SUMMARY

§ 1 is concerned with the term model of the λ-calculus. It is proved that Church’s δ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several λ-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which λ-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a λ-algebra (i.e. a model of the λ-calculus). Elements of \( M \) are thought of as functions. Arbitrary \( f : M \rightarrow M \) are called external functions. Such a function is representable (by an element \( a \in M \)) if \( \forall b \in M \ f(b) = a \cdot b \). A function \( f \) is definable in \( \mathcal{M} \) if \( f \) is representable by \( \llbracket F \rrbracket^\mathcal{M} \) for some closed term \( F \). Here \( \llbracket F \rrbracket^\mathcal{M} \) denotes the value of \( F \) in the model \( \mathcal{M} \).

Other notations:

- \( x, y, ... \) denote variables of the λ-calculus.
- \( a, b, ... \) denote variables ranging over the elements of a λ-algebra.
- \( F, G, ... \) denote λ-terms.

The numerals 0, 1, 2, ... denote some adequate representation of the natural numbers as λ-terms e.g. those of Church: \( n = \lambda f x. f^n(x) \).

If \( \mathcal{M} = \langle M, \cdot \rangle \) is a λ-algebra, then \( \mathcal{M}^0 \) is the sub-λ-algebra \( \langle M^0, \cdot \rangle \) where \( M^0 = \{ \llbracket F \rrbracket^\mathcal{M} \in M \mid F \text{ closed term} \} \).

If \( T \) is a consistent extension of the λ-calculus, \( \mathcal{M}(T) \) is the term-model of \( T \), i.e. the set of all λ-terms modulo provable equality in \( T \). The closed term-model of \( T \), notation \( \mathcal{M}^0(T) \), is defined as \( (\mathcal{M}(T))^0 \). A λ-algebra \( \mathcal{M} \) is hard if \( \mathcal{M} = \mathcal{M}^0 \). In such an \( \mathcal{M} \) a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model \( \mathcal{M} = \mathcal{M}(\lambda \eta) \).
Church's $\delta$ is an external function satisfying

\[ \delta MM' = \begin{cases} 0 & \text{if } M \text{ is a closed normal form (nf)} \\ 1 & \text{if } M, M' \text{ are different closed nf's.} \end{cases} \]

In Böhm [1972] it is proved that $FN_1 \ldots N_n$ different $\beta\eta$-nf's $AF \vdash FN_1 = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying $\star$ for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_\infty$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim \mathcal{M} g$, if $f(a) \cdot b = g(b) \cdot a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a) b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

\[ f \sim \mathcal{M} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}. \]

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda \eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. Definition. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k^{th}$ approximate normal form of $M$.

1.2. Definition. (i) A selector is a term of the form

\[ U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i < n. \]

A permutator is a term of the form

\[ C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)} \]

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$. 

\[ a \in M_{\eta} \quad \text{iff} \quad a \in M \quad \text{and} \quad \eta \text{ is applied in } a. \]
1.3. **Lemma.** Simple terms have a normal form (nf).

**Proof.** Realize that each simple term is of the form $x\overrightarrow{P}$, $U\overrightarrow{P}$, $C\overrightarrow{P}$ with $\overrightarrow{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf.

1.4. **Theorem.** Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\overrightarrow{P}, \overrightarrow{Q}$, with $x \notin FV(\overrightarrow{P})$, $\lambda \vdash M\overrightarrow{P} = x\overrightarrow{Q}$ ("$x$ is Böhm'd out").

(ii) Moreover $\overrightarrow{P}$ can be chosen as a sequence of simple terms.

**Proof.** Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1$, $x$ occurs in $BT(M^{\pi_1})$ at depth $k - 1$. Iterating this leads to $M^{\pi_2} = \lambda y \cdot x\overrightarrow{Q}$, hence $M^{\pi_2}y = x\overrightarrow{Q}$, for a Böhm transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that

$$M^{\pi_2}y = M ... x_i ... [x_j/Cx_j] ... [x_k/Ux_k] ... y = M\overrightarrow{P}$$

for some simple terms $\overrightarrow{P}$ with $x \notin FV(\overrightarrow{P})$ (where $C$ is a permutator and $U$ a selector).

1.5. **Lemma.** Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \not\vdash FX_1 = FX_2$ for some $X_1$, $X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

**Proof.** Note that if $P, P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin FV((Fx)^k)$ $(N^k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k = (Fx)^k [x/M] = (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM =Fx$. Since $F, M$ are closed it follows that for all $\lambda$-terms $N$, $\lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction.

**Remark.** 1.5 also holds for $F, M$ not necessarily closed.

1.6. **Definition.** $\emptyset = I$, $n + 1 = K \cdot n$.

1.7. **Lemma.** The function $sg$ is not $\lambda$-definable with respect to $\{n | n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash F \emptyset = \emptyset$, $\vdash F n + 1 = I$.

**Proof.** Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\overrightarrow{P} = x\overrightarrow{Q}$ for some $\overrightarrow{P}, \overrightarrow{Q} = Q_1 ... Q_m$. But then for all $n > m$,

$$\vdash I\overrightarrow{P} = F n \overrightarrow{P} = n Q_1 ... Q_m = n - m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's.
1.8. Definition. A system of terms \( \{M_n | n \in \omega \} \) is an adequate system of numerals iff

(i) Each \( M_n \) has a \( \text{n}f \).
(ii) Each recursive function can be \( \lambda \)-defined with respect to the \( M_n \).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \( sg \) functions can be \( \lambda \)-defined with respect to the \( M_n \).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. Corollary.

(i) \( \{n | n \in \omega \} \) is not an adequate system of numerals. (ii) Church’s \( \delta \) is not \( \lambda \)-definable.

Proof. (i) Immediate. (ii) If \( \delta \) were \( \lambda \)-definable, then so would be \( sg \), viz. by \( \lambda x \cdot \delta x \equiv 0 \equiv 1 \).

Remark. (i) Although not definable, \( \delta \) can consistently be added to the \( \lambda \)-calculus, see Church [1941].
(ii) Contrary to this, the corresponding \( \delta \) for open \( \lambda \)-terms would be inconsistent at once. For let \( x \not\equiv y \), then

\[
(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot 1(KK)S)x = (\lambda y \cdot KKS)x = KKS = K
\]

but also

\[
(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = 0(KK)S = S.
\]

(iii) One could also consider the definability of a \( \delta \) for all closed terms, i.e.: \( \delta MM = 0 \) for \( M \) closed

\[
\delta MN = 1 \quad \text{for} \quad M, N \quad \text{closed such that} \quad \not\models M = N.
\]

But then the following version of the Russell paradox would result.

Define \( X = \delta x 1 \). If \( \not\models \delta = 1 \) then \( \not\models X = 1 \iff \models \neg X = 1 \).

Now let \( A = FP \) (i.e. the fixed point of \( \neg \): \( \models \neg A = \neg A \)).
Then \( \models \neg A = 1 \iff \models A = 1 \). Thus \( \models \neg 0 = 1 \).

To see the relation with the Russell paradox, note that \( A = BB \) with \( B = \lambda x. \neg (xx) \). (In illative combinatory logic \( MN \) is interpreted as \( N \in M \) and \( \lambda x \cdot P \) as \( \{x|P\} \).

1.10. Theorem. Let \( \omega = \{n | n \in \omega \} \) be an adequate system of numerals and let \( f \) be a map into \( \omega \) definable by \( F \). Then \( f \) is constant.

Proof. First assume \( \omega \) is Church’s system of numerals.
Suppose \( f \) is not constant, then by 1.5 \( x \in BT(Fx) \). Hence for some simple \( \vec{P} \) and \( \vec{Q}, \lambda \models Fx\vec{P} = x\vec{Q} \).

Hence \( \lambda \models FM\vec{P} = M\vec{Q} \) for all \( M \). But \( M\vec{Q} \) can take arbitrary values and not \( FM\vec{P} \), since \( \models \vec{P} = P_1^n(P_2)P_3 \ldots P_k \) always has a \( \text{n}f \) by 1.3.
Now let ω be an arbitrary system of numerals. It is well-known how to define a term G such that \( G^n = n \).

Suppose a non-constant \( f : \text{terms} \to \omega \) would be definable, then \( G \circ f \) were a definable non-constant mapping into \( \omega \).

First alternative proof (due to the referee).

Suppose \( F \) is not constant, i.e. let \( n_1 \neq n_2 \in Ra(F) \). Define \( G \) as the \( \lambda \)-defining term of the recursive function

\[
g(x) = \begin{cases} 
0 & \text{if } x = n_1, \\
1 & \text{else}.
\end{cases}
\]

Then the range of \( G \circ F \) is \( \{0, 1\} \) contrary to 2.3.

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if \( \Omega \) is unsolvable and \( N \) a \( nf \), then \( F\Omega = N \Rightarrow Fx = N \) for all \( x \). (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of \( F \) are numerals it follows that \( F\Omega \) has a \( nf \), i.e. \( F \) is constant.

1.11. Corollary. There is no \( F \) such that

\[
FM = \emptyset \quad \text{if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n) \\
\{1\} \text{ else}
\]

for any adequate system.

1.12. Question. Is there a term \( F \) such that

\[
FM \text{ has a } nf \text{ (is solvable) if } M \text{ is a numeral} \\
\text{has no } nf \text{ (is unsolvable) else.}
\]

§ 2. THE RANGE PROPERTY

2.1. Definition. Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra. For each \( f \in M \), we define \( Ra^\mathcal{M}(f) \), the range of \( f \) in \( \mathcal{M} \), as follows:

\[
Ra^\mathcal{M}(f) = \{ f \cdot x \mid x \in M \}.
\]

Notation. \( Ra^\mathcal{M}(F) = Ra^\mathcal{M}(\langle F \rangle^\mathcal{M}) \) for terms \( F \).

When possible, the superscript \( \mathcal{M} \) will be dropped in \( Ra^\mathcal{M} \).

2.2. Definition. A \( \lambda \)-algebra \( \mathcal{M} \) satisfies the range property if for all \( f \in M \), the cardinality of \( Ra^\mathcal{M}(f) \) is 1 or \( \aleph_0 \).

2.3. Range Theorem. (Barendregt; Myhill). Let \( T \) be a r.e. \( \lambda \)-theory. Then \( \mathcal{M}(T) \) (and also \( \mathcal{M}^\mathcal{O}(T) \)) has the range property.

Proof. Suppose \( f \in M \) and \( Ra(f) = \{ m_0, \ldots, m_k \} \), \( k > 0 \). Define

\[
N_t = \{ x \mid f \cdot x = m_t \} \subset M.
\]
Every such \( N_i \) is r.e. Therefore \( N = \bigcup_1^k N_i \), the complement of \( N_0 \) is also r.e. Hence \( N_0 \) is recursive.

On the other hand \( N_0 \) is non-trivial and closed under equality, which contradicts Scott’s theorem (Barendregt [1976] 2.21).

The proof for \( \mathcal{M}_0(T) \) is the same.

2.4. Corollary. \( \mathcal{M}(\lambda), \mathcal{M}_0(\lambda), \mathcal{M}(\lambda \eta) \) and \( \mathcal{M}_0(\lambda \eta) \) have the range property.

The range property, however, is not satisfied in every \( \lambda \)-algebra.

2.5. Theorem. \( P\omega \) and \( D_{\infty} \) do not satisfy the range property.

Proof. Since the proof is similar in both cases, let \( \mathcal{S} = (S, \prec) \) denote either \( (P_{\infty}, \subseteq) \) or \( (D_{\infty}, \sqsubseteq) \). We define the following function \( f : S \to S \) by \( f(x) = \top \) if \( x \neq \bot \) else \( \bot \) (\( \top \) and \( \bot \) are the largest respectively smallest element of \( S \)).

Claim: \( f \) is continuous. Then by Scott [1972], [1975] \( f \) is representable and since \( f \) has range of cardinality two we are done.

For open \( O \) in \( S \) one has: \( x \in O \) and \( x \prec y \Rightarrow y \in O \).


Hence for open \( O \), \( \bot \in O \Rightarrow O = S \), and \( O \neq \emptyset \Rightarrow \top \in O \).

Now for every open set \( O \), \( f^{-1}(O) \) is open:

Case 1. \( \bot \in O \). Then \( O = S \) so \( f^{-1}(S) = S \) which is open.

Case 2. \( \bot \notin O \). If \( O = \emptyset \), then we are done. Else \( \top \in O \) and hence \( f^{-1}(O) = S - \{ \bot \} = \{ x | x \notin \bot \} \). \( \boxed{\text{Claim proved}} \).

\( U_\bot \) is open in \( D_{\infty} \), see e.g. Barendregt [1976] 1.2.

\( U_\bot \) is open in \( P\omega \): Let \( O_k = \{ x \in \mathcal{S} \} \). Note \( e_0 = \emptyset = \bot \) and that the \( O_k \) form a base for the topology on \( P\omega \).

Now:

\[
x \in U_\bot \Leftrightarrow x \notin O \Leftrightarrow \exists k \geq 0 \quad e_k \subseteq x \Leftrightarrow x \in \bigcup_{k=0}^{\infty} O_k
\]

which is, as a union of elements of a base, indeed open. \( \boxed{\text{Claim proved}} \).

The following theorem was announced in Wadsworth [1973] for the \( D_{\infty} \) case.

2.6. Theorem. Let \( \mathcal{S} \) be \( D_0^0 \) or \( P^0\omega \). Then \( \mathcal{S} \) satisfies the range property.

Proof. Let \( F \) be a closed term. Consider \( BT(Fx) \).

Case 1. \( x \notin BT(Fx) \). Then \( BT(FM) = BT(FM') \) for all \( M, M' \). Since terms with equal Böhm trees are equal in \( \mathcal{S} \) (see Barendregt [1976], Hyland [1976]), it follows that \( Ra^{\mathcal{S}}(F) \) has cardinality 1.

Case 2. \( x \in BT(Fx) \). Then by 1.4 \( \lambda \vdash Fx\vec{F} = x\vec{Q} \).

Since \( \text{[NQ]}^{\mathcal{S}} \) can take arbitrary values in \( \mathcal{S} \) when \( N \) ranges over the closed terms, \( Ra^{\mathcal{S}}(F) \) is infinite. \( \boxed{\text{Claim proved}} \).
2.7. Conjecture. $\mathcal{M}(\mathcal{H})$ satisfies the range property.

2.8. Question. Does every hard $\lambda$-algebra $\mathcal{M}$ (i.e. $\mathcal{M} = \mathcal{M}^0$) satisfy the range theorem?

§ 3. Duality

3.1. Definition. Let $f, g$ be two external functions on a $\lambda$-algebra $\mathcal{M} = \langle M, \cdot \rangle$.

$f, g$ are dual iff $\forall a, b \in M : f(a) \cdot b = g(b) \cdot a$. Notation $f \sim _{\mathcal{M}} g$, or simply $f \sim g$.

3.2. Definition. $\mathcal{M}$ is rich iff all dual functions on $\mathcal{M}$ are representable in $\mathcal{M}$.

Remarks. (i) Let $f$ be an external function on $\mathcal{M}$. $f$ is locally representable iff for each $b \in M$ the function $h$ defined by $h(a) = f(a) \cdot b$ is representable. Then $f$ is locally representable iff $f$ has a dual. A model is rich iff all locally representable functions are representable.

(ii) If $f$ is representable (by $f_0 \in M$, say), then $f$ has a dual $g$ which is also representable (by $g_0 = \lambda ab \cdot f_0 ba$).

(iii) Let $\mathcal{M}$ be extensional. Then $f$ has at most one dual. Hence if $f \sim _{\mathcal{M}} g$ and $f$ is representable, then by (ii) $g$ is representable.

3.3. Theorem. If $\mathcal{M}$ is rich, then $\mathcal{M}$ is extensional.

Proof. Suppose $\mathcal{M}$ is not extensional. Then there exist $b, b' \in M$ such that for all $c \in M$ $b \cdot c = b' \cdot c$ and $b \neq b'$.

Define

$$f(a) = \begin{cases} b' & \text{if } a = b \\ b & \text{else.} \end{cases}$$

and

$$g = [\lambda y \cdot K(by)]^\mathcal{M},$$

then for all $a, a' \in M : f(a) \cdot a' = b \cdot a' = g(a') \cdot a$, hence $f \sim g$. But $f$ cannot be representable since it has no fixed point. Thus $\mathcal{M}$ is not rich. \[\square\]

3.4. Corollary. The following $\lambda$-algebras are not rich: $P\omega$; $P^0\omega$; $\mathcal{M}(\lambda)$; $\mathcal{M}^0(\lambda)$; $\mathcal{M}^0(\lambda \eta)$.

Proof. 

1. $P\omega$ is not extensional:

   Take for example $a = \{(0, 0)\}$ and $b = \{(0, 0), (1, 0)\}$.

   Then $\forall c \in P\omega \ a \cdot c = b \cdot c$ but $a \neq b$.

2. $P^0\omega$ is not extensional: Let $1 = \lambda xy \cdot xy$, then $P^0\omega \models Ixy = 1xy$, but $P^0\omega \models I = 1$ for otherwise $P\omega \models I = 1$, so $P\omega \models Vx \ x = \lambda y \cdot xy$ which implies that $P\omega$ were extensional.

3. By the Church Rosser property $\lambda \models I = 1$. So $\mathcal{M}(\lambda)$, $\mathcal{M}^0(\lambda)$ are not extensional.
4. $\mathcal{M}^0(\lambda \eta)$ is not extensional because the $\lambda$-calculus is $\omega$-incomplete, see Plotkin [1974].

3.5. Theorem. $D_\infty$ is rich.

Proof. Suppose that $f, g$ are dual i.e.:

$$\forall a, b \in D_\infty: \ f(a) \cdot b = g(b) \cdot a.$$  

We have to show that $f, g$ are representable.

It is sufficient to show that $f, g$ are continuous. Take a directed $X \subset D_\infty$.

For all $b \in D_\infty$

$$f(\bigsqcup X) \cdot b = g(b) \cdot \bigsqcup X = \bigsqcup \{g(b) \cdot a \mid a \in X\} =$$

$$\bigsqcup \{f(a) \cdot a \in X\} = \bigsqcup \{f(a) \mid a \in X\} \cdot b$$

by the duality condition and the continuity of application.

Thus by extensionality in $D_\infty$ for all directed $X$ $f(\bigsqcup X) = \bigsqcup \{f(a) \mid a \in X\}$ i.e. $f$ is continuous. The proof for $g$ is dual. $\blacksquare$

3.6. Theorem. $\mathcal{M}(\lambda \eta)$ is rich.

Proof. Define

$$M = \lambda \eta \ N \ \text{iff} \ \lambda \eta \vdash M = N,$$

$$x \in_{\lambda \eta} M \ \text{iff for all} \ M' = \lambda \eta \ M \ \text{one has} \ x \in FV(M').$$

Let $f, g$ be dual functions on $\mathcal{M}(\lambda \eta)$.

3.6.0. Lemma. (i) $x \in_{\lambda \eta} M \iff V N(\lambda \eta \vdash M \rightarrow N \Rightarrow x \in FV(N))$.

(ii) Let $M' \equiv M[z/y]$ and $\lambda \vdash M' \rightarrow N'$. Then $\forall N \lambda \vdash M \rightarrow N$ and $N' \equiv N[z/y]$.

(iii) $x \in_{\lambda \eta} M \Rightarrow x \in_{\lambda \eta} M[z/y]$, for $z \neq x$.

Proof. (i) $\Rightarrow$ Trivial. $\Leftarrow$ Suppose $M = \lambda \eta M'$. By the Church-Rosser theorem $\lambda \eta \vdash M \rightarrow N$, $M' \rightarrow N'$ for some $N$. By assumption $x \in FV(N)$. But then $x \in FV(M')$.

(ii) Induction on the length of proof of $M' \rightarrow N'$. In the case that $M' \equiv (\lambda a \cdot P)Q$, $N' \equiv P[a/Q]$ it may be assumed that $a \notin z, y$. Therefore one can apply the well-known substitution lemma

$$A[u/B][v/C] = A[v/C][u/B[v/C]] \ \text{if} \ u \neq v \ \text{and} \ u \notin FV(C).$$

(iii) Suppose $\lambda \eta \vdash P[z/y] \rightarrow R'$. By (ii) for some $R \lambda \eta \vdash P \rightarrow R$ and $R' \equiv R[z/y]$. By assumption and (i), $x \in FV(R)$. Since $x \neq z$ also $x \in FV(R')$. Therefore by (i) $x \in_{\lambda \eta} P[z/y]$. $\blacksquare$ 3.6.0
3.6.1. **Lemma.** (i) If \( x \in_{\eta} \lambda y \cdot P \) then \( x \in_{\eta} P \) and \( x \neq y \).

(ii) If \( x \neq y \), then \( x \in_{\eta} M \iff x \in_{\eta} M y \).

**Proof.** (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P =_{\eta} N \), then \( \lambda y \cdot P =_{\eta} \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subset FV(N) \). Thus \( x \in_{\eta} P \).

(ii) \( \implies \) Suppose \( \lambda y \vdash M y \to N \) in order to prove \( x \in FV(N) \).

Case 1. \( N = M' y \) with \( \lambda y \vdash M \to M' \). Since \( x \in_{\eta} M \), also \( x \in FV(M') \subset FV(N) \).

Case 2. \( M \to \lambda z \cdot M_1 \) and \( \lambda y \vdash M y \to (\lambda z \cdot M_1)y \to M_1[z/y] \to N \).

Since \( x \in_{\eta} M \), also \( x \in_{\eta} \lambda z \cdot M_1 \) and by (i) \( x \in_{\eta} M_1 \) and \( z \neq x \), so by 3.6.0. (iii) \( x \in_{\eta} M_1[z/y] \). Therefore \( x \in FV(N) \). ■

3.6.2. **Lemma.** If \( \exists y \neq x \ x \in_{\eta} f(y) \), then \( \forall y \neq x \ x \in_{\eta} g(y) \) (and hence \( \forall y \neq x \ x \in_{\eta} f(y) \)).

**Proof.** Suppose \( x \in_{\eta} f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in_{\eta} f(y) \cdot y' =_{\eta} g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in_{\eta} g(y') \). (The rest follows by applying the statement to \( x \in_{\eta} g(y) \)). ■

3.6.3. **Main Lemma.** There is a variable \( x \) such that for all terms \( M : f(x)[x/M] = f(M) \).

**Proof.** Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \notin_{\eta} f(v) \). Then \( x \notin_{\eta} g(z) \) for all \( z \neq x \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \notin_{\eta} M \), \( f(M), x, f(x) \). Hence \( x \notin_{\eta} g(y) \). Now since \( y \neq x \) and \( x \notin_{\eta} g(y) \), \( (f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = g(y) \cdot x = g(y) \cdot M = f(M) \cdot y \).

Since \( y \neq f(x) \), \( M, f(M) \), extensionality yields \( f(x)[x/M] = f(M) \). ■

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similar for \( g \). ■

The following construction is needed for the proof of 3.10.

3.7. **Definition.** Let \( \neq \) be a Gödel numbering of terms. \( \lceil M \rceil \) is the numeral \( \neq M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \neq M_n \) is a recursive function.

3.8. **Lemma.** (Coding of infinite sequences). Let \( \{M_n\} \) be a recursive sequence of terms such that \( FV(M_n) \subseteq \{x\} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term.

Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

**Proof.**

(1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:

\[
E(\lceil M \rceil) = M, \text{ for } M \text{ with } FV(M) = \{x\}.
\]
(2) Let $[M, N]$ be a pairing of terms defined by $\lambda z \cdot zMN$. Then $[M, N]K = M$ and $[M, N](KI) = N$. Define ordered tuples as follows:

$$[M] = M, \ [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]].$$

(3) Let $M_n$ with $FV(M_n) \subseteq \{x\}$ be a recursive sequence of terms. We want to code the sequence $\langle M_n \rangle_{n \in \omega}$ as a $\lambda$-term. Let $S^+$ be such that $S^+ \eta \xrightarrow{\beta} \eta + 1$ and let $b \equiv \lambda y \cdot [E(Fy), (x(S^+ y))]$, where $F \lambda$-defines $f$, and $B \equiv FP b$ (i.e. the fixed point of $b$). Then

$$B \eta \xrightarrow{\beta} bB \eta \xrightarrow{\beta} [E(F \eta), B \eta + 1] \xrightarrow{\beta} [M \eta, B \eta + 1].$$

So $B_0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots$. Hence by setting $\langle M_n \rangle_{n \in \omega} = B_0$ we have a coding for infinite sequences of terms with one fixed free variable.

(4) It is easy to construct a term $p$ such that $pM = M_1$, (take e.g. $pxa$ if zero $x$ then $aK$ else $p(x - 1)(a(KI))$, using the fixed point theorem).

3.9. Lemma. For all closed $Z$ there is an $n$ such that $Z \Omega^n = \Upsilon \Omega$. ($Z \Omega^n$ is short for $Z \Omega \Omega \ldots \Omega$).

Proof.

Case 1. $Z$ is unsolvable; then $Z = \Upsilon \Omega$, so $n = 0$.

Case 2. $Z$ is solvable; then $Z$ has a $\lambda$-term, $Z = \lambda x \cdot x_1 A_1 \ldots A_m \ (x_1 e \equiv x)$.

Take $n = i$, so $Z \Omega^i = \lambda x^i \cdot \Omega A_1 \ldots A_m = \Upsilon \Omega$.  

3.10. Theorem. If $\mathcal{H}$ is hard and sensible, then $\mathcal{H}$ is not rich.

Proof. If $\mathcal{H}$ is hard, then $\mathcal{H}$ is isomorphic to $\mathcal{H}^0(T)$, where $T = Th(\mathcal{H})$. We reason in $\mathcal{H}^0(T)$. Since $\mathcal{H}$ is sensible, $\mathcal{H} \subseteq T$.

Let $h : \omega \rightarrow \omega$ be a function not definable in $\mathcal{H}$. Such an $h$ exists since a hard model is countable.

Let $A_n(x, y)$ be the term $x \Omega^n(y \Omega^n(h \eta)), n \in \omega$. For closed $M$ the sequence $A_0(M, y), A_1(M, y), \ldots$ is by 3.9

$$M(y(h \eta)), M \Omega(y \Omega(h \eta)), \ldots, M \Omega^n(y \Omega^n(h \eta)), \Omega, \Omega, \ldots,$$

where $n$ is such that $M \Omega^{n+1} = \Omega$. Thus $\lambda n \cdot A_n(M, y)$ is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define $f(M) = \lambda y \cdot \langle A_n(M, y) \rangle_{n \in \omega}$. Similarly for closed $N$

$$\lambda n \cdot A_n(x, N)$$

is recursive and it is possible to define $g(N) = \lambda x \cdot \langle A_n(x, N) \rangle_{n \in \omega}$. Then for all closed $M, N : f(M)$ and $g(N)$ are well defined and $f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N) \rangle_{n \in \omega}$ by construction. So $f$ and $g$ are dual.

Suppose now that $\mathcal{H}$ is rich, i.e. $f$ were representable by some closed $F$. Then for all closed $M, N : FMN = f(M)N = \langle A_n(M, N) \rangle_{n \in \omega}$.

But then $p\eta F(K^n I)(KI) = p\eta \langle h(\eta) \rangle_{n \in \omega} = h(\eta)$, hence $h$ were definable, contradiction. Thus $\mathcal{H}$ is not rich.
3.11. **Corollary.** $D^\omega_\infty$ and $\mathcal{M}(T)$ for $T \supseteq \mathcal{H}$ are not rich.

3.12. **Questions.** (i) Is every extensional term model $\mathcal{M}(T)$ rich? (ii) Is $\mathcal{M}(\lambda\omega)$ rich?

Here $\lambda\omega$ is the $\lambda$-theory obtained by adding the $\omega$-rule to the theory, see Barendregt [1974].

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