A NOTE ON POSSIBLE COUNTEREXAMPLES TO THE ABHYANKAR-SATHAYE CONJECTURE CONSTRUCTED BY SHPILRAIN AND YU

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In [SY99], Shpilrain and Yu construct a class of candidate counterexamples to the Embedding Conjecture of Abhyankar and Sathaye (see [Sat76] and [Abh78]).

**Proposition 1 ([SY99], Proposition 1.5).** Let $q$ be a polynomial over $\mathbb{C}$ in four variables and let $k$ be a positive integer. In $\mathbb{C}[x,y,z,t,u]$, define $\omega := x^k - tz^2 - uz$, $\alpha := \omega - (\omega^2 + yz)z$, and $\beta := y - (\omega^2 + yz)^2 z + 2\omega(\omega^2 + yz)$ and consider the following polynomial $p$:

$$p := x - q(\beta + \alpha^2, z, t + \beta, u - \beta).$$

Then the zero fiber of this polynomial is isomorphic to a coordinate hyperplane, i.e., $\mathbb{C}[x,y,z,t,u]/(p) \cong \mathbb{C}[x,y,z,t]$. If the Embedding Conjecture were true, then these polynomials $p$ are all coordinates, i.e., component of a polynomial automorphism $\mathbb{C}[x,y,z,t,u] \to \mathbb{C}[x,y,z,t,u]$. Shpilrain and Yu explain why this class of polynomials $p$ (for $k \geq 2$) could contain counterexamples to this conjecture.

However, this note shows that these polynomials $p$ are all coordinates. They even turn out to be tame.

**Proposition 2.** In the situation of the previous proposition, $p$ is a tame coordinate.

**Proof.** Write $\nu := \omega^2 + yz$, $A := \beta + \alpha^2$, $B := z$, $C := t + \beta$, and $D := u - \beta$. Note that $\alpha = \omega - \nu z$, $\beta = y - \nu^2 z + 2\omega \nu$, and $\omega = x^k - z(u + tz)$. Applying the (elementary) polynomial automorphism

$$u \mapsto u - tz \quad \text{(and} \quad x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t)$$

(1)
to the polynomials $\omega, \nu, \alpha, \beta, A, B, C,$ and $D$ transforms them into

\[
\begin{align*}
\omega_1 &:= x^k - zu, \\
\nu_1 &:= \omega_1^2 + yz = x^{2k} + z(y - 2x^ku + zu^2), \\
\alpha_1 &:= \omega_1 - \nu_1 z, \\
\beta_1 &:= y - \nu_1^2 z + 2\omega_1\nu_1, \\
A_1 &:= \beta_1 + \alpha_1^2, \\
B_1 &:= z, \\
C_1 &:= t + \beta_1, \quad \text{and} \\
D_1 &:= u - tz - \beta_1
\end{align*}
\]

respectively. Now applying the polynomial automorphism

\[y \mapsto y + 2x^k u - zu^2,\]

(2)

transforms these polynomials into

\[
\begin{align*}
\omega_2 &:= x^k - zu, \\
\nu_2 &:= x^{2k} + zy, \\
\alpha_2 &:= \omega_2 - \nu_2 z \\
&= x^k - z(u + x^{2k} + zy), \\
\beta_2 &:= y - \nu_2^2 z + 2\omega_2\nu_2, \\
A_2 &:= \beta_2 + \alpha_2^2, \\
B_2 &:= z, \\
C_2 &:= t + \beta_2, \quad \text{and} \\
D_2 &:= u - tz - \beta_2
\end{align*}
\]

respectively. Now applying the polynomial automorphism

\[u \mapsto u - x^{2k} - zy\]

(3)
transforms these polynomials into
\[
\begin{align*}
\omega_3 &:= x^k - z(u - x^2k - zy), \\
\nu_3 &:= x^2k + zy, \\
\alpha_3 &:= x^k - zu, \\
\beta_3 &:= y + 2x^k(u - x^2k - zy) - z(u - x^2k - zy)^2 - \nu_3^2z + 2\omega_3\nu_3, \\
A_3 &:= \beta_3 + \alpha_3^2, \\
B_3 &:= z, \\
C_3 &:= t + \beta_3, \quad \text{and} \\
D_3 &:= (u - x^2k - zy) - tz - \beta_3 \\
&= u - y - x^2k - 2x^ku - yz - tz + u^2z \\
&= u - y - x^2k - 2x^ku - z(t + y - u^2),
\end{align*}
\]
respectively. Now applying the polynomial automorphism
\[t \mapsto t - y + u^2\] transforms these polynomials into
\[
\begin{align*}
\omega_4 &:= x^k - z(u - x^2k - zy), \\
\nu_4 &:= x^2k + zy, \\
\alpha_4 &:= x^k - zu, \\
\beta_4 &:= y + 2x^k(u - x^2k - zy) - z(u - x^2k - zy)^2 - \nu_4^2z + 2\omega_4\nu_4, \\
A_4 &:= \beta_4 + \alpha_4^2, \\
B_4 &:= z, \\
C_4 &:= t - y + u^2 + \beta_4, \quad \text{and} \\
D_4 &:= u - y - x^2k - 2x^ku - zt,
\end{align*}
\]
respectively. Now applying the polynomial automorphism
\[y \mapsto -y + u - x^2k - tz - 2x^ku\] transforms the polynomials \(A_4, B_4, C_4, \) and \(D_4\) into
\[
\begin{align*}
A_5 &:= -y + u - 2x^ku - tz + u^2z^2 - u^2z, \\
B_5 &:= z, \\
C_5 &:= t + 2x^ku + u^2 - u^2z, \quad \text{and} \\
D_5 &:= y,
\end{align*}
\]
respectively. Now applying the polynomial automorphism
\[t \mapsto t - 2x^ku - u^2 + u^2z\]
transforms these polynomials into
\[
A_6 := -y + u - 2x^k uz - tz + 2x^k uz + u^2 z - u^2 z^2 + u^2 z^2 - u^2 z = u - y - tz,
\]
\[B_6 := z,\]
\[C_6 := t,\] and
\[D_6 := y,\]
respectively. Now applying the polynomial automorphism
\[
u \mapsto u + y + tz\] (7)
transforms these polynomials into \(u, z, t,\) and \(y\) respectively. Hence the polynomial
\[p = x - q(A,B,C,D)\]
is transformed into \(x - q(u,z,t,y)\) by successively applying these automorphisms. Finally applying the polynomial automorphism
\[x \mapsto x + q(u,z,t,y)\] (8)
then transforms it into \(x\). Since the polynomial automorphisms (1)-(8) are all elementary, \(p\) is a tame coordinate. □

References


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