Finite rank modules over a valuation ring

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Abstract

Let \( K = (K, | \cdot |) \) be a spherically (= maximally) complete non-archimedean rank 1 valued field with valuation ring \( B_K := \{ \lambda \in K : |\lambda| \leq 1 \} \). It is proved (Theorem 3.8) that a \( B_K \)-module of finite rank is a direct sum of \( B_K \)-modules of rank 1. The proof uses convexity techniques and seminorms. However to obtain the announced result it is not sufficient to use only real-valued seminorms, (see §2), so we are led to allow a more general range, a so-called \( G \)-module (see §3).

Introduction

Let \( K, B_K \) be as above. A subset \( A \) of a \( K \)-vector space \( E \) is called absolutely convex if \( 0 \in A \) and if \( x, y \in A, \lambda, \mu \in B_K \) implies \( \lambda x + \mu y \in A \) i.e. if \( A \) is a \( B_K \)-submodule of \( E \). A \( B_K \)-module \( B \) is said to be of finite rank if there is an \( n \in \mathbb{N} \), an absolutely convex \( A \subset K^n \) and a surjective \( B_K \)-module homomorphism \( A \to B \). The smallest \( n \) for which this is true is called the rank of \( B \). (One can prove easily that it is the same as the Fleischer rank introduced in [1].) The following natural question was stated in [2], p. 35 as an open problem.

Q. Is every rank \( n \) \( B_K \)-module a direct sum of \( n \) rank 1 submodules?

For a non-spherically complete base field, a twodimensional indecomposable absolutely convex set is constructed in [3], p. 68 so the condition of spherical completeness of \( K \) is necessary to obtain a positive answer.

In this note we prove that Q has a positive answer. During preparation of this note, it was kindly pointed out by Prof. L. Fuchs that there is a direct purely algebraic proof using the theory of [1], sketched as follows. Let \( B \) be a finite rank \( B_K \)-module. It is a surjective image of a finite rank torsion-free module \( A \). As every rank one submodule of \( A \) is pure-injective, \( A \) is completely decomposable. By [1], Th. 5.5, \( B \) is polyserial, by spherical completeness and [1], Th. 5.1 all uniserials are pure-injective and therefore \( A \) is a direct sum of uniserials.

Now we present an alternative proof, using techniques of convexity and seminorms. To this end we write a \( B_K \)-module of rank \( n \) as \( T/S \) where \( S \subset T \) are absolutely convex sets in \( K^n \) and study orthogonality properties of the Minkowski seminorms of
$S$ and $T$. As we will see in §2 this method yields the result only for special, so-called "edged sets," $S$ and $T$. To obtain the full answer we extend the notion of Minkowski function by admitting a range set different from $[0, \infty)$, see §3.

1 Preliminaries

Throughout $K$, $B_K$ are as above. For a subset $X$ of a $K$-vector space $E$ we denote by $[X]$ the $K$-linear span of $X$. An absolutely convex set $A \subseteq E$ is called absorbing if $[A] = E$.

Let $p$ be a (non-archimedean) seminorm on a $K$-vector space $E$. Two subspaces $D_1, D_2$ of $E$ are called $p$-orthogonal if $D_1 \cap D_2 = \{0\}$ and $p(d_1 + d_2) = \max\{p(d_1), p(d_2)\}$ for all $d_1 \in D_1$, $d_2 \in D_2$. If, in addition, $E = D_1 \oplus D_2$ we call $D_2$ ($D_1$) a $p$-orthocomplement of $D_1$ ($D_2$).

A finite linearly independent sequence $e_1, \ldots, e_n$ in $E$ is called $p$-orthogonal if

$$p(\sum_{i=1}^{n} \lambda_i e_i) = \max_{1 \leq i \leq n} p(\lambda_i e_i)$$

for all $\lambda_1, \ldots, \lambda_n \in K$, i.e., if $K e_i$ is $p$-orthogonal to $\sum_{j \neq i} K e_j$, for each $i$.

**Proposition 1.1** Let $E$ be an $n$-dimensional space over $K$ ($n \in \mathbb{N}$), let $p$ be a seminorm. Then each subspace of $E$ has a $p$-orthocomplement. In particular, each $p$-orthogonal sequence can be extended to a $p$-orthogonal base of $E$.

**Proof.** The statements are well-known for norms $p$, ([3], 5.5, 5.15). We leave the extension to the case of seminorms $p$ to the reader.

2 The edged case

Recall that for an absolutely convex subset $A$ of a $K$-vector space, $A^e := \bigcap_{r>0} \{\lambda a : \lambda \in K, |\lambda| \leq r, a \in A\}$ i.e., $A^e = A$ if the valuation of $K$ is discrete, $A^e = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$ if the valuation of $K$ is dense. $A$ is called edged if $A^e = A$. The following is well-known.

**Proposition 2.1** For an absolutely convex subset $A$ of a $K$-vector space the formula

$$p_A(x) = \inf\{|\lambda| : \lambda \in K, x \in \lambda A\}$$

defines a seminorm $p_A$ on $[A]$. We have

$$\{x \in [A] : p_A(x) < 1\} \subset A \subset \{x \in [A] : p_A(x) \leq 1\}.$$  

$A$ is edged if and only if $A = \{x \in [A] : p_A(x) \leq 1\}$.

**Proposition 2.2** Let $n \in \mathbb{N}$, let $p, q$ be seminorms on $K^n$. Then there exists a base $e_1, \ldots, e_n$ of $K^n$ that is both $p$- and $q$-orthogonal.
Proof. (After [4], 1.10). It suffices to prove the existence of an \( e \in K^n \setminus \{0\} \) and a subspace \( D \) of \( K^n \) such that \( K^n = Ke \oplus D \), and \( Ke \) and \( D \) are both \( p \)- and \( q \)-orthogonal. If \( p(e) = 0 \) for some nonzero \( e \), let \( D \) be any \( q \)-orthocomplement of \( Ke \). Then trivially \( D \) and \( Ke \) are \( p \)-orthogonal. So, we may assume that \( p \) is a norm. Let \( e_1, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see 1.1). Set \( t := \max_i q(e_i)/p(e_i) = q(e_k)/p(e_k) \) for some \( k \in \{1, \ldots, n\} \). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then trivially \( D \) and \( Ke \) are \( p \)-orthogonal. So, we may assume that \( p \) is a norm.

Let \( e_1, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see 1.1). Set \( t := \max_{i \in \mathbb{Z}} g(e_i)/p(e_i) = q(e_k)/p(e_k) \) for some \( k \in \{1, \ldots, n\} \). Then \( tp(x) > q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( g \)-orthocomplement of \( Ke \) (see 1.1). Then

As a corollary we obtain

**Proposition 2.3** Let \( S \subset T \) be edged absolutely convex subsets of \( K^n \) where \( n \geq 1 \). Then there exists a base \( e_1, \ldots, e_n \) of \( K^n \), and absolutely convex \( C_1, \ldots, C_n \) and \( D_1, \ldots, D_n \) in \( K \) such that

\[
S = C_1 e_1 \oplus \cdots \oplus C_n e_n \quad \text{ and } \quad T = D_1 e_1 \oplus \cdots \oplus D_n e_n.
\]

**Proof.** By 2.2 there is a base \( e_1, \ldots, e_m \) of \( [S] \) that is both \( p_S \)- and \( p_T \)-orthogonal. Extend it to a \( p_T \)-orthogonal base \( e_1, \ldots, e_s \) of \( [T] \) (see 1.1) and further extend it to a base \( e_1, \ldots, e_n \) of \( K^n \). Set

\[
C_i := \begin{cases} \{ \lambda \in K : p_S(\lambda e_i) \leq 1 \} & \text{if } i \in \{1, \ldots, m\} \\ \{0\} & \text{if } i \in \{m+1, \ldots, n\} \end{cases}
\]

\[
D_i := \begin{cases} \{ \lambda \in K : p_T(\lambda e_i) \leq 1 \} & \text{if } i \in \{1, \ldots, s\} \\ \{0\} & \text{if } i \in \{s+1, \ldots, m\}. \end{cases}
\]

To prove that \( S = \sum_{i=1}^{n} C_i e_i = C_1 e_1 + \cdots + C_m e_m \) first observe that for each \( x \in C_1 e_1 + \cdots + C_m e_m \) we have \( p_S(x) \leq 1 \), so \( x \in S \) by the last statement of 2.1 (here we use that \( S \) is edged). Hence, \( C_1 e_1 + \cdots + C_m e_m \subset S \). Conversely, if \( x \in S \), \( x = \sum_{i=1}^{n} \lambda_i e_i \) where \( \lambda_i \in K \), then, by orthogonality and 2.1, \( 1 \geq p_S(x) = \max_{i=1}^{n} p_S(\lambda_i e_i) \), so \( \lambda_i \in C_i \) for each \( i \in \{1, \ldots, m\} \) i.e. \( S \subset C_1 e_1 + \cdots + C_m e_m \). That \( T = \sum D_i e_i \) is proved similarly.

**Corollary 2.4** Let \( B \) be a \( B_K \)-module of finite rank. If \( B \) has the form \( T/S \), where \( S \subset T \) are edged absolutely convex sets in some finite-dimensional \( K \)-vector space then \( B \) is the direct sum of submodules of rank \( \leq 1 \).

**Proof.** Let \( e_i, C_i, D_i \) be as in 2.3. Obviously, \( C_i \subset D_i \) for each \( i \) and we find \( T/S \cong \bigoplus_{i=1}^{n} (D_i/C_i) \).

In the next section we will remove the edgedness condition. Notice that if the valuation of \( K \) is discrete each absolutely convex set is edged, so we may assume that the valuation of \( K \) is dense.
3 The general case

From now on in §3, let $G := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$. It is a multiplicative subgroup of $(0, \infty)$. The following notion has been used successfully in Functional Analysis over infinite rank valued fields to define (semi)norms, see [6], [5] for a discussion.

**Definition 3.1** A $G$-module is a linearly ordered set $X$ together with an action $G \times X \to X$ (i.e. $g_1(g_2 x) = (g_1 g_2) x$, $1.x = x$ for all $g_1, g_2 \in G$, $x \in X$) such that $g_1 \geq g_2$, $x_1 \geq x_2$ ($g_1, g_2 \in G$, $x_1, x_2 \in X$) implies $g_1 x_1 \geq g_2 x_2$, and such that for each $\varepsilon \in X$ and $x \in X$ there exists a $g \in G$ and that $gx < \varepsilon$.

**Lemma 3.2** Let $X$ be a $G$-module, let $x \in X$. If $g \in G$, $gx = x$ then $g = 1$.

**Proof.** The set $\{g \in G : gx = x\}$ is easily seen to be a proper subgroup $H$ of $G$. If $h \in H$, $h > 1$ and $g \in G$, $g \geq 1$ then $1 \leq g \leq h^n$ for some $n$. It follows that $H = G$, a contradiction.

Obvious examples of $G$-modules are $G$ itself, the group $(0, \infty)$ or any union of multiplicative cosets of $G$ in $(0, \infty)$. For a more interesting example, let $X$ be a $G$-module, let $Y$ be a totally ordered set. Then $X \times Y$ becomes a $G$-module under the lexicographic ordering and the action

$$g(x, y) = (gx, y) \quad (g \in G, \ x \in X, \ y \in Y).$$

We adjoin an element $0_X$ to $X$ for which $0_X < x, \ 0.x = 0_X = 0.0_X$ for every $x \in X$ but from now on we will write 0 instead of $0_X$.

**Definition 3.3** Let $E$ be a $K$-vector space, let $X$ be a $G$-module. An $X$-seminorm is a map $p : E \to X \cup \{0\}$ such that $p(0) = 0$, $p(\lambda x) = |\lambda| p(x)$, $p(x + y) \leq \max(p(x), p(y))$ for all $\lambda \in K, \ x, y \in E$.

**Remark.** It is not hard to see that Proposition 1.1 remains valid if we replace $p$ by an $X$-seminorm. (For a formal proof for norms, see [6], 3.3.)

To define the kind of seminorms we are interested in, let $X := (0, \infty) \times \{0, 1\}$ with the lexicographic ordering. Then for each $r \in (0, \infty)$ the element $(r, 1)$ is an immediate successor of $(r, 0)$ which suggests the notation $r$ for $(r, 0)$ and $r^+$ for $(r, 1)$. The action defined above now reads as $|\lambda|^r^+ = (|\lambda|)^r^+$ ($\lambda \in K, \lambda \neq 0$). Thus, we have ‘doubled’ every positive real number $r$ by giving it a successor $r^+$, and we write $X = (0, \infty) \cup (0, \infty)^+$ where $(0, \infty)^+ := \{r^+ : r \in (0, \infty)\}$.

From now on in this note we assume that the valuation of $K$ is dense and let $X_K := G \cup (0, \infty)^+$ (which is a $G$-submodule of $(0, \infty) \times (0, \infty)^+$ we have just introduced).

**Theorem 3.4** Let $A$ be an absolutely convex subset of a $K$-vector space. Then the formula

$$q_A(x) = \begin{cases} p_A(x) & \text{if } p_A(x) = \min\{|\lambda| : x \in \lambda A\} \\ p_A(x)^+ & \text{otherwise} \end{cases}$$

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defines an $X_K$-seminorm $q_A \geq p_A$ on $[A]$ for which $A = \{ x \in [A] : q_A(x) \leq 1 \}$.

**Proof.** We first prove

$$(*) \quad q_A(x) \leq |\lambda| \iff x \in \lambda A \quad (x \in [A], \lambda \in K, \lambda \neq 0)$$

yielding the desired identity $A = \{ x \in [A] : q_A(x) \leq 1 \}$.

Let $q_A(x) \leq |\lambda|$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A \subseteq \lambda A$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $p_A(x) \leq q_A(x) < |\lambda|$ so $p_A(\lambda^{-1} x) < 1$ hence $\lambda^{-1} x \in A$ by 2.2.

If, conversely, $x \in \lambda A$ and $q_A(x) = |\mu|$ for some $\mu \in K$ then $|\mu| = \min\{|\nu| : x \in \nu A\} \leq |\lambda|$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $r < |\nu|$ for all $\nu$ for which $x \in \nu A$, so $r < |\lambda|$, hence $q_A(x) = r^+ < |\lambda|$.

To show that $q_A$ is a seminorm, let $x \in [A]$, $\lambda \in K$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A$ so that $\lambda x \in \lambda \mu A$ so that by $(*)$ $q_A(\lambda x) \leq |\lambda \mu| = |\lambda| q_A(x)$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $x \in \mu A$ for all $|\mu| > r$ so $\lambda x \in \nu A$ for all $|\nu| > r |\lambda|$, hence $q_A(\lambda x) \leq |\lambda|$ for all $|\nu| > r |\lambda|$ i.e. $q_A(\lambda x) \leq (r |\lambda|)^+ = |\lambda| r^+ = |\lambda| q_A(x)$. So we have proved $q_A(\lambda x) \leq |\lambda| q_A(x)$.

To prove the converse inequality (which is only needed for $\lambda \neq 0$) we observe that $|\lambda| q_A(x) = |\lambda| q_A(\lambda^{-1} x) \leq |\lambda| |\lambda^{-1} q_A(x) = q_A(x)$. Finally we prove the strong triangle inequality $q_A(x+y) \leq \max (q_A(x), q_A(y))$.

**Lemma 3.5** Let $p, q$ be $X_K$-seminorms on a $K$-vector space $E$. If $\{ x \in E : p(x) \leq 1 \} \subseteq \{ x \in E : q(x) \leq 1 \}$ then $p \geq q$.

**Proof.** By obvious scalar multiplication we have

$$\{ x \in E : p(x) \leq |\lambda| \} \subseteq \{ x \in E : q(x) \leq |\lambda| \}$$

for each $\lambda \in K^\times$. Then the above inclusion is also true for $\lambda = 0$. Now let $r^+ \in (0, \infty)^+$. From

$$\{ x \in E : p(x) \leq r^+ \} = \bigcap_{\lambda \in K, |\lambda| > r} \{ x \in E : p(x) \leq |\lambda| \}$$

and a similar formula for $q$ we obtain

$$\{ x \in E : p(x) \leq s \} \subseteq \{ x \in E : q(x) \leq s \}$$

for every $s \in X_K \cup \{0\}$. It follows that $q \leq p$.

**Corollary 3.6** Let $E$ be a $K$-vector space, let $p$ be an $X_K$-seminorm.

(i) If $A := \{ x \in E : p(x) \leq 1 \}$ then $p = q_A$.

(ii) Let $B : E \to E$ be a linear map. If $p(x) \leq 1$ implies $p(Bx) \leq 1$ for all $x \in E$ then $p(Bx) \leq p(x)$ for all $x \in E$. 

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Proof. (i) is a direct consequence of \( \{ x \in E : p(x) \leq 1 \} = \{ x \in E : p_A(x) \leq 1 \} \) and Lemma 3.5. For (ii) apply 3.5 to the seminorms \( p \) and \( p \circ B \).

**Proposition 3.7** Let \( n \in \mathbb{N} \), let \( p \) and \( q \) be \( X_K \)-seminorms on \( K^n \). Then there is a base of \( E \) that is both \( p \)- and \( q \)-orthogonal.

**Proof.** Like in the proof of 2.2 we prove the existence of an \( e \in K^n \setminus \{ 0 \} \) and an \((n-1)\)-dimensional subspace \( D \) such that \( K^n = Ke \oplus D \) where \( Ke \) and \( D \) are both \( p \)- and \( q \)-orthogonal, and we may assume that \( p \) is a norm. Let \( e_1, e_2, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see Remark following 3.3). For each \( i \in \{1, \ldots, n\} \) let \( C_i := \{ \lambda \in K : p(\lambda e_i) \leq 1 \} \) and \( A_i := C_i e_i \). Then by \( p \)-orthogonality

\[
\{ x \in K^n : p(x) \leq 1 \} = A_1 + \cdots + A_n.
\]

Now set \( l(A_i) := \{ t \in X_K \cup \{ 0 \} : \text{there is an } a \in A_i \text{ with } t \leq q(a) \} \). Then \( l(A_i) \) is an initial part of \( X_K \cup \{ 0 \} \), so \( l(A_1), \ldots, l(A_n) \) are linearly ordered by inclusion; let \( l(A_1) \) be the largest one. Set \( e := e_1 \). If \( l(A_1) = \{ 0 \} \) then \( q = 0 \) and we can take \( D = [e_2, \ldots, e_n] \), so assume \( q \neq 0 \) on \( A_1 \). Now let \( D \) be a \( q \)-orthogonal complement of \( Ke \) (Remark following 3.3) and let \( P : D + Ke \to D \) be the natural projection.

We finish the proof by showing that \( Ke \) and \( D \) are \( p \)-orthogonal, i.e. that \( p(x) \leq 1 \) implies \( p(Px) \leq 1 \) (**3.6** (ii)). Let \( x \in K^n, p(x) \leq 1 \). Then \( x = a_1 + \cdots + a_n \) where \( a_i \in A_i \) for each \( i \). We have, for each \( i \), \( q(a_i) \in l(A_i) \subset l(A_1) \), so \( q(a_i) \leq q(b) \) for some \( b \in A_1 \) and \( q(b) \neq 0 \). Then \( q(Pa_i) \leq q(a_i) \leq q(b) \). Now \( Pa_i \in [b] \) so \( Pa_i = \lambda b \) for some \( \lambda \in K \). We see that \( |\lambda| q(b) \leq q(b) \) implying \( |\lambda| \leq 1 \) by 3.2, so \( Pa_i \in A_i \). Then \( Px = \sum Pa_i \in A_1 \) i.e., \( p(Px) \leq 1 \), and we are done.

**Remark.** The above proof is valid for an \( X_K \)-seminorm \( p \) and an \( X \)-seminorm \( q \) for any \( G \)-module \( X \). I do not know whether the conclusion of 3.7 holds for an \( X \)-seminorm \( p \) and a \( Y \) seminorm \( q \) where \( X \) and \( Y \) are arbitrary \( G \)-modules.

The following corollary obtains.

**Theorem 3.8** (Let \( K \) be spherically complete and) let \( B \) be a \( B_K \)-module of finite rank. Then \( B \) is a direct sum of submodules of rank \( \leq 1 \).

**Proof.** The proofs of Proposition 2.3 and Corollary 2.4 can formally be taken over, where \( p_S \) and \( p_T \) are replaced by the \( X_K \)-seminorms \( q_S \) and \( q_T \) respectively.
References


