RESIDUAL PROPERTIES
IN THE THEORY OF POLYNOMIAL MAPS

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Abstract

This paper provides tools to reduce questions about polynomial maps over a
ring $R$ to questions about polynomial maps over the residue fields $R_p/pR_p$ of
the ring $R$. As an application of this theory, the Abhyankar-Moh Theorem is
generalized to arbitrary $\mathbb{Q}$-algebras.

In order to solve questions about polynomial maps, derivations, or other objects
in the theory of polynomial maps, over arbitrary commutative rings$^1$ $R$, it is often
convenient to reduce such questions to the case that $R$ is a field. For example, if one
has a polynomial map $F$ over a domain $R$ with $\det J(F) = 1$ and one wants to know
if it is invertible, then one only needs to check if $F$ is invertible over the quotient field
$Q(R)$ of $R$.

This paper tries to reduce such questions over a ring $R$ to questions about fields
by looking at the residue fields $R_p/pR_p$ for all prime ideals $p$ of $R$.

Section 1 gives an inventarisation of these so-called “residual properties”. There
are two main results here. The first one is that “being locally nilpotent” is a residual
property of a derivation for a Noetherian ring $R$ and in dimension two also for arbitrary
$\mathbb{Q}$-algebras (Propositions 4 and 6). The second one is that “being a coordinate” is
a residual property of a polynomial over an arbitrary $\mathbb{Q}$-algebra in dimension two
(Proposition 12). This generalizes a result from Bhatwadelar and Dutta in [BD93].

Section 2 shows the power of these properties by generalizing the Abhyankar-Moh
Theorem (see [AM75]) to arbitrary $\mathbb{Q}$-algebras.

1 Residual Properties

Notation 1. Let $R$ be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $a$ be an
ideal of $R[X]$ and let $p$ be a prime ideal of $R$. Then $a_p$ denotes the ideal generated
by the image of the ideal $a$ in $R_p/pR_p[X]$ under the map induced by the natural
homomorphism $R \rightarrow R_p/pR_p$.

A similar notation will in the sequel be used for polynomials, polynomial maps,
and derivations.

$^1$ In this paper, all rings will be commutative and have a unit element.
**Proposition 2.** Let $R$ be a ring, $n \in \mathbb{N}^+$, and $R[X] := R[X_1, \ldots, X_n]$. Let $\mathfrak{a}$ be an ideal of $R[X]$. Then the following two statements are equivalent:

1. $\mathfrak{a} = R[X]$;
2. for every $\mathfrak{p} \in \text{Spec}(R)$, $\bar{\mathfrak{a}}_{\mathfrak{p}} = R_{\mathfrak{p}}/pR_{\mathfrak{p}}[X]$.

**Proof.** The implication $1 \Rightarrow 2$ is trivial. So assume that for every $\mathfrak{p} \in \text{Spec}(R)$, $\bar{\mathfrak{a}}_{\mathfrak{p}} = R_{\mathfrak{p}}/pR_{\mathfrak{p}}[X]$.

Assume that $\mathfrak{a} \neq R[X]$. Then there is some maximal ideal $\mathfrak{m}$ of $R[X]$ such that $\mathfrak{a} \subseteq \mathfrak{m}$. Let $\mathfrak{p} := \mathfrak{m} \cap R$. This is a prime ideal of $R$.

Using the natural isomorphism between $R_{\mathfrak{p}}/pR_{\mathfrak{p}}$ and $Q(R/p)$, one can easily see that $1 \in \bar{\mathfrak{a}}_{\mathfrak{p}}$ means that there are an $r \in R \setminus \mathfrak{p}$, polynomials $g_1, \ldots, g_s \in R[X]$, and polynomials $f_1, \ldots, f_s \in \mathfrak{a}$ such that

$$r \equiv g_1f_1 + \cdots + g_sf_s \pmod {p[X]}.$$ 

Because $\mathfrak{a} \subseteq \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$ this implies, however, that $r \in \mathfrak{m}$. Because also $r \in R$, this contradicts the fact that $r \not\in \mathfrak{p} = \mathfrak{m} \cap R$.

Therefore $\mathfrak{a} = R[X]$. \hfill \qed

The following lemma is Lemma 2.1.15 from [Eps00]. It shows that in order to check if an $R$-derivation $D$ of $R[X]$ is locally nilpotent, the ring $R$ can assumed to be reduced.

**Lemma 3.** Let $R$ be a ring, $n \in \mathbb{N}^+$, and $R[X] := R[X_1, \ldots, X_n]$. Let $D \in \text{Der}_R(R[X])$. Let $\eta$ be the nilradical of $R$ and denote by $D/\eta$ the derivation on $R/\eta[X]$ induced by $D$. Assume that $D/\eta$ is locally nilpotent. Then $D$ is locally nilpotent as well.

**Proof.** Let $R'$ be the subring of $R$ generated by all coefficients appearing in the polynomials $D(X_1), \ldots, D(X_n)$. Then $R'$ is Noetherian and $D$ restricts to a derivation $D'$ on $R'[X]$. Note that $D$ is locally nilpotent if and only if $D'$ is. Also note that the nilradical $\eta'$ of $R'$ equals $\eta \cap R'$ and that hence $D/\eta$ is locally nilpotent if and only if $D'/\eta'$ is. Therefore it is possible to assume, without loss of generality, that the ring $R$ is Noetherian.

By induction on $n$ it will follow that for every $n \in \mathbb{N}$ and every $g \in R[X]$, there exists an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^N[X]$. For $n = 1$, this is just the assumption that $D/\eta$ is locally nilpotent. Now assume that the claim holds for $n$ and consider $g \in R[X]$. By induction hypothesis $D^N(g) \in \eta^n[X]$ for some $N \in \mathbb{N}$, say $D^N(g) = \sum c_\alpha X^\alpha$, for certain $c_\alpha \in \eta^n$. Since $D/\eta$ is locally nilpotent, there are $M_n \in \mathbb{N}$ such that $D^{M_n}(X^\alpha) \in \eta[X]$. Taking $M := M + \max_{\alpha \in A} M_n$, it follows that $D^M(g) \in \eta^{n+1}[X]$.

Because $R$ is Noetherian, its nilradical $\eta$ is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Consequently, for every $g \in R[X]$ there is an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^e[X] = (0)$. Therefore $D$ is locally nilpotent. \hfill \qed
Proposition 4. Let $R$ be a Noetherian ring, $n \in \mathbb{N}^*$, and consider the polynomial ring $R[X] := R[X_1, \ldots, X_n]$. Let $D \in \text{Der}_R(R[X])$. Then the following two statements are equivalent:

1. $D$ is locally nilpotent;

2. for every $p \in \text{Spec}(R)$, the derivation $\bar{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X])$ is locally nilpotent.

Proof. The implication 1 $\Rightarrow$ 2 is clear. So assume that for every $p \in \text{Spec}(R)$, the derivation $\bar{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X])$ is locally nilpotent.

Let $p \in \text{Spec}(R)$. Consider the ring $R/p$ and the derivation $D/p$ induced by $D$ on $R/p[X]$. Because $Q(R/p) = R_p/pR_p$ and the derivation $\bar{D}_p$ on $R_p/pR_p[X]$ is locally nilpotent, $D/p$ is locally nilpotent as well. So $D/p$ is locally nilpotent for all $p \in \text{Spec}(R)$.

Since $R$ is Noetherian, its nilradical $\eta$ is a finite intersection of prime ideals, say $\eta = p_1 \cap \cdots \cap p_s$. Because $D/p_i$ is locally nilpotent for every $i \in \{1, \ldots, s\}$, $D/\eta$ is locally nilpotent too. Namely, let $g \in R[X]$. Then there is an $N_i \in \mathbb{N}$ such that $D^{N_i}(g) \in p_i$, for every $i \in \{1, \ldots, s\}$. Taking $N := \max_{i \in \{1, \ldots, s\}} N_i$ it follows that $D^N(g) \in p_1 \cap \cdots \cap p_s = \eta$.

By Lemma 3, $D$ is locally nilpotent. $\square$

In dimension two, the condition that the ring $R$ is Noetherian can be avoided. In order to prove this, the following result is needed ([Ess00], Theorem 1.3.49).

Lemma 5. Let $k$ be a field of characteristic 0 and take $D \in \text{Der}_k(k[X,Y])$. Assume that $D \neq 0$ and let

$$d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}.$$ 

(Here, by convention, the degree of 0 is taken to be $-\infty$.) Then $D$ is locally nilpotent if and only if $D^{d+2}(X) = D^{d+2}(Y) = 0$. $\square$

Proposition 6. Let $R$ be a $k$-algebra and $D \in \text{Der}_R(R[X,Y])$. Then the following two statements are equivalent:

1. $D$ is locally nilpotent;

2. for every $p \in \text{Spec}(R)$, the derivation $\bar{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X,Y])$ is locally nilpotent.

Proof. The implication 1 $\Rightarrow$ 2 is once again clear, so assume that for every $p \in \text{Spec}(R)$, the derivation $\bar{D}_p \in \text{Der}_{R_p/pR_p}(R_p/pR_p[X,Y])$ is locally nilpotent. Take $d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}$.

Let $p \in \text{Spec}(R)$. Just as in the proof of Proposition 4, consider the ring $R/p$ and the locally nilpotent derivation $D/p$ induced by $D$ on $R/p[X,Y]$ and on
\[ Q(R/p)[X,Y] \]. Note that \( Q(R/p) \) is a field of characteristic 0, since \( R \) is a \( \mathbb{Q} \)-algebra. Hence, by the previous lemma,

\[ (D/p)^{d+2}(X) = (D/p)^{d+2}(Y) = 0, \]

or, differently said, \( D^{d+2}(X) \in p[X,Y] \) and \( D^{d+2}(Y) \in p[X,Y] \).

Hence \( D^{d+2}(X) \in \bigcap_{p \in \text{Spec}(R)} p = \eta \), the nilradical of \( R \), and also \( D^{d+2}(Y) \in \eta \). This means that \( D/\eta \) is locally nilpotent and hence, by Lemma 3, \( D \) is locally nilpotent too. \( \square \)

In dimension two, the concept of coordinate over a \( \mathbb{Q} \)-algebra is also a residual concept. In order to prove this, a characterization of coordinates in terms of locally nilpotent derivations is needed. The general form presented here as Proposition 11 comes essentially from [BEM99].

**Definition 7.** Let \( R \) be a ring and \( n \in \mathbb{N}^* \). A polynomial \( f \in R[X] := R[X_1, \ldots, X_n] \) is called a coordinate over \( R \) if it is a component of a polynomial automorphism over \( R \), i.e., if there are polynomials \( f_1, \ldots, f_{n-1} \in R[X] \) such that \( (f_1, \ldots, f_{n-1}, f) \in \text{Aut}_R(R[X]) \).

**Definition 8.** Let \( R \) be a ring, \( n \in \mathbb{N}^* \), and \( R[X] := R[X_1, \ldots, X_n] \). Let \( D \) be an \( R \)-derivation of \( R[X] \). A polynomial \( s \in R[X] \) is called a slice of \( D \) if \( D(s) = 1 \).

The following lemma is a part of Theorem 3.7 of [BEM99].

**Lemma 9.** Let \( R \) be a \( \mathbb{Q} \)-algebra. Then any locally nilpotent \( R \)-derivation on \( R[X,Y] \) with divergence 0 (i.e., with \( \partial_X(D(X)) + \partial_Y(D(Y)) = 0 \)) and 1 in the ideal generated by \( D(X) \) and \( D(Y) \) has a slice. \( \square \)

**Lemma 10.** Let \( R \) be a \( \mathbb{Q} \)-algebra, \( f \in R[X,Y] \), and let \( D \) be the \( R \)-derivation \( f_X \partial_X - f_X \partial_Y \) on \( R[X,Y] \). Assume that \( f \) is a coordinate in \( R[X,Y] \). Then \( D \) is locally nilpotent and has a slice.

**Proof.** Let \( g \in R[X,Y] \) be a polynomial such that \( (f,g) \) is an invertible polynomial map over \( R \). Let \( \eta \) denote the nilradical of \( R \). To avoid notational clutter, reduction modulo this nilradical \( \eta \) of \( R \) or modulo the ideal \( \eta[X,Y] \) of \( R[X,Y] \) will be denoted by an overline.

Then \((\tilde{f}, \tilde{g})\) is an invertible polynomial map over \( \tilde{R} \) and hence \( \tilde{R}[X,Y] = \tilde{R}[\tilde{f}, \tilde{g}] \).

Now note that

\[ \tilde{D}(\tilde{g}) = \det J(\tilde{f}, \tilde{g}) \in \tilde{R}[X,Y]^* = \tilde{R}^* \]

and so \( \tilde{D}^2(\tilde{g}) = 0 \). Also \( \tilde{D}(\tilde{f}) = 0 \) and therefore \( \tilde{D} \) is locally nilpotent. By Lemma 3, \( D \) is locally nilpotent too.

So in view of Lemma 9, the only thing left to show is that 1 is an element of the ideal generated by \( D(X) \) and \( D(Y) \). Now \( \det J(f,g) \in R[X,Y]^* \). So \( g_x f_y - g_y f_x \) is invertible. Hence the ideal generated by \( D(X) = f_X \) and \( D(Y) = -f_Y \) contains an invertible element and consequently contains 1. Therefore, as observed, \( D \) has a slice. \( \square \)
Proposition 11. Let $R$ be a $\mathbb{Q}$-algebra, $f \in R[X,Y]$, and $D := f_Y \partial_X - f_X \partial_Y \in \text{Der}_R(R[X,Y])$. Then the following three statements are equivalent:

1. $D$ is locally nilpotent and $(f_X, f_Y) = R[X,Y]$;
3. $f$ is a coordinate.

Proof. The equivalence of 1 and 2 is Theorem 3.7 of [BEM99] and the equivalence of 2 and 3 follows from Proposition 2.1 of [Wri81] and Lemma 10. \qed

Now it is possible to prove that “being a coordinate” is a residual property for arbitrary $\mathbb{Q}$-algebras. This extends the result from [BD93], which proves the following statement for a Noetherian domain of characteristic 0 that either contains $\mathbb{Q}$ or for which $R/\pi$ is seminormal.

Proposition 12. Let $R$ be a $\mathbb{Q}$-algebra and $f \in R[X,Y]$. Then the following two statements are equivalent:

1. $f$ is a coordinate over $R$ in $R[X,Y]$;
2. for every $p \in \text{Spec}(R)$, $\tilde{f}_p$ is a coordinate over $R_p/pR_p$ in $R_p/pR_p[X,Y]$.

Proof. This is now an easy consequence of the equivalence $1 \Leftrightarrow 3$ from the previous proposition and from Propositions 2 and 6. \qed

The condition that $R$ is a $\mathbb{Q}$-algebra cannot simply be dropped in this proposition. For Bhatwadekar and Dutta have constructed the following example in [BD93]. Take $R := \mathbb{Z}[\sqrt{2}]$ and take

$$f := X - 2Y(\sqrt{2}X - Y^2) + \sqrt{2}(\sqrt{2}X - Y^2)^2 - \sqrt{2}(Y - \sqrt{2}(\sqrt{2}X - Y^2))^4.$$  

Then $\tilde{f}_p$ is a coordinate over $R_p/pR_p$, for every prime ideal $p$ of $R$ (there are only two), but $f$ itself is not a coordinate over $R$.

2 The Abhyankar-Moh Theorem

Using the theory in the previous section, the Abhyankar-Moh Theorem ([AM75], Theorem 1.1) can be generalized to arbitrary $\mathbb{Q}$-algebras.

The Abhyankar-Moh Theorem states the following.

Theorem 13. Let $k$ be a field of characteristic 0 and let $f, g \in k[T]$. Assume that $k[f,g] = k[T]$. Then $\deg(f) \mid \deg(g)$ or conversely.

(Actually, the theorem as formulated in [AM75] is slightly more general. It also holds for $\text{char}(k) \neq 0$ provided that either $\deg(f)$ or $\deg(g)$ is not divisible by $\text{char}(k)$. For the present situation, that information is not needed.) It can also be formulated in the following way ([AM75], Theorem 1.2).
**Theorem 14.** Let $k$ be a field of characteristic 0 and let $f \in k[X,Y]$. Assume that $k[X,Y]/(f) \cong k[T]$. Then $f$ is a coordinate in $k[X,Y]$.

This is the formulation that will actually be generalized.

**Theorem 15.** Let $R$ be a $\mathbb{Q}$-algebra and let $f \in R[X,Y]$. Assume that $R[X,Y]/(f) \cong R[T]$. Then $f$ is a coordinate in $R[X,Y]$.

*Proof.* Let $D$ be the derivation $f_1 \partial_X - f_2 \partial_Y$ on $R[X,Y]$.

Let $p \in \text{Spec}(R)$. Then $R_p / pR_p[X,Y]/(\tilde{f}_p) \cong R_p / pR_p[T]$. Since $R$ is a $\mathbb{Q}$-algebra, $R_p / pR_p$ is a field of characteristic 0. So Theorem 14 implies that $\tilde{f}_p$ is a coordinate over $R_p / pR_p$ in $R_p / pR_p[X,Y]$.

Now Proposition 12 implies that $f$ itself is a coordinate over $R$. \hfill $\square$

Note that [BD93] also gives such a generalization of the Abhyankar-Moh Theorem, but only for Noetherian domains $R$ of characteristic 0 which either contain $\mathbb{Q}$ or which satisfy that $R/\bar{\eta}$ is seminormal.

**References**


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