N - d - 1) \cdots (N - q + 1) \binom{s}{q} \leq \binom{t}{q} ((q - d)!), \]
contradicting the condition in the theorem. □

For many triples \((t, s, n)\) the lowest bound for \(N\) will appear if we take \(d = 3\) and \(q = s\); if \(s = 4\) this is the only choice. So we state:

**Corollary 10.** A set with \(R(t, s)\) and with \(N\) points contains a round \(n\)-set if
\[ \binom{N - 3}{s - 3} > \left( \frac{n - 3}{s - 3} - 1 \right) \binom{t}{s}. \]

In particular a set with \(R(t, 4)\) and with \(N\) points contains a round \(n\)-set if
\[ N > 3 + (n - 4) \binom{t}{4}. \]

However to guarantee a round 100-set in a set \(V\) with \(R(14, 7)\), for instance, we need \(|V| \geq 736\) according to the corollary but the theorem with \(d = 3\) and \(q = 6\) gives the better bound \(|V| \geq 729\).

If we take \(d = q - 1\) in the theorem we get a simpler, but not a better, bound:
\[ N > q - 1 + (n - q) \binom{t}{q} / \binom{N}{q}. \] The right hand side increases with \(q\) so we better take \(q = 4\): \(N > 3 + (n - 4) \binom{t}{4} / \binom{N}{4}\). By using that \(R(t, s)\) entails \(R(t - s - 4, 4)\): \(N > 3 + (n - 4) \binom{t - s + 4}{4}\), but \(\binom{t - s + 4}{4} \geq \binom{4}{4}\) if \(s > 4\).

Probably our bound is far too strong. Indeed, whereas, as mentioned in the Introduction, a set \(V\) with \(R(5, 4)\) contains a round \((|V| - 1)\)-set if \(|V| \geq 7\), the theorem only promises a round \(k\)-set if \(|V| \geq 5k - 16\). By the theorem with \(d = 3\) and \(q = 6\) a 29-set with \(R(10, 7)\) has a round 12-set, but by Proposition 5 this is already true for a 15-set. The proof in [1] that a set \(V\) with \(R(6, 4)\) and \(|V| \geq 78\) has \(C(2)\) is based on a lemma (Lemma 2) stating that such a set contains a round 9-set. The corollary guarantees a round 9-set if \(|V| > 78\).

The following theorem however will tell us that \(|V| \geq 78\) is sufficient, although the bound in it again is not very explicit.
**Theorem 11.** Let $V$ be a set with property $R(t, s)$ and $|V| \geq N$. Let $3 < q \leq s$. Then $V$ contains a round subset of cardinality

$$\left\lceil \left( \begin{array}{c} N \cr q \end{array} \right) \left( \begin{array}{c} s \cr q \end{array} \right) \left( \begin{array}{c} t \cr q \end{array} \right)^{-1} \right\rceil \left[ \begin{array}{c} q - 1 \\ N - 1 \\ N - 2 \\ \vdots \\ 2 \\ N - q + 2 \end{array} \right] + q - 1.$$

**Proof.** See (1) in the proof of Theorem 9: the number of round $q$-subsets is

$$\left( \begin{array}{c} N \cr q \end{array} \right) \left( \begin{array}{c} s \cr q \end{array} \right) \left( \begin{array}{c} t \cr q \end{array} \right)^{-1} \geq \left( \begin{array}{c} N - q \cr q - 1 \end{array} \right) \left( \begin{array}{c} s - q \cr q - 1 \end{array} \right) \left( \begin{array}{c} t - q \cr q - 1 \end{array} \right)^{-1},$$

so it is at least $T_q := \left\lceil \left( \begin{array}{c} N \cr q \end{array} \right) \left( \begin{array}{c} s \cr q \end{array} \right) \left( \begin{array}{c} t \cr q \end{array} \right)^{-1} \right\rceil$. So there is a $T_q \times N$ 0,1-matrix $M_0$ of which the rows are the characteristic vectors of different round $q$-subsets. Since that matrix contains $T_q \cdot q$ 1’s, there is a column, say the first, with $\geq T_{q-1} := \left\lceil \frac{T_q}{N-1} \right\rceil$ 1’s. Deleting the rows with a 0 in the first column and then also the first column we get a submatrix $M_1$ with $\geq T_{q-2} := \left\lceil \frac{T_{q-1}}{N-2} \right\rceil$ 1’s. Continuing in the same way we finally find a submatrix $M_{q-1}$ with $T_1 := \left[ \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ \left( \begin{array}{c} N \cr q \end{array} \right) \left( \begin{array}{c} s \cr q \end{array} \right) \left( \begin{array}{c} t \cr q \end{array} \right)^{-1} \right] \left[ \begin{array}{ccc} q - 1 \\ N - 1 \\ N - 2 \\ \vdots \\ 2 \\ N - q + 2 \end{array} \right]$ rows each containing a 1 in a different column. They correspond to $T_1$ round $q$-subsets sharing $q - 1$ ($\geq 3$) points. The union of these sets is a round $T_1 + q - 1$ set. □

Here also it is not apparent what is the best choice for $q$. For instance for a 100-set with $R(20, 10)$ the theorem with $q = 10$ guarantees a round 10-subset, whereas $q = 4$ or $q = 8$ promise only a round 8-subset. With $R(14, 7)$ and $N = 50$ the best choice is $q = 7$ (a round 7-set), but for $N = 120$ the best choice is $q = 4$ (a round 8-set). A less precise proof, omitting the inner “ceil”’s, yields the cardinality

$$\left\lceil \left( \begin{array}{c} N - q + 1 \cr q \end{array} \right) \left( \begin{array}{c} s \cr q \end{array} \right) \left( \begin{array}{c} t \cr q \end{array} \right)^{-1} \right\rceil + q - 1,$$

so a round $n$-set if $N \geq (n - q) + (n - q) + (n - q) + \cdots + (n - q) + q - 1$; this is precisely the bound we get by taking $d = q - 1$ in Theorem 9, what suggests that generally as with that theorem the best choice is $q = 4$, but also that Theorem 9 will give a better result if $s > 4$. But if $s = 4$ using the ceil’s we may gain a little. For instance with $R(6, 4)$ a round set of 9 points is guaranteed by Theorem 9 if $N \geq 79$ and by Theorem 11 if $N \geq 78$. With $R(9, 5)$ a round set of 6 points asks for $N \geq 144$ and $N \geq 143$, respectively.

5. Some Particular Cases

$R(7, 4)$ is the “smallest” case in which $k$ (as in Lemma 1) can be 2. The second example in Figure 1 has $R(7, 4)$ with $N = 10$, but not $RR(7, 4)$.

**Proposition 12.** A set with $R(7, 4)$ having a round 27-subset or a cardinality $\geq 809$ has $RR(7, 4)$.

**Proof.** Such a set $V$ contains a round subset with 27 points, by Corollary 10. Let $C$ be its support. Let $P = V \cap C$, so $|P| \geq 27$, and $Q = V - P$. If $Q$ is a round set we are ready. If not, then $Q$ has a non-round subset $T = \{A, B, C, D\}$. A round set through 3 points of $T$ contains at most 2 points of $P$, so $P$ has a subset $U$ of (at least) 19 points none of which form a round set with 3 points from $T$. This
gives us \( (\begin{array}{l} 9 \end{array}) = 969 \) 7-tuples \( \{i,j,k, A, B, C, D\} \) with \( \{i,j,k\} \subset U \), each containing a round 4-tuple. Such a 4-tuple must contain 2 points from \( U \) and 2 points from \( T \). Since in \( T \) there are only 6 pairs, there is a pair, \( \{A,B\} \) say, that is part of a round 4-tuple in \( \geq 969/6 \), so in at least 162, of our 7-tuples. But for instance \( \{1,2, A, B\} \) and \( \{1,3, A, B\} \) can not both be round, since then \( \{1,2,3, A, B\} \) would be round. So there are at most 9 round 4-tuples \( \{i,j,A,B\} \) with \( i,j \in U \), and one of them must occur in \( \geq 162/9 = 18 \) of our 7-tuples. Since, however, for given \( i \) and \( j \) there are only \( 17 \) triples \( \{i,j,k\} \) in \( U \) this is impossible. □

With the stronger property \( R(8,5) \) a smaller cardinality is sufficient:

**Proposition 13.** A set with \( R(8,5) \) having a round 7-subset or a cardinality \( \geq 28 \) has \( C(3) \).

**Proof.** By Corollary 10 such a set \( V \) has a round subset \( P \) with 7 points, 1, 2, ..., 7, say. Let \( C \) be its support and \( Q \) the set of points of \( V \) not on \( C \). Suppose \( |Q| > 3 \) and let \( U = \{A, B, C, D\} \) be a 4-subset of \( Q \). If \( U \) is round its support has at most 2 points in \( P \), say 6 and 7, if any. But then there is no round 5-set in \( \{1,2,3,4, A, B, C, D\} \), so \( U \) is not round. The triples from \( U \) determine 4 circles (or 3 circles and one line) of which no two can pass through the same point of \( P \) (if, e.g., \( \{A,B,C,1\} \) and \( \{A,B,D,1\} \) would be round, then \( \{A,B,C,D,1\} \) would be round). So at most 3 of these circles contain a pair of points of \( P \). Deleting from \( P \) one point of every such pair we see that there is a 4-tuple in \( P \) having no 2 of its points on one of these 4 circles; let \( \{1,2,3,4\} \) be such a 4-tuple. Then \( \{1,2,3,4, A, B, C, D\} \) would not contain a round 5-tuple. So \( |Q| < 4 \) and we have \( C(3) \). □

One may note that the proof only uses \( R(8,5) \) and the existence of a round 7-set, and that \( R(9,6) \) entails \( R(8,5) \) and guarantees a round 7-set already if there are 16 points. But the case \( R(9,6) \) is better treated by Proposition 6.

**Proposition 14.** A set with \( R(9,5) \) having a round 12-subset or a cardinality \( \geq 98 \) has \( RR(9,5) \).

**Proof.** Such a set \( V \) has a round subset \( P \) with 12 points by Corollary 10. Let \( C \) be its support and \( Q \) the set of points of \( V \) not on \( C \). We are ready if \( |Q| < 5 \), so we suppose \( |Q| \geq 5 \). First assume \( Q \) has a 5-subset \( T = \{A, B, C, D, E\} \) not containing a round 4-tuple. There are \( \binom{12}{4} = 495 \) 9-tuples \( \{h,i,j,k,A,B,C,D,E\} \). The round 5-set in such a 9-set can not have 3 points in \( P \) since \( C \cap T = \emptyset \), nor can it have 4 points in \( T \), so it is a set \( \{a,b,X,Y,Z\} \) with \( a,b \in P \) and \( X,Y,Z \in T \). Since \( \{a,b,X,Y,Z\} \) and \( \{c,d,X,Y,Z\} \), \( c,d \in P \), \( \{a,b\} \neq \{c,d\} \), can not both be round, there are at most 10 such round 5-tuples. So one of these must be in \( \geq 495/10 \), so in at least 50 of our 9-tuples. But only \( \binom{10}{2} = 45 \) of these contain a given pair \( a,b \in P \) and thus our assumption is false. So all 5-tuples in \( Q \) contain a round 4-tuple. If a 5-tuple contains two round 4-tuples it is itself round, and if all 5-tuples in \( Q \) are round \( Q \) itself is round, so then \( V \) lies on two circles and we have \( RR(9,5) \). Otherwise we have a 5-subset \( T = \{A,B,C,D,E\} \subset Q \) in which \( U = \{A,B,C,D\} \) is round and the other 4-tuples are not. The support of \( U \) has at most 2 points in common with \( P \), so in \( P \)

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we can take a subset \( P' \) of 10 points not on the support of \( U \). There are \( \binom{10}{4} = 210 \) 9-tuples \( \{h, i, j, k, A, B, C, D, E\} \) with \( \{h, i, j, k\} \subset P' \). Their round 5-subsets can not contain 3 points of \( P' \) and neither 3 points of \( U \), so are of type \( \{a, b, X, Y, E\} \) with \( a, b \in P' \) and \( X, Y \in U \). As above a fixed triple \( \{X, Y, E\} \) can serve only once, so there are at most 6 such round 5-subsets; therefore one of them must be contained in \( \geq 210/6 = 35 \) of our 9-sets. But since for given \( a, b \) there are only \( \binom{5}{2} = 28 \) 9-sets containing \( a \) and \( b \) this is impossible. □

6. Remarks on the 3-Dimensional Case

Generalisation to Euclidean 3-space is not a simple matter. The reason is that, whereas two circles in the plane are the same if they have 3 points in common, two spheres in 3-space sharing 4 (or even more) points need not be the same.

Let a set (always: of points in Euclidean 3-space) have property \( S(t, s) \) if every \( t \)-subset contains a spherical \( s \)-subset. Again we have the “standard” examples like: a set has \( S(7, 5) \) if all points with the exception of at most two are on a sphere. But there are other more or less trivial configurations: take a sphere \( Q \), points \( A, B, C \) not on \( Q \) and 3 spheres each through \( A, B \) and \( C \) and intersecting \( Q \) in (disjoint) circles. Every set containing \( A, B \) and \( C \) and moreover only points of these circles has \( S(7, 5) \). If this set has cardinality \( \geq 13 \) one of the circles contains 4 points of it, and we could eliminate this example by forbidding concyclic 4-tuples.

When we restrict ourselves to sets without concyclic 4-tuples (which includes sets in which every 4-tuple is in general position) it is not difficult to prove analogues of some of the above results. We give some examples, with condensed or omitted proofs. The proofs of the following three theorems are almost copies of those of the Propositions 5, 6 and 7.

**Proposition 15.** Let \( 2s \geq t + 5 \). A set with \( S(t, s) \) and containing no concyclic 4-set has all points except for at most \( t - s \) on a sphere.

**Proposition 16.** Let \( 2s = t + 4 \) and \( s \geq 7 \). A set with \( S(t, s) \) and containing no concyclic 4-set has all points except for at most \( t - s \) on a sphere.

**Proposition 17.** Let \( 2s = t + 3 \) and \( s \geq 8 \). A set with \( S(t, s) \) and containing no concyclic 4-set has all points except for at most \( t - s \) on a sphere.

**Proof.** Let \( V \) be such a set, let \( S \) be the support of a largest spherical subset of \( V \) and take \( W = \{1, 2, \ldots, s\} \) on \( V \cap S \). Suppose we have a subset \( U = \{s + 1, s + 2, \ldots, t + 1\} \) of points not on \( S \); \( |U| = s - 2 \geq 6 \). Suppose \( U \) is not spherical, then \( (W - \{1\}) \cup U \) contains a spherical \( s \)-subset with 3 points in \( W \) and \( s - 3 \) points in \( U \), say \( \{1, 2, 3\} \cup U' \). The spherical \( s \)-set in \( (W - \{2\}) \cup U \) then contains an \( s - 3 \)-subset \( U'' \) of \( U \) with \( U'' \neq U' \). But \( |U' \cap U''| = s - 4 \geq 4 \) and \( U' \cup U'' = U \), so \( U \) is spherical. But then \( (W - \{1\}) \cup U \) shows that \( W \) has at least two points on the support of \( U \), say 1 and 2. The same goes for \( (W - \{2\}) \cup U \), so we may suppose that 1, 2 and 3 are on the support of \( U \). Since \( U \cup \{1, 2, 3\} \) is spherical and has \( s + 1 \) points, by definition of \( S \) there is a point \( x \) in \( V \cap S, x \not\in W \). Now \( \{x, 3, 4, \ldots, s\} \cup U \) also has two points on the support of \( U \). So we have 4 points on the intersection of \( S \) and the support of \( U \), a contradiction. □
Theorem 18. Of a set with $S(t,t - 1)$ and containing no concyclic 4-set all points except for at most one lie on a sphere.

Proof. The set has $S(6,5)$ (moreover the cases with $t > 6$ are covered by Proposition 15). Take a spherical 5-set $\{1,2,3,4,5\}$. Suppose the set has two points $A, B$ not on its support. The 6-set $\{1,2,3,4,A,B\}$ yields a spherical 5-set $\{1,2,3,A,B\}$ (after renumbering $\{1,2,3,4\}$). The 6-set $\{2,3,4,5,A,B\}$ yields a spherical 5-set $\{3,4,5,A,B\}$ ($\{2,3,4,A,B\}$ is impossible). Now $\{1,2,4,5,A,B\}$ has no spherical 5-set, contradiction. □

As to the analogue of Theorem 9 we restrict ourselves to the case $S(7,5)$.

Theorem 19. A set of cardinality $N$ with $S(7,5)$ and containing no concyclic 4-set contains a spherical $m$-set ($m > 5$) if $N > 21m - 100$.

Proof. Let $r$ be the number of spherical quintuples in the set. The number of pairs $(Q,S)$ with $Q$ a spherical quintuple, $S$ a 7-subset and $Q \subseteq S$ is $r\binom{N-5}{2}$, and also at least $\binom{N}{7}$. So

$$\binom{N}{7} \leq r\binom{N-5}{2}.$$  \hfill (*)

The number of pairs $(F,Q)$ with $F$ a 4-subset, $Q$ a spherical quintuple and $F \subseteq Q$ is $5r$. If every 4-subset would be in no more than $m - 5$ spherical quintuples, then we would have

$$5r \leq (m - 5)\binom{N}{4}.$$  \hfill (**)

From (*) and (**) however it would follow that

$$N \leq 21m - 101.$$  

So there is a (non-planar) 4-subset belonging to at least $m - 4$ spherical quintuples. Their union is a spherical $m$-set. □

References
