The version of the following full text has not yet been defined or was untraceable and may differ from the publisher's version.

For additional information about this publication click this link.
http://hdl.handle.net/2066/18679

Please be advised that this information was generated on 2017-07-22 and may be subject to change.
Symmetric functions, I:

D.C. van Leijenhorst

Computing Science Institute/

CSI-R9803 January 1998
Symmetric functions, I:

“How symmetric can a function be?”

D.C. van Leijenhorst

March 23, 1998

ABSTRACT

The symmetric complexity of a polynomial $f$ in $n$ variables is defined as the number of times the symmetric function theorem is applicable. In this paper a sharp upper bound on this measure is derived by a matrix method.

1 Introduction

Consider a field $K$ of characteristic 0, and let $R$ be the ring $K[x_1, \ldots, x_n]$ where $n$ is > 0.

A symmetric function is any element of $R$ invariant under the symmetric group acting as coordinate permutations. Examples are the elementary symmetric functions: $a_0 = 1$, $a_i = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} x_{j_1} x_{j_2} \cdots x_{j_i}$, $(1 \leq i \leq n)$; $a_i = 0$ ($i < 0$ or $i > n$).
The Symmetric Function Theorem [4], [5] states that any symmetric function \( f \) can be uniquely written as \( g(a_1, \ldots, a_n) \) for some \( g = g(x_1, \ldots, x_n) \) from \( \mathbb{R} \), called the symmetric representation of \( f \).

Here we shall address the question of what happens when this \( g \) is symmetric again. This is of course perfectly possible and if it occurs \( k - 1 \) times, \( f \) is called \( k \)-fold symmetric. That is:

**Definition 1** A polynomial \( f \) in \( n \) variables is 0-fold symmetric if \( f \) is not symmetric; and \( k \)-fold symmetric with \( k > 0 \) if \( f \) is symmetric and the symmetric representation of \( f \) is \( k - 1 \)-fold symmetric. The number \( k \) is called the symmetric complexity of \( f \).

A \( k \)-fold symmetric function \( f \) possesses a high degree of symmetry indeed, and it is an interesting complexity problem to find a bound on \( k \) expressed in the coefficients and exponents of \( f \). Such a result is given in Theorem 1. Our method is based on term orderings and the like, familiar from Groebner basis theory [3]. Thus it is possible to translate the problem into linear algebra, involving the explicit calculation of the spectrum and eigenvectors of a matrix.

Another interesting question that arises in a natural way in this context is: how can we describe the behavior (e.g., fixpoints) of the iteration \((x_1, \ldots, x_n) \rightarrow (a_1, \ldots, a_n)\)? We shall restrict ourselves to a numerical example for \( n = 4 \) (see Section 4).
2 Notations and generalities

Put $x = (x_1, \ldots, x_n)$; let $a_i = a_i(x)$ be defined as above, and let $a = (a_1, \ldots, a_n)$.

Stretching notation a bit, we can view $\{c_1, \ldots, c_n\} \rightarrow a(c)$ as a mapping from the unordered lists of length $n$ over $K$ to $K^n$, which is a bijection if $K$ is algebraically closed. Indeed, one has

$$\sum_{i=0}^{n} a_i(c_1, \ldots, c_n)T^i = \prod_{i=0}^{n}(c_iT + 1).$$

Instead of this however we shall consider the simpler mappings $c \rightarrow a(c)$ from $K^n$ to $K^n$ and $a : x \rightarrow a(x)$ from $R$ to $R$.

**Definition 2** Let $a^0 = (x_1, \ldots, x_n)$; and for $k > 0$ define $a^k = (a_1^k, a_2^k, \ldots, a_n^k)$ where $a_i^k = a_i^k(x) = a_i(a_1^{k-1}, a_2^{k-1}, \ldots, a_n^{k-1})$, $1 \leq i \leq n$.

The $a_i^k$ are called the *iterated elementary symmetric functions* (iesf’s.) An interesting fact is given by

**Lemma 1** For all $k \geq 1$, the iesf’s $a_1^k, a_2^k, \ldots, a_n^k$ are algebraically independent.

**Proof.** Induction w.r.t. $k$. For $k = 1$ this is well-known [5]. Now let $f(y_1, \ldots, y_n)$ be such that $f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0$ in $R$.

By definition of the $a_i^k$’s, there exists a symmetric polynomial $g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z))$ with $g(a_1^k, a_2^k, \ldots, a_n^k) = f(a_1^{k+1}, a_2^{k+1}, \ldots, a_n^{k+1}) = 0$; hence $g(z_1, \ldots, z_n) = 0$ by the induction hypothesis. But now we are in the case $k = 1$ again, since $g(z_1, \ldots, z_n) = f(a_1(z), a_2(z), \ldots, a_n(z))$ and it follows that $f(y_1, \ldots, y_n) = 0$. ■
A term is any monomial \( t = x_1^{i_1}x_2^{i_2} \ldots x_n^{i_n} \). Its total degree is \( tdeg(t) = \sum_{j=1}^n i_j \) and the total degree \( tdeg(f) \) of \( f \in R \) is \max_{t \in f} tdeg(t) \) (which of course is equal to \( tdeg(t) \), any \( t \) in \( f \) if \( f \) is symmetric.)

An admissible ordering [3] on the set \( T \) of terms in \( R \) is a total order on \( T \) that satisfies:

\[ 1 < t; \text{ and } t < t' \Rightarrow st < st' \text{ for all terms } s, t, t'. \]

The latter property is called monotonicity of term multiplication.

An admissible ordering is a well-ordering. Admissible orders abound and have been classified; well-known examples are the lexicographic orders and various total degree orderings like the "grevlex" [3].

For a given ordering, the leading term \( lt(f) \) of \( f \) is the highest term occurring in \( f \).

3 The main theorem

Our main result is given by

**Theorem 1** Let \( f \) be any non-constant \( k \)-fold symmetric polynomial in \( n \geq 2 \) variables. Then the symmetric complexity \( k \) is bounded by:

\[
tdeg(f) \geq \frac{\binom{2n+1}{k-1}}{\pi^{k-1}} \cdot 1.149 - 1.048(0.53)^{k-1}
\]
Remark: This bound is fairly precise: it is an approximation of a more complex bound, which is sharp in the sense that it is reached by $f = a_1^k$. This will follow from the proof.

First let us give an outline of the proof. The idea is very simple and consists of three steps.

**i.** If $k$ increases, one observes that the iesf's $a_1^k$ grow very quickly in "size".
To measure this size, we consider the highest term $t_1^k$ of $a_1^k$ in an admissible ordering.

Remark: Explicit calculation of the complete $a_1^k$'s in Maple, say, leads to considerable memory problems. A piece of code to experiment with is given on the WWW at http://www.cs.kun.nl/bolke/ksymmaple.

**ii.** Next, we shall be able to estimate the exponents occurring in $t_1^k$; this is the technical part.

**iii.** Finally, for a given $f$ of complexity $k$ we shall show that for some $i$, a term $t_1^k$ actually occurs in $f$ as $lt(f)$. Hence, $k$ is bounded as a function of $lt(f)$, and this ends the proof.

As an admissible ordering on $T$, take the lexicographic order with $x_1 > x_2 > \ldots > x_n$. Let $t_1^k$ be $lt(a_1^k)$. We shall derive a recursion for $t_1^k$.

**Lemma 2 a.**

$t_1^k = t_{n-1}^k \cdots t_{i+1}^k$ (if $k > 1$)

**b.** If $p > q$, $t_p^k > t_q^k$ (if $k \geq 1$).

**Proof:**

For $k = 1$, statement $b.$ holds. Indeed, $t_1^k = lt(a_1) = x_1x_2\ldots x_i$. Also,
a. holds trivially. Now if for any \( k \) \( a \) and \( b \) are true, then by definition one has \( a_{i}^{k+1} = \sum_{1 \leq j_{1} < j_{2} < \cdots < j_{k} \leq n} a_{i_{1}}^{k} a_{i_{2}}^{k} \cdots a_{i_{k}}^{k} \). All coefficients are positive, so no terms cancel. By the monotonicity property, \( lt(a_{j_{1}}^{k} a_{j_{2}}^{k} \cdots a_{j_{k}}^{k}) = lt(a_{j_{1}}^{k}) lt(a_{j_{2}}^{k}) \cdots lt(a_{j_{k}}^{k}) = t_{j_{1}}^{k} t_{j_{2}}^{k} \cdots t_{j_{k}}^{k} \). Since \( b \) holds and, again, by monotonicity, this is maximal if \( j_{1} = n, j_{i-1} = n - 1, \ldots, j_{1} = n - i + 1 \). This proves \( a \) for index \( k + 1 \). But then, if \( p > q \) one has \( t_{p}^{k+1} > t_{q}^{k+1} \) since the r.h.s. divides the l.h.s. Hence \( b \) holds as well.

In part ii. of the proof, we shall estimate the size of the exponents in \( t_{i}^{k} \).

**Definition 3** The exponents vector \( ev(t) \) of a term \( t = x_{1}^{i_{1}} \ldots x_{n}^{i_{n}} \) is \( i = (i_{1}, i_{2}, \ldots, i_{n}) \). We denote \( ev(t_{i}^{k}) \) by \( e_{i}^{k} = (e_{i_{1}}^{k}, \ldots, e_{i_{n}}^{k}) \).

One has \( e_{i}^{0} = (1, 1, \ldots, 1, 0, \ldots, 0) \) \( (i \ \text{ones}) \). Define \( E_{k} \) to be the matrix having the \( e_{i}^{k} \)'s as its columns; note that \( E_{1} = U \), the upper triangular all-one matrix.

**Lemma 3** a. Let \( t = x_{1}^{i_{1}} \ldots x_{n}^{i_{n}} \) be any term; then for all \( k \geq 1 \) the exponents vector of \( lt(t(a_{1}^{k}, \ldots, a_{n}^{k})) \) equals \( E_{k}(i) \).

b. Let \( D \) be the symmetric matrix with ones below and on the antidiagonal and zeroes above; put \( D^{k} = (d_{i,j}), \; 1 \leq i,j \leq n \). Let \( U \) be the upper triangular all-one matrix. Then \( E_{k} = U D^{k-1} \). Hence \( E_{k} \) is nonsingular and for \( k \geq 1 \) one has: \( e_{i}^{k} = \sum_{j=1}^{n} d_{k-1,j} \).

**Proof:**

By monotonicity, \( lt((a_{1}^{k})^{i_{1}} \ldots (a_{n}^{k})^{i_{n}}) = (t_{1}^{k})^{i_{1}} \ldots (t_{n}^{k})^{i_{n}}, \) the exponents vector of
which is $E_k(i)$ by linearity. This proves part $a$. 

For part $b.$, note that statement $a.$ of Lemma 2 can be written as: $\xi^k = \xi_{n-1}^k + \xi_{n-2}^k + \ldots + \xi_{n-i+1}^k$, which is equivalent to $E_k = E_{k-1}D$. So $E_k = E_1D^{k-1} = UD^{k-1}$. 

In the Corollary to Proposition 2 we shall find an explicit solution to this recursion.

Before analyzing this, let us first proceed to part $iii$. Suppose that $f$ is not constant and $k$-fold symmetric, $k \geq 1$. We wish to prove that some $t^k_i$ really occurs in $f$.

By definition, there exists $f_k \in R$ such that $f_k(a_1^k, \ldots, a_n^k) = f$ (though we shall not need it, note that $f_k$ is unique by Lemma 1. ) Let $t = x_{i_1}^1 \ldots x_{i_n}^k$ be a term of the polynomial $f_k(x)$ such that $\tau =_{D \in f} \text{lt}((a_1^k)^{i_1}, \ldots, (a_n^k)^{i_n})$ is maximal in the term ordering. By Lemma 3, $ev(\tau) = E_k(i_1, \ldots, i_n) = E_k(i)$. 

First note that $\tau$ is unique. Indeed, suppose that besides $t$ there is another term $s = x_{j_1}^1 \ldots x_{j_n}^k$ yielding the same $\tau$, then by lemma 3 one would have $E_k(i) = E_k(j)$ (with $i = (j_1, \ldots, j_n)$); hence $E_k(i - j) = 0$. But $E_k$ was nonsingular so $i = j$ and $s = t$. 

7
Also, $\tau$ does not cancel when $f_k(a^1, \ldots, a^n)$ is expanded to $f$. Otherwise, there would be some term $s$ in $f_k$ and a term $\sigma$ from $s(a^1, \ldots, a^n)$ such that $\tau = \sigma$. (N.b. all these terms are in $R$, i.e. of the form $x_1^{p_1} \ldots x_n^{p_n}$.) Then however, $\sigma \leq \ell t(s(a^k)) < \ell t(t(a^k))$. This contradicts the unicity of $t$ and the maximality of $\tau$.

We conclude that $\tau = \ell t(f)$. This shows what we wanted, namely that some $t_k^j$ occurs in $t$, hence in $f$. ■

In fact we have proven more, namely:

**Proposition 1** Let $U$ be the upper triangular all-one matrix and $D$ the (symmetric) lower antitriangular all-one matrix. Then for any $k$-fold symmetric function $f$ and $k \geq 1$,

$$\text{ev}(\ell t(f)) \in UD^{k-1}((\mathbb{N} \cup \{0\})^n).$$

How good is this? In order to answer this question let us give an estimate of the entries of powers of $D$.

For $p = 1, 2, \ldots, n$ let us define the following quantities:

$$w_p = -\frac{2^n}{2^{n+1}};$$

$$\alpha_p = w_p + w_p^{-1} = -2\cos\left(\frac{2p\pi}{2n+1}\right); \quad V_p = w_p - w_p^{-1} = 2\sin\left(\frac{2p\pi}{2n+1}\right);$$

$$\lambda_p = 4\cos^2\left(\frac{p\pi}{2n+1}\right); \quad \mu_p = (-1)^n / 2\cos\left(\frac{2p\pi}{2n+1}\right);$$

$$x_m^p = 2(-1)^{m+1} \frac{\sin\left(\frac{2m\pi}{2n+1}\right)}{\sqrt{2n+1}} \quad (m = 1, 2, \ldots, n); \quad \vec{x}^p = (x_1^p, x_2^p, \ldots, x_n^p).$$
These numbers satisfy the relations:

\[ \lambda_p = 2 + \alpha_p; \quad V_p^2 = \alpha_p^2 - 4; \quad w_p^{2n+1} = -1; \]

\[ \mu_p = \frac{1}{(e_p^2 + e_p^{-2})}; \quad \mu_p^{-2} = \lambda_p; \quad x_p^m = \frac{(w_p^{m} - w_p^{-m})}{i\sqrt{2m+1}} \quad (m = 1, 2, \ldots, n); \text{ also,} \]

\[ w_p = \frac{(\alpha_p + V_p)}{2} \quad \text{and} \quad w_p^{-1} = \frac{(\alpha_p - V_p)}{2} \quad \text{are the roots of} \quad X^2 - \alpha_p X + 1 = 0. \]

Let \( \langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i} \) be the standard Hermitian inner product. It is elementary to verify that the \( x_p \) are perpendicular of length 1. Now one has:

**Proposition 2** The vectors \( x_p \) form an orthonormal basis upon which the matrix \( D \) assumes a diagonal form \( \Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n) \).

*Proof.*

Since the proof is fairly standard, let us just outline it. One easily verifies that the inverse of \( D \) is the matrix with ones on the antidiagonal, -1’s just above it, and zeroes elsewhere. Next, its square \( D^{-2} \) is seen to be tridiagonal:

\[ (D^{-2})_{i,i} = 2 \quad (i < n); \quad (D^{-2})_{n,n} = 1; \quad (D^{-2})_{i,j} = -1 \quad (|i - j| = 1). \]

Tridiagonal matrices have been studied extensively in the theory of orthogonal polynomials [2] and the numerical theory of parabolic differential equations.

\[ D^{-2}, \text{ being symmetric, can be diagonalized on a real orthonormal basis. Let} \]
\[ z = (z_1, z_2, \ldots, z_n) \text{ be an eigenvector of} \ D^{-2} \ \text{with eigenvalue} \ \lambda. \ \text{Put} \ z = z(\alpha), \]

again with \( \alpha = 2 - \lambda \). W.l.o.g, let \( z_1 = 1 \) and let \( z_0 = Dz \ 0 \). Then \( (D^{-2} - \lambda I)z = 0 \) amounts to the recursion
\[ z_0 = 0; \quad z_1 = 1; \]
\[ z_m = \alpha z_{m-1} - z_{m-2} \quad (1 < m \leq n); \]
\[ -z_{n-1} + (\alpha - 1)z_n = 0 \]

(the latter being the characteristic equation.)

**Remark:** this is the familiar recursion of the Tchebycev polynomials \( T_{m-1}(x) \) in \( x = \frac{\alpha}{2} \), though these have initial values \( T_0 = 1, T_1 = x \). In fact it is not difficult to prove that \( z_m = \frac{\frac{\pi}{2} T_m(y) - T_{m-1}(y))}{((\frac{\alpha}{2})^2 - 1)}. \)

Let \( V = \sqrt{(\alpha^2 - 4)} \) and \( w = (\alpha + V)/2, \; w' = (\alpha - V)/2 \), the roots of \( X^2 - \alpha X + 1 = 0 \). If \( w = w', \; \alpha = \pm 2 \); but then \( z_m = (\pm1)^{m-1} m, \; -z_{n-1} + (\alpha - 1)z_n \neq 0 \), and there are no eigenvalues. So suppose \( w \neq w' \).

Solving the recursion by standard techniques yields \( z_m = \frac{w^{m+w^{-m}}}{V}; \; 1 \leq m \leq n \). By some easy calculations, the eigenvalue equation \( -z_{n-1} + (\alpha - 1)z_n = 0 \) reduces to \( w^{2n+1} = -1 \) (where \( w \neq -1 \) since \( w \neq w' \)). From this, \( w = -e^\frac{2\pi}{2n+1}, \; p = 1, 2, \ldots, n \). We shall now take this \( p \) as an index (i.e., use \( \alpha_p, \; \lambda_p, \; \mu_p, \; w_p, \; V_p, \; z_p, \; x_p, \; x_p \)).

The numbers and vectors \( \alpha_p, \; \lambda_p, \; \mu_p, \; w_p, \; V_p, \; x_p, \; x_p \) are in fact those defined earlier. Normalization of \( V_p z_p \) yields the \( p^{th} \) eigenvector \( \mathbf{z}^p \) as \( x^p_m = 2(-1)^{m+1} \frac{\sin(\frac{2\pi p}{2n+1})}{\sqrt{2n+1}} \). Similarly, one finds the formulas for \( \alpha_p, \; \lambda_p \) etc.

The \( \mathbf{z}^p(\alpha) \) form an orthogonal eigenbasis over which the symmetric matrix \( D^{-2} \) diagonalizes. But in fact by an easy calculation, \( D^{-1} \mathbf{z}^p = \mu_p^{-1} \mathbf{z}^p \); hence
$D^{-1}$ and $D$ diagonalize as well. This ends the proof.

Note that the eigenvalues $\mu_p$ of $D$ are all different and $\max_p |\mu_p| = |\mu_n| = \frac{1}{2\cos(\frac{2\pi}{2n+1})}$. Also, $\text{sign} \mu_p = (-1)^{n+p}$ (consider $pn \mod 2n+1$ for $p$ odd and $p$ even).

Corollary

The (nonnegative integral) entries of $D^k$ are given in closed form by the formula

$$(D^k)_{i,j} = \sum_{p=1}^{n} (-1)^{i+j+(n+p)k} \frac{\sin(\frac{2p\pi}{2n+1})\sin(\frac{2p\pi i}{2n+1})}{(2n+1)2^{k-2}\cos^k(\frac{p\pi}{2n+1})}$$

Proof:

As before, let $\Delta = \text{Diag}(\mu_1, \mu_2, \ldots, \mu_n)$. Let $S$ be the orthogonal basis transformation matrix with columns $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ and let $S^T (= S^{-1})$ be its transpose. Then $D^k = S\Delta^k S^T$ and, thus, $(D^k)_{i,j} = \sum_{p=1}^{n} (\mu_p^k x_i^p x_j^p)$. Substitution of our earlier expressions now yields the desired formula.

This also is the explicit solution of the recursion for the exponents vectors $e_i^k$.

Remark: The following very nice graph-theoretic argument to find the eigenvalues of the matrix $D$ was communicated by A. Blokhuis, A.E. Brouwer and R. Riebeek [7].
Let \( N = (-1)^n D^{-1} \). We can write \( N = A - B \), where both \( A \) and \( B \) are 0–1 matrices (and \( A \) and \( B \) are zero wherever \( N \) is zero). With \( P = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \) we see that \( P \) is the adjacency matrix of a path of length \( 2n \). Each eigenvector \( u \) of \( N \) with eigenvalue \( \theta \) yields an antisymmetric eigenvector \( \begin{pmatrix} u \\ -u \end{pmatrix} \) of \( P \) with eigenvalue \( \theta \), and conversely. But the antisymmetric eigenvectors of \( P \) are precisely those that can be extended to eigenvectors of a \((2n + 1)\)-cycle by defining it to be zero on the additional point. It follows that the eigenvalues are \( \theta = 2 \cos \frac{2\pi j}{2n+1} \), where \( 1 \leq j \leq n \) from which those of \( D \) follow.

All calculations involving the matrix \( D \) have been checked for specific cases using Maple. A collection of appropriate Maple statements can be found on the WWW at http://www.cs.kun.nl/ bolke/ksymmaple.

In order to prove Theorem 1, we have to estimate the total degree of \( h(f) \), which in view of Proposition 1 can be written as \( \langle UD^{k-1}(i), j \rangle \) for some nonzero vector \( i \) over \( \mathbb{N} \cup \{0\} \) and with \( j \) the all-one vector.

Write this as \( \langle D(i), D^{k-2}U^T(j) \rangle = \langle D(i), D^{k-2}(1, 2, \ldots, n) \rangle \). Note that \( D(i) \) has at least one positive entry, namely the \( n^{th} \). Hence,

\[
\text{tdeg } f \geq \sum_{q=1}^{\nu} q\langle D^{k-2}\rangle_{u,q}.
\]

(equality occurs if \( i = (1, 0, \ldots, 0) \); e.g, if \( f = a^1_i \).)
Put $t = k - 2$. By the Corollary,

$$\sum_{q=1}^{n} q(D^{k-2})_{n,q} = \sum_{q=1}^{n} q \sum_{p=1}^{n} (-1)^{n+q+(n+p)t} \frac{\sin \left( \frac{2\pi n \pi}{2n+1} \right) \sin \left( \frac{2\pi \pi}{2n+1} \right)}{(2n+1)2^{n-2} \cos^{2} \left( \frac{\pi}{2n+1} \right)}.$$  

The summation over the index $q$ can easily, though tediously, be calculated explicitly (e.g., using the complex form of the sine or with the help of a computer algebra package like Maple).

The double sum then reduces to:

$$\frac{(-1)^{n} t}{(2n+1)2^{t}} \sum_{p=1}^{n} (-1)^{t} \frac{\sin \left( \frac{2\pi n \pi}{2n+1} \right)^{2} \cos^{t+2} \left( \frac{\pi}{2n+1} \right)}{(2n+1)2^{n+1}}.$$  

The largest term occurs for $p = n$ and we shall see that in fact this term dominates. Indeed, since $\cos x \geq 1 - \frac{2x}{\pi}$ on the interval $[0, \frac{\pi}{2}]$, one has $\cos \left( \frac{\pi}{2n+1} \right) \geq \frac{2(n-p)+1}{2n+1}$. Also, $\sin \left( \frac{2\pi n \pi}{2n+1} \right)^{2} \leq 1$. Hence, the sum of the first $n - 1$ terms can be estimated as

$$\sum_{p=1}^{n-1} \frac{(-1)^{n+p} \sin^{2} \left( \frac{\pi}{2n+1} \right)}{2^{t}(2n+1)\cos^{t+2} \left( \frac{\pi}{2n+1} \right)} \leq \sum_{r=1}^{n-1} \frac{(2n+1)^{t+1}}{2^{t}(2r+1)^{t+2}} \text{ (where } r = n - p \text{)}.$$  

Thus,

$$\frac{(2n+1)^{t+1}}{2^{t}} \sum_{r=1}^{\infty} \frac{1}{(2r+1)^{t+2}} \leq \frac{(2n+1)^{t+1}}{2^{t}} \left\{ \frac{1}{3^{t+2}} + \int_{1}^{\infty} \frac{dx}{(2x+1)^{t+2}} \right\} \leq \frac{(2n+1)^{t+1}}{2^{t}3^{t+1}}.$$
Let \( H = \frac{\sin^2\left(\frac{\pi}{2(2n+1)}\right)}{2^{2(2n+1)}\cos^2\left(\frac{\pi}{2n+1}\right)} \) be the largest \((n^{th})\) term. By Taylor expansion around \( \frac{\pi}{2} \) one has, for some \( |\varepsilon| \leq 1 \),

\[
\sin\left(\frac{n\pi}{2n+1}\right) = 1 - \left(\frac{\pi}{2(2n+1)}\right)^2 \varepsilon^2 \geq \frac{15}{26} \text{ if } n \geq 2.
\]

Similarly, \( \cos\left(\frac{n\pi}{2n+1}\right) \leq \left(\frac{\pi}{2(2n+1)}\right)^2 \). Thus, \( H \geq \frac{4(2n+1)!+1}{\pi^2} \left(\frac{10}{26}\right)^2 \) from which Theorem 1 immediately follows. \( \blacksquare \)

4 An example of a "fixpoint polynomial"

In the introduction we mentioned the fixpoints of the iteration \((x_1, \ldots, x_n) \rightarrow (a_1, \ldots, a_n)\) An amusing and perhaps intriguing numerical example for \( n = 4 \) is the following:

\[
(-T + 1)(-1.324717957T + 1)(.7548776668T + 1)(.5698402906T + 1) \approx 1 - .9999999994T - 1.324717957T^2 + .7548776668T^3 + .5698402912T^4
\]

The relevant equations were solved in the obvious way using Maple, by first constructing a Groebner basis of the ideal \( I(x_1 + x_2 + x_3 + x_4 - x_1, x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 - x_2, x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - x_3, x_1x_2x_3x_4 - x_4) \).

\textbf{Acknowledgement.} I wish to thank A. Blokhuis, A.E. Brouwer and R. Riebeek for a stimulating discussion, and more in particular Andries Brouwer for his comments on a draft version of this article.
REFERENCES.


5. B.L. Van der Waerden, Algebra 1 & 2; Springer, 1971.