RESIDUAL PROPERTIES
IN THE THEORY OF POLYNOMIAL MAPS

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Abstract

This paper provides tools to reduce questions about polynomial maps over a ring $R$ to questions about polynomial maps over the residue fields $R_p/pR_p$ of the ring $R$. As an application of this theory, the Abhyankar-Moh Theorem is generalized to arbitrary $\mathbb{Q}$-algebras.

In order to solve questions about polynomial maps, derivations, or other objects in the theory of polynomial maps, over arbitrary commutative rings $R$, it is often convenient to reduce such questions to the case that $R$ is a field. For example, if one has a polynomial map $F$ over a domain $R$ with $\det J(F) = 1$ and one wants to know if it is invertible, then one only needs to check if $F$ is invertible over the quotient field $Q(R)$ of $R$.

This paper tries to reduce such questions over a ring $R$ to questions about fields by looking at the residue fields $R_p/pR_p$ for all prime ideals $p$ of $R$.

Section 1 gives an inventory of these so-called “residual properties”. There are two main results here. The first one is that “being locally nilpotent” is a residual property of a derivation for a Noetherian ring $R$ and in dimension two also for arbitrary $\mathbb{Q}$-algebras (Propositions 4 and 6). The second one is that “being a coordinate” is a residual property of a polynomial over an arbitrary $\mathbb{Q}$-algebra in dimension two (Proposition 10). This generalizes a result from Bhatwadekar and Dutta in [BD93].

Section 2 shows the power of these properties by generalizing the Abhyankar-Moh Theorem (see [AM75]) to arbitrary $\mathbb{Q}$-algebras.

1 Residual Properties

Notation 1. Let $R$ be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $a$ be an ideal of $R[X]$ and let $p$ be a prime ideal of $R$. Then $a_p$ denotes the ideal generated by the image of the ideal $a$ in $R_p/pR_p[X]$ under the map induced by the natural homomorphism $R \rightarrow R_p/pR_p$.

A similar notation will be used in the sequel be used for polynomials, polynomial maps, and derivations.

\footnote{In this paper, all rings will be commutative and have a unit element.}
**Proposition 2.** Let \( R \) be a ring, \( n \in \mathbb{N}^* \), and \( R[X] := R[X_1, \ldots, X_n] \). Let \( \mathfrak{a} \) be an ideal of \( R[X] \). Then the following two statements are equivalent:

1. \( \mathfrak{a} = R[X] \);
2. for every \( \mathfrak{p} \in \text{Spec}(R) \), \( \tilde{\mathfrak{a}}_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X] \).

**Proof.** The implication 1 \( \Rightarrow \) 2 is trivial. So assume that for every \( \mathfrak{p} \in \text{Spec}(R) \), \( \tilde{\mathfrak{a}}_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X] \).

Assume that \( \mathfrak{a} \neq R[X] \). Then there is some maximal ideal \( \mathfrak{m} \) of \( R[X] \) such that \( \mathfrak{a} \subseteq \mathfrak{m} \). Let \( \mathfrak{p} := \mathfrak{m} \cap R \). This is a prime ideal of \( R \).

Using the natural isomorphism between \( R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \) and \( Q(R/\mathfrak{p}) \), one can easily see that \( 1 \in \tilde{\mathfrak{a}}_\mathfrak{p} \) means that there are an \( r \in R \setminus \mathfrak{p} \), polynomials \( g_1, \ldots, g_s \in R[X] \), and polynomials \( f_1, \ldots, f_s \in \mathfrak{a} \) such that

\[
r \equiv g_1f_1 + \cdots + g_sf_s \pmod{\mathfrak{p}[X]}.\]

Because \( \mathfrak{a} \subseteq \mathfrak{m} \) and \( \mathfrak{p} \subseteq \mathfrak{m} \) this implies, however, that \( r \in \mathfrak{m} \). Because also \( r \in R \), this contradicts the fact that \( r \not\in \mathfrak{p} = \mathfrak{m} \cap R \).

Therefore \( \mathfrak{a} = R[X] \). \( \square \)

The following lemma is Lemma 2.11.5 from [Ess00]. It shows that in order to check if an \( R \)-derivation \( D \) of \( R[X] \) is locally nilpotent, the ring \( R \) can assumed to be reduced.

**Lemma 3.** Let \( R \) be a ring, \( n \in \mathbb{N} \), and \( R[X] := R[X_1, \ldots, X_n] \). Let \( D \in \text{Der}_R(R[X]) \). Let \( \eta \) be the nilradical of \( R \) and denote by \( D/\eta \) the derivation on \( R/\eta[X] \) induced by \( D \). Assume that \( D/\eta \) is locally nilpotent. Then \( D \) is locally nilpotent as well.

**Proof.** Let \( R' \) be the subring of \( R \) generated by all coefficients appearing in the polynomials \( D(X_1), \ldots, D(X_n) \). Then \( R' \) is Noetherian and \( D \) restricts to a derivation \( D' \) on \( R'[X] \). Note that \( D \) is locally nilpotent if and only if \( D' \) is. Also note that the nilradical \( \eta' \) of \( R' \) equals \( \eta \cap R' \) and that hence \( D/\eta \) is locally nilpotent if and only if \( D'/\eta' \) is. Therefore it is possible to assume, without loss of generality, that the ring \( R \) is Noetherian.

By induction on \( n \) it will follow that for every \( n \in \mathbb{N} \) and every \( g \in R[X] \), there exists an \( N \in \mathbb{N} \) such that \( D^N(g) \in \eta^n[X] \). For \( n = 1 \), this is just the assumption that \( D/\eta \) is locally nilpotent. Now assume that the claim holds for \( n \) and consider \( g \in R[X] \). By induction hypothesis \( D^N(g) \in \eta^n[X] \) for some \( N \in \mathbb{N} \), say \( D^N(g) = \sum_{a \in A} c_a X^a \), for certain \( c_a \in \eta^n \). Since \( D/\eta \) is locally nilpotent, there are \( M_a \in \mathbb{N} \) such that \( D^{M_a}(X^a) \in \eta[X] \). Taking \( M := N + \max_{a \in A} M_a \), it follows that \( D^M(g) \in \eta^{n+1}[X] \).

Because \( R \) is Noetherian, its nilradical \( \eta \) is finitely generated and hence there is an \( e \in \mathbb{N} \) such that \( \eta^e = \{0\} \). Consequently, for every \( g \in R[X] \) there is an \( N \in \mathbb{N} \) such that \( D^N(g) \in \eta^e[X] = \{0\} \). Therefore \( D \) is locally nilpotent. \( \square \)
Proposition 4. Let $R$ be a Noetherian ring, $n \in \mathbb{N}^*$, and consider the polynomial ring $R[X] := R[X_1, \ldots, X_n]$. Let $D \in \text{Der}_R(R[X])$. Then the following two statements are equivalent:

1. $D$ is locally nilpotent;

2. for every $p \in \text{Spec}(R)$, the derivation $\tilde{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X])$ is locally nilpotent.

Proof. The implication $1 \Rightarrow 2$ is clear. So assume that for every $p \in \text{Spec}(R)$, the derivation $\tilde{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X])$ is locally nilpotent.

Let $p \in \text{Spec}(R)$. Consider the ring $R/p$ and the derivation $D/p$ induced by $D$ on $R/p[X]$. Because $Q(R/p) = R_p/pR_p$ and the derivation $\tilde{D}_p$ on $R_p/pR_p[X]$ is locally nilpotent, $D/p$ is locally nilpotent as well.

So $D/p$ is locally nilpotent for all $p \in \text{Spec}(R)$.

Since $R$ is Noetherian, its nilradical $\eta$ is a finite intersection of prime ideals, say $\eta = p_1 \cap \cdots \cap p_s$. Because $D/p_i$ is locally nilpotent for every $i \in \{1, \ldots, s\}$, $D/\eta$ is locally nilpotent too. Namely, let $g \in R[X]$. Then there is an $N_i \in \mathbb{N}$ such that $D^{N_i}(g) \in p_i$, for every $i \in \{1, \ldots, s\}$. Taking $N := \max_{i \in \{1, \ldots, s\}} N_i$ it follows that $D^N(g) \in p_1 \cap \cdots \cap p_s = \eta$.

By Lemma 3, $D$ is locally nilpotent. \qed

In dimension two, the condition that the ring $R$ is Noetherian can be avoided. In order to prove this, the following result is needed ([Ess00], Theorem 1.3.49).

Lemma 5. Let $k$ be a field of characteristic 0 and take $D \in \text{Der}_k(k[X,Y])$. Assume that $D \neq 0$ and let

$$d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}.$$

(Here, by convention, the degree of 0 is taken to be $-\infty$.) Then $D$ is locally nilpotent if and only if $D^{d+2}(X) = D^{d+2}(Y) = 0$. \qed

Proposition 6. Let $R$ be a Q-algebra and $D \in \text{Der}_R(R[X,Y])$. Then the following two statements are equivalent:

1. $D$ is locally nilpotent;

2. for every $p \in \text{Spec}(R)$, the derivation $\tilde{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X,Y])$ is locally nilpotent.

Proof. The implication $1 \Rightarrow 2$ is once again clear, so assume that for every $p \in \text{Spec}(R)$, the derivation $\tilde{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X,Y])$ is locally nilpotent. Take $d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}$.

Let $p \in \text{Spec}(R)$. Just as in the proof of Proposition 4, consider the ring $R/p$ and the locally nilpotent derivation $D/p$ induced by $D$ on $R/p[X,Y]$ and on
$Q(R/p)[X,Y]$. Note that $Q(R/p)$ is a field of characteristic 0, since $R$ is a $ℚ$-algebra. Hence, by the previous lemma,

$$(D/p)^{d+2}(X) = (D/p)^{d+2}(Y) = 0,$$

or, differently said, $D^{d+2}(X) \in p[X,Y]$ and $D^{d+2}(Y) \in p[X,Y]$. Hence $D^{d+2}(X) \in \bigcap_{p \in \text{Spec}(R)} p = \eta$, the nilradical of $R$, and also $D^{d+2}(Y) \in \eta$. This means that $D/\eta$ is locally nilpotent and hence, by Lemma 3, $D$ is locally nilpotent too.

In dimension two, the concept of coordinate over a $ℚ$-algebra is also a residual concept. In order to prove this, a characterization of coordinates in terms of locally nilpotent derivations is needed. The general form presented here as Proposition 11 comes essentially from [BEM99].

**Definition 7.** Let $R$ be a ring and $n \in \mathbb{N}^*$. A polynomial $f \in R[X] := R[X_1, \ldots, X_n]$ is called a coordinate over $R$ if it is a component of a polynomial automorphism over $R$, i.e., if there are polynomials $f_1, \ldots, f_{n-1} \in R[X]$ such that $(f_1, \ldots, f_{n-1}, f) \in \text{Aut}_R(R[X])$.

**Definition 8.** Let $R$ be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $D$ be an $R$-derivation of $R[X]$. A polynomial $s \in R[X]$ is called a slice of $D$ if $D(s) = 1$.

The following lemma is a part of Theorem 3.7 of [BEM99].

**Lemma 9.** Let $R$ be a $ℚ$-algebra. Then any locally nilpotent $R$-derivation on $R[X,Y]$ with divergence 0 (i.e., with $\partial_X(D(X)) + \partial_Y(D(Y)) = 0$) and 1 in the ideal generated by $D(X)$ and $D(Y)$ has a slice. □

**Lemma 10.** Let $R$ be a $ℚ$-algebra, $f \in R[X,Y]$, and let $D$ be the $R$-derivation $f_Y\partial_X - f_X\partial_Y$ on $R[X,Y]$. Assume that $f$ is a coordinate in $R[X,Y]$. Then $D$ is locally nilpotent and has a slice.

**Proof.** Let $g \in R[X,Y]$ be a polynomial such that $(f,g)$ is an invertible polynomial map over $R$. Let $\eta$ denote the nilradical of $R$. To avoid notational clutter, reduction modulo this nilradical $\eta$ of $R$ or modulo the ideal $\eta[X,Y]$ of $R[X,Y]$ will be denoted by an overline.

Then $(\bar{f}, \bar{g})$ is an invertible polynomial map over $\bar{R}$ and hence $\bar{R}[X,Y] = \bar{R}[\bar{f}, \bar{g}]$. Now note that

$$\bar{D}(\bar{g}) = \det J(\bar{f}, \bar{g}) \in \bar{R}[X,Y]^* = \bar{R}^*$$

and so $\bar{D}^2(\bar{g}) = 0$. Also $\bar{D}(\bar{f}) = 0$ and therefore $\bar{D}$ is locally nilpotent. By Lemma 3, $D$ is locally nilpotent too.

So in view of Lemma 9, the only thing left to show is that 1 is an element of the ideal generated by $D(X)$ and $D(Y)$. Now $\det J(f,g) \in R[X,Y]^*$. So $g_X f_Y - g_Y f_X$ is invertible. Hence the ideal generated by $D(X) = f_X$ and $D(Y) = -f_X$ contains an invertible element and consequently contains 1. Therefore, as observed, $D$ has a slice. □
Proposition 11. Let $R$ be a $\mathbb{Q}$-algebra, $f \in R[X,Y]$, and $D := f_Y \partial_X - f_X \partial_Y \in \text{Der}_R(R[X,Y])$. Then the following three statements are equivalent:

1. $D$ is locally nilpotent and $(f_X, f_Y) = R[X,Y]$;
3. $f$ is a coordinate.

Proof. The equivalence of 1 and 2 is Theorem 3.7 of [BEM99] and the equivalence of 2 and 3 follows from Proposition 2.1 of [Wri81] and Lemma 10. □

Now it is possible to prove that “being a coordinate” is a residual property for arbitrary $\mathbb{Q}$-algebras. This extends the result from [BD93], which proves the following statement for a Noetherian domain of characteristic 0 that either contains $\mathbb{Q}$ or for which $R/\eta$ is seminormal.

Proposition 12. Let $R$ be a $\mathbb{Q}$-algebra and $f \in R[X,Y]$. Then the following two statements are equivalent:

1. $f$ is a coordinate over $R$ in $R[X,Y]$;
2. for every $p \in \text{Spec}(R)$, $\tilde{f}_p$ is a coordinate over $R_p / pR_p$ in $R_p / pR_p[X,Y]$.

Proof. This is now an easy consequence of the equivalence 1 $\iff$ 3 from the previous proposition and from Propositions 2 and 6. □

The condition that $R$ is a $\mathbb{Q}$-algebra cannot simply be dropped in this proposition. For Bhatwadelkar and Dutta have constructed the following example in [BD93]. Take $R := \mathbb{Z}[2\sqrt{2}]$ and take

$$f := X - 2Y(\sqrt{2}X - Y^2) + \sqrt{2}(\sqrt{2}X - Y^2)^2 - \sqrt{2}(Y - \sqrt{2}(\sqrt{2}X - Y^2))^4.$$ 

Then $\tilde{f}_p$ is a coordinate over $R_p / pR_p$, for every prime ideal $p$ of $R$ (there are only two), but $f$ itself is not a coordinate over $R$.

2 The Abhyankar-Moh Theorem

Using the theory in the previous section, the Abhyankar-Moh Theorem ([AM75], Theorem 1.1) can be generalized to arbitrary $\mathbb{Q}$-algebras.

The Abhyankar-Moh Theorem states the following.

Theorem 13. Let $k$ be a field of characteristic 0 and let $f, g \in k[T]$. $f, g \neq 0$. Assume that $k[f, g] = k[T]$. Then $\deg(f) | \deg(g)$ or conversely.

(Actually, the theorem as formulated in [AM75] is slightly more general. It also holds for $\text{char}(k) \neq 0$ provided that either $\deg(f)$ or $\deg(g)$ is not divisible by $\text{char}(k)$. For the present situation, that information is not needed.) It can also be formulated in the following way ([AM75], Theorem 1.2).
Theorem 14. Let $k$ be a field of characteristic $0$ and let $f \in k[X,Y]$. Assume that $k[X,Y]/(f) \cong k[T]$. Then $f$ is a coordinate in $k[X,Y]$.

This is the formulation that will actually be generalized.

Theorem 15. Let $R$ be a $\mathbb{Q}$-algebra and let $f \in R[X,Y]$. Assume that $R[X,Y]/(f) \cong R[T]$. Then $f$ is a coordinate in $R[X,Y]$.

Proof. Let $D$ be the derivation $f_Y \partial_X - f_X \partial_Y$ on $R[X,Y]$.

Let $p \in \text{Spec}(R)$. Then $R_p / pR_p[X,Y]/(\tilde{f}_p) \cong R_p / pR_p[T]$. Since $R$ is a $\mathbb{Q}$-algebra, $R_p / pR_p$ is a field of characteristic $0$. So Theorem 14 implies that $\tilde{f}_p$ is a coordinate over $R_p / pR_p$ in $R_p / pR_p[X,Y]$.

Now Proposition 12 implies that $f$ itself is a coordinate over $R$.

Note that [BD93] also gives such a generalization of the Abhyankar-Moh Theorem, but only for Noetherian domains $R$ of characteristic $0$ which either contain $\mathbb{Q}$ or which satisfy that $R/\eta$ is seminormal.

References


