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THEORETICAL PEARLS

Applications of Plotkin-terms: partitions and morphisms for closed terms

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Abstract
This theoretical pearl is about the closed term model of pure untyped lambda-terms modulo $\beta$-convertibility. A consequence of one of the results is that for arbitrary distinct combinators (closed lambda terms) $M, M', N, N'$ there is a combinator $H$ such that

$$HM = HM' \neq HN = HN'.$$

The general result, which comes from Statman [1998], is that uniformly r.e. partitions of the combinators, such that each "block" is closed under $\beta$-conversion, are of the form $\{H^{-1}\{M\}\}_{M \in \mathcal{B}}$. This is proved by making use of the idea behind the so-called Plotkin-terms, originally devised to exhibit some global but non-uniform applicative behavior. For expository reasons we present the proof below. The following consequences are derived: a characterization of morphisms and a counter-example to the perpendicular lines lemma for $\beta$-conversion.

1. Introduction

We use notations from recursion theory and lambda calculus, see Rogers [1987] and Barendregt [1984].

NOTATION. (i) $\varphi_e$ is the $e$-th partial recursive function of one argument.

(ii) $W_e = \text{dom}(\varphi_e) \subseteq \mathbb{N}$ is the r.e. set with index $e$.

(iii) $\mathcal{A}$ is the set of lambda-terms and $\mathcal{A}^0$ is the set of closed-lambda terms (combinators).

(iv) $W_e = \{M \in \mathcal{A}^0 \mid \#M \in W_e\} \subseteq \mathcal{A}^0$; here $\#M$ is the code of the term $M$.

1.1. DEFINITION. (i) Inspired by Visser [1980] we define a Visser-partition ($V$-partition) of $\mathcal{A}^0$ to be a family $\{W_e\}_{e \in S}$ such that
1.2. DEFINITION. Let \( \{ W_e \} \subseteq S \) be a V-partition.

1. The partition is said to be covering if \( \bigcup_{e \in S} W_e = \emptyset \).
2. The partition is said to be inhabited if \( \forall e \in S \ W_e \neq \emptyset \).
3. A V-partition \( \{ W_e \} \subseteq S \) is said to be (extensionally) equivalent with \( \{ W'_e \} \) if these families define the same collection of non-empty sets, i.e. if

\[ \{ W_e | e \in S \land W_e \neq \emptyset \} = \{ W'_e | e \in S' \land W'_e \neq \emptyset \} . \]

1.3. EXAMPLE. Let \( H \) be some given combinator. Define

\[ W_{e(M, H)} = \{ N \in A^\emptyset | HN = HM \} , \]

Then \( \{ W_e \} \subseteq S_H \), with \( S_H = \{ e(M, H) | M \in A^\emptyset \} \), is an example of a covering and inhabited V-partition. We denote this V-partition by \( \{ W_{e(M, H)} \} \subseteq M \in A^\emptyset \).

1.4. PROPOSITION. (i) Every V-partition is effectively equivalent to an inhabited one.

(ii) Every V-partition can effectively be extended to a covering one.

PROOF. (i) Given \( \{ W_e \} \subseteq S \) define \( S' = \{ e \in S | W_e \neq \emptyset \} \). Then \( \{ W_e \} \subseteq S' \) is the required modified partition.

(ii) Given \( \{ W_e \} \subseteq S \) define

\[ W_{e(M)} = \{ N | N = M \lor \exists e \in S M, N \in W_e \} . \]

Then \( \{ W_{e(M)} \} \subseteq M \in A^\emptyset \) is the required V-partition. \( \blacksquare \)

The main theorem comes in two versions. The second more sharp version is needed for the construction of so called inevitably consistent equations, see Statman [1999].

1.5. THEOREM (Main theorem). (i) Let \( \{ W_e \} \subseteq S \) be a V-partition. Then one can construct effectively a combinator \( H \) such that for all \( M, N \in A^\emptyset \)

\[ HM = HN \Leftrightarrow M = N \lor \exists e \in S M, N \in W_e . \]

The construction of \( H \) is effective in the code of the underlying r.e. set \( S \).

(ii) Let \( \{ W_e \} \subseteq S \) be a pseudo-V-partition. Then one can construct effectively a combinator \( H \) such that if \( \{ W_e \} \subseteq S \) is an actual V-partition, then (*) holds.

The theorem will be proved in §2. It has several consequences. In order to state these we have to formulate the notion of morphism on \( A^\emptyset \) and the so-called perpendicular lines lemma.

1.6. DEFINITION. Let \( \varphi : A^\emptyset \rightarrow A^\emptyset \) be a map. Then \( \varphi \) is a morphism if

1. \( \varphi(M) = Ec_f(M) \), for some recursive function \( f \).
2. \( M = N \Rightarrow \varphi(M) = \varphi(N) \).
1.7. Lemma. (i) Let \( F \) be a combinator and define \( \varphi_H(M) \equiv H M \). Then \( \varphi_H \) is a morphism.

(ii) Let \( F, G \) be combinators such that for all \( M \in \Lambda^\theta \) there exists a unique \( N \in \Lambda^\theta \) with \( FM = GN \). Then there is a map \( \varphi_{F,G} \) such that \( FM = G \varphi_{F,G}(M) \), for all \( M \), which is a morphism.

Proof. (i) For the coding \( \# \) let \( app \) be the recursive function such that \( \#(PQ) = app(\#P, \#Q) \). Define \( f(m) = app(\#H, m) \). Then \( \varphi_H(M) = Ec_f(\#M) \). It is obvious that \( \varphi_H \) preserves \( \beta \)-equality.

(ii) Let \( R(m, n) \) be an r.e. relation. Then we have \( R(m, n) \Leftrightarrow \exists z T(m, n, z) \), for some recursive \( T \). Let \( < n, z > \) be a recursive pairing with recursive inverses \( < n, z > .0 = n, < n, z > .1 = z \). Define (\( \mu \) is the least number operator)

\[
\iota_n. R(m, n) = (\mu p. T(m, p.0, p.1)).0.
\]

Then \( \exists n \in R(m, n) \Rightarrow R(m, \iota_n. R(m, n)) \). In order to construct the morphism \( \varphi_{F,G} \), define

\[
f(m) = \iota_n. F(Ec_m) = G(Ec_n).
\]

By the assumption (existence) \( f \) is total. Define \( \varphi_{F,G}(M) = Ec_f(\#M) \). Now \( f(\#M) = \Rightarrow F(Ec_m) = G(Ec_n) \). Therefore \( FM = G \varphi_{F,G}(M) \), for all \( M \). The condition

\[
M = M' \Rightarrow \varphi_{F,G}(M) = \varphi_{F,G}(M')
\]

holds by the assumption (unicity). \( \blacksquare \)

One may wonder whether dropping the unicity condition in lemma 1.7 (ii) one may obtain a morphism by making a right uniformisation. This is not the case.

1.8. Proposition. There exists combinators \( F, G \) such that \( \forall M \exists N \quad FM = GN \) but without any morphism satisfying \( \forall M \quad FM = G \varphi(N) \).

Proof. Let \( \Delta = \chi \Omega \) and define \( F = \chi x.(x, \Delta, 1) \) and \( G = \chi y.(E y, y \Omega \Delta, y!) \). Then, see Statman [1986],

\[
FM = \beta GN \iff (N = \beta c_n \vee N = \beta 1) \& EN = \beta M. \tag{1}
\]

Any morphism \( \varphi \) such that \( FM = G \varphi(M) \) would solve the convertibility problem recursively: one has by (1)

\[
M = M' \iff \varphi(M) = \varphi(M'), \tag{2}
\]

and since \( \varphi(M), \varphi(M') \) have nf's by (1), the RHS of (2) is decidable. \( \blacksquare \)

1.9. Proposition. Not every morphism is of the form \( \varphi_H \).

Proof. Let \( F, G \in \Lambda^\theta \) be such that \( F \circ G = 1 \). Then \( F, G \) determine a so-called inner model \( \llbracket \cdot \rrbracket = \llbracket \cdot \rrbracket^{F,G} \) as follows.

\[
\llbracket x \rrbracket = x;
\llbracket PQ \rrbracket = F[\llbracket P \rrbracket][\llbracket Q \rrbracket];
\llbracket \lambda x. P \rrbracket = G(\lambda x.[\llbracket P \rrbracket]).
\]
Using the condition on \( F, G \) it can be proved that
\[
M =_\beta N \Rightarrow [M] = [N].
\]
Therefore defining \( \varphi(M) = [M] \) we obtain a morphism.

Now take \( F \equiv \lambda y. ul, \Gamma \equiv \lambda z. yz \). Then indeed \( F \circ G = 1 \) and for the resulting inner model one has \([I] = \lambda y. yI]\) and \([\Pi] = (\lambda y. (\lambda z. zI))((\lambda y. (\lambda z. zI)))).\)

Suppose towards a contradiction that the resulting \( \varphi \) is of the form \( \varphi_H \). Then \( H! = \lambda y. yI \), so \( H \) is solvable and hence has a hnf \( \lambda x_1 \ldots x_n . M_1 \ldots M_m \). But \( H\Pi = (\lambda y. (\lambda z. zI))((\lambda y. (\lambda z. zI))) \), which is unsolvable. Therefore the head-variable \( x_i \) is \( x_1 \). But then \( H\Pi = \lambda x_2 \ldots x_n . \Omega M_1^* \ldots M_m^* \) which is not of the correct form.

The following is a corollary to the main theorem.

1.10. **Corollary.** Every morphism \( \varphi \) is of the form \( \varphi_{F,G} \).

**Proof.** Let \( \varphi \) be a given morphism. Define
\[
\mathcal{W}_{\varphi}(N) = \{ Z \mid \exists M \in A^0 [\varphi(M) = N \land (Z = \langle c_0, M \rangle \lor Z = \langle c_1, N \rangle)] \}.
\]
Then \( \{\mathcal{W}_{\varphi}(N)\} \) is a V-partition. By the main theorem there exists an \( H \) such that
\[
H(c_0, M) = H(c_1, N) \iff (c_0, M) = (c_1, N) \lor N = \varphi(M) \iff N = \varphi(M).
\]
Define
\[
F = \lambda m. H(c_0, m); \\
G = \lambda n. H(c_1, n).
\]
Then \( FM = GN \iff N = \varphi(M) \). Therefore \( \varphi = \varphi_{F,G} \).

Note that for a given morphism \( \varphi \) one can define by
\[
\mathcal{W}_{\varphi}(M) = \{ N \in A^0 \mid \varphi(M) = \varphi(N) \}.
\]
This is an inhabited V-partition. It is not difficult to show that each V-partition is equivalent to one of the form \( \mathcal{W}_{\varphi}(M_H) \). Note that \( \{\mathcal{W}_{\varphi}(M_H)\} = \{\mathcal{W}_{\varphi}(M_H)\} \), see lemma 1.7. The following result shows that covering V-partitions are always of this more restricted form.

1.11. **Corollary.** If \( \{\mathcal{W}_e\} \) is a covering V-partition, then \( \{\mathcal{W}_e\} \) is equivalent to \( \{\mathcal{W}_{e(M,H)}\}_{M \in A^0} \) for some \( H \), effectively found from \( \{\mathcal{W}_e\} \).

**Proof.** Let \( H \) be the combinator constructed effectively from \( \{\mathcal{W}_e\} \). We will show that \( \mathcal{W}_{e(M,H)} = \{ N \mid HN = HM \} \) is equivalent to \( \{\mathcal{W}_e\} \). Claim. For \( N \in \mathcal{W}_e \) one has \( \mathcal{W}_e = \mathcal{W}_{e(M,H)} \). Indeed,
\[
N \in \mathcal{W}_e \iff M = N \lor M, N \in \mathcal{W}_e \iff HN = HM \iff N \in \mathcal{W}_{e(M,H)}.
\]
Therefore, noting that $M \in \mathcal{W}_e(M,H)$,
\[ \{W_e \mid M \in \Lambda^0, W_e \neq \emptyset\} \subseteq \{W_e(M,H) \mid W_e(M,H) \neq \emptyset, M \in \Lambda^0\}. \]

The converse inclusion holds also, since every $M$ belongs to some $\mathcal{W}_e$ and hence $\mathcal{W}_e(M,H) = \mathcal{W}_e$ for this $e$. ■

The following theorem states that if a combinator, seen as function of $n$ arguments, is constant—modulo Böhm-tree equality—on $n$ perpendicular lines, then it is constant everywhere.

1.12. THEOREM (Perpendicular lines lemma). Let $F$ be a combinator. Suppose that for $n \in \mathbb{N}$ there are combinators $M_{ij}$, $1 \leq i \neq j \leq n$, and $N_1, \ldots, N_n$ such that for all combinators $Z$ one has ($\simeq$ denotes Böhm-tree equality, i.e. $M \simeq N \iff BT(M) = BT(N)$)
\[
\begin{align*}
F &\quad Z \quad M_{12} \quad \ldots \quad M_{1n-1} \quad M_{1n} \simeq N_1; \\
F &\quad M_{21} \quad Z \quad \ldots \quad M_{2n-1} \quad M_{2n} \simeq N_2; \\
& \vdots \\
F &\quad M_{n1} \quad M_{n2} \quad \ldots \quad M_{nn-1} \quad Z \simeq N_n.
\end{align*}
\]
Then for all $P_1, \ldots, P_n \in \Lambda^0$ one has
\[
FP_1 \ldots P_n \simeq N_1(\simeq N_2 (\simeq \cdots (\simeq N_n))).
\]

PROOF. This is the restriction to closed terms of a theorem in Barendregt [1984], theorem 14.4.12, having the same proof. ■

1.13. COROLLARY. The perpendicular lines lemma is false for any $n > 1$, if $\simeq$ is replaced by $=_{\beta}$.

PROOF (For $n = 1$ the perpendicular lines lemma is trivially true for $=_{\beta}$,). Let $n > 1$. For notational simplicity we assume $n = 2$ and give a counter example. Define
\[
\begin{align*}
\mathcal{W}_{e_1} &\quad = \{N \in \Lambda^0 \mid N = (S, S)\} \\
\mathcal{W}_{e_2} &\quad = \{N \in \Lambda^0 \mid \exists Z \in \Lambda^0 [N = (1, Z) \lor N = (Z, 1)]\}
\end{align*}
\]
Then $\{\mathcal{W}_e\}_{e \in \{e_1, e_2\}}$ is a V-partition. Let $H$ be the combinator obtained from this partition by the main theorem. Then for all $Z \in \Lambda^0$
\[
H(S, S) \neq H(1, Z) = H(Z, 1).
\]
Now define $F \equiv \lambda xy. H(x, y)$. Then for all $Z \in \Lambda^0$
\[
FSS \neq FZ = FZ1.
\]
This is indeed a counterexample. ■

We do believe the conjecture in Barendregt [1984], stating that the perpendicular line lemma with $\simeq$ replaced by $=_{\beta}$ is correct for open terms.
2. Proof of the main theorem

In order to prove the main theorem 1.5, let a V-partition determined by \( S \) be fixed in this section. By proposition 1.4 it may be assumed that the partition is inhabited.

2.1. LEMMA. Let \( \{ W_e \}_{e \in S} \) be an inhabited V-partition.

(i) There exists a total recursive function \( f = f_S \) such that
\[
\forall e \in S \; W_e = \{ f((2e + 1)2^n) \mid n \in \mathbb{N} \}.
\]

(ii) There exists a combinator \( E_S \) such that
\[
\forall e \in S \; W_e = \{ E_s C((2e + 1)2^n) \mid n \in \mathbb{N} \}.
\]

PROOF. (i) By elementary recursion theory there exists a recursive function \( h \) such that \( W_e = \text{Range}(\varphi_{h(e)}) \) and \( \varphi_{h(e)} \) is total, for all \( e \in S \). Observing that \( e, n \) are uniquely determined by \( k = (2e + 1)2^n \), define \( f \) by \( f(0) = 0, \; f((2e + 1)2^n) = \varphi_{h(e)}(n) \).

(ii) Take \( E_S = E \circ F_S \), where \( F_S \) lambda defines \( f_S \) and \( E c_{\#M} = M \) for all \( M \in \Lambda^S \).

2.2. DEFINITION. (i) Define
\[
\text{odd}(0) = 0; \quad \text{odd}((2e + 1)2^n) = 2e + 1.
\]

(ii) Define \( M \sim N \) iff \( M = N \lor M = E_m, N = E_n \) and \( \text{odd}(m) = \text{odd}(n) \), for some \( m, n \).

Notice that \( M \sim N \) iff \( M = N \) or \( \exists e \in SM, N \in W_e \). Therefore we have to prove that there exists a combinator \( H \) such that
\[
HM = HN \iff M \sim N.
\]

The proof consists in constructing a combinator \( H = H^S \) such that

1. \( M \sim N \Rightarrow HM = HN \), proposition 2.4;
2. \( HM = HN \Rightarrow M \sim N \), proposition 2.9.

The second part of the main theorem easily follows by inspecting the proof.

2.3. DEFINITION. (i) Define
\[
T \equiv \lambda xyz.xy(xyz);
A \equiv \lambda fgyzx.fx(a(Ex))[f(S^+x)y(g(S^+x))z];
B \equiv \lambda fgyx.(Sx)(a(E(Tx))(g(S^+x))(gx)).
\]

(ii) By the double fixed-point theorem there exists terms \( F, G \) such that
\[
F \Rightarrow AFG; \quad G \Rightarrow BFG.
\]
To be explicit, write
\[ D \equiv (\lambda xy.y(xxy)); \]
\[ Y \equiv DD; \]
\[ G \equiv Y(\lambda u.B(Y(\lambda v.Auv))u); \]
\[ F \equiv Y(\lambda u.AuG). \]

(iii) Finally define
\[ H \equiv \lambda xa.Fc_1(ax)(Gc_1). \]

**NOTATION.** Write
\[ F_k \equiv Fc_k; \]
\[ G_k \equiv Gc_k; \]
\[ E_k \equiv Ec_k; \]
\[ a_k \equiv aE_k; \]
\[ H_k[ ] \equiv F_k[ ]G_k; \]
\[ C_k[ ] \equiv F_k a_k([ ]G_k). \]

Note that by construction
\[ F_k M N \rightarrow F_k a_k(F_{k+1}MG_{k+1}N); \]
\[ G_k \rightarrow F_{k+1}a_2kG_{k+1}G_k. \]

By reducing \( F \), respectively \( G \), it follows that
\[ H_k[a_p] \equiv F_k a_p G_k \rightarrow C_k[H_{k+1}[a_p]] \quad \text{(1)} \]
\[ H_k[a_k] \equiv F_k a_k G_k \rightarrow C_k[H_{k+1}[a_2k]] \quad \text{(2)} \]

2.4. **Proposition.** \( M \sim N \Rightarrow HM = HN. \)

**Proof.** By lemma 2.1 it suffices to show \( HE_k = HE_{2k} \) for all \( k \).

\[ HE_k = \lambda a.H_1[a_k] \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[H_k[a_k]]\ldots]], \quad \text{by (1),} \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a_2k]]\ldots]]], \quad \text{by (2),} \]
\[ HE_{2k} = \lambda a.H_1[a_2k] \]
\[ = \lambda a.C_1[C_2[\ldots C_{k-1}[C_k[H_k[a_2k]]\ldots]]], \quad \text{by (1).} \]
As a piece of art we exhibit in more detail the reduction flow (contracted redexes are underlined).

\[
\begin{align*}
H_{\xi_k} & \\
\lambda a.F_1 a_k G_1 \\
\lambda a.F_1 a_1(F_2 a_2 G_2 G_1) \\
\lambda a.F_1 a_1(F_2 a_2(G_3 a_3 G_2) G_1) \\
\cdots \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(...(F_k a_k G_{k-1})...G_2) G_1) = \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(...(F_k a_k G_{k-1})...G_2) G_1) \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(...(F_k a_k(F_{k+1} a_2 G_{k+1} G_k) G_{k-1})...G_2) G_1) \\
\end{align*}
\]

And also

\[
H_{\xi_{2k}} \to \cdots \to \\
\lambda a.F_1 a_1(F_2 a_2(F_3 a_3(...(F_k a_k(F_{k+1} a_2 G_{k+1} G_k) G_{k-1})...G_2) G_1)
\]

For the converse implication we need the fine structure of the reduction.

2.5. DEFINITION. Define

\[
\begin{align*}
D_{\xi_k}^0 [M] & \equiv F_k(aM) \equiv Y(\lambda u. AuG)c_k(aM) \\
D_{\xi_k}^1 [M] & \equiv (\lambda y. y(DDy))(\lambda u. AuG)c_k(aM) \\
D_{\xi_k}^2 [M] & \equiv (\lambda u. AuG)F_k(aM) \\
D_{\xi_k}^3 [M] & \equiv AFGc_k(aM) \\
D_{\xi_k}^4 [M] & \equiv (\lambda xz. F_z(xE_z))(F_{S+z} y(G_{S+z} z))c_k(aM) \\
D_{\xi_k}^5 [M] & \equiv (\lambda yz. F_z(aE_x))(F_{S+x} y G_{S+x} z))c_k(aM) \\
D_{\xi_k}^6 [M] & \equiv (\lambda z. F_k(aE_z)(F_{S+z} c_a(aM))G_{S+z} z) \\
D_{\xi_k}^7 [M] & \equiv (\lambda xz. F_k(aE_x)(F_{S+z} c_a(aM))G_{S+z} z) \\
\end{align*}
\]

2.6. LEMMA. Let \( F_k(aM)N \) head-reduce in \( 8p + q \) steps to \( W \). Then

\[
\begin{align*}
W & \equiv D_{\xi_k}^p[M]N, & \text{if } p = 0; \\
& \equiv D_{\xi_k}^p[E_k][(H_{k+1}[E_k])^{p-1}(H_{k+1}[M]N)], & \text{else.}
\end{align*}
\]

PROOF. Note that \( F_k(aM)N \equiv D_{\xi_k}^0[M]N \). Moreover,

\[
\begin{align*}
D_{\xi_k}^q[M]N & \to_h D_{\xi_k}^{q+1}[M]N, & \text{for } q < 7; \\
D_{\xi_k}^q[M]N & \to_h D_{\xi_k}^q[E_k][H_{k+1}[M]N].
\end{align*}
\]

The rest is clear. At steps 16, 24 we obtain for example

\[
\begin{align*}
D_{\xi_k}^6[E_k][H_{k+1}[M]N] & \to_h D_{\xi_k}^6[E_k][(H_{k+1}[E_k])(H_{k+1}[M]G_k)]. \\
D_{\xi_k}^6[E_k][(H_{k+1}[E_k])(H_{k+1}[M]G_k)] & \to_h D_{\xi_k}^6[E_k][(H_{k+1}[E_k])^2(H_{k+1}[M]G_k)].
\end{align*}
\]

Remember that a standard reduction \( \sigma:M \to_N N \) always consists of a head-reduction followed by an internal reduction:

\[
\sigma:M \to_h W \to_i N.
\]
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NOTATION. Write $M =_{s \leq n} N$ if there are standard reductions of length $\leq n$ from $M$ respectively $N$ to a common reduct $Z$. Similarly $M =_{i \leq n} N$ for internal standard reductions. Also the notations $=_{s < n}$ and $=_{i < n}$ will be used.

2.7. LEMMA. (i) $D_k^q[M]N =_{i \leq n} D_k^q[M']N' \Rightarrow q = q' & N =_{s \leq n} N'$.
(ii) $D_k^q[M]N =_{i \leq n} D_k^q[M']N' & q < 7 \Rightarrow M =_{s \leq n} M'$.
(iii) $D_k^q[M]N =_{i \leq n} D_k^q[M']N' \Rightarrow H_{k+1}[M] =_{s \leq n} H_{k+1}[M']$.

PROOF. (i) Suppose $D_k^q[M]N =_{i \leq n} D_k^q[M']N'$. Then By observing where the free variable $a$ occurs one can conclude that $q = q'$. Since the reductions to a common reduct are internal, the positions of $N, N'$ are not changed and hence $N =_{s \leq n} N'$.
(ii) Obvious from the definition of $D_k^q$.
(iii) In this case it follows that $D_k^q[E_k](H_{k+1}[M]z) =_{i \leq n} D_k^q[E_k](H_{k+1}[M']z)$.

The conclusion $H_{k+1}[M] =_{s \leq n} H_{k+1}[M']$ depends on the fact that there are the free variables $z$ to mark the residuals.

2.8. LEMMA. Suppose $G_k =_{s \leq n} (H_{k+1}[E_k])^d(H_{k+1}[M]G_k)$. Then $H_{k+1}[E(Tc_k)] =_{s \leq n} H_{k+1}[M]$.

PROOF. By induction on $d$. If $d = 0$, then we have $G_k =_{s \leq n} H_{k+1}[M]G_k$. So there are standard reductions of these two terms to a common reduct. Observe that the head-reduction starting with $G_k$ begins as follows.

$$
G_k \equiv Y(\lambda u. B(Y(\lambda v. Au))u)c_k \\
\rightarrow_h (\lambda x. x(Y x))(\lambda u. B(Y(\lambda v. Au))u)c_k \\
\rightarrow_h (\lambda u. B(Y(\lambda v. Au)))c_k \\
\rightarrow_h BF(c_k) \\
\rightarrow_h (\lambda x. F(S^+ k)(a(E^S(T x)))(g(S^+ k))(g x)c_k \\
\rightarrow_h (\lambda x. F(S^+ k)(a(E^S(T x)))(G(S^+ k))(G x)c_k \\
\rightarrow_h F(S^+ k)(a(E^S(T c_k)))(G(S^+ k))(G c_k).
$$

The heads of these terms are not of order 0 except the last one. But $H_{k+1}[X]$ is always of order 0. Therefore the mentioned standard reduction of $G_k$ goes at least to this last term $H_{k+1}[E^S(T c_k)]G_k$. But then $H_{k+1}[E^S(T c_k)] =_{s \leq n} H_{k+1}[M]$.

If $d > 0$, then start the same argument as above, but at the intermediate conclusion $H_{k+1}[E^S(T c_k)]G_k =_{s \leq n} (H_{k+1}[E_k])^d(H_{k+1}[M]G_k)$, one proceeds by concluding that $G_k =_{s \leq n} H_{k+1}[E_k]^{d-1}(H_{k+1}[M]G_k)$ and uses the induction hypothesis.

2.9. PROPOSITION. $H_k[M] = H_k[N] \Rightarrow M \sim N$. 

PROOF. By the standardization theorem it suffices to show for all \( n \) that
\[
\forall k \in \mathbb{N} [H_k[M] =_{s \leq n} H_k[N] \Rightarrow M \sim N].
\]
This will be done by induction on \( n \). From \( H_k[M] =_{s \leq n} H_k[N] \) it follows that
\[
H_k[M] \to_h W_M \to_i Z
\]
\[
H_k[N] \to_h W_N \to_i Z.
\]
for some \( W_M, W_N, Z \).

Case 1. \( W_M, W_N \) are both reached after \( < 8 \) steps. Then by lemma 2.6 \( W_M \equiv D^*_k[M]G_k, W_N \equiv D^*_k[N]G_k \). By lemma 2.7(i) it follows that \( q = q' \). If \( q < 7 \), then by 2.7(ii) one has \( M = N \) so \( M \sim N \). If \( q = 7 \), then by 2.7(iii) one has \( H_{k+1}[M] =_{s \leq n} H_{k+1}[N] \) and by the induction hypothesis one has \( M \sim N \).

Case 2. \( W_M \) is reached after \( p \geq 8 \) steps and \( W_N \) after \( q < 8 \) steps. Then \( p = 8d + q \) and, keeping in mind lemma 2.7(i), it follows that \( W_M \equiv D^*_k[M]G_k, W_N \equiv D^*_k[G_k]R, G_k =_{s \leq n} R \), where \( R \equiv (H_{k+1}[E_k])^{d-1}(H_{k+1}[N]G_k) \). Then as in case 1 it follows that \( M \sim E_k \). Moreover, by lemma 2.8 \( H_{k+1}[E_{2k}] =_{s \leq n} H_{k+1}[N] \), so by the induction hypothesis \( E_{2k} \sim N \). So \( M \sim E_k \sim E_{2k} \sim N \).

Case 3. Both \( W_M, W_N \) are reached after \( \geq 8 \) steps. Then
\[
W_M \equiv D^*_k[E_k][(H_{k+1}[E_k])^d(H_{k+1}[M]G_k)];
W_N \equiv D^*_k[E_k][(H_{k+1}[E_k])^d(H_{k+1}[N]G_k)].
\]
If \( d = d' \), then by lemma 2.7
\[
(H_{k+1}[E_k])^d(H_{k+1}[M]G_k) =_{s \leq n} (H_{k+1}[E_k])^d(H_{k+1}[N]G_k),
\]
so
\[
H_{k+1}[M] =_{s \leq n} H_{k+1}[N],
\]
since \( H_{k+1}[X] \) is always of order 0. Therefore by the induction hypothesis \( M \sim N \).

If on the other hand, say, \( d < d' \), then (writing \( d' = d + e \))
\[
W_M \equiv D^*_k[E_k][(H_{k+1}[E_k])^d(H_{k+1}[M]G_k)];
W_N \equiv D^*_k[E_k][(H_{k+1}[E_k])^d(H_{k+1}[E_k])^e-1(H_{k+1}[N]G_k)].
\]
so
\[
H_{k+1}[M] =_{s \leq n} H_{k+1}[E_k];
G_k =_{s \leq n} (H_{k+1}[E_k])^{e-1}(H_{k+1}[N]G_k),
\]
since \( H_{k+1}[X] \) is always of order 0. Therefore by lemma 2.8
\[
H_{k+1}[E_{2k}] =_{s \leq n} H_{k+1}[N],
\]
Therefore by the induction hypothesis twice we obtain \( M \sim E_k \sim E_{2k} \sim N \). ■

References

Theoretical pearls


