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On the continuity property for an attractor of a semidynamical system with a parameter

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On the continuity property for an attractor of a semidynamical system with a parameter

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Abstract

An approach to verify continuity of a global attractor of a semidynamical system with a parameter is presented. This approach makes it possible to establish a connection between upper and lower semicontinuity of a global attractor and boundedness as a function of rate of attraction to an attractor. The obtained results were used for the scalar Lorenz wave equation in 3D space, the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

1 Introduction

In the present paper for a semigroup \( \{S_{\lambda}(t, \cdot)\} \) corresponding, for example, to a evolution equation, where \( \lambda \) is a problem parameter, and having global attractor the problem of stability of \( M_{\lambda} \) as \( \lambda \to \lambda_0 \) is considered. The most powerful test for the attractor \( M_{\lambda} \) being in \( O_\varepsilon(M_{\lambda_0}) \), with \( O_\varepsilon(M_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( M_{\lambda_0} \), was proved by Kapitanskii and Kostin.

The main purpose of the current work is to consider a criterion when in addition \( M_{\lambda_0} \subset O_\varepsilon(M_{\lambda}) \) and \( \varepsilon \to 0 \) as \( \lambda \to \lambda_0 \). The basis of this theorem are the characteristics of the function of rate of attraction \( \Psi(\lambda, t) \) in some small \( \delta \)-neighborhood of \( \lambda_0 \)

\[
\text{dist}(S_{\lambda}(t, B_\delta), M_{\lambda}) \leq \Psi(\lambda, t)\text{dist}(B_\delta, M_{\lambda}), \quad t \geq 0, \ \lambda \in O_\varepsilon(\lambda_0).
\]

Here \( B_\delta \) is a bounded absorbing set and \( S_{\lambda}(t, \cdot) \) is an approximation of the given nonlinear operator \( S_{\lambda_0}(t, \cdot) \).

These results are applied in Section 4 to the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

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2 Asymptotically compact semigroups

Let \( X \) be a Banach space with norm \( \| \cdot \| \), \( \Theta \) be a nontrivial subgroup of real number \( R \) and let \( \Theta_+ = \Theta \cap [0, +\infty[ \) be the intersection of \( \Theta \) and \( R_+ \). We shall deal with the abstract semigroup \( \{ X, \Theta_+, S(\cdot) \} \) of nonlinear operator \( S : X \times \Theta_+ \to X \). The term semigroup or semidynamical system refers to any family of single-valued continuous operator \( S \) depending on a parameter \( t \in \Theta_+ \) and enjoying the semigroup property:

\[
S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \Theta_+, \forall u \in X
\]

A Banach space \( X \) is a phase space of a semigroup, \( \Theta_+ \) is a time space and \( S(\cdot) \) is an evolution operator. When \( \Theta = R \) a semigroup is a semigroup with continuous time.

Let \( B \) and \( M \) be bounded subsets of \( X \). We say that \( B \) is attracted to \( M \) by the semigroup \( S(\cdot) \) if

\[
\text{dist}(S(t, B), M) \to 0 \quad \text{as} \quad t \to \infty
\]

Here

\[
\text{dist}(A, B) = \sup_{y \in A} \{ \text{dist}(y, B) \}, \quad \text{dist}(y, B) = \inf_{x \in B} \| x - y \|
\]

A set \( M \) is called an attracting set of the semigroup if \( M \) attracts each bounded \( B \subseteq X \). The minimal among the closed attracting sets is called the global attractor [9] (minimal global \( B \)-attractor [3]). The global attractor of a semigroup is defined as the set \( \mathcal{M} \) which is compact in \( X \), invariant for \( S(\cdot) \), i.e.

\[
S(t, \mathcal{M}) = \mathcal{M}, \quad t \geq 0
\]

and which attracts all the bounded sets of \( X \).

We need the following lemmas (see [5, 6]):

**Lemma 1** Let \( \mathcal{M} \) be a compact attractor of the semigroup \( \{ X, \Theta_+, S(\cdot) \} \), \( \hat{\mathcal{M}} \) be a compact attractor of the semigroup \( \{ X, \hat{\Theta}_+, \hat{S}(\cdot) \} \). Let \( \Theta_+ \) be a subsemigroup of \( \Theta_+ \) and let for some points \( t, \hat{t} \)

\[
S(t, u) = \hat{S}(\hat{t}, u) \quad \text{for all} \quad u \in X
\]

Then \( \mathcal{M} = \hat{\mathcal{M}} \).

Lemma 1 implies

**Lemma 2** Let \( \mathcal{M} \) be a compact attractor of the semigroup \( \{ X, R_+, S(\cdot) \} \) and let \( t_0 > 0 \). Then \( \mathcal{M} \) is attractor of the semigroup \( \{ X, t_0 Z_+, S(\cdot) \} \). Here \( t_0 Z_+ \equiv \{ kt_0, k \in Z_+ \} \).

Later on we need the following definitions, see [3].

A set \( B_\varepsilon \) is called absorbing if for each bounded \( B \subset X \) and for each \( \varepsilon > 0 \) there exists \( T = T(\varepsilon, B) \) such that

\[
S(t, B) \subset B_\varepsilon, \quad \forall t \geq T
\]
If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_\varepsilon(\mathcal{M})$ is an absorbing set. Here $O_\varepsilon(\mathcal{M})$ is the $\varepsilon$-neighbourhood of $\mathcal{M}$, i.e.

$$O_\varepsilon(\mathcal{M}) = \{ u : \exists v \in \mathcal{M}, \|u - v\| < \varepsilon \}$$

A semigroup is called bounded if for each bounded $B$ the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set $B_0$

$$\forall x \in X, \exists T(x) : S(t, x) \subset B_0, \text{ for any } t \geq T(x)$$

A semigroup is called asymptotically compact if for each bounded $B$ such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [3].

**Theorem 1** Let the semigroup $\{X, \Theta_+, S(\cdot)\}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_0)]_X$$

$\mathcal{M}$ is compact and invariant. If $X$ is connected then $\mathcal{M}$ is also connected.

We now summarize the results:

**Lemma 3** Let $S(t) : X \to X, t \in \mathbb{R}_+$ be a continuous semigroup possessing a non-empty compact attractor $\mathcal{M}$. Then this semigroup is an asymptotically compact semigroup.

**Proof.** We shall prove that each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

can be covered by a finite $\varepsilon$-network where $\varepsilon$ is arbitrary small positive number. The set $O_{\varepsilon/2}(\mathcal{M})$ is an absorbing set and for any bounded $B$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset O_{\varepsilon/2}(\mathcal{M}) \text{ for any } t \geq T.$$ 

Let us choose $N$ so large that the set $\{S(t_k, u_k)\}_{k=1}^N, u_k \in B$ belongs to $O_{\varepsilon/2}(\mathcal{M})$. The set

$$V^\mathcal{M} = \{ v^\mathcal{M} \in \mathcal{M} : \|v^\mathcal{M} - u_k\| \leq \varepsilon/2, k \geq N \} \subset \mathcal{M}$$

is a precompact set and may be covered by a finite $\varepsilon/2$-network $\bar{V}^\mathcal{M} = \{v_i \in V^\mathcal{M}\}_{i=1}^n$ and for any $k$ there exists $i$ such that:

$$\|v^\mathcal{M}_k - \bar{v}_i\| \leq \varepsilon/2, \quad k \geq N$$

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This implies
\[ \|u_k - \tilde{v}_i\| \leq \|u_k - v^M_k\| + \|v^M_k - \tilde{v}_i\| \leq \varepsilon. \]
Thus the set \( \mathcal{V}^M \) is a finite \( \varepsilon \)-network of the set \( \{S(t_k, u_k)\}^\infty_{k=1} \) and then the set \( \{S(t_k, u_k)\}^\infty_{k=1} \) is a compact set.

Note, if a semigroup possesses a nonempty attractor \( \mathcal{M} \) then the semigroup is pointwise dissipative.

3 Semigroups with a parameter

The current paragraph deals with the problem of stability of \( \mathcal{M} \) with respect to perturbations of the original operator \( S(\cdot) \).

Let us consider a semigroup \( S(\cdot) : X \to X \) which depends on a parameter \( \lambda \in \Lambda \). The main purpose is to consider a criterion when the set \( \mathcal{M}_{\lambda_0} \) and \( \mathcal{M}_\lambda \) are close to each other in the Hausdorff metric, that is

\[ \lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_{\lambda}, \mathcal{M}_{\lambda_0}) = 0 \]
\[ \lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = 0 \]

The relations (1) and (2) are usually referred to as the upper and lower semicontinuity of the attractor \( \mathcal{M}_\lambda \) in \( \lambda_0 \).

The most powerful test for the attractor \( \mathcal{M}_\lambda \) being in \( O_\varepsilon(\mathcal{M}_{\lambda_0}) \) with \( O_\varepsilon(\mathcal{M}_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( \mathcal{M}_{\lambda_0} \), was proved in [5].

We consider when in addition \( \mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_\lambda) \) and \( \varepsilon \to 0 \) as \( \lambda \to \lambda_0 \). The basis of this theorem [2] are the characteristics of the function of rate of attraction in a some small \( \delta \)-neighborhood of \( \lambda_0 \).

We assume that the following conditions (a) hold:

1. \( \Lambda \) is compact with metric \( \| \cdot \| \_\Lambda \) and \( \lambda_0 \) is a nonisolated point of \( \Lambda \).
2. For each \( \lambda \in \Lambda \) the semigroup \( \{X, \Theta_+, S(\cdot)\} \) possesses a pointwise absorbing set \( B_\lambda \) and non-empty attractor \( \mathcal{M}_\lambda \).
3. There exists a bounded absorbing set \( B_\alpha \) and for each \( \lambda \in \Lambda \) a set \( B_\lambda \) belongs to the set \( B_\alpha \).

By definition, each \( \varepsilon \)-neighbourhood \( O_\varepsilon(\mathcal{M}_\lambda) \) is an absorbing set. Assume that we know a function \( \Theta(\lambda, \varepsilon) = \Theta(\lambda, \varepsilon, B_\alpha) \) such that

\[ \text{dist}(S(\lambda, t, B_\alpha), \mathcal{M})) \leq \varepsilon, \quad \text{as} \ t \geq \Theta(\varepsilon, \lambda) \]

Lemma 4 Under the assumptions (a) let there exist \( \varepsilon > 0 \), \( \lambda_1, \lambda_2 \in \Lambda \) and a point \( T \geq \Theta(\lambda_1, \varepsilon) \) such that for the operators \( S(\lambda_1, t) \) and \( S(\lambda_2, t) \) the following estimate is valid

\[ \|S(\lambda_1, T, u) - S(\lambda_2, T, u)\| < \varepsilon \quad \text{for any} \ u \in B_\alpha \]

Then

\[ \mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_1}) \]
Remark. Without loss of generality, assume that $O_{2\varepsilon}(\mathcal{M}_\lambda) \subset B_a$.

Proof. Suppose that estimate (3) holds for a $T \geq \Theta(\lambda_1, \varepsilon)$ with some $\varepsilon > 0$. The set $\mathcal{M}_\lambda$ is the attractor for $\{X, \Theta, S_\lambda(\cdot)\}$ and for the $B_a$ we can find an attraction time $T = \Theta(\lambda_1, \varepsilon)$ to the $\varepsilon$-neighbourhood of $\mathcal{M}_\lambda$.

$$S_{\lambda_1}(T, u) \subset O_{\varepsilon}(\mathcal{M}_\lambda), \forall u \in B_a$$

Furthermore, we obtain $S_{\lambda_1}(T + t, u) \subset O_{\varepsilon}(\mathcal{M}_\lambda)$ for any $t \geq 0$. Due to (3) we have $\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon$. Hence

$$S_{\lambda_2}(T, u) \in O_{2\varepsilon}(\mathcal{M}_\lambda)$$

The above injection and $S_{\lambda_2}(T, u) \in B_a$ gives us

$$\|S_{\lambda_1}(T, S_{\lambda_2}(T, u)) - S_{\lambda_2}(T, S_{\lambda_2}(T, u))\| < \varepsilon \quad \text{for any} \ u \in B_a$$

Since $S_{\lambda_1}(T, v) \in O_{\varepsilon}(\mathcal{M}_\lambda)$ for each $v \in B_a$ we have $S_{\lambda_2}(T, S_{\lambda_2}(T, u)) = S_{\lambda_2}(2T, u) \in O_{2\varepsilon}(\mathcal{M}_\lambda)$. After a finite number of steps we obtain $S_{\lambda_2}(nT, u) \in O_{2\varepsilon}(\mathcal{M}_\lambda)$ as $n = 1, 2, ...$ for any $u \in B_a$. This, together with Lemma 1, implies

$$\mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_\lambda).$$

This completes the proof.

The next theorem provides an estimate for the distance between two attractors.

Theorem 2 Under the assumptions (a)

(i) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ and a point $T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon)$ such that

$$\|S_{\lambda}(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)\| < \varepsilon \quad \forall u \in B_a, \forall \lambda \in O_{\delta}(\lambda_0) \quad (4)$$

Then the attractor $\mathcal{M}_\lambda$ is upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon. \quad (5)$$

(ii) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

for arbitrary $\lambda \in O_{\delta}(\lambda_0)$ there exists a point $T_{\lambda} = T(\lambda) \geq \Theta(\lambda, \varepsilon)$ satisfies the following estimate

$$\|S_{\lambda}(T_{\lambda}, u) - S_{\lambda_0}(T_{\lambda}, u)\| < \varepsilon \quad \forall u \in B_a \quad (6)$$

Then the attractor $\mathcal{M}_{\lambda}$ is lower and upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\max \{\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda), \text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0})\} \leq 2\varepsilon. \quad (7)$$

Proof. According to (4) and assumption (i) we have that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for a $\lambda \in O_{\delta}(\lambda_0)$ we have $\mathcal{M}_\lambda \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_0})$. Thus,

$$\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) = \sup_{u \in \mathcal{M}_\lambda} \|u - \mathcal{M}_{\lambda_0}\| \leq 2\varepsilon.$$
As \( \varepsilon \to 0 \) we obtain \( \text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \to 0 \). Thus, inequality (5) is proved.

In order to prove (7), from (5) and (ii) we have

\[
\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon
\]

To obtain the reverse inequality

\[
\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) \leq 2\varepsilon \quad \text{as} \quad \lambda \to \lambda_0
\]

use the assumption (ii) for arbitrary \( \varepsilon > 0 \) and give \( O_\delta(\lambda_0) \) such that for any \( \lambda \in O_\delta(\lambda_0) \) inequality (6) holds in a point \( T_\lambda \geq \Theta(\lambda, \varepsilon) \). This, together with Lemma 4 implies

\[
\mathcal{M}_{\lambda_0} \subset O_{2\varepsilon}(\mathcal{M}_\lambda) \quad \forall \lambda \in O_\delta(\lambda_0).
\]

As \( \varepsilon \to 0 \) we have

\[
\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = \sup_{u \in \mathcal{M}_{\lambda_0}} (u, \mathcal{M}_\lambda) \to 0.
\]

This completes the proof.

**Remark 1.** For a semigroup with compact attractor upper semicontinuity was proved in [5] when (5) is valid in an some point \( T_{\lambda_0} > 0 \). But, we can not find the rate of convergence for \( \mathcal{M}_\lambda \to \mathcal{M}_{\lambda_0} \). When inequality (4) holds for \( t \in [\tau, \tau + T_{\lambda_0}] \) for any \( \tau > 0 \), uniformly in \( \tau \), then (5) was proved in [10].

**Remark 2.** The assumption (ii) allows \( T_\lambda \to \infty \) as \( \lambda \to \lambda_0 \).

**Remark 3.** A function rate of attraction \( \Theta(\cdot) \) can be approximate, for example see [11], for some nontrivial PDE system with nontrivial attractor.

A sequence of operators \( S_\lambda(\cdot) \) is called locally converging in a point \( \lambda_0 \) on a bounded \( B \) if for any \( \varepsilon > 0 \) and for each \( T > 0 \) there exists \( \delta \) such that

\[
||S_\lambda(T, u) - S_{\lambda_0}(T, u)|| \leq \varepsilon \quad \forall \lambda \in O_\delta(\lambda_0), \forall u \in B.
\]

Theorem 2 and the above definition imply

**Theorem 3** Under the assumptions (a) let the sequence of operators \( S_\lambda(\cdot) \) be locally converging in a point \( \lambda_0 \) on a set \( B_0 \). Let the function \( \Theta(\lambda, \varepsilon) \) be uniformly bounded on \( \lambda \) for each fixed \( \varepsilon > 0 \), i.e.

\[
\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \ \sup_{\lambda \in O_\delta(\lambda_0)} \Theta(\lambda, \varepsilon, B_\delta) \leq T_\varepsilon < \infty.
\]

Then, the attractor \( \mathcal{M}_\lambda \) depends on \( \lambda \) in the point \( \lambda_0 \) continuously.

**Remark.** In this way, if we cannot calculate a function \( \Theta(\lambda, \varepsilon) \), but in a some small neighbourhood of the \( \lambda_0 \) we may find above estimate for \( \Theta(\cdot) \) and sequence of operators \( S_\lambda \) is locally converging to the \( S_{\lambda_0} \) as \( \lambda \to \lambda_0 \) then the attractor \( \mathcal{M}_{\lambda_0} \) is continuous in \( \lambda_0 \).

A sequence of operators \( S_\lambda(\cdot) \) is called globally converging in a point \( \lambda_0 \) on a bounded set \( B \) if for any \( \varepsilon > 0 \) and for any \( T > 0 \) there exists \( \delta \) such that the following estimate holds

\[
||S_\lambda(t, u) - S_{\lambda_0}(t, u)|| \leq \varepsilon \quad \forall \lambda \in O_\delta(\lambda_0), \forall u \in B, \ 0 \leq t \leq T.
\]
Remark. For the ODE \( y'(t) = -\alpha_n y(t) \), as \( \alpha_n \to 0^+ \), we have globally converging operators. It is easy to verify that the attractor \( \mathcal{M}_0 \) in \( \lambda_0 = 0 \) is upper semicontinuous but not lower semicontinuous.

Let us prove a criterion (see [8]) when an attractor for sequence of globally converging operators is continuous (lower and upper semicontinuous) in a point \( \lambda_0 \).

**Theorem 4** Under the assumptions (a) assume, that a sequence of operators \( S_{\lambda}(\cdot) \) is globally converging in \( \lambda_0 \) on the set \( B_\varepsilon \). Then the attractor \( \mathcal{M}_\lambda \) depends continuously on \( \lambda \) in the point \( \lambda_0 \) if and only if a function \( \Theta(\lambda, \varepsilon) \) is uniformly bounded on \( \lambda \) for each fixed \( \varepsilon > 0 \), i.e.

\[
\forall \varepsilon > 0 \exists \varepsilon > 0 : \sup_{\lambda \in O_{\delta_2}(\lambda_0)} \Theta(\lambda, \varepsilon, B_\varepsilon) \leq T_\varepsilon < \infty. \tag{9}
\]

**Proof.** Theorem 2 together with (8), (9) implies the continuity of the attractor. Suppose, that attractor depends continuously on \( \lambda \) in \( \lambda_0 \). This implies

\[
\forall \varepsilon > 0 \exists \delta_1(\varepsilon) > 0 : \mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_{\lambda_0}), \mathcal{M}_{\lambda} \subset O_\varepsilon(\mathcal{M}_{\lambda_0}), \forall \lambda \in O_{\delta_1}(\lambda_0). \tag{10}
\]

Under the definition of a function \( \Theta(\cdot) \) for \( T = \Theta(\lambda_0, \varepsilon) \) we have

\[
S_{\lambda_0}(T + \tau, u) \subset O_\varepsilon(\mathcal{M}_{\lambda_0}), \quad \forall \tau > 0.
\]

This, together with (8), implies that there exists \( \delta_2(\varepsilon, T) \) such that

\[
\|S_{\lambda}(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \quad \forall \lambda \in O_{\delta_2}(\lambda_0), \forall u \in B, t \leq T.
\]

Choose \( \delta \) in the following way \( \delta = \min\{\delta_1, \delta_2\} \). Then

\[
S_{\lambda_0}(T, B_\delta) \subset O_{\delta}(\mathcal{M}_{\lambda_0}) \subset O_{2\delta}(\mathcal{M}_{\lambda}) \forall \lambda \in O_\delta(\lambda_0).
\]

This, together with (10), gives us

\[
S_{\lambda_0}(T, B_\delta) \subset O_{\delta}(\mathcal{M}_{\lambda}) \forall \lambda \in O_\delta(\lambda_0). \tag{11}
\]

Combine (11) with (10) for any \( \tau > 0 \) and use the same arguments as in the proof of Theorem 2 to obtain

\[
S_{\lambda}(T + \tau, B_\delta) \subset O_{\delta}(\mathcal{M}_{\lambda}) \forall \lambda \in O_\delta(\lambda_0), \tau > 0.
\]

The last estimate means that function \( \Theta(\lambda_0, \varepsilon) \) is a function of attraction to the \( 3\varepsilon \)-neighbourhood of the attractor \( \mathcal{M}_\lambda \) for any \( \lambda \in O_\delta(\lambda_0) \). Thus, the function \( \Theta(\lambda, \varepsilon) \leq \Theta(\lambda_0, \varepsilon/3) < \infty \) for \( \forall \lambda \in O_\delta(\lambda_0) \), which proved the Theorem.

4 On a function of rate of attraction to a attractor

Here we consider the example of a semigroup having global attractor and known function of rate of attraction to attractor, namely, 1D Chafee-Infante problem

\[
u_t = u_{xx} + bu - f(u), \quad b > 0, \quad f(\cdot) \in C^1(\cdot)
\]
\[ u(0,t) = u(\pi,t) = 0; \quad u(x,0) = u_0(x) \in H^1_0(0,\pi) \]
\[ f(0) = f'(0) = 0, \quad s^{-1}f(s) < f'(s), s \neq 0; \quad S^{-1}f(s) \to +\infty, \text{ as } |s| \to \infty. \]

The Chafee-Infante problem generated semigroup in the phase space \( H^1_0(0,\pi) \). The following theorem was given in [6].

**Theorem 5** Under the above assumptions, if \( b \neq m^2, m = 1, 2, \ldots \), then the attractor \( \mathcal{M} \) attracts its neighbourhood exponentially

\[ \text{dist}(S(t,B),\mathcal{M}) \leq C e^{-at}, \quad C, a = \text{const} > 0 \]

If \( b = m^2 \) and in addition

\[ K^{-1}(|s| + |t|)^q(s-t)^2 \leq (f(s) - f(t))(s-t) \leq K(|s| + |t|)^q(s-t)^2 \]

for some positive \( K, q, s_0 \) and all \( s, t \in [-s_0, s_0] \), then the attractor \( \mathcal{M} \) attracts its neighbourhood polynomially

\[ \text{dist}(S(t,B),\mathcal{M}) \leq D(1 + t)^{-\alpha}, \quad D, \alpha = \text{const} > 0. \]

This, together with Theorem 4, implies continuity of the attractor \( \mathcal{M} \) under different types of perturbations of the original operator \( S(\cdot) \). For example

\[ u_t = u_{xx} + (b + \delta)u - f(u) + \delta_1(u) \]

or difference approximations with steps \( \tau, h \), when the constant \( a, \alpha, C, D \) are bounded as \( \delta, \delta_1(\cdot), \tau, h \to 0 \) (see [12]).

Let us consider the Navier-Stokes equations

\[ \frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u = -\nabla p + f, \quad \nu \neq 0 \]
\[ \text{div } u = 0 \]
\[ u|_{\partial \Omega} = 0; \quad u(t = 0, x) = u_0(x) \]

in an arbitrary bounded \( \Omega \subset \mathbb{R}^2 \). Here \( x = (x_1, x_2) \), \( u = (u_1, u_2) \) is the velocity vector, \( p \) is the pressure, \( \nu = \text{const} > 0 \). Existence of the global attractor \( \mathcal{M} \) for (12),(13) was first studied by Ladyzhenskaya in [2]. In [1] it was proved, that some approximations of (12),(13) have global attractors \( \mathcal{M}_{\tau,h} \) in \( \varepsilon \)-neighborhoods of \( \mathcal{M} \). Babin an Vishik in [4] established a theorem on upper semicontinuity of the attractor \( \mathcal{M}_\lambda \) for \( \lambda = (\nu, f) \). Following [2, 1] it is easy to verify the assumptions (a) when \( \lambda = (\nu, f)(\omega \lambda = (\tau, h)) \) and globally converging \( S_\lambda(\cdot) \) to \( S_{\lambda_0} \) when \( \lambda \to \lambda_0 \). The properties of the function \( \Theta(\cdot) \) for the problem (12),(13) is unknown. However, when the operator \( S_\lambda(\cdot) \) is upper semicontinuous on a compact set \( \Lambda \) then it is continuous on a some everywhere dense subset \( \Lambda \subset \Lambda \). Thus, for any \( \lambda \) and each \( \varepsilon > 0 \) there exists \( \lambda_0 \) and \( \delta = \delta(\varepsilon) \) such that

\[ ||\lambda - \lambda_0|| < \varepsilon \quad \text{and} \quad \sup_{\lambda_0 \in \mathcal{O}_\delta(\lambda_0)} \Theta(\lambda_0, \varepsilon) \leq T_\varepsilon < \infty. \]
The above results are valid (see [7]) for some modifications of the systems (12), (13) in 3D case.

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