Finite rank modules over a valuation ring

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Report No. 9932 (July 1999)
Abstract

Let $K = (K, | \cdot |)$ be a spherically (= maximally) complete non-archimedean rank 1 valued field with valuation ring $B_K := \{ \lambda \in K : |\lambda| \leq 1 \}$. It is proved (Theorem 3.8) that a $B_K$-module of finite rank is a direct sum of $B_K$-modules of rank 1. The proof uses convexity techniques and seminorms. However to obtain the announced result it is not sufficient to use only real-valued seminorms, (see §2), so we are led to allow a more general range, a so-called $G$-module (see §3).

Introduction

Let $K, B_K$ be as above. A subset $A$ of a $K$-vector space $E$ is called absolutely convex if $0 \in A$ and if $x, y \in A$, $\lambda, \mu \in B_K$ implies $\lambda x + \mu y \in A$ i.e. if $A$ is a $B_K$-submodule of $E$. A $B_K$-module $B$ is said to be of finite rank if there is an $n \in \mathbb{N}$, an absolutely convex $A \subset K^n$ and a surjective $B_K$-module homomorphism $A \to B$. The smallest $n$ for which this is true is called the rank of $B$. (One can prove easily that it is the same as the Fleischer rank introduced in [1].) The following natural question was stated in [2], p. 35 as an open problem.

Q. Is every rank $n$ $B_K$-module a direct sum of $n$ rank 1 submodules?

For a non-spherically complete base field, a two-dimensional indecomposable absolutely convex set is constructed in [3], p. 68 so the condition of spherical completeness of $K$ is necessary to obtain a positive answer.

In this note we prove that Q has a positive answer. During preparation of this note, it was kindly pointed out by Prof. L. Fuchs that there is a direct purely algebraic proof using the theory of [1], sketched as follows. Let $B$ be a finite rank $B_K$-module. It is a surjective image of a finite rank torsion-free module $A$. As every rank one submodule of $A$ is pure-injective, $A$ is completely decomposable. By [1], Th. 5.5, $B$ is polyserial, by spherical completeness and [1], Th. 5.1 all uniserials are pure-injective and therefore $A$ is a direct sum of uniserials.

Now we present an alternative proof, using techniques of convexity and seminorms.

To this end we write a $B_K$-module of rank $n$ as $T/S$ where $S \subset T$ are absolutely convex sets in $K^n$ and study orthogonality properties of the Minkowski seminorms of $T$.
S and T. As we will see in §2 this method yields the result only for special, so-called edged sets, S and T. To obtain the full answer we extend the notion of Minkowski function by admitting a range set different from \([0, \infty)\), see §3.

1 Preliminaries

Throughout \(K, B_K\) are as above. For a subset \(X\) of a \(K\)-vector space \(E\) we denote by \([X]\) the \(K\)-linear span of \(X\). An absolutely convex set \(A \subseteq E\) is called absorbing if \([A] = E\).

Let \(p\) be a (non-archimedean) seminorm on a \(K\)-vector space \(E\). Two subspaces \(D_1, D_2\) of \(E\) are called \(p\)-orthogonal if \(D_1 \cap D_2 = \{0\}\) and \(p(d_1 + d_2) = \max\{p(d_1), p(d_2)\}\) for all \(d_1 \in D_1, d_2 \in D_2\). If, in addition, \(E = D_1 \oplus D_2\) we call \(D_2\) (\(D_1\)) a \(p\)-orthocomplement of \(D_1\) (\(D_2\)).

A finite linearly independent sequence \(e_1, \ldots, e_n\) in \(E\) is called \(p\)-orthogonal if \(p(\sum_{i=1}^n \lambda_i e_i) = \max_{1 \leq i \leq n} p(\lambda_i e_i)\) for all \(\lambda_1, \ldots, \lambda_n \in K\) i.e. if \(Ke_i\) is \(p\)-orthogonal to \(\sum_{j \neq i} Ke_j\), for each \(i\).

**Proposition 1.1** Let \(E\) be an \(n\)-dimensional space over \(K\) \((n \in \mathbb{N})\), let \(p\) be a seminorm. Then each subspace of \(E\) has a \(p\)-orthocomplement. In particular, each \(p\)-orthogonal sequence can be extended to a \(p\)-orthogonal base of \(E\).

**Proof.** The statements are well-known for norms \(p\), ([3], 5.5, 5.15). We leave the extension to the case of seminorms \(p\) to the reader.

2 The edged case

Recall that for an absolutely convex subset \(A\) of a \(K\)-vector space, \(A^e := \bigcap_{r > 1} \{\lambda a : \lambda \in K, |\lambda| \leq r, a \in A\}\) i.e., \(A^e = A\) if the valuation of \(K\) is discrete, \(A^e = \bigcap \{\lambda A : \lambda \in K, |\lambda| > 1\}\) if the valuation of \(K\) is dense. \(A\) is called edged if \(A^e = A\). The following is well-known.

**Proposition 2.1** For an absolutely convex subset \(A\) of a \(K\)-vector space the formula

\[
p_A(x) = \inf \{|\lambda| : \lambda \in K, x \in \lambda A\}
\]

defines a seminorm \(p_A\) on \([A]\). We have

\[
\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \leq 1\}.
\]

\(A\) is edged if and only if \(A = \{x \in [A] : p_A(x) \leq 1\}\).

**Proposition 2.2** Let \(n \in \mathbb{N}\), let \(p, q\) be seminorms on \(K^n\). Then there exists a base \(e_1, \ldots, e_n\) of \(K^n\) that is both \(p\)- and \(q\)-orthogonal.
Proof. (After \cite{4}, 1.10). It suffices to prove the existence of an \(e \in K^n \setminus \{0\}\) and a subspace \(D\) of \(K^n\) such that \(K^n = Ke \oplus D\), and \(Ke\) and \(D\) are both \(p\) and \(q\)-orthogonal. If \(p(e) = 0\) for some nonzero \(e\), let \(D\) be any \(q\)-orthocomplement of \(Ke\). Then trivially \(D\) and \(Ke\) are \(p\)-orthogonal. So, we may assume that \(p\) is a norm. Let \(e_1, \ldots, e_n\) be a \(p\)-orthogonal base of \(K^n\) (see 1.1). Set \(t := \max \{q(e_i)/p(e_i) : i \leq \min\{1, \ldots, n\}\}\). Then \(tp(x) > q(x)\) for all \(x \in K^n\). Choose \(e := e_k\), let \(D\) be a \(q\)-orthocomplement of \(Ke\) (see 1.1). To show that \(Ke\) and \(D\) are also \(p\)-orthogonal let \(x \in D\). Then \(tp(e + x) > q(e + x) \geq q(e) = tp(e)\), so \(p(e + x) > p(e)\) implying orthogonality.

As a corollary we obtain

**Proposition 2.3** Let \(S \subset T\) be edged absolutely convex subsets of \(K^n\) where \(n \geq 1\). Then there exists a base \(e_1, \ldots, e_n\) of \(K^n\), and absolutely convex \(C_1, \ldots, C_n\) and \(D_1, \ldots, D_n\) in \(K\) such that

\[
S = C_1 e_1 \oplus \cdots \oplus C_n e_n \\
T = D_1 e_1 \oplus \cdots \oplus D_n e_n.
\]

**Proof.** By 2.2 there is a base \(e_1, \ldots, e_m\) of \([S]\) that is both \(p_S\)- and \(p_T\)-orthogonal. Extend it to a \(p_T\)-orthogonal base \(e_1, \ldots, e_s\) of \([T]\) (see 1.1) and further extend it to a base \(e_1, \ldots, e_n\) of \(K^n\). Set

\[
C_i := \begin{cases} 
\{ \lambda \in K : p_S(\lambda e_i) \leq 1 \} & \text{if } i \in \{1, \ldots, m\} \\
\{0\} & \text{if } i \in \{m+1, \ldots, n\}
\end{cases}
\]

\[
D_i := \begin{cases} 
\{ \lambda \in K : p_T(\lambda e_i) \leq 1 \} & \text{if } i \in \{1, \ldots, s\} \\
\{0\} & \text{if } i \in \{s+1, \ldots, m\}.
\end{cases}
\]

To prove that \(S = \sum_{i=1}^n C_i e_i = C_1 e_1 + \cdots + C_m e_m\) first observe that for each \(x \in C_1 e_1 + \cdots + C_m e_m\) we have \(p_S(x) \leq 1\), so \(x \in S\) by the last statement of 2.1 (here we use that \(S\) is edged). Hence, \(C_1 e_1 + \cdots + C_m e_m \subset S\). Conversely, if \(x \in S\), \(x = \sum_{i=1}^m \lambda_i e_i\) where \(\lambda_i \in K\), then, by orthogonality and 2.1, \(1 \geq p_S(x) = \max_{i} p_S(\lambda_i e_i)\), so \(\lambda_i \in C_i\) for each \(i \in \{1, \ldots, m\}\) i.e. \(S \subset C_1 e_1 + \cdots + C_m e_m\). That \(T = \sum D_i e_i\) is proved similarly.

**Corollary 2.4** Let \(B\) be a \(B_K\)-module of finite rank. If \(B\) has the form \(T/S\), where \(S \subset T\) are edged absolutely convex sets in some finite-dimensional \(K\)-vector space then \(B\) is the direct sum of submodules of rank \(\leq 1\).

**Proof.** Let \(e_i, C_i, D_i\) be as in 2.3. Obviously, \(C_i \subset D_i\) for each \(i\) and we find \(T/S \cong \bigoplus_{i=1}^n (D_i/C_i)\).

In the next section we will remove the edgedness condition. Notice that if the valuation of \(K\) is discrete each absolutely convex set is edged, so we may assume that the valuation of \(K\) is dense.
3 The general case

From now on in §3, let \( G := \{|\lambda| : \lambda \in K, \lambda \neq 0\} \). It is a multiplicative subgroup of \((0, \infty)\). The following notion has been used successfully in Functional Analysis over infinite rank valued fields to define (semi)norms, see [6], [5] for a discussion.

**Definition 3.1** A G-module is a linearly ordered set \( X \) together with an action \( G \times X \to X \) (i.e. \( g_1(g_2 x) = (g_1 g_2) x \), \( 1 x = x \) for all \( g_1, g_2 \in G, x \in X \)) such that \( g_1 \geq g_2, x_1 \geq x_2 \) \((g_1, g_2 \in G, x_1, x_2 \in X)\) implies \( g_1 x_1 \geq g_2 x_2 \), and such that for each \( \varepsilon \in X \) and \( x \in X \) there exists a \( g \in G \) and that \( g x < \varepsilon \).

**Lemma 3.2** Let \( X \) be a G-module, let \( x \in X \). If \( g \in G, gx = x \) then \( g = 1 \).

**Proof.** The set \( \{g \in G : gx = x\} \) is easily seen to be a proper subgroup \( H \) of \( G \). If \( h \in H, h > 1 \) and \( g \in G, g \geq 1 \) then \( 1 \leq g \leq h^n \) for some \( n \). It follows that \( H = G \), a contradiction.

Obvious examples of G-modules are \( G \) itself, the group \((0, \infty)\) or any union of multiplicative cosets of \( G \) in \((0, \infty)\). For a more interesting example, let \( X \) be a G-module, let \( Y \) be a totally ordered set. Then \( X \times Y \) becomes a G-module under the lexicographic ordering and the action \( g(x, y) = (gx, y) \) (\( g \in G, x \in X, y \in Y \)).

We adjoin an element \( 0_x \) to \( X \) for which \( 0_x < x, 0_x = 0_x = 0_0 x \) for every \( x \in X \) but from now on we will write 0 instead of \( 0_X \).

**Definition 3.3** Let \( E \) be a K-vector space, let \( X \) be a G-module. An X-seminorm is a map \( p : E \to X \cup \{0\} \) such that \( p(0) = 0, p(\lambda x) = |\lambda| p(x), p(x+y) \leq \max(p(x), p(y)) \) for all \( \lambda \in K, x, y \in E \).

**Remark.** It is not hard to see that Proposition 1.1 remains valid if we replace \( p \) by an X-seminorm. (For a formal proof for norms, see [6], 3.3.)

To define the kind of seminorms we are interested in, let \( X := (0, \infty) \times \{0, 1\} \) with the lexicographic ordering. Then for each \( r \in (0, \infty) \) the element \((r, 1)\) is an immediate successor of \((r, 0)\) which suggests the notation \( r^+ \) for \((r, 0)\) and \( r^+ \) for \((r, 1)\). The action defined above now reads as \( |\lambda| r^+ = (|\lambda| r)^+ \) \((\lambda \in K, \lambda \neq 0)\). Thus, we have ‘doubled’ every positive real number \( r \) by giving it a successor \( r^+ \), and we write \( X = (0, \infty) \cup (0, \infty)^+ \) where \((0, \infty)^+ := \{r^+ : r \in (0, \infty)\}\).

From now on in this note we assume that the valuation of \( K \) is dense and let \( X_K := G \cup (0, \infty)^+ \) (which is a G-submodule of \((0, \infty) \times (0, \infty)^+ \) we have just introduced).

**Theorem 3.4** Let \( A \) be an absolutely convex subset of a K-vector space. Then the formula

\[
q_A(x) = \begin{cases} 
p_A(x) & \text{if } p_A(x) = \min\{|\lambda| : x \in \lambda A\} 
\lambda A(x)^+ & \text{otherwise}
\end{cases}
\]
defines an $X_K$-seminorm $q_A \geq p_A$ on $[A]$ for which $A = \{ x \in [A] : q_A(x) \leq 1 \}$.

**Proof.** We first prove

\[ (*) \quad q_A(x) \leq |\lambda| \iff x \in \lambda A \quad (x \in [A], \lambda \in K, \lambda \neq 0) \]

yielding the desired identity $A = \{ x \in [A] : q_A(x) \leq 1 \}$.

Let $q_A(x) \leq |\lambda|$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A \subseteq \lambda A$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $p_A(x) \leq q_A(x) < |\lambda|$ so $p_A(\lambda^{-1} x) < 1$ hence $\lambda^{-1} x \in A$ by 2.2.

If, conversely, $x \in \lambda A$ and $q_A(x) = |\mu|$ for some $\mu \in K$ then $|\mu| = \min\{|\nu| : x \in \nu A\} \leq |\lambda|$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $r < |\nu|$ for all $\nu$ for which $x \in \nu A$, so $r < |\lambda|$, hence $q_A(x) = r^+ < |\lambda|$. To show that $q_A$ is a seminorm, let $x \in [A], \lambda \in K$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A$ so that $\lambda x \subseteq \mu A$ so that by $(*)$ $q_A(\lambda x) \leq |\lambda \mu| = |\lambda| q_A(x)$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $x \in \mu A$ for all $|\nu| > r$ so $\lambda x \in \nu A$ for all $|\nu| > r|\lambda|$, hence $q_A(\lambda x) \leq |\lambda|$ for all $|\nu| > r|\lambda|$ i.e. $q_A(\lambda x) \leq (r|\lambda|)^+ = |\lambda| r^+ = |\lambda| q_A(x)$. So we have proved $q_A(\lambda x) \leq |\lambda| q_A(x)$. To prove the converse inequality (which is only needed for $\lambda \neq 0$) we observe that $|\lambda| q_A(x) = |\lambda| q_A(\lambda^{-1} \lambda x) \leq |\lambda| \lambda^{-1} q_A(\lambda x) = q_A(\lambda x)$. Finally we prove the strong triangle inequality $q_A(x+y) \leq \max(q_A(x), q_A(y))$. Suppose $q_A(x) \leq q_A(y)$. If $q_A(y) = |\lambda|$ for some $\lambda \in K$ then by $(*)$ $y \in \lambda A$ and also $x \in \lambda A$ so $x+y \in \lambda A$, implying $q_A(x+y) \leq |\lambda|$. If $q_A(y) = r^+$ for some $r \in (0, \infty)$ then for all $\lambda \in K$ with $|\lambda| > r$ we have $y \in \lambda A$ and also $x \in \lambda A$ so $x+y \in \lambda A$. We see that $q_A(x+y) \leq |\lambda|$ for all $|\lambda| > r$ i.e. $q_A(x+y) \leq r^+$.

**Lemma 3.5** Let $p, q$ be $X_K$-seminorms on a $K$-vector space $E$. If $\{ x \in E : p(x) \leq 1 \} \subseteq \{ x \in E : q(x) \leq 1 \}$ then $p \geq q$.

**Proof.** By obvious scalar multiplication we have

\[ \{ x \in E : p(x) \leq |\lambda| \} \subseteq \{ x \in E : q(x) \leq |\lambda| \} \]

for each $\lambda \in K^\times$. Then the above inclusion is also true for $\lambda = 0$. Now let $r^+ \in (0, \infty)^+$. From

\[ \{ x \in E : p(x) \leq r^+ \} = \bigcap_{|\lambda| > r} \{ x \in E : p(x) < |\lambda| \} \]

and a similar formula for $q$ we obtain

\[ \{ x \in E : p(x) \leq s \} \subseteq \{ x \in E : q(x) \leq s \} \]

for every $s \in X_K \cup \{0\}$. It follows that $q \leq p$.

**Corollary 3.6** Let $E$ be a $K$-vector space, let $p$ be an $X_K$-seminorm.

(i) If $A := \{ x \in E : p(x) \leq 1 \}$ then $p = q_A$.

(ii) Let $B : E \to E$ be a linear map. If $p(x) \leq 1$ implies $p(Bx) \leq 1$ for all $x \in E$ then $p(Bx) \leq p(x)$ for all $x \in E$. 

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Proof. (i) is a direct consequence of \( \{ x \in E : p(x) \leq 1 \} = \{ x \in E : p_A(x) \leq 1 \} \) and Lemma 3.5. For (ii) apply 3.5 to the seminorms \( p \) and \( p \circ B \).

**Proposition 3.7** Let \( n \in \mathbb{N} \), let \( p \) and \( q \) be \( X_K \)-seminorms on \( K^n \). Then there is a base of \( E \) that is both \( p \)– and \( q \)-orthogonal.

**Proof.** Like in the proof of 2.2 we prove the existence of an \( e \in K^n \setminus \{0\} \) and an \((n-1)\)-dimensional subspace \( D \) such that \( K^n = Ke \oplus D \) where \( Ke \) and \( D \) are both \( p \)- and \( q \)-orthogonal, and we may assume that \( p \) is a norm. Let \( e_1,e_2,\ldots,e_n \) be a \( p \)-orthogonal base of \( K^n \) (see Remark following 3.3). For each \( i \in \{1,\ldots,n\} \) let \( C_i := \{ \lambda \in K : p(\lambda e_i) \leq 1 \} \) and \( A_i := C_i e_i \). Then by \( p \)-orthogonality

\[
\{ x \in K^n : p(x) \leq 1 \} = A_1 + \cdots + A_n.
\]

Now set \( l(A_i) := \{ t \in X_K \cup \{0\} : \text{there is an } a \in A_i \text{ with } t \leq q(a) \} \). Then \( l(A_i) \) is an initial part of \( X_K \cup \{0\} \), so \( l(A_i) \), \( i \), \( l(A_n) \) are linearly ordered by inclusion; let \( l(A_1) \) be the largest one. Set \( e := e_1 \). If \( l(A_1) = \{0\} \) then \( q = 0 \) and we can take \( D = [e_2,\ldots,e_n] \), so assume \( q \neq 0 \) on \( A_1 \). Now let \( D \) be a \( q \)-orthogonal complement of \( Ke \) (Remark following 3.3) and let \( P : D + Ke \to D \) be the natural projection. We finish the proof by showing that \( Ke \) and \( D \) are \( p \)-orthogonal, i.e. that \( p(x) \leq 1 \) implies \( p(Px) \leq 1 \) (3.6 (ii)). Let \( x \in K^n \), \( p(x) \leq 1 \). Then \( x = a_1 + \cdots + a_n \) where \( a_i \in A_i \) for each \( i \). We have, for each \( i \), \( q(a_i) \in l(A_i) \subset l(A_1) \), so \( q(a_i) \leq q(b) \) for some \( b \in A_1 \) and \( q(b) \neq 0 \). Then \( q(Pa_i) \leq q(a_i) \leq q(b) \). Now \( Pa_i \in [b] \) so \( Pa_i = \lambda b \) for some \( \lambda \in K \). We see that \( |\lambda|q(b) \leq q(b) \) implying \( |\lambda| \leq 1 \) by 3.2, so \( Pa_i \in A_i \). Then \( Px = \sum Pa_i \in A_i \) i.e., \( p(Px) \leq 1 \), and we are done.

**Remark.** The above proof is valid for an \( X_K \)-seminorm \( p \) and an \( X \)-seminorm \( q \) for any \( G \)-module \( X \). I do not know whether the conclusion of 3.7 holds for an \( X \)-seminorm \( p \) and a \( Y \) seminorm \( q \) where \( X \) and \( Y \) are arbitrary \( G \)-modules.

The following corollary obtains.

**Theorem 3.8** (Let \( K \) be spherically complete and) let \( B \) be a \( B_K \)-module of finite rank. Then \( B \) is a direct sum of submodules of rank \( \leq 1 \).

**Proof.** The proofs of Proposition 2.3 and Corollary 2.4 can formally be taken over, where \( p_S \) and \( p_T \) are replaced by the \( X_K \)-seminorms \( q_S \) and \( q_T \) respectively.
References


