SUMMARY

§ 1 is concerned with the term model of the \( \lambda \)-calculus. It is proved that Church's \( \delta \) is not definable and that the definable functions into the numerals are constant. In § 2 it is proved that for several \( \lambda \)-algebras the range of a representable function is either a singleton or infinite. In § 3 it is examined in which \( \lambda \)-algebras the local representability of external functions implies the global representability.

INTRODUCTION

Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra (i.e. a model of the \( \lambda \)-calculus). Elements of \( M \) are thought of as functions. Arbitrary \( f : M \to M \) are called external functions. Such a function is representable (by an element \( a \in M \)) if \( \forall b \in M \ f(b) = a \cdot b \). A function \( f \) is definable in \( \mathcal{M} \) if \( f \) is representable by \( [F]^\mathcal{M} \) for some closed term \( F \). Here \( [F]^\mathcal{M} \) denotes the value of \( F \) in the model \( \mathcal{M} \).

Other notations:
- \( x, y, \ldots \) denote variables of the \( \lambda \)-calculus.
- \( a, b, \ldots \) denote variables ranging over the elements of a \( \lambda \)-algebra.
- \( F, G, \ldots \) denote \( \lambda \)-terms.
- The numerals 0, 1, 2, \ldots denote some adequate representation of the natural numbers as \( \lambda \)-terms e.g. those of Church: \( n = \lambda x. \ \lambda f. f^n(x) \).

If \( \mathcal{M} = \langle M, \cdot \rangle \) is a \( \lambda \)-algebra, then \( \mathcal{M}^0 \) is the sub-\( \lambda \)-algebra \( \langle M^0, \cdot \rangle \) where \( M^0 = \{ [F]^\mathcal{M} \in M \mid F \text{ closed term} \} \).

If \( T \) is a consistent extension of the \( \lambda \)-calculus, \( \mathcal{M}(T) \) is the term-model of \( T \), i.e. the set of all \( \lambda \)-terms modulo provable equality in \( T \). The closed term-model of \( T \), notation \( \mathcal{M}^0(T) \), is defined as \( (\mathcal{M}(T))^0 \). A \( \lambda \)-algebra \( \mathcal{M} \) is hard if \( \mathcal{M} = \mathcal{M}^0 \). In such an \( \mathcal{M} \) a function is representable iff it is definable.

For other terminology see Barendregt [1976].

The three sections of the paper treat different aspects of the notion of representability.

In § 1 attention is restricted to the standard extensional term model \( \mathcal{M} = \mathcal{M}(\lambda \eta) \).
Church's $\delta$ is an external function satisfying

($\star$) $\delta MM = 0$ if $M$ is a closed normal form (nf)

$\delta MM' = 1$ if $M, M'$ are different closed nf's.

In Böhm [1972] it is proved that $FN_1 \ldots N_n$ different $\beta\eta$-nf's $\mathcal{AF} \vdash FN_i = i$. As a consequence it follows that for every finite set $A$ of nf's there is a term $\delta$ satisfying ($\star$) for $M, M' \in A$.

At the Orléans logic conference (1972) the question was raised whether the general Church's $\delta$ is definable as a $\lambda$-term.

We will give a negative answer which was already established in Barendregt [1972] and independently in Wadsworth [1972] (see also Hindley and Mitschke [1975]). All three proofs of the non-existence of $\delta$ are different.

Furthermore it is proved that the only definable functions from the terms into the numerals are the constant functions.

In § 2 it will be proved that definable functions in various $\lambda$-algebras have a range of cardinality 1 or $\aleph_0$. For representable functions this is not true in $D_\infty$ and $P_\omega$.

Two external functions $f$ and $g$ on $\mathcal{M}$ are dual, notation $f \sim \mathcal{M} g$, if $f(a).b = g(b).a$ for all $a, b \in \mathcal{M}$. In that case for each $b$ the map $\lambda a. f(a).b$ is representable and $f$ is said to be locally representable, similarly for $g$.

A model $\mathcal{M}$ is rich if for all $f, g$:

$$f \sim \mathcal{M} g \Rightarrow f \text{ and } g \text{ are representable in } \mathcal{M}.$$ 

The results of § 3 are: $D_\infty$ and $\mathcal{M}(\lambda\eta)$ are rich; rich models are extensional; hard sensible models (e.g. the interior of $D_\infty$) are not rich.

We would like to draw the proof of 3.6 to the reader's attention. There variables of the $\lambda$-calculus are not just used in the usual way, but also serve as separate entities.

§ 1. NON-DEFINABILITY RESULTS

The main tool in this section is the "Böhm out" technique 1.4. This result is also of use in § 2.

1.1. Definition. Let $BT(M)$ be the Böhm tree of $M$, see Barendregt [1976], § 6. $x \in BT(M)$ iff $x \in FV(M^k)$ for some $k$, where $M^k$ is the $k^{th}$ approximate normal form of $M$.

1.2. Definition. (i) A selector is a term of the form

$$U \equiv \lambda x_1 \ldots x_n \cdot x_i, \quad 1 < i < n.$$ 

A permutator is a term of the form

$$C \equiv \lambda x_1 \ldots x_n \cdot x_{\pi(1)} \ldots x_{\pi(n)}$$

for some permutation $\pi$.

(ii) Simple terms are inductively defined by: Any variable, selector or permutator is a simple term. If $P, Q$ are simple terms, so is $PQ$. 

1.3. Definition. A permutator is $\pi$-representable if

$$\lambda x \cdot C \cdot x \equiv \lambda x \cdot C \cdot x_{\pi(1)} \ldots x_{\pi(n)}.$$
1.3. **Lemma.** Simple terms have a normal form (nf).

**Proof.** Realize that each simple term is of the form $x\overrightarrow{P}$, $U\overrightarrow{P}$, $C\overrightarrow{P}$ with $\overrightarrow{P}$ simple, $U$ a selector and $C$ a permutator. Then it can be shown by induction on the term length that they have a nf. ■

1.4. **Theorem.** Let $FV(M) = \{x\}$ and $x \in BT(M)$. Then

(i) For some $\overrightarrow{P}, \overrightarrow{Q}$, with $x \notin FV(\overrightarrow{P})$, $\lambda \vdash M\overrightarrow{P} = x\overrightarrow{Q}$ ("$x$ is Bohmed out").

(ii) Moreover $\overrightarrow{P}$ can be chosen as a sequence of simple terms.

**Proof.** Let $x$ occur in $BT(M)$ at depth $k > 0$. By a similar construction as in Barendregt [1976] 6.14, 6.15, for some Böhm-transformation $\pi_1$, $x$ occurs in $BT(M^{\pi_1})$ at depth $k-1$. Iterating this leads to $M^{\pi_2} = \lambda\overrightarrow{y}.x\overrightarrow{Q}$, hence $M^{\pi_2}\overrightarrow{y} = x\overrightarrow{Q}$, for a Böhm transformation $\pi_2$.

Checking the details of the construction of $\pi_2$ one verifies that $M^{\pi_2}\overrightarrow{y} = M \ldots x_i \ldots [x_i/Cx_j] \ldots [x_k/Ux_k] \ldots y = M\overrightarrow{P}$ for some simple terms $\overrightarrow{P}$ with $x \notin FV(\overrightarrow{P})$ (where $C$ is a permutator and $U$ a selector). ■

1.5. **Lemma.** Let $F$ be a closed $\lambda$-term such that $F$ is not constant, i.e. $\lambda \not\vdash FX_1 = FX_2$ for some $X_1$, $X_2$, and suppose that for some closed $\lambda$-term $M$, $FM$ has a nf. Then $x \in BT(Fx)$ for all $x$.

**Proof.** Note that if $P$, $P'$ have equal finite $\Omega$-free Böhm-trees, then $\lambda \not\vdash P = P'$. Now suppose $x \notin BT(Fx)$ for some $x$. Then for all $k$, $x \notin FV((Fx)^k)$ ($N^k$ is the $k$-th approximate normal form of $N$, cf. Barendregt [1976] 7.4 (iv)). Hence $(FM)^k = (Fx)^k [x/M] = (Fx)^k$ for all $k$, and it follows that $BT(FM) = BT(Fx)$. But since $FM$ has a nf, $BT(FM)$ is finite and $\Omega$-free and therefore $\lambda \vdash FM = Fx$. Since $F$, $M$ are closed it follows that for all $\lambda$-terms $N$, $\lambda \vdash FN = FM$, i.e. $F$ is constant, a contradiction. ■

**Remark.** 1.5 also holds for $F$, $M$ not necessarily closed.

1.6. **Definition.** $0 = I$, $n + 1 = K \ n$.

1.7. **Lemma.** The function $sg$ is not $\lambda$-definable with respect to $\{n | n \in \omega\}$, i.e. for no closed $\lambda$-term $F \vdash F \ 0 = 0$, $\vdash F \ n + 1 = 1$.

**Proof.** Suppose $F$ exists. Then by 1.5 $x \in BT(Fx)$. Hence by 1.4 $Fx\overrightarrow{P} = x\overrightarrow{Q}$ for some $\overrightarrow{P}, \overrightarrow{Q} = Q_1 \ldots Q_m$. But then for all $n > m$,

$$\vdash I\overrightarrow{P} = F \ n \ \overrightarrow{P} = n \ Q_1 \ldots Q_m = n - m$$

contradicting the Church-Rosser theorem since the $k$ are different nf's. ■
1.8. **Definition.** A system of terms \( \{ M_n | n \in \omega \} \) is an *adequate system of numerals* iff

(i) Each \( M_n \) has a \( \text{nf} \).

(ii) Each recursive function can be \( \lambda \)-defined with respect to the \( M_n \).

In Barendregt [1977] is shown that the second condition can be replaced by (ii'): The successor, predecessor and \( \text{sg} \) functions can be \( \lambda \)-defined with respect to the \( M_n \).

The following corollary was proved independently by Barendregt [1972] and Wadsworth [1972].

1.9. **Corollary.**

(i) \( \{ n | n \in \omega \} \) is not an adequate system of numerals. (ii) Church's \( \delta \) is not \( \lambda \)-definable.

**Proof.** (i) Immediate. (ii) If \( \delta \) were \( \lambda \)-definable, then so would be \( \text{sg} \), viz. by \( \lambda x \cdot \delta x = 0 = 1 \).

**Remark.** (i) Although not definable, \( \delta \) can consistently be added to the \( \lambda \)-calculus, see Church [1941].

(ii) Contrary to this, the corresponding \( \delta \) for open \( \lambda \)-terms would be inconsistent at once. For let \( x \not= y \), then

\[
(\lambda y \cdot \delta xy(KK)S)x = (\lambda y \cdot I(KK)S)x = (\lambda y \cdot KK)Sx = KK = K
\]

but also

\[
(\lambda y \cdot \delta xy(KK)S)x = \delta xx(KK)S = 0(KK)S = S.
\]

(iii) One could also consider the definability of a \( \delta \) for *all* closed terms, i.e.: \( \delta M M = 0 \) for \( M \) closed

\[
\delta M N = I \quad \text{for} \quad M, N \quad \text{closed such that} \quad \models M = N.
\]

But then the following version of the Russell paradox would result. Define \( \neg X = \delta X I \). If \( \not\models 0 = I \) then \( \not\models X = I \iff \models \neg X = I \).

Now let \( A = FP \neg \) (i.e. the fixed point of \( \neg \cdot \): \( \models \neg A = \neg \neg A \)).

Then \( \models \neg A = I \iff \models \neg A \equiv I \). Thus \( \models \neg 0 = I \).

To see the relation with the Russell paradox, note that \( A = BB \) with \( B = \lambda x. \neg \ (xx) \). (In illative combinatory logic \( M N \) is interpreted as \( N \in M \) and \( \lambda x \cdot P \) as \( \{ x | P \} \).)

1.10. **Theorem.** Let \( \omega = \{ n | n \in \omega \} \) be an adequate system of numerals and let \( f \) be a map into \( \omega \) definable by \( F \). Then \( f \) is constant.

**Proof.** First assume \( \omega \) is Church's system of numerals.

Suppose \( f \) is not constant, then by 1.5 \( x \in BT(Fx) \). Hence for some simple \( \vec{P} \) and \( \vec{Q}, \lambda \models Fx \vec{P} = x \vec{Q} \).

Hence \( \lambda \models FM \vec{P} = M \vec{Q} \) for all \( M \). But \( M \vec{Q} \) can take arbitrary values and not \( FM \vec{P} \), since \( \models \vec{P} = P_1 n(P_2)P_3 \ldots P_k \) always has a \( \text{nf} \) by 1.3.
Now let \( \omega \) be an arbitrary system of numerals. It is well-known how to define a term \( G \) such that \( Gn = n \).
Suppose a non-constant \( f : \text{terms} \rightarrow \omega \) would be definable, then \( G \circ f \) were a definable non-constant mapping into \( \omega \).

First alternative proof (due to the referee).
Suppose \( F \) is not constant, i.e. let \( n_1 \neq n_2 \in Ra(F) \). Define \( G \) as the \( \lambda \)-defining term of the recursive function

\[
g(x) = \begin{cases} 
0 & \text{if } x = n_1, \\
1 & \text{else.}
\end{cases}
\]

Then the range of \( G \circ F \) is \( \{0, 1\} \) contrary to 2.3.

Second alternative proof. By Barendregt's lemma in de Boer [1975] it follows that if \( \Omega \) is unsolvable and \( N \) a \( \text{nf} \), then \( F\Omega = N \Rightarrow Fx = N \) for all \( x \). (General genericity lemma; see also Barendregt [1977a] for a proof.) Now if the values of \( F \) are numerals it follows that \( F\Omega \) has a \( \text{nf} \), i.e. \( F \) is constant.

1.11. Corollary. There is no \( F \) such that

\[
FM = 0 \text{ if } M \text{ is a numeral (i.e. } \vdash M = n \text{ for some } n) \\
1 \text{ else}
\]

for any adequate system.

1.12. Question. Is there a term \( F \) such that

\[
FM \text{ has a } \text{nf} \text{ (is solvable) if } M \text{ is a numeral} \\
\text{has no } \text{nf} \text{ (is unsolvable) else.}
\]

§ 2. THE RANGE PROPERTY

2.1. Definition. Let \( \mathcal{M} = \langle M, \cdot \rangle \) be a \( \lambda \)-algebra. For each \( f \in M \), we define \( Ra^{\mathcal{M}}(f) \), the range of \( f \) in \( \mathcal{M} \), as follows:

\[
Ra^{\mathcal{M}}(f) = \{ f \cdot x | x \in M \}.
\]

Notation. \( Ra^{\mathcal{M}}(F) = Ra^{\mathcal{M}}([[F]]^{\mathcal{M}}) \) for terms \( F \).

When possible, the superscript \( \mathcal{M} \) will be dropped in \( Ra^{\mathcal{M}} \).

2.2. Definition. A \( \lambda \)-algebra \( \mathcal{M} \) satisfies the range property if for all \( f \in M \), the cardinality of \( Ra^{\mathcal{M}}(f) \) is 1 or \( \aleph_0 \).

2.3. Range theorem. (Barendregt; Myhill). Let \( T \) be a r.e. \( \lambda \)-theory. Then \( \mathcal{M}(T) \) (and also \( \mathcal{M}^0(T) \)) has the range property.

Proof. Suppose \( f \in M \) and \( Ra(f) = \{ m_0, \ldots, m_k \} \), \( k > 0 \). Define

\[
N_t = \{ x | f \cdot x = m_t \} \subseteq M.
\]
Every such \( N_t \) is r.e. Therefore \( N = \bigcup_N N_t \), the complement of \( N_0 \) is also r.e. Hence \( N_0 \) is recursive.

On the other hand \( N_0 \) is non-trivial and closed under equality, which contradicts Scott’s theorem (Barendregt [1976] 2.21).

The proof for \( \mathcal{M}^0(T) \) is the same.

2.4. Corollary. \( \mathcal{M}(\lambda), \mathcal{M}^0(\lambda), \mathcal{M}(\lambda \eta) \) and \( \mathcal{M}^0(\lambda \eta) \) have the range property.

The range property, however, is not satisfied in every \( \lambda \)-algebra.

2.5. Theorem. \( P_0 \) and \( D_\infty \) do not satisfy the range property.

Proof. Since the proof is similar in both cases, let \( \mathcal{S} = (S, <) \) denote either \( (P_\infty, \subseteq) \) or \( (D_\infty, \sqsubseteq) \). We define the following function \( f : S \to S \) by \( f(x) = \top \) if \( x \neq \bot \) else \( \bot \) (\( \top \) and \( \bot \) are the largest respectively smallest element of \( S \)).

Claim: \( f \) is continuous. Then by Scott [1972], [1975] \( f \) is representable and since \( f \) has range of cardinality two we are done.

For open \( O \) in \( S \) one has: \( x \in O \) and \( x < y \Rightarrow y \in O \).


Hence for open \( O, \bot \in O \Rightarrow O = S \), and \( O \neq \emptyset \Rightarrow \top \in O \).

Now for every open set \( O, f^{-1}(O) \) is open:

Case 1. \( \bot \in O \). Then \( O = S \) so \( f^{-1}(S) = S \) which is open.

Case 2. \( \bot \notin O \). If \( O = \emptyset \), then we are done. Else \( \top \in O \) and hence \( f^{-1}(O) = S - \{ \bot \} = \{ x \mid x \leq \bot \} = \bigcup_k U_k \).

\( U_k \) is open in \( D_\infty \), see e.g. Barendregt [1976] 1.2.

\( U_k \) is open in \( P_0 \): Let \( O_k = \{ x \mid e_k \subseteq x \} \). Note \( e_0 = \emptyset = \bot \) and that the \( O_k \) form a base for the topology on \( P_0 \).

Now:

\[ x \in U_k \iff x \notin O \iff \exists k \neq 0 \mid e_k \subseteq x \iff x \in \bigcup_{k=0}^{\infty} O_k \]

which is, as a union of elements of a base, indeed open.

The following theorem was announced in Wadsworth [1973] for the \( D_\infty \) case.

2.6. Theorem. Let \( \mathcal{S} \) be \( D_\infty ^0 \) or \( P_0 ^0 \). Then \( \mathcal{S} \) satisfies the range property.

Proof. Let \( F \) be a closed term. Consider \( BT(Fx) \).

Case 1. \( x \notin BT(Fx) \). Then \( BT(FM) = BT(FM') \) for all \( M, M' \). Since terms with equal Böhm trees are equal in \( \mathcal{S} \) (see Barendregt [1976], Hyland [1976]), it follows that \( R_{\mathcal{S}^0}(F) \) has cardinality 1.

Case 2. \( x \in BT(Fx) \). Then by 1.4 \( \lambda \mid Fx \beta = xQ \).

Since \( [NQ]_{\mathcal{S}} \) can take arbitrary values in \( \mathcal{S} \) when \( N \) ranges over the closed terms, \( R_{\mathcal{S}^0}(F) \) is infinite.
2.7. Conjecture. $H(\mathcal{H})$ satisfies the range property.

2.8. Question. Does every hard $\lambda$-algebra $H$ (i.e. $H = H^0$) satisfy the range theorem?

§ 3. Duality

3.1. Definition. Let $f, g$ be two external functions on a $\lambda$-algebra $\mathcal{H} = (\mathcal{M}, \cdot)$.

$f, g$ are dual iff $\forall a, b \in \mathcal{M}: f(a) \cdot b = g(b) \cdot a$. Notation $f \sim \mathcal{H} g$, or simply $f \sim g$.

3.2. Definition. $\mathcal{H}$ is rich iff all dual functions on $\mathcal{H}$ are representable in $\mathcal{H}$.

Remarks. (i) Let $f$ be an external function on $\mathcal{H}$. $f$ is locally representable iff for each $b \in \mathcal{M}$ the function $h$ defined by $h(a) = f(a) \cdot b$ is representable. Then $f$ is locally representable iff $f$ has a dual. A model is rich iff all locally representable functions are representable.

(ii) If $f$ is representable (by $f_0 \in \mathcal{M}$, say), then $f$ has a dual $g$ which is also representable (by $g_0 = \lambda ab f_0 ba$).

(iii) Let $\mathcal{H}$ be extensional. Then $f$ has at most one dual. Hence if $f \sim \mathcal{H} g$ and $f$ is representable, then by (ii) $g$ is representable.

3.3. Theorem. If $\mathcal{H}$ is rich, then $\mathcal{H}$ is extensional.

Proof. Suppose $\mathcal{H}$ is not extensional. Then there exist $b, b' \in \mathcal{M}$ such that for all $c \in \mathcal{M}$ $b \cdot c = b' \cdot c$ and $b \neq b'$.

Define

$$f(a) = \begin{cases} b' & \text{if } a = b \\ b & \text{else.} \end{cases}$$

and

$$g = [\lambda y \cdot K(by)]^\mathcal{H},$$

then for all $a, a' \in \mathcal{M}: f(a) \cdot a' = b \cdot a' = g(a') \cdot a$, hence $f \sim g$. But $f$ cannot be representable since it has no fixed point. Thus $\mathcal{H}$ is not rich.

3.4. Corollary. The following $\lambda$-algebras are not rich: $P_\omega$; $P^0_\omega$; $H(\lambda)$; $H_0^0(\lambda)$; $H_0(\lambda \eta)$.

Proof.

1. $P_\omega$ is not extensional:

Take for example $a = \{(0, 0)\}$ and $b = \{(0, 0), (1, 0)\}$.

Then $V c \in P_\omega a \cdot c = b \cdot c$ but $a \neq b$.

2. $P^0_\omega$ is not extensional: Let $1 = \lambda xy \cdot xy$, then $P^0_\omega \models Ixy = 1xy$, but $P^0_\omega \models I = 1$ for otherwise $P_\omega \models I = 1$, so $P_\omega \models V x x = \lambda \cdot xy$ which implies that $P_\omega$ were extensional.

3. By the Church Rosser property $\lambda \not\models I = 1$. So $H(\lambda)$, $H_0^0(\lambda)$ are not extensional.
4. $\mathcal{M}^0(\lambda \eta)$ is not extensional because the $\lambda$-calculus is $\omega$-incomplete, see Plotkin [1974].

3.5. Theorem. $D_\infty$ is rich.

Proof. Suppose that $f, g$ are dual i.e.:

$$\forall a, b \in D_\infty: f(a) \cdot b = g(b) \cdot a.$$ 

We have to show that $f, g$ are representable. It is sufficient to show that $f, g$ are continuous. Take a directed $X \subseteq D_\infty$.

For all $b \in D_\infty$

$$f(\bigsqcup X) \cdot b = g(b) \cdot \bigsqcup X = \bigsqcup \{g(b) \cdot a | a \in X\} = \bigsqcup \{f(a) \cdot a | a \in X\} \cdot b$$

by the duality condition and the continuity of application.

Thus by extensionality in $D_\infty$: for all directed $X$, $f(\bigsqcup X) = \bigsqcup \{f(a) | a \in X\}$ i.e. $f$ is continuous. The proof for $g$ is dual. ■

3.6. Theorem. $\mathcal{M}(\lambda \eta)$ is rich.

Proof. Define

$$M = _{\lambda \eta}N \iff \lambda \eta \vdash M = N,$$

$$x \in _{\lambda \eta}M \iff \text{for all } M' = _{\lambda \eta}M \text{ one has } x \in FV(M').$$

Let $f, g$ be dual functions on $\mathcal{M}(\lambda \eta)$.

3.6.0. Lemma. (i) $x \in _{\lambda \eta}M \iff VN(\lambda \eta \vdash M \rightarrow N) \Rightarrow x \in FV(N)$.

(ii) Let $M' = M[z/x]$ and $\lambda \vdash M' \rightarrow N'$. Then $\forall N \lambda \vdash M \rightarrow N$ and $N' = N[z/x]$.

(iii) $x \in _{\lambda \eta}M \Rightarrow x \in _{\lambda \eta}M[z/x]$, for $z \not\equiv x$.

Proof. (i) $\Rightarrow$ Trivial. $\Leftarrow$ Suppose $M = _{\lambda \eta}M'$. By the Church-Rosser theorem $\lambda \eta \vdash M \rightarrow N$, $M' \rightarrow N'$ for some $N$. By assumption $x \in FV(N)$. But then $x \in FV(M')$.

(ii) Induction on the length of proof of $M' \rightarrow N'$. In the case that $M' \equiv (\lambda a \cdot P)Q$, $N' \equiv P[a/Q]$ it may be assumed that $a \not\equiv z, y$. Therefore one can apply the well-known substitution lemma

$$A[u/B][v/C] = A[v/C][u/B[v/C]] \text{ if } u \not\equiv v \text{ and } u \not\equiv FV(C).$$

(iii) Suppose $\lambda \eta \vdash P[z/y] \rightarrow R'$. By (ii) for some $R \lambda \eta \vdash P \rightarrow R$ and $R' \equiv R[z/y]$. By assumption and (i), $x \in FV(R)$. Since $x \not\equiv z$ also $x \in FV(R')$. Therefore by (i) $x \in _{\lambda \eta}P[z/y]$. ■ 3.6.0
3.6.1. Lemma. (i) If \( x \in_{\eta} \lambda y \cdot P \) then \( x \in_{\eta} P \) and \( x \neq y \).
(ii) If \( x \neq y \), then \( x \in_{\eta} M \iff x \in_{\eta} M y \).

Proof. (i) Since \( x \in FV(\lambda y \cdot P) \) clearly \( x \neq y \). Suppose \( P =_{\eta} N \), then \( \lambda y \cdot P =_{\eta} \lambda y \cdot N \). By assumption \( x \in FV(\lambda y \cdot N) \subseteq FV(N) \). Thus \( x \in_{\eta} P \).
(ii) \( \Rightarrow \) Suppose \( \eta y \vdash M y \rightarrow N \) in order to prove \( x \in FV(N) \).
Case 1. \( N = M' y \) with \( \eta y \vdash M \rightarrow M' \). Since \( x \in_{\eta} M \), also \( x \in FV(M') \subseteq FV(N) \).
Case 2. \( M \rightarrow \lambda z \cdot M_1 \) and \( \eta y \vdash M y \rightarrow (\lambda z \cdot M_1) y \rightarrow M_1[z/y] \rightarrow N \).
Since \( x \in_{\eta} M \), also \( x \in_{\eta} \lambda z \cdot M_1 \) and by (i) \( x \in_{\eta} M_1 \) and \( z \neq x \), so by 3.6.0. (iii) \( x \in_{\eta} M_1[z/y] \). Therefore \( x \in FV(N) \). 

3.6.2. Lemma. If \( \exists y \neq x \ x \in_{\eta} f(y) \), then \( \forall y \neq x \ x \in_{\eta} g(y) \) (and hence \( \forall y \neq x \ x \in_{\eta} f(y) \)).

Proof. Suppose \( x \in_{\eta} f(y) \), \( y \neq x \). Let \( y' \neq x \). Then by 3.6.1. (ii) \( x \in_{\eta} f(y) \cdot y' =_{\eta} g(y') \cdot y \). Hence, by 3.6.1. (ii), \( x \in_{\eta} g(y') \). (The rest follows by applying the statement to \( x \in_{\eta} g(y') \)). 

3.6.3. Main Lemma. There is a variable \( x \) such that for all terms \( M \): \( f(x)[x/M] = f(M) \).

Proof. Let \( v \) be any variable. Choose \( x \neq v \) such that \( x \notin_{\eta} f(v) \). Then \( x \notin_{\eta} g(z) \) for all \( z \neq x \), by the dual of 3.6.2.
Given \( M \), one can find a \( y \) such that \( y \notin_{\eta} M \), \( f(M) \), \( x \), \( f(x) \). Hence \( x \notin_{\eta} g(y) \). Now since \( y \neq x \) and \( x \notin_{\eta} g(y) \), \( (f(x)[x/M]) \cdot y = (f(x) \cdot y)[x/M] = (g(y) \cdot x)[x/M] = g(y) \cdot M = f(M) \cdot y \).
Since \( y \notin f(x) \), \( M \), \( f(M) \), extensionality yields \( f(x)[x/M] = f(M) \). 

Now it follows by 3.6.3. that \( f \) can be represented by the term \( \lambda x \cdot f(x) \) and similary for \( g \).

The following construction is needed for the proof of 3.10.

3.7. Definition. Let \( \# \) be a Gödel numbering of terms. \( \langle M \rangle \) is the numeral \( \# M \). A sequence of terms \( M_n \) is recursive if \( \lambda n \cdot \# M_n \) is a recursive function.

3.8. Lemma. (Coding of infinite sequences). Let \( \{ M_n \} \) be a recursive sequence of terms such that \( FV(M_n) \subseteq \{ x \} \) for all \( n \). Then there exists a term \( X \) such that \( p_i X = M_i \), for all \( i \), where \( p \) is some fixed closed term. Par abus de langage we write \( \langle M_n \rangle_{n \in \omega} \) for \( X \).

Proof.
(1) As in Curry et al. [1972], 13 B3 there is a term \( E \) which enumerates all terms with \( x \) as only free variable:

\[ E(\langle M \rangle) = M, \text{ for } M \text{ with } FV(M) = \{ x \}. \]
(2) Let \([M, N]\) be a pairing of terms defined by \(\lambda z \cdot zMN\). Then 
\([M, N]K = M\) and \([M, N](KI) = N\). Define ordered tuples as follows:
\([M] = M, [M_1, ..., M_{n+1}] = [M_1, [M_2, ..., M_{n+1}]].\)

(3) Let \(M_n\) with \(FV(M_n) \subseteq \{x\}\) be a recursive sequence of terms.
We want to code the sequence \([M_n]\) as a \(\lambda\)-term. Let \(\Sigma^{+}\) be such that
\(\Sigma^{+} \eta \to \eta + 1\) and let \(b = \lambda xy \cdot [E(Fy), (x(S^{+}y))]\), where \(F\) \(\lambda\)-defines \(f\), and 
\(B = F\Sigma^{+} b\) (i.e. the fixed point of \(b\)). Then 
\[B_{n} \to \Sigma^{+} B_{n} \to [E(F_{n}), B_{n+1}] \to [M_{n}, B_{n+1}].\]
So \(B_{0} = [M_{0}, B_{1}] = [M_{0}, M_{1}, B_{2}] = ...\). Hence by setting \(\langle M_{n}\rangle_{n \in \omega} = B_{0}\) we have a coding for infinite sequences of terms with one fixed free variable.

(4) It is easy to construct a term \(p\) such that 
\(p_{n} \{M_{n}\} \in \{0\}\) if zero \(x\) then \(aK\) else \(p(x-1)(a(KI))\), using the fixed point theorem. 

3.9. Lemma. For all closed \(Z\) there is an \(n\) such that \(Z\Omega^{n} = \not\exists \Omega\). 
\((Z\Omega^{n}\) is short for \(Z\Omega^{n} ... \Omega\). 

Proof:
Case 1. \(Z\) is unsolvable; then \(Z = \not\exists \Omega\), so \(n = 0\).
Case 2. \(Z\) is solvable; then \(Z\) has a \(h\) \(\exists\) \(y\), \(Z = \lambda x \cdot x_{1}A_{1} ... A_{m} (x_{i} \in \not\exists x)\).
Take \(n = i\), so \(Z\Omega^{i} = \lambda x^{i} \cdot \Omega A_{1} ... A_{m} = \not\exists \Omega\). 

3.10. Theorem. If \(\mathcal{M}\) is hard and sensible, then \(\mathcal{M}\) is not rich.

Proof. If \(\mathcal{M}\) is hard, then \(\mathcal{M}\) is isomorphic to \(\mathcal{M}(T)\), where \(T = \text{Th}(\mathcal{M})\). 
We reason in \(\mathcal{M}(\bar{T})\). Since \(\mathcal{M}\) is sensible, \(\mathcal{H} \subseteq T\).
Let \(h: \omega \to \omega\) be a function not definable in \(\mathcal{M}\). Such an \(h\) exists since a hard model is countable.
Let \(A_{n}(x, y)\) be the term \(x\Omega^{n}(y_{\Omega^{n}(h_{y})})\), \(n \in \omega\). For closed \(M\) the sequence 
\(A_{0}(M, y), A_{1}(M, y), ...\) is by 3.9 
\(M(y(h_{y})), M_{\Omega^{n}}(y_{\Omega^{n}(h_{y})}), ..., M\Omega^{n}(y_{\Omega^{n}(h_{y})}), \Omega, \Omega, \...\),
where \(n\) is such that \(M\Omega^{n+1} = \Omega\). Thus \(\lambda n \cdot A_{n}(M, y)\) is up to convertibility 
a recursive sequence containing one fixed free variable and hence representable as a term. Define \(f(M) = \lambda y \cdot \langle A_{n}(M, y) \rangle_{n \in \omega}\). Similarly for closed \(N\) 
\(\lambda n \cdot A_{n}(x, N)\) is recursive and it is possible to define \(g(N) = \lambda x \cdot \langle A_{n}(x, N) \rangle_{n \in \omega}\). 
Then for all closed \(M, N:\) \(f(M)\) and \(g(N)\) are well defined and \(f(M) \cdot N = g(N) \cdot M = \langle A_{n}(M, N) \rangle_{n \in \omega}\) by construction. So \(f\) and \(g\) are dual.

Suppose now that \(\mathcal{M}\) is rich, i.e. \(f\) were representable by some closed \(F\).
Then for all closed \(M, N:\) \(FMN = f(M)N = \langle A_{n}(M, N) \rangle_{n \in \omega}\).
But then \(p_{n}f(F(K_{n}I)(K_{n}I)) = p_{n}\langle h(y) \rangle_{n \in \omega} = h(y)\), hence \(h\) were definable, contradiction. Thus \(\mathcal{M}\) is not rich.
3.11. Corollary. $D^\infty_\omega$ and $\mathcal{M}(T)$ for $T \subseteq \mathcal{H}$ are not rich.

3.12. Questions. (i) Is every extensional term model $\mathcal{M}(T)$ rich?
(ii) Is $\mathcal{M}(\lambda\omega)$ rich?

Here $\lambda\omega$ is the $\lambda$-theory obtained by adding the $\omega$-rule to the theory, see Barendregt [1974].

Mathematical Institute
Budapestlaan 6
Utrecht, The Netherlands

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