SUMMARY

§ 1 is concerned with the term model of the λ-calculus. It is proved that Church’s δ is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several λ-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which λ-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let $\mathcal{M} = <M, \cdot>$ be a λ-algebra (i.e. a model of the λ-calculus). Elements of $M$ are thought of as functions. Arbitrary $f : M \rightarrow M$ are called external functions. Such a function is representable (by an element $a \in M$) if $\forall b \in M \; f(b) = a - b$. A function $f$ is definable in $\mathcal{M}$ if $f$ is representable by $\lfloor F \rfloor^\mathcal{M}$ for some closed term $F$. Here $\lfloor F \rfloor^\mathcal{M}$ denotes the value of $F$ in the model $\mathcal{M}$.

Other notations:

$x, y, \ldots$ denote variables of the λ-calculus.
$a, b, \ldots$ denote variables ranging over the elements of a λ-algebra.
$F, G, \ldots$ denote λ-terms.

The numerals 0, 1, 2, \ldots denote some adequate representation of the natural numbers as λ-terms e.g. those of Church: $n = \lambda f x. \, f^n(x)$.

If $\mathcal{M} = <M, \cdot>$ is a λ-algebra, then $\mathcal{M}^0$ is the sub-λ-algebra $<M^0, \cdot>$ where $M^0 = \{ \lfloor F \rfloor^\mathcal{M} \in M \mid F \text{ closed term} \}$.

If $T$ is a consistent extension of the λ-calculus, $\mathcal{M}(T)$ is the term-model of $T$, i.e. the set of all λ-terms modulo provable equality in $T$. The closed term-model of $T$, notation $\mathcal{M}^0(T)$, is defined as $(\mathcal{M}(T))^0$. A λ-algebra $\mathcal{M}$ is hard if $\mathcal{M} = \mathcal{M}^0$. In such an $\mathcal{M}$ a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model $\mathcal{M} = \mathcal{M}(\lambda \eta)$. 25 Indagationes
Church's $\delta$ is an external function satisfying

$$(\star) \delta MM = 0 \text{ if } M \text{ is a closed normal form (nf)}$$

$\delta MM' = 1 \text{ if } M, M' \text{ are different closed nf's.}$$

In Böhm [1972] it is proved that $FN_1 \ldots N_n$ different $\beta\eta$-nf's $AF \vdash FN_i = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying $(\star)$ for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_\infty$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim_{\mathcal{M}} g$, if $f(a) \cdot b = g(b) \cdot a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a) \cdot b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

$$f \sim_{\mathcal{M}} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.$$ 

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda\eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. DEFINITION. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k^{\text{th}}$ approximate normal form of $M$.

1.2. DEFINITION. (i) A selector is a term of the form

$$U = \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i < n.$$ 

A permutator is a term of the form

$$C = \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}$$

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$. 

1.3. **Lemma.** Simple terms have a normal form (nf).

**Proof.** Realize that each simple term is of the form $x\vec{P}$, $U\vec{P}$, $C\vec{P}$ with $\vec{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf.

1.4. **Theorem.** Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\vec{P}, \vec{Q}$, with $x \notin FV(\vec{P})$, $\lambda \vdash M\vec{P} = x\vec{Q}$ ("$x$ is Böhmd out").

(ii) Moreover $\vec{P}$ can be chosen as a sequence of simple terms.

**Proof.** Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1$, $x$ occurs in $BT(M\pi_1)$ at depth $k - 1$. Iterating this leads to $M\pi_2 = \lambda y \cdot x\vec{Q}$, hence $M\pi_2 y = x\vec{Q}$, for a Böhm transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that

$$M\pi_2 y = M \ldots x_i \ [x_j/C x_j] \ldots [x_k/U x_k] \ldots y = M\vec{P}$$

for some simple terms $\vec{P}$ with $x \notin FV(\vec{P})$ (where $C$ is a permutator and $U$ a selector).

1.5. **Lemma.** Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \nmid FX_1 = FX_2$ for some $X_1, X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

**Proof.** Note that if $P, P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin FV((Fx)^k)$ ($N^k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k \equiv (Fx)^k \ [x/M] \equiv (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM = Fx$. Since $F, M$ are closed it follows that for all $\lambda$-terms $N$, $\lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction.

**Remark.** 1.5 also holds for $F, M$ not necessarily closed.

1.6. **Definition.** $\underline{0} = I$, $n + \underline{1} = K n$.

1.7. **Lemma.** The function $sg$ is not $\lambda$-definable with respect to $\{n | n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash F \underline{0} = \underline{0}$, $\vdash F n + \underline{1} = \underline{1}$.

**Proof.** Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\vec{P} = x\vec{Q}$ for some $\vec{P}, \vec{Q} = Q_1 \ldots Q_m$. But then for all $n > m$,

$$\vdash I\vec{P} = F n \vec{P} = n \ Q_1 \ldots Q_m = n - m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's.
1.8. **Definition.** A system of terms \{M_n | n \in \omega\} is an *adequate system of numerals* iff

(i) Each \(M_n\) has a *nf*.
(ii) Each recursive function can be \(\lambda\)-defined with respect to the \(M_n\).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \(sg\) functions can be \(\lambda\)-defined with respect to the \(M_n\).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. **Corollary.**

(i) \([n | n \in \omega]\) is not an adequate system of numerals. (ii) Church's \(\delta\) is not \(\lambda\)-definable.

**Proof.** (i) Immediate. (ii) If \(\delta\) were \(\lambda\)-definable, then so would be \(sg\), viz. by \(\lambda x. \delta x = 0 \, 0 \, 1\).

**Remark.** (i) Although not definable, \(\delta\) can consistently be added to the \(\lambda\)-calculus, see Church [1941].

(ii) Contrary to this, the corresponding \(\delta\) for *open* \(\lambda\)-terms would be inconsistent at once. For let \(x \neq y\), then

\[(\lambda y. \delta xy(KK)S)x = (\lambda y. 1(KK)S)x = (\lambda y. KKS)x = KKS = K\]

but also

\[(\lambda y. \delta xy(KK)S)x = \delta xx(KK)S = 0(KK)S = S.\]

(iii) One could also consider the definability of a \(\delta\) for *all* closed terms, i.e.: \(\delta MM = 0\) for \(M, N\) closed

\(\delta MN = 1\) for \(M, N\) closed such that \(\n M = N\).

But then the following version of the Russell paradox would result. Define \(\neg X = \delta X1\). If \(\n X = 1\) then \(\n X = 1 \iff \neg A = A\).

Now let \(A = FP\) (i.e. the fixed point of \(\neg: \neg A = \neg A\)). Then \(\n A = 1 \iff A = 1\). Thus \(0 = 1\).

To see the relation with the Russell paradox, note that \(A = BB\) with \(B = \lambda x. \neg (xx)\). (In illative combinatory logic \(MN\) is interpreted as \(N \in M\) and \(\lambda x. P\) as \(\{x | P\}\).)

1.10. **Theorem.** Let \(\omega = [n | n \in \omega]\) be an adequate system of numerals and let \(f\) be a map into \(\omega\) definable by \(F\). Then \(f\) is constant.

**Proof.** First assume \(\omega\) is Church's system of numerals.

Suppose \(f\) is not constant, then by 1.5 \(x \in BT(Fx)\). Hence for some simple \(\tilde P\) and \(\tilde Q, \lambda \vdash Fx\tilde P = x\tilde Q\).

Hence \(\lambda \vdash FM\tilde P = M\tilde Q\) for all \(M\). But \(M\tilde Q\) can take arbitrary values and not \(FM\tilde P\), since \(\lambda \tilde P = P_1^n(P_2)P_3 \ldots P_k\) always has a *nf* by 1.3.
Now let $\omega$ be an arbitrary system of numerals. It is well-known how to define a term $G$ such that $Gn = n$.

Suppose a non-constant $f: \text{terms} \to \omega$ would be definable, then $G \circ f$ were a definable non-constant mapping into $\omega$.

First alternative proof (due to the referee).

Suppose $F$ is not constant, i.e. let $n_1 \neq n_2 \in Ra(F)$. Define $G$ as the $\lambda$-defining term of the recursive function

$$g(x) = \begin{cases} 0 & \text{if } x = n_1, \\ 1 & \text{else.} \end{cases}$$

Then the range of $G \circ F$ is $\{0, 1\}$ contrary to 2.3.

Second alternative proof. By Barendregt’s lemma in de Boer [1975] it follows that if $\Omega$ is unsolvable and $N$ a $nf$, then $F\Omega = N \Rightarrow Fx = N$ for all $x$ (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of $F$ are numerals it follows that $F\Omega$ has a $nf$, i.e. $F$ is constant.

1.11. Corollary. There is no $F$ such that

$$FM = 0 \text{ if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n)$$

$$1 \text{ else}$$

for any adequate system.

1.12. Question. Is there a term $F$ such that

$FM$ has a $nf$ (is solvable) if $M$ is a numeral

has no $nf$ (is unsolvable) else.

§ 2. THE RANGE PROPERTY

2.1. Definition. Let $\mathcal{M} = \langle M, \cdot \rangle$ be a $\lambda$-algebra. For each $f \in M$, we define $Ra^{\mathcal{M}}(f)$, the range of $f$ in $\mathcal{M}$, as follows:

$$Ra^{\mathcal{M}}(f) = \{ f \cdot x \mid x \in M \}.$$  

Notation. $Ra^{\mathcal{M}}(F) = Ra^{\mathcal{M}}([F]^{\mathcal{M}})$ for terms $F$.

When possible, the superscript $\mathcal{M}$ will be dropped in $Ra^{\mathcal{M}}$.

2.2. Definition. A $\lambda$-algebra $\mathcal{M}$ satisfies the range property if for all $f \in M$, the cardinality of $Ra^{\mathcal{M}}(f)$ is 1 or $\aleph_0$.

2.3. Range theorem. (Barendregt; Myhill). Let $T$ be a r.e. $\lambda$-theory. Then $\mathcal{M}(T)$ (and also $\mathcal{M}^0(T)$) has the range property.

Proof. Suppose $f \in M$ and $Ra(f) = \{ m_0, \ldots, m_k \}$, $k > 0$. Define

$$N_i = \{ x \mid f \cdot x = m_i \} \subseteq M.$$
Every such $N_t$ is r.e. Therefore $N = \bigcup_t N_t$, the complement of $N_0$ is also r.e. Hence $N_0$ is recursive.

On the other hand $N_0$ is non-trivial and closed under equality, which contradicts Scott’s theorem (Barendregt [1976] 2.21).

The proof for $\mathcal{M}_0^0(T)$ is the same.

2.4. COROLLARY. $\mathcal{M}(\lambda), \mathcal{M}_0^0(\lambda), \mathcal{M}(\lambda\eta)$ and $\mathcal{M}_0^0(\lambda\eta)$ have the range property.

The range property, however, is not satisfied in every $\lambda$-algebra.

2.5. THEOREM. $P\omega$ and $D_\infty$ do not satisfy the range property.

PROOF. Since the proof is similar in both cases, let $\mathcal{S} = (S, \subset)$ denote either $(P_\infty, \subseteq)$ or $(D_\infty, \subseteq)$. We define the following function $f : S \to S$ by $f(x) = \top$ if $x \neq \bot$ else $\bot$ ($\top$ and $\bot$ are the largest respectively smallest element of $S$.)

Claim: $f$ is continuous. Then by Scott [1972], [1975] $f$ is representable and since $f$ has range of cardinality two we are done.

For open $O$ in $S$ one has: $x \in O$ and $x \subset y \Rightarrow y \in O$.


Hence for open $O$, $\bot \in O \Rightarrow O = S$, and $O \neq \emptyset \Rightarrow \top \in O$.

Now for every open set $O$, $f^{-1}(O)$ is open:

Case 1. $\bot \in O$. Then $O = S$ so $f^{-1}(S) = S$ which is open.

Case 2. $\bot \notin O$. If $O = \emptyset$, then we are done. Else $\top \in O$ and hence $f^{-1}(O) = S - \{\bot\} = \{x \in S \mid x \subset \bot\}$ which is, as a union of elements of a base, indeed open.

The following theorem was announced in Wadsworth [1973] for the $D_\infty$ case.

2.6. THEOREM. Let $\mathcal{S}$ be $D_\infty^0$ or $P_\omega^0$. Then $\mathcal{S}$ satisfies the range property.

PROOF. Let $F$ be a closed term. Consider $BT(Fx)$.

Case 1. $x \notin BT(Fx)$. Then $BT(FM) = BT(FM')$ for all $M, M'$. Since terms with equal Böhm trees are equal in $\mathcal{S}$ (see Barendregt [1976], Hyland [1976]), it follows that $Ra_{\mathcal{S}}(F)$ has cardinality 1.

Case 2. $x \in BT(Fx)$. Then by 1.4 $\lambda \vdash Fx\tilde{F} = x\tilde{Q}$.

Since $[N\tilde{Q}]_{\mathcal{S}}$ can take arbitrary values in $\mathcal{S}$ when $N$ ranges over the closed terms, $Ra_{\mathcal{S}}(F)$ is infinite.
2.7. Conjecture. $\mathcal{M}(\mathcal{H})$ satisfies the range property.

2.8. Question. Does every hard $\lambda$-algebra $\mathcal{M}$ (i.e. $\mathcal{M} = \mathcal{M}^0$) satisfy the range theorem?

§ 3. Duality

3.1. Definition. Let $f, g$ be two external functions on a $\lambda$-algebra $\mathcal{M} = (M, \cdot)$.

$f, g$ are dual iff $\forall a, b \in M : f(a) \cdot b = g(b) \cdot a$. Notation $f \sim \mathcal{M} g$, or simply $f \sim g$.

3.2. Definition. $\mathcal{M}$ is rich iff all dual functions on $\mathcal{M}$ are representable in $\mathcal{M}$.

Remarks. (i) Let $f$ be an external function on $\mathcal{M}$. $f$ is locally representable iff for each $b \in M$ the function $h$ defined by $h(a) = f(a) \cdot b$ is representable. Then $f$ is locally representable iff $f$ has a dual. A model is rich iff all locally representable functions are representable.

(ii) If $f$ is representable (by $f_0 \in M$, say), then $f$ has a dual $g$ which is also representable (by $g_0 = \lambda ab \cdot f_0 ba$).

(iii) Let $\mathcal{M}$ be extensional. Then $f$ has at most one dual. Hence if $f \sim g$ and $f$ is representable, then by (ii) $g$ is representable.

3.3. Theorem. If $\mathcal{M}$ is rich, then $\mathcal{M}$ is extensional.

Proof. Suppose $\mathcal{M}$ is not extensional. Then there exist $b, b' \in M$ such that for all $c \in M$ $b \cdot c = b' \cdot c$ and $b \neq b'$.

Define

$$f(a) = \begin{cases} b' & \text{if } a = b \\ b & \text{else.} \end{cases}$$

and

$$g = \mu y. K(b y) \upharpoonright \mathcal{M},$$

then for all $a, a' \in M : f(a) \cdot a' = b \cdot a' = g(a') \cdot a$, hence $f \sim g$. But $f$ cannot be representable since it has no fixed point. Thus $\mathcal{M}$ is not rich.

3.4. Corollary. The following $\lambda$-algebras are not rich: $P\omega; P^0\omega; \mathcal{M}(\lambda); \mathcal{M}^0(\lambda); \mathcal{M}^0(\lambda)$. 

Proof.

1. $P\omega$ is not extensional:

Take for example $a = \{(0, 0)\}$ and $b = \{(0, 0), (1, 0)\}$. Then $V c \in P\omega a \cdot c = b \cdot c$ but $a \neq b$.

2. $P^0\omega$ is not extensional: Let $1 = \lambda xy \cdot xy$, then $P^0\omega \models I xy = 1 xy$, but $P^0\omega \not\models 1 = 1$ for otherwise $P\omega \models 1 = 1$, so $P\omega \models V x x = \lambda y \cdot xy$ which implies that $P\omega$ were extensional.

3. By the Church Rosser property $\lambda \not\models 1 = 1$. So $\mathcal{M}(\lambda), \mathcal{M}^0(\lambda)$ are not extensional.
4. \( \mathcal{M}^0(\lambda \eta) \) is not extensional because the \( \lambda \)-calculus is \( \omega \)-incomplete, see Plotkin [1974].

3.5. **Theorem.** \( D_{\infty} \) is rich.

**Proof.** Suppose that \( f, g \) are dual i.e.:

\[
\forall a, b \in D_{\infty}: f(a) \cdot b = g(b) \cdot a.
\]

We have to show that \( f, g \) are representable.

It is sufficient to show that \( f, g \) are continuous. Take a directed \( X \subset D_{\infty} \).

For all \( b \in D_{\infty} \)

\[
f(\sqcup X) \cdot b = g(b) \cdot \sqcup \{g(b) \cdot a | a \in X\} = \sqcup \{f(a) \cdot a | a \in X\} \cdot b
\]

by the duality condition and the continuity of application.

Thus by extensionality in \( D_{\infty} \) for all directed \( X \)

\[
f(\sqcup X) = \sqcup \{f(a) | a \in X\}
\]

i.e. \( f \) is continuous. The proof for \( g \) is dual. ■

3.6. **Theorem.** \( \mathcal{M}(\lambda \eta) \) is rich.

**Proof.** Define

\[
M = _{\lambda \eta} N \iff \lambda \eta \vdash M = N,
\]

\[
x \in _{\lambda \eta} M \iff \text{for all } M' = _{\lambda \eta} M \text{ one has } x \in FV(M')
\]

Let \( f, g \) be dual functions on \( \mathcal{M}(\lambda \eta) \).

3.6.0. **Lemma.** (i) \( x \in _{\lambda \eta} M \iff VN[\lambda \eta \vdash M = N \Rightarrow x \in FV(N)]. \)

(ii) Let \( M' = M[z/y] \) and \( \lambda \vdash M' \rightarrow N' \). Then \( \forall N \lambda \vdash M \rightarrow N \) and \( N' \equiv N[z/y] \).

\[
\begin{array}{c}
M \\
x
\end{array} \quad \Rightarrow \quad \begin{array}{c}
N \\
x
\end{array} \quad \Rightarrow \quad \begin{array}{c}
M' \\
x
\end{array} \quad \Rightarrow \quad \begin{array}{c}
N' \\
x
\end{array}
\]

(iii) \( x \in _{\lambda \eta} M \Rightarrow x \in _{\lambda \eta} M[z/y], \) for \( z \not\equiv x \).

**Proof.** (i) \( \Rightarrow \) Trivial. \( \Leftarrow \) Suppose \( M = _{\lambda \eta} M' \). By the Church-Rosser theorem \( \lambda \eta \vdash M \rightarrow N, M' \rightarrow N' \) for some \( N \). By assumption \( x \in FV(N) \). But then \( x \in FV(M') \).

(ii) Induction on the length of proof of \( M' \rightarrow N' \). In the case that \( M' \equiv (\lambda a \cdot P)Q, N' \equiv P[a/Q] \) it may be assumed that \( a \not\equiv z, y \). Therefore one can apply the well-known substitution lemma

\[
A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \not\equiv v \text{ and } u \not\in FV(C).
\]

(iii) Suppose \( \lambda \eta \vdash P[z/y] \rightarrow R' \). By (ii) for some \( R \lambda \eta \vdash P \rightarrow R \) and \( R' \equiv R[z/y] \). By assumption and (i), \( x \in FV(R) \). Since \( x \not\equiv z \) also \( x \in FV(R') \). Therefore by (i) \( x \in _{\lambda \eta} P[z/y]. \) ■
3.6.1. Lemma. (i) If \( x \in_{\eta} \lambda y \cdot P \) then \( x \in_{\eta} P \) and \( x \not\equiv y \).
(ii) If \( x \not\equiv y \), then \( x \in_{\eta} \mathcal{M} \iff x \in_{\eta} M y \).

Proof. (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \not\equiv y \). Suppose \( P =_{\eta} \lambda y \cdot N \), then \( \lambda y \cdot P =_{\eta} \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subset FV(N) \). Thus \( x \in_{\eta} P \).
(ii) \( \implies \) Suppose \( \lambda y \cdot M y \rightarrow N \) in order to prove \( x \in FV(N) \).
Case 1. \( N = M' y \) with \( \lambda y \cdot M \rightarrow M' \). Since \( x \in_{\eta} M \), also \( x \in FV(M') \subset FV(N) \).
Case 2. \( M \rightarrow \lambda z \cdot M_1 \) and \( \lambda \eta \cdot M y \rightarrow (\lambda y \cdot M_1) y \rightarrow M_1[z/y] \rightarrow N \).
Since \( x \in_{\eta} M \), also \( x \in_{\eta} \lambda z \cdot M_1 \) and by (i) \( x \in_{\eta} M_1 \) and \( z \not\equiv x \), so by 3.6.0. (iii) \( x \in_{\eta} M_1[z/y] \). Therefore \( x \in FV(N) \).

3.6.2. Lemma. If \( \forall y \not\equiv x \ x \in_{\eta} f(y) \), then \( \forall y \not\equiv x \ x \in_{\eta} g(y) \) (and hence \( \forall y \not\equiv x \ x \in_{\eta} f(y) \)).

Proof. Suppose \( x \in_{\eta} f(y) \), \( y \not\equiv x \). Let \( y' \not\equiv x \). Then by 3.6.1. (ii) \( x \in_{\eta} f(y) \cdot y' =_{\eta} g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in_{\eta} g(y') \). (The rest follows by applying the statement to \( x \in_{\eta} g(y) \)). \( \square_{3.6.2} \)

3.6.3. Main Lemma. There is a variable \( x \) such that for all terms \( M : f(x)[x/M] = f(M) \).

Proof. Let \( v \) be any variable. Choose \( x \not\equiv v \) such that \( x \not\equiv l_{\eta} f(v) \). Then \( x \not\equiv l_{\eta} g(z) \) for all \( z \not\equiv x \), by the dual of 3.6.2.

Given \( M \), one can find a \( y \) such that \( y \not\equiv l_{\eta} M, f(M), x, f(x) \). Hence \( x \not\equiv l_{\eta} g(y) \). Now since \( y \not\equiv x \) and \( x \not\equiv l_{\eta} g(y) \), \( (f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = (g(y) \cdot y)[x/M] = g(y) \cdot M = f(M) \cdot y \).

Since \( y \not\equiv f(x), M, f(M) \), extensionality yields \( f(x)[x/M] = f(M) \). \( \square_{3.6.3} \)

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similar for \( g \). \( \square_{3.6} \)

The following construction is needed for the proof of 3.10.

3.7. Definition. Let \( \#$ \) be a Gödel numbering of terms. \( \# M \) is the numeral \( \# M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \# M_n \) is a recursive function.

3.8. Lemma. (Coding of infinite sequences). Let \( \{ M_n \} \) be a recursive sequence of terms such that \( FV(M_n) \subset \{ x \} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

Proof.
(1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:
\[
E(\langle M \rangle) = M, \text{ for } M \text{ with } FV(M) = \{ x \}.
\]
Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot zMN\). Then \([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:

\([M] = M, [M_1, \ldots, M_{n+1}] = [M_1, [M_2, \ldots, M_{n+1}]]\).

Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms.

We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(S^+\) be such that
\[
S^+ n \xrightarrow{\beta} n + 1
\]
and let \(b = \lambda xy \cdot [E(Fy), (x(S^+ y))]\), where \(F\) \(\lambda\)-defines \(f\), and
\(B = FP b\) (i.e. the fixed point of \(b\)). Then
\[
B_n \xrightarrow{\beta} bB_n \xrightarrow{\beta} [E(Fn), B_{n+1}] \xrightarrow{\beta} [M_n, B_{n+1}].
\]
So \(B_0 = [M_0, B_1] = [M_0, M_1, B_2] = \ldots\). Hence by setting \(\langle M_n\rangle_{n \in \omega} = B_0\) we have a coding for infinite sequences of terms with one fixed free variable.

It is easy to construct a term \(p\) such that
\[
\langle M_n \rangle_{n \in \omega} = p\langle x \rangle p(x \mapsto \lambda k \cdot \langle A_k \rangle_{k \in \omega}).
\]
(e.g. \(pxa = 0\) if \(x = 0\) then \(a)\) else \(p(x - 1)(a(KI))\), using the fixed point theorem.)

3.9. Lemma. For all closed \(Z\) there is an \(n\) such that \(Z\Omega^n = \not\in \Omega\). (\(Z\Omega^n\) is short for \(Z\Omega \Omega \ldots \Omega\), \(n\) times.)

Proof.

Case 1. \(Z\) is unsolvable; then \(Z = \not\in \Omega\), so \(n = 0\).

Case 2. \(Z\) is solvable; then \(Z\) has a \(hnj\), \(Z = \overrightarrow{x} \cdot x_i A_1 \ldots A_m (x_i \in \overrightarrow{x})\).

Take \(n = i\), so \(Z\Omega^i = \overrightarrow{x} \cdot \Omega A_1 \ldots A_m = \not\in \Omega\).

3.10. Theorem. If \(\mathcal{H}\) is hard and sensible, then \(\mathcal{H}\) is not rich.

Proof. If \(\mathcal{H}\) is hard, then \(\mathcal{H}\) is isomorphic to \(\mathcal{H}^0(T)\), where \(T = Th(\mathcal{H})\). We reason in \(\mathcal{H}^0(T)\). Since \(\mathcal{H}\) is sensible, \(\mathcal{H} \subseteq T\).

Let \(h: \omega \to \omega\) be a function not definable in \(\mathcal{H}\). Such an \(h\) exists since a hard model is countable.

Let \(A_n(x, y)\) be the term \(x\Omega^n(y\Omega^n(hy)), n \in \omega\). For closed \(M\) the sequence
\[
A_0(M, y), A_1(M, y), \ldots
\]
\(\mathcal{M}(y(hy)), \Omega(y\Omega^n(hy)), \ldots, \mathcal{M}^{\Omega^n(hy)}, \Omega, \ldots\),
where \(n\) is such that \(\mathcal{M}^{\Omega^{n+1}} = \Omega\). Thus \(\lambda y \cdot A_n(M, y)\) is up to convertibility a recursive sequence containing one fixed free variable and hence representable as a term. Define \(\langle f(M) = \lambda y \cdot \langle A_n(M, y)\rangle_{n \in \omega}\). Similarly for closed \(N\)
\(\lambda n \cdot A_n(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_n(x, N)\rangle_{n \in \omega}\).

Then for all closed \(M, N:\) \(f(M)\) and \(g(N)\) are well defined and \(f(M) \cdot N = g(N) \cdot M = \langle A_n(M, N)\rangle_{n \in \omega}\) by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{H}\) is rich, i.e. \(f\) were representable by some closed \(F\). Then for all closed \(M, N:\)
\(FMN = f(M)N = \langle A_n(M, N)\rangle_{n \in \omega}\).

But then \(p_1(F(K^n I)(K^n I)) = p_2(h(y))\), hence \(h\) were definable, contradiction. Thus \(\mathcal{H}\) is not rich.
3.11. Corollary. $D^\infty_\omega$ and $\mathcal{M}^\emptyset(T)$ for $T \subseteq \mathcal{H}$ are not rich.

3.12. Questions. (i) Is every extensional term model $\mathcal{M}(T)$ rich?  
(ii) Is $\mathcal{M}(\lambda_\omega)$ rich?

Here $\lambda_\omega$ is the $\lambda$-theory obtained by adding the $\omega$-rule to the theory, see Barendregt [1974].

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