§0. Introduction. The theory of Uniformly Reflexive Structures (URS) studied by Wagner and Strong ([8], [6], [1]), is an elegant axiomatization of parts of recursion theory. The theory abstracts some properties of the function \( n \downarrow (m) \) (i.e. the \( n \)th partial recursive function applied to \( m \)) by considering arbitrary domains with a binary operation application. The standard URS is \( \mathcal{X} \) with domain \( \omega \cup \{\ast\} \) and application \( n \uparrow m = (n \downarrow (m)) \) if defined, else.

However the URS are not completely adequate for the description of recursion theory. Real computations do have a length, a feature which is missing in the URS. In fact there are sentences in the language of URS undecided by the axioms. E.g. let \( e = \lambda x.xx \), i.e. \( ex = xx \) for all \( x \), then \( ee = \ast \) is such a sentence. But this sentence holds in the intended interpretation \( \mathcal{X} \) as follows from an argument using length of computation.

Moreover in a URS it is not always possible to represent the partial recursive functions.

To overcome these defects we introduce a concept of a norm.

A Normed Uniformly Reflexive Structure (NURS) is a URS which a norm \(|x; y|\) can be defined satisfying:

1. \(|x; y| \in \omega \cup \{\ast\}\)
2. \(|x; y| = \ast \iff x.y = \ast\)
3. \(|s.x.y; z| > |x.z; y; z| + |x;z| + |y;z|\), if \(|s.x.y; z| \neq \ast\)

The intended interpretation of \(|x; y|\) is "the length of computation of \( x.y \)".

The following facts motivate the introduction of NURS. As was intended \( \mathcal{X} \) is a NURS. Wagners (highly) constructible URS are NURS. In every NURS \( ee = \ast \) holds. More generally, for a NURS \( \mathcal{X} \) and a term \( M \) of the theory, \( M \) has no normal form \( \iff \mathcal{X} \vdash M = \ast \).

In a NURS all splinters are semi-computable, and hence can be used to represent the partial recursive functions.
The use of length of computation in recursion theory has also been stressed by Y. Moschovakis [3]. In fact the axioms of the norm in a URS imply Moschovakis' condition on the length of computation.

Familiarity with URS is assumed. See e.g. Wagner [8] and Strong [6].

In §1 the defects of URS mentioned above are shown. A formal theory WS, convenient for the study of URS, is introduced in §2. The term model of an extension of WS provides some counter examples for the relation between semi-computable and recursively enumerable. The results about the NURS are proved in §3.

§1. The definition of a URS given below is not exactly the same as those of Wagner and Strong. The axioms are written down in a way showing the correspondence with combinatory logic. Axiom 7 is added; it implies that we may assume that terms with different normal forms are unequal in a URS (2.10).

1.1. Def. A URS is a structure \( \mathcal{V}_t = \langle U, *, i, k, s, \delta, ' \rangle \) such that the following holds where \( a, b, c \) are variables ranging over \( U - \{ * \} \):

1. \( *a = a, * = * = * \)
2. \( i.a = a \)
3. \( k.a.b = a \)
4. \( s.a.b.c = (a.c)(b.c) \); \( s.a.b \neq * \)
5. \( a = b \rightarrow \delta.a.b = k \); \( a \neq b \rightarrow \delta.a.b = k.i \)
6. \( i \neq k \)
7. \( s.a.b = s.a'.b' \rightarrow a = a' \wedge b = b' \).

1.2. Def. Kleen's URS, \( \mathcal{K} \), is the structure \( \langle \omega, *, i, k, s, \delta, ' \rangle \) such that \( \omega = \omega \cup \{ * \} \) with, \( * \notin \omega \), \( n.m = \{ n \}(m) \) if defined

\[ * \in \text{else} \]

\[ *.n = n.* = *.* = * \], and \( i, k, s, \delta \) are to be found by the s-m-n theorem such that axioms 2, ..., 7 hold. As an example we construct k. Let \( \psi(x, y) = x. \) Then \( \psi \) is partial recursive. Hence

\[ x = \psi(x, y) \]
\[ = \{ e \}(x, y) \quad \text{for some index } e \text{ of } \psi. \]
\[ = \{ s \} \{ e, x \}(y) \]
\[ = \{ \{ k \}(x) \}(y) \quad \text{k index of } \lambda x. s \{ e, x \}. \]
\[ = k \cdot x \cdot y. \]
By pumping up the indices, cf. Rogers [4], p.83, we can assure that axiom 7 holds.

1.3. **Theorem.** Let e = s.i.i. Then e.e = * is independent in the theory of the URS.  

Proof. It will be shown that e.e = * is true in \( \mathcal{X} \) but false in a modification \( \mathcal{X}^* \).

We have \( \mathcal{X} \models e.a = (i.a)(i.a) = a.a \), i.e. \( \{e\}(a) = \{a\}(a) \).

The computation of \( \{e\}(a) \) runs as follows:

Read a; compute \( \{a\}(a) \). Hence the computation of \( \{e\}(e) \) is:

Read e; compute \( \{e\}(e) \); Read e; compute \( \{e\}(e) \); ...

Therefore \( \{e\}(e) \) is undefined. Hence \( \mathcal{X} \models e.e = * \).

Let \( \mathcal{X}^* = (\omega^*, \ast, i, k, s^*, \delta, *) \) be the following modification of \( \mathcal{X} \).

1) Compare this with the following: Let \( E = \{x|x \in x\} \). Then \( E \in E \) is independent in \( \text{ZF} \) without foundation, but refusable in \( \text{ZF} \) itself.

Another defect of the URS is the following. The partial recursive functions can be represented in a URS provided one has an infinite semi-computable (SC) splinter, Strong [6], 3.2. However, H. Friedman has shown that there is a URS without infinite SC splinter.

1.4. **Def.** Let \( \alpha \) be a non-standard model of Peano arithmetic with universe A. Let \( \mathcal{X}_\alpha \) be the structure \( (A^*, \ast, i, k, s, \delta, \ast) \) where \( \ast \notin A \), \( i, k, s, \delta \) are as in 1.2 and \( i \) is defined by

\[
\ast \ast a = a \ast \ast = \ast \ast = \ast \ast
\]

\( a \ast b = c \) if \( \alpha \models \{a\}(b) = c \) i.e. \( \alpha \models \exists z[T(a, b, z) \land U(z) = c] \)

\( U \) and \( T \) are the components of Kleene's normal form theorem. Then \( \mathcal{X}_\alpha \) is a URS; e.g. \( \mathcal{X}_\alpha \models k.a.b = a \) holds since \( \{(k)(a)\}(b) = a \) is provable in Peano arithmetic, hence \( \alpha \models \{(k)(a)\}(b) = a \).

\[
1) \text{Compare this with the following: Let } E = \{x|x \in x\}. \text{ Then } E \notin E \text{ is independent in } \text{ZF} \text{ without foundation, but refusable in } \text{ZF} \text{ itself.}
\]
1.5. **Theorem** (H. Friedman). \( \mathcal{A} \) is a URS without infinite SC splinter.

Proof. If \( \mathcal{A} \) would contain an infinite SC splinter, each splinter would be SC, Strong [6] 3.11. Therefore the set of standard numbers would be SC. But this is absurd since SC sets are definable \((x \in A \iff f(x) \neq \ast)\), and the set of standard numbers is not. \( \Box \)

1.6. **Cor.** There exists a URS with an infinite non SC splinter on which the partial recursive functions can be represented.

Proof. Let \( \mathbb{N} \) be the standard model of Peano arithmetic. Let \( \mathcal{A} \cong \mathbb{N} \) be a non-standard model. For each partial recursive function \( \psi \) with index \( e \) we have

\[
\mathbb{N} \vDash [e](\bar{e},\bar{n}) = m \iff \psi(n) = m
\]

\[
\mathcal{A} \vDash \exists z T(e,n,z) \iff \psi(n) \text{ is undefined.}
\]

Therefore, since \( \mathcal{A} \cong \mathbb{N} \), \( \mathcal{A} \vDash [e](\bar{e},\bar{n}) = m \iff \psi(n) = m \)

\[
\forall [e]\exists [e] \bar{n} = \bar{m} \iff \psi(n) \text{ is undefined.}
\]

However, there exists a URS such that only partial recursive functions with recursive domain can be represented on any of the infinite splinters.

1.7. **Theorem.** There exists a URS such that for no infinite splinter \( X \) the partial recursive functions can be represented on \( X \).

Proof. Let \( \mathcal{A} \) be a non-standard model of Peano arithmetic in which only the recursive r.e. sets are definable on \( \omega \), see [2], Exc.7,p123. Let \( \psi \) be a partial recursive function with non recursive domain \( A \). Then \( \psi \) is not representable on the splinter of standard integers for otherwise \( A \) would be definable on \( \omega \). But then \( \psi \) is not representable on any infinite splinter \( X \), since all infinite splinters are in bijective computable correspondence, [6],3-7. \( \Box \)

§2. The following theory WS is convenient for the study of URS.

2.1. **Def.** WS has the following language.

**Alphabet:** \( x_0, x_1, \ldots \) variables

\( I, K, S, A, \ast \) constants

\( >, = \) reduction, equality

\( (, ) \) brackets
Terms are inductively defined by
1. A variable or constant is a term
2. If \( M, N \) are terms, so is \( (MN) \).

Formulas are \( M \geq N \) and \( M = N \) where \( M, N \) are terms.

Notation: \( x, y, z, \ldots \) denote arbitrary variables
\( M, N, L \) denote arbitrary terms
\( M_1 M_2 \ldots M_n \) stands for \((\ldots(M_1 M_2)\ldots M_n)\)
\( M \subseteq M' \) if \( M \) is a subterm of \( M' \)
\( x \in M \) if \( x \) occurs in \( M \)
\( M \) is closed if for no \( x \) \( x \in M \)
\( \equiv \) denotes syntactic equality.

If \( M \) is a closed WS term and \( \forall x (\cdot, \cdot, i, k, s, \delta, \cdot) \) is a URS, then \( M^\forall \) is the obvious interpretation of \( M \) in \( \forall x : \cdot \equiv \cdot, i^\forall = i, \) etc, \( (MN)^\forall = M^\forall N^\forall \); \( \forall^\forall M = N \iff M^\forall = N^\forall \).

A term \( M \) is in normal form (nf) if it has no subterms of the form \( *, IA, KAB, SABC \) or \( \Delta AB \).

WS is defined by the following axioms and rules:

**I**

1. \( *M \gg * \)
2. \( IM \gg M \)
3. \( KNM \gg M \) if \( N \) is in nf
4. \( SMNL \gg ML(NL) \)
5. \( a \Delta MM \gg K \) if \( M \) is in nf
   \( b \Delta MN \gg KI \) if \( M, N \) are in nf and \( M \neq N \)

**II**

1. \( M \gg M \)
2. \( M \gg M' \Rightarrow ZM \gg ZM' \), \( MZ \gg M'Z \)
3. \( M \gg N, N \gg L \Rightarrow M \gg L \)

**III**

1. \( M \gg N \Rightarrow M = N \)
2. \( M = N \Rightarrow N = M \)
3. \( M = N, N = L \Rightarrow M = L \)

2.2. (Church-Rosser theorem) If \( WS \vdash M = N \), then for some term \( Z \)
\( WS \vdash M \gg Z \) and \( WS \vdash N \gg Z \).
Proof. Well-known. See e.g. [5], T. 11, p. 144.

2.3. Def. A WS-term \( M \) has a nf if \( WS \vdash M = M' \) and \( M' \) is in nf.

By 2.2 the normal form of a term is unique if it exists. If \( M \) has a nf, all its reduction sequences terminate, by the restriction in axioms I2, 4.
2.4. Def. Let \( \mathcal{U} \) be a URS with domain \( U \). \( WS(\mathcal{U}) \) is the theory \( WS \) modified as follows. For each \( a \in U, a \) is an additional constant. A term of \( WS(\mathcal{U}) \) is in nf, if it does not contain a subterm \( a, IA, \text{etc.} \) or \( aM \). \( WS(\mathcal{U}) \) has the additional axioms
\[
\Delta MN \Rightarrow KI \quad \text{if} \ M, N \text{ are nf's and } \forall \not\vdash M \neq N.
\]
Clearly \( \forall \not\vdash WS(\mathcal{U}) \).
2.2 and 2.3 apply also to \( WS(\mathcal{U}) \).

2.5. (Abstraction) Let \( M \) be a \( WS(U) \) term not containing \( \ast \).
Then there exists a \( WS(U) \) term \( \lambda x. M \) such that
1. \( \lambda x. M \) is in nf; \( x \notin \lambda x. M \)
2. \( WS(U) \vdash (\lambda x. M)N = [x/N]M \) for \( N \) in nf.
Proof. As in combinatory logic.

2.6. Def. Let \( M \sim M' \) denote \( Mx = M'x \) for \( x \notin MM' \).

2.7. (Fixed Point Theorem) There exists a \( WS(\mathcal{U}) \) term \( FP \) such that
1. \( WS \vdash FP f \sim f(FP f) \)
2. \( FP f \) is in nf.
Proof. Let \( \omega_f = \lambda x. f(xx)z \) and \( FP f = \omega_f \omega_f \).

2.8. Lemma. Let \( M \) be a \( WS(\mathcal{U}) \) term. Then \( M \) is a nf \( \Rightarrow \)
\( \forall \not\vdash M \neq \ast \).
Proof. The set of normal forms NF can be defined inductively
by 1. \( a, I, K, S, \Delta \in NF \). 2. \( AB \in NF \Rightarrow KA, SA, \Delta A \) and \( SAB \in NF \). Then
the result follows inductively realizing that in a URS
\( k.a, s.a, \delta.a, s.a.b \neq \ast \).

2.9. Lemma.
Then there exists a term \( P \) such that for all \( \not\vdash \)
1. \( \not\vdash Pab \neq \ast \)
2. \( \not\vdash Pab \sim a \)
3. \( \not\vdash Pab = Pab' \rightarrow a = a' \land b = b' \).
Proof. Let \( P = \lambda abx. K(ax)b \). Clearly \( P \) satisfies 1 and 2. By writing out \( P \) in terms of \( I, K \) and \( S \), one sees that \( P \) satisfies 3 due to axiom 7.

2.10. Cor. Let \( M \neq M' \) be WS terms in nf. Then we may assume \( \forall \not{\Downarrow} M \neq M' \) for all \( \forall \).

Proof. By changing if necessary the basic constants \( i,k,s, \) and \( \delta \), using \( P \). See e.g. [3], p. 133 bottom.

What we may we will.

2.11. Cor. WS(\(A\)) is a conservative extension of WS.

Proof. The only axiom of WS not in WS(\(A\)) is \( I4b \). However, this follows from the modified axiom by 2.10. Hence WS(\(A\)) is an extension of WS. If \( M,N \) are WS terms and WS(\(A\)) \( \vdash M = N \) (or \( \vdash M \not{=} N \)), then the proof involves only WS terms (unless WS \( \vdash M = N = * \)). The WS(\(A\)) axioms only can hold for \( A \not{=} B \), by 2.10. Hence WS \( \vdash M = N \) (or \( M \not{=} N \)).

2.12. Theorem 1. WS(\(A\)) \( \vdash M = N \Rightarrow \forall \not{\Downarrow} M = N \)

Proof. 1. Induction on the length of proof of \( M = N \) using 2.10.

2. By 1. and 2.6.

The converse of 2.12. 1,2 are false. E.g. in \( \forall \not{\Downarrow} EE \not{=} * \) where \( E = SII \). But EE has no nf. However, if \( \forall \) is a NURS the converse of 2.12.2 is true. See 3.3.

2.13. Def. Let \( WS^* \) be WS augmented by the axioms:

\( M \not{=} * \) if \( M \) has no nf.

For each NURS \( \forall \) we will have the completeness result:

\( WS^* \vdash M = N \Rightarrow \forall \not{\Downarrow} M = N \), for closed \( M,N \);

see 3.5.

2.14. Def. \( \forall (WS^*_0) \) (respectively \( \forall (WS^*_c) \)) is the term model consisting of arbitrary (respectively closed) WS terms modulo provable equality in WS*. Clearly they are URS.

Similarly we define \( \forall (WS^*_0,c(\forall)) \).

These term models can be used for some counter-examples.
2.15. **Def.** A subset $X$ of a URS $\mathcal{M}$ is RE if $X = \emptyset$ or $X = Ra f = \{a \mid \exists x ((fx = a))\}$ for some total $f$ in $\mathcal{M}$ (i.e. $\forall a \, fa \neq \ast$). In $\mathcal{M}$, $X$ is RE $\iff$ $X$ is SC.

2.16. **Theorem.** For $\mathcal{M}(\text{WS}^*)$ we have
1. $X$ is SC $\iff$ $X$ is RE
2. $X$ is RE $\iff$ $X$ is SC
3. $X$ is computable $\iff$ $X$ is finite or cofinite.

**Proof.**

2.16.1 **Def.** The family of $F$, $\mathcal{F}(F)$, is the set
\[
\{N \mid \exists F' \supseteq F' \land N \subseteq F'\}.
\]
If $F$ has a nf, $\mathcal{F}(F)$ is finite. Each reduction of $FA$ to a nf can be written in the form
\[
FA \geq_\beta M_n[A] \geq_\delta M_1[A] \geq_\delta M_1[A] \geq_\delta \ldots \geq M[A]
\]
where $\geq_\beta$ is axiomatized leaving out the $\Delta$ reduction axioms and $\geq_\delta$ is axiomatized leaving out the $\ast, I, K, S$ axioms. $A$ may not actually occur in $M[A]$. Referring to the sequence ($\ast$) we define:

2.16.2 **Def.** $\text{Diag}_n(F, A) = \{AC_1[A]C_2[A] \mid AC_1[B]C_2[B] \subseteq M_n\}$.

$B$ satisfies $\text{Diag}_n(F, A) \iff AC_1[A]C_2[A] = AC_1[A]C_2[B]$, for all members of $\text{Diag}_n(FA)$.

2.16.3 **Lemma.** Let $FA$ have a nf for all $A$. Let $xa \notin F$. Consider the sequence ($\ast$) for $F(xa)$. Then
\[B\] satisfies $\text{Diag}_n(F, xa) = M_n[B] \geq_\delta M'_n[B]$.

1. $xa$ is never "active" (i.e. in a subterm of the form $(xa)P$) in $M_n[xa]$, $M'_n[xa]$.

2. For almost all, i.e. all except finitely many, $B$ satisfies $\text{Diag}_n(F, xa)$.

**Proof.** 0 is obvious.

1. follows by substituting for $xa$ a nf $\omega$ such that $\omega P$ has no nf for all $P$.

2. follows by realizing that the only possible exceptions are in $\mathcal{F}(F)$.

3. was proved by Strong [7] for the URS $\mathcal{M}(\text{WS}^*_c)$.

1) A different example of 1. was given in Wagner [8], 6.13.
2.16.4 Cor. Let $FA$ have a nf for all $A$. Let $xa \notin F$ and $xa \notin M$, the nf of $F(xa)$. Then for almost all $B$

$$F(B) = F(xa).$$

Proof. Let $\text{Diag}(F, xa) = \cup \text{Diag}_n(F, xa)$ which is finite. This is satisfied by almost all $B$ (2.16.3.2). Thus (2.16.3.0)

$$FB \supseteq M[B].$$

Also $F(xa) \supseteq M[xa]$. But then, since $xa \notin M[xa]$

$$FB = F(xa).$$

More easily one can prove the following.

2.16.5 Cor. Let $F(xa)$ have a nf, where $xa \notin F$, $xa \notin$ the nf of $F(xa)$. Then for $x' \notin F$ $F(x'a) = F(xa)$.

Proof. Since $x'a$ is a non-active term, it does not matter if it occurs in an active place.

2.16.6 Cor. Suppose $RA F \subseteq$ closed normal forms. Then $Ra F$ is finite.

Proof. Take $xa \notin F$. By the assumption, never $xa \in M$, the nf of $FA$. Hence for almost all $B$, $FB = F(xa)$.

Now we can prove 2.16.

1. Take $X = \{K^n I_n | n \in \omega\}$. Then $X$ is an infinite splinter

hence SC (since $\forall \in (\text{WS}_{\omega}^*)$ is a NURS, see §3). Suppose $X$ were RE, say $X = Ra F$. Then $F$ satisfies the assumption of 2.16.6, but $Ra F = X$

is not finite. Contradiction.

2. Take $X = Ra F$, with $Fa = xa$. Suppose $X$ were SC, i.e.

$$GM = I \quad \text{if } M \in X$$

$$* \quad \text{else}$$

for some $G$. Take $a \notin G$. Then $xa \notin G$. Also $xa \notin I$ which is the nf of $G(xa)$. Hence for $x' \notin G$ it follows by 2.16.5 that

$$G(x'a) = G(xa) = I,$$

i.e. $x'a \in X$, a contradiction.

3. Let $X = \emptyset$ be computable. Define

$$GM = M \quad \text{if } M \in X$$

$$M_o \quad \text{else}$$

for some $M_o \in X$.

Then $X = Ra G$. Suppose the complement of $X$ is not finite. Then there is a variable $x \notin Ra G \cup f(G)$. Then $xa \notin G$, $xa \notin$ the nf of $G(xa)$. Hence by 2.16.4 $GB = G(xa)$ for almost all $B$, i.e.

$$X = Ra G \text{ is finite.}$$
§3. For NURS it is convenient to define for elements of
\(w \cup \{\infty\}: p \geq q \iff p = \infty \lor p > q.\) Then \(\geq\) is transitive and
axiom 3 for a norm can be stated as
\[|s.a.b;c| \geq |a.c;b.c| + |a;c| + |b;c|.\]

3.1. Examples of NURS.
1. \(\mathcal{X}\) becomes a NURS by defining

\[|e;x| = \mu z \ T(e,x,z) \text{ if defined}
\]

\[\infty \text{ else.}\]

Then an examination of the properties of the \(T\) predicate shows
that this defines a norm on \(\mathcal{X}\).

2. \(\mathcal{V}(\text{WS}^\ast)\) are NURS by defining

\[|F;x| = \text{the length of the inside out reduction of } FX \text{ to } \text{nf}
\]

\[\infty \text{ if } FX \text{ has no } \text{nf.}\]

The inside out reduction only reduces reduces SABC, etc. when
A, B and C are normal forms.

3. Let \(\mathcal{V}\) be a (highly) constructible URS in the sense of [8]. Then \(\mathcal{V}\)
is a NURS:

Let

\[f(e;x) = \mu n[(e,x) \in \Lambda_n] \text{ if defined}
\]

\[\infty \text{ else.}\]

Take \(|e;x| = f^\ast(e;n)\). This is a norm on \(\mathcal{V}\), for let \(f(sxy,z) = n,\)
then \(sxy = \phi_0(x,y), n > 0\) and \((x,z),(y,z),(xz,zy) \in \Lambda_{n-1}\) (see
[8], p. 20–21 for the notation). Then \(f(x,z), f(y,z), f(xz,zy) \in n-1,\)
and \(|sxy;z| = 4^n > 3.4^{n-1} \geq |x;z| + |y;z| + |xz;yz|\).

4. Let \(\mathcal{V}\) be a non-standard model of Peano arithmetic. Then \(\mathcal{V}_{\mathcal{V}}\)
is not a NURS. This follows from 1.5 and 3.4. Similarly it follows
from 1.3 and 3.2 that \(\mathcal{V}^\ast\) is not a NURS.

The sentence \(EE = \ast\), with \(E = \text{SII}\), which was independent in the
theory of URS becomes true in all NURS.

3.2. Let \(E = \text{SII}\) and \(\mathcal{V}\) be a NURS. Then

\[\mathcal{V} \vdash EE = \ast.\]

Proof. Suppose \(EE \neq \ast\). Then \(|E;E| \neq \infty\). But then

\[|E;E| = |\text{SII};E| > |E;E| = |E;E|, \quad \text{a contradiction}.\]

\[\blacksquare\]

More general

3.3. Theorem. Let \(\mathcal{V}\) be a NURS and \(M\) a \(\text{WS(\mathcal{V})}\) term. Then

\(M\) has no nf \(\iff\) \(\mathcal{V} \vdash M = \ast\).
Proof.  

This will be proved in a number of steps.

3.3.1 Def. $\text{SC}(M)$, the set of subcomputations of $M$, is defined inductively by:

If $M$ is in normal form $\text{SC}(M) = \emptyset$; else $M \equiv AB$ and $\text{SC}(AB) = \text{SC}(A) \cup \text{SC}(B) \cup \{|A^X;B^Y|\}$. Below we often omit the superscript $\forall$.

Clearly $\text{SC}(M)$ is a finite set $\emptyset \cup \ldots \cup \{\infty\}$ and if $M \supset M'$, then $\text{SC}(M) \supset \text{SC}(M')$.

3.3.2 Def. $\|M\| = \text{Max}\{\text{SC}(M)\}$. If $\text{SC}(M)$ contains $\infty$, $\|M\| = \infty$.

3.3.3 Lemma. If $M \supset M'$, then $\|M\| > \|M\'|$.

3.3.4 Lemma. $\|M\| = \infty \iff \forall M = *$.

Proof. $\|M\| = \infty \iff \exists \in \text{SC}(M)$

$\iff$ for some $AB \subset M \quad |A;B| = \infty$

$\iff$ for some $AB \subset M \quad \forall M = *$

$\iff \forall M = *$.

3.3.5 Lemma. Let $M \supset M'$ be an axiom of $\text{WS}(A)$, then $\|M\| > \|M\'|$.

Proof. Let $M \equiv SABC$ and $M' \equiv AC(BC)$. Then $\text{SC}(M) = \{S;A|,|SA;B|,|SAB;C|\} \cup \text{SC}(A) \cup \text{SC}(B) \cup \text{SC}(C)$. $\text{SC}(M') = \{|A;C|,|B;C|,|AC;BC|\} \cup \text{SC}(A) \cup \text{SC}(B) \cup \text{SC}(C)$.

Since $|SAB;C| > \text{Max}\{|A;C|,|B;C|,|AC;BC|\}$

$\|M\| > \|M\'|$. Equality may occur, e.g. if $\text{SC}(C)$ contains the largest subcomputation.

If $M \equiv KAB$, $M \equiv IA$ or $M \equiv M'$, then $M' \equiv A$ or $M' \equiv M$, hence $M \supset M'$ and the result follows by 3.3.3.

If $M \equiv \Delta AB$, then $M' \equiv K$ or $\equiv KI$, so $\text{SC}(M) \supset \text{SC}(M') = \emptyset$, hence $\|M\| > \|M\'|$. Similarly if $M \equiv aN$.

3.3.6 Cor. If $\text{WS}(A) \supset M \supset M'$, then $\|M\| > \|M\'|$.

Proof. Induction on the length of proof of $M \supset M'$.

Let us consider only the case that $M \supset M'$ is $ZA \supset ZA'$ and is a direct consequence of $A \supset A'$. Then $\text{SC}(ZA) = \text{SC}(Z) \cup \text{SC}(A) \cup \{|Z;A|\}$ and similarly for $\text{SC}(ZA')$. Now $\forall A = A'$, hence $|Z;A| = |Z;A'|$. Hence $\|ZA\| > \|ZA\'|$ by the induction hypothesis $\|A\| > \|A\'|$.

3.3.7 Def. A special redex is a WS$^*$(A) term $SABC$, where $A$, $B$ and $C$ are in normal form.
3.3.8 Lemma. If SABC is a special redex, then
\[ ||SABC|| > |AC(BC)|.\]
Proof. Since SC(A) = SC(B) = SC(C) = ∅
\[ ||SABC|| = \text{Max}(|S;A|,|SA;B|,|SAB;C|) > |SAB;C| \geq \]
\[ > \text{Max}(|A;C|,|B;C|,|AC;BC|) = ||AC(BC)||.\]

3.3.9 Lemma. Let M be a WSO^1 term without normal form. Then there
exists a special redex N without normal form in the family (see
2.16.1) of M, or else ∀x Mx *.
Proof. Consider the finite set T of subterms of M partially
ordered by c. Let N be a minimal element of T without a normal
form. Then all subterms of N have a normal form. Checking all
possibilities it follows that N is of the form SABC. Let A*, B* and C*
be the normal forms of A, B and C. Now we have
\[ M \equiv -(SABC) \geq -(SA*B*C*) \]
and SA*B*C* is a special redex without normal form.

3.3.10 Cor. If M has no normal form, then there exists a term
M' without normal form and ||M|| > ||M'||.
Proof. Let N be as in 3.3.9, then ||M|| > ||N|| by 3.3.6 and 3.3.3.
Let N > M'. Then ||N|| > ||M'|| by 3.3.8. Since N has no normal form,
neither has M'.

Now the proof of 3.3.10 can be given.
Let M be a term without normal form. Suppose ∀x Mx *.
Then ||M|| ≠ ∞ by 3.3.4. Hence by 3.3.10 there exists a sequence
M,M',M'',... such that ||M|| > ||M'|| > ||M''|| > ... is an infinite
descending chain of integers.

3.4. Theorem. In a NURS ∀x all infinite splinters are SC.
Proof. Let X = {f^n} be an infinite splinter. Define by the
fixed point lemma a WS(V) term H such that
\[ H_yx = I \quad \text{if } y = x \]
\[ H(fy)x \text{ else.} \]
Then \( h = (H_0)^{1/2} \) is a semi-characteristic function of X:
If a ∈ X, clearly \( H \circ a = I, \) hence ha ≠ •.
If a ∉ X, then \( H \circ a > H(f(o)a > ... \), i.e.
H ∘ a has no nf. Hence ha = • by 3.3.
WS* is a complete axiomatization for the equations true in all NURS.

3.5. **Theorem.** Let $\mathcal{U}$ be a NURS. Then for closed WS terms:

$$\text{WS}^* \vdash M = N \iff \mathcal{U} \models M = N.$$ 

**Proof.** $\Rightarrow$ By 2.12.1, 3.3. $\Leftarrow$ By 2.10, 3.3. $\blacksquare$

3.6. **Theorem.** Each URS can be embedded in a NURS (cf. Wagner [8], p. 31, 6.2), if the similarity type has no constants.

**Proof.** Clearly $\forall \mathcal{U} \subseteq \mathcal{V}((\text{WS}^*, c(\mathcal{U})))$ which is a NURS by 3.1.2. $\blacksquare$

**Concluding remarks.**

A URS is almost a precomputation theory in the sense of Moschovakis [3]¹. Restricting the attention to single-valued functions, his computation theories have an additional length of computation $|e;x|$ satisfying

$$(+) \quad |S^n_m(e,x);y| > |e;x,y|, \quad \text{if defined.}$$

Define in a NURS $|e;x| = |e;x_1| + |e.x_1;x_2| + \ldots + |e.x_1 \ldots x_{n-1};x_n|$. Then it follows readily from the definition of $S^n_m$ in a URS ([8], 2.6) that this norm satisfies Moschovakis' axiom $(+)$. As suggested in [6], there is another way of extending a URS. A selection² URS is an URS containing a "selection operator" $c$ such that

$$\exists a[f.a \neq \cdot] = f.(c.f) \neq \cdot.$$ 

1) Not quite, because a URS does not need to contain a computable successor set.

2) In [6] such a URS is called "well-ordered". This name is a little absurd as can be argued as follows. Let $\mathcal{M}$ be a model of Peano arithmetic of power continuum. Then $\mathcal{M}_{\omega}$ is a selection URS but cannot be well-ordered in ZF. On the other hand $\mathcal{V}(\text{WS}^*_\omega)$ is countable and hence well-ordered, but has no selection operator.
In a selection URS a set is computable iff it is SC and co SC, \([6], 3.4\). This is not true in a general URS, \([8], p.39 \) bottom.

Having a norm or a selection operator are independent of each other. \( \mathcal{X} \) has a selection operator \( (c)(e) = (\nu x T(e, (x)_0, (x)_1))_0 \). Since this is provably in arithmetic a selection operator, \( \mathcal{X} \) is a selection URS but not a NURS. Conversely, it is not difficult to show that \( \mathcal{U}(WS^*_o) \) is not a selection URS, although it is a NURS,

In a NURS it would be natural to require for a selection operator \( c \)

\[ |c; a| \geq |a; c; a| \]

cf.\([3], p.225, (6-4)\).

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References.


