COMBINATORY LOGIC AND THE AXIOM OF CHOICE

BY

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INTRODUCTION

The equational calculus combinatory logic \((CL)\) can be extended by embedding it in the first order intuitionistic (classical) predicate logic. The resulting theories are denoted by \((CL)_I\), \((CL)_C\) respectively. We can treat \(K=S\) as absurdity, because from \(K=S\) any formula can be derived.

In his Bucharest lecture (1971) Scott considered the following axiom scheme of choice:

\[(AC) \forall x \forall y A(x, y) \rightarrow \exists z \forall x A(x, zx)\]

and showed that \((CL)_C + (AC)\) is inconsistent. He suggested that \((CL)_I + (AC)\) might be consistent. At another occasion Scott suggested the use combinators as a tool for realizability. By a combinatory version of Kleene's \((\Gamma|\rightarrow\text{-})\)-realizability the following results are proved 2):

1) \((CL)_I + (AC)\) is a conservative extension of \(CL\) (and hence consistent).
2) \((CL)_I\) is closed under the rule of choice.
3) \((CL)_I\) and \((CL)_I + (AC)\) are disjunctive and existential for arbitrary formulas (containing possibly variables).

Result 1) and some other conservative extension results can be summarized as follows in fig. 1 (see also 4.7).

The method is extended in an obvious way to include extensionality in the results mentioned in 1), 2), 3) and fig. 1.

We see that the situation is somewhat different from intuitionistic arithmetic with Church's thesis. In that context 1) and 2) hold, but 3) is obviously false, e.g. \(x = 0 \lor x \neq 0\), but neither \(x = 0\) nor \(x \neq 0\) (the disjunctive and existential property only holds for closed sentences).

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1) The author is supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).
2) Independently Statman (1972) gives a normalization theorem for the intuitionistic theory of functionals of finite type with the axiom of choice from which 1), 2) and 3) follow implicitly.
conservative

\[
\begin{array}{c}
\text{conservative} \\
\text{not conservative}
\end{array}
\]

\[
\begin{array}{c}
CL \\
(CL)_I \\
(CL)_I + (AC) \\
(CL)_c \\
\text{conservative}
\end{array}
\]

Fig. 1.

The reason that for \((CL)_I + (AC) 3)\) is provable, is that it is possible to define a realizability concept which satisfies

\[(*) \quad MrVxA(x) \iff \text{for all } N[MNrA(N)],\]

where \(N\) ranges over terms possibly containing free variables.

The fact that the existential property holds for arbitrary formulas, i.e. that \(\vdash \exists x A(\overline{a}, x) \Rightarrow \vdash A(\overline{a}, M)\) for some \(M\), is a stronger form of the so-called "combinatory completeness": all Skolem functions are definable.

On the other hand, the disjunctive property for arbitrary formulas, i.e. \(\vdash A(\overline{a}) \lor B(\overline{a}) \Rightarrow \vdash A(\overline{a})\) or \(\vdash B(\overline{a})\), shows a certain weakness of the theories considered: the union of two definable sets can only be proved to be the universe of one of them is the universe. As an attempt to strengthen \((CL)_I\) one could add an axiom or rule expressing that the objects in the range of a quantifier behave like the objects given by the closed terms. This can be done in two ways, by an axiom of term induction \((TI)\), like the induction axiom in arithmetic, or by a term rule \((TR)\), like the \(\omega\)-rule in arithmetic:

\[TI: \quad A(K) \land A(S) \land Vxy(A(x) \land A(y) \rightarrow A(xy)) \rightarrow VxA(x)\]

\[TR: \quad \text{From } A(Z) \text{ for all closed terms } Z, \text{ conclude } VxA(x).\]

In contrast to what might be expected, there is no obvious way of realizing \(TI\), even if the realizability concept does not have the special property \((*)\) mentioned above but is defined as usual. This is due to the fact that the components of a compound term \(Z_1Z_2\) are not retracable, i.e. there is no term \(M\) such that \(M(Z_1Z_2) = Z_1\) or \(M(Z_1Z_2) = Z_2\). This in contrast to the case in arithmetic, where the predecessor of a numeral is retracable. Since for this reason it is doubtful that \(TI\) is of interest, it is not considered any further in this paper.

Concerning \(TR\) it is shown that \((CL)_c + TR + \text{extensionality}\) is consistent and that \((CL)_I + TR(+\text{ext.})\) is disjunctive and existential for sentences (the later by using Kleene's \(\Gamma|\text{realizability concept}\)). It is open whether \((CL)_I + (TR) + (AC)\) is consistent.
Finally it is shown that \((CL)_C + (TR) + \text{extensionality}\) is not a complete theory (which shows another difference with arithmetic).

The plan of this paper is as follows.

§ 1 is an introduction to \(CL\) proving its consistency by a Church-Rosser technique.

In § 2 \(CL\) is embedded in the first order predicate logic formulated as a system of natural deduction. This treatment simplifies the verification of the realizability of the provable formulas.

In § 3 combinatory realizability is defined and is applied to prove the results 1), 2) and 3).

Finally in § 4 we consider extensions of the theory, consisting of extensionality and the term rule.

At the end some open problems are stated.

§ 1. Combinatory Logic

In this § we will describe the equational theory combinatory logic and derive some of its elementary properties.

1.1. Definition

\(CL\) is a theory with the following language.

Alphabet: \(K, S\) constants
\(a_0, a_1, a_2, ...\) parameters
(,) improper symbols
\(=\) equality

Terms: Terms are defined inductively by
1) Any parameter or constant is a term.
2) If \(M, N\) are terms, then \((MN)\) is a term.

Formulas: If \(M, N\) are terms, then \(M = N\) is a formula.

1.2. Notation

As a syntactic notation to refer to arbitrary parameters we use the letters \(a, b, c\) etc.

\(M, N, L\) etc. is a syntactic notation for arbitrary terms.
\(M_1M_2 ... M_n\) stands for \((. (M_1 M_2) ... M_n)\) (association to the left).
If \(\vec{N} = N_1, ..., N_n\) then \(M\vec{N}\) is \(MN_1 ... N_n\).
\(M \equiv N\) means that \(M\) and \(N\) are identical terms.

1.3. Definition

The theory \(CL\) is defined by the following axioms and rules:

I 1. \(KMN = M\)
2. \(SMNL = ML(NL)\)

1) Or free variables. The name variable itself is reserved for bound variables.
II 1. \( M = M \)

2. \( \frac{M = N}{N = M} \)

3. \( \frac{M = N, \ N = L}{M = L} \)

4. \( \frac{M = M', \ \ M = M'}{ZM = ZM', \ MZ = M'Z} \)

In the above \( \ldots_1 \) expresses that \( \ldots_2 \) is a direct consequence of \( \ldots_1 \).

1.4. \textbf{Definition}

\( \text{Par} (M) \) is the set of parameters that occur in \( M \).

\( M \) is closed iff \( \text{Par} (M) = \emptyset \).

1.5. \textbf{Definition}

Let \( \text{Par} (M) \subseteq \{a_1, \ldots, a_n\} = \{\overrightarrow{a}\} \), then sometimes we will write \( M(\overrightarrow{a}) \).

We say that \( \overrightarrow{N} \) corresponds to \( \overrightarrow{a} \) if \( \overrightarrow{N} = N_1, \ldots, N_n \). \( M(\overrightarrow{N}) \) denotes the results of substituting simultaneously \( N_i \) for \( a_i \) in \( M \).

1.6. \textbf{Lemma}

If \( CL \vdash M_1(\overrightarrow{a}) = M_2(\overrightarrow{a}) \), then \( CL \vdash M_1(\overrightarrow{N}) = M_2(\overrightarrow{N}) \).

\textbf{Proof}

Induction on the length of proof of \( M_1 = M_2 \).

The following consistency theorem for \( CL \) was proved first in \textsc{Curry} [1930]. We will show it by proving the so called Church-Rosser property for \( CL \) (imitating the proof of \textsc{Rosser} [1935], T12, p–144 for a slightly different system).

1.7. \textbf{Theorem} (consistency)

Not \( CL \vdash K = S \).

\textbf{Remark}

If \( K = S \) would be provable, the \( P = Q \) for arbitrary \( P, Q \) would be provable: \( ML = KMNL = SMNL = ML(NL) \) for all \( M, N, L \).

Take \( M = L = I \) and \( N = KP, \ P \) arbitrary (where \( I = SKK \) and has the property \( CL \vdash IM = M \)). Then \( I = KPI = P \) for all \( P \).

Hence for all \( P, Q \) \( P = I = Q \).

\textbf{Proof}

We extend \( CL \) to a theory \( CL' \) as follows: to the alphabet of \( CL \) we adjoin two binary predicate symbols \( >_1 \) and \( > \) and we extend the definition of formulas accordingly.
Furthermore we add the following axioms and rules.

\[ KMN >_1 M \]

\[ SMNL >_1 ML(NL) \]

\[ M >_1 M \]

\[ M >_1 M' \quad M > M' \]

\[ \frac{M > M', M = M'}{M > L} \]

\[ M > N, N > L \]

\[ \frac{M >_1 M', N >_1 N'}{MN >_1 M'N'} \]

**Lemma 1.7.1.**

If \( CL' \vdash M_1 >_1 M_2 \) and \( CL' \vdash M_1 >_1 M_3 \), then there exists a term \( M_4 \) such that \( CL' \vdash M_2 >_1 M_4 \) and \( CL' \vdash M_3 >_1 M_4 \) (see fig. 2).

![Fig. 2.](image)

**Proof**

Induction on the length of proof of \( M_1 >_1 M_2 \).

**case 1.** \( M_1 >_1 M_2 \) is \( KMN >_1 M \).

**subcase 1.1.** \( M_3 \equiv KMN \) or \( M_3 \equiv M \). Then take \( M_4 \equiv M \).

**subcase 1.2.** \( M_3 \equiv KM'N' \), where \( CL' \vdash M >_1 M' \) and \( CL' \vdash N >_1 N' \)

Then take \( M_4 \equiv M' \).

**case 2.** \( M_1 >_1 M_2 \) is \( SMNL >_1 ML(NL) \). This case is treated similarly to case 1.

**case 3.** \( M_1 \equiv M_2 \). Then take \( M_4 \equiv M_3 \).

**case 4.** \( M_1 >_1 M_2 \) is \( MN >_1 M'N' \) and is a direct consequence of \( M >_1 M', N >_1 N' \).

**subcase 4.1.** \( M_1 >_1 M_3 \) is \( KPQ >_1 P \). Then go to case 1.

**subcase 4.2.** \( M_1 >_1 M_3 \) is \( SPQR >_1 PR(QR) \). Then go to case 2.

**subcase 4.3.** \( M_1 \equiv M_3 \). Then go to case 3.

**subcase 4.4.** \( M_1 >_1 M_3 \) is \( MN >_1 M''N'' \) and is a direct consequence of \( M >_1 M'' \) and \( N >_1 N'' \).

By the induction hypothesis there are terms \( M'', N'' \) such that \( M' >_1 M'' \), \( M'' >_1 M''' \), \( N' >_1 N''' \) and \( N'' >_1 N''' \). Then take \( M_4 \equiv M''N'' \). ■
Lemma 1.7.2.
If $CL' \vdash M_1 > M_2$ and $CL' \vdash M_1 > M_3$, then there exists a term $M_4$ such that $CL' \vdash M_2 > M_4$ and $CL' \vdash M_3 > M_4$.

Proof
Realizing that
$$CL' \vdash M > N \iff \exists N_1 \ldots N_k \ CL' \vdash M >_1 N_1 >_1 \ldots >_1 N_k >_1 N$$
we can complete a diagram like:

\[
\begin{array}{c}
M_1 \\
M_2 \\
M_3
\end{array}
\]

by means of lemma 1.7.1 to a diagram like:

Lemma 1.7.3. (Church-Rosser property for $CL$)
If $CL \vdash M = N$, then for some term $Z$ $CL' \vdash M > Z$ and $CL' \vdash N > Z$.

Proof
If $CL \vdash M = N$, then clearly $CL' \vdash M = N$. By induction on the length of proof in $CL'$ we show the conclusion, using 1.7.2. in the case of the transitivity of $\equiv$.

From 1.7.3. it follows immediately that not $CL \vdash K = S$, since $CL' \vdash K > Z \Rightarrow Z \equiv K$ and similarly for $S$. ■ 1.7.

1.8. Theorem (Combinatory completeness)
For every term $M$ and every parameter $a$, there exists a term $\lambda a \cdot M$ such that $\text{Par}(\lambda a \cdot M) = \text{Par}(M) - \{a\}$ and $CL \vdash (\lambda a \cdot M)a = M$ for arbitrary $N$.

Proof
Induction on the complexity of $M$.

case 1. $M$ is a parameter or constant.
  subcase 1.1. $M \neq a$. Take $\lambda a \cdot M \equiv KM$.
  subcase 1.2. $M \equiv a$. Take $\lambda a \cdot M \equiv SKK$.

case 2. $M \equiv N_1 N_2$. Take $(\lambda a \cdot M) \equiv S(\lambda a \cdot N_1)(\lambda a \cdot N_2)$. ■
1.9. **Notation**

\( \lambda a_1 \ldots a_n \cdot M \) stands for \( (\lambda a_1 \cdot (\ldots (\lambda a_n \cdot M \ldots ))) \).

1.10. **Definition** (Ordered pairs)

\[ [M_0, M_1] \equiv \lambda z \cdot zM_1M_2. \]

\( (M)_0 \equiv MK \)

\( (M)_1 \equiv MK_*, \) where \( K_* = \lambda ab.b \).

Then \( CL \vdash ([M_0, M_1])_i \equiv M_i, \ i \in \{0, 1\} \).

1.11. **Theorem** (Fixed point theorem)

For every term \( F \) there exists a term \( \Omega \) such that \( CL \vdash F\Omega = \Omega \).

**Proof**

Let \( \omega = \lambda a \cdot (F(aa)) \) and \( \Omega = \omega \omega \).

Then \( CL \vdash \Omega = \omega \omega = (\lambda a \cdot (F(aa)))\omega = F(\omega \omega) = F\Omega \).

\[ \Box \]

§ 2. **First order theories extending combinatory logic**

We will extend the equational theory \( CL \) by making it a first order theory. The intuitionistic first order extension is called \( (CL)_I \), the classical \( (CL)_C \).

The common language of \( (CL)_I, (CL)_C \) is called \( (CL) \).

2.1. **Definition**

\( (CL) \) is the following language

**Alphabet:** the symbols of the alphabet of \( CL \)

- \( x_0, x_1, \ldots \) variables
- \( \land, \lor, \top \) propositional connectives
- \( \forall, \exists \) quantifiers.

**Terms:** Terms are the same as the terms of \( CL \).

**Formulas:** Formulas are defined inductively by

1) Every formula of \( CL \) is a formula.
2) If \( A, B \) are formulas, then \( (A \land B), (A \lor B) \) and \( (A \top B) \) are formulas.
3) If \( A \) is a formula, \( x \) a variable not occurring in \( A \), then \( \forall x A^* \) and \( \exists x A^* \) are formulas, where \( A^* \) is obtained from \( A \) by replacing occurrences of a parameter by \( x \).

2.2. **Definition**

\( \bot \) (falsity) is an abbreviation for \( K = S \).

\( \neg A \) is an abbreviation for \( A \top A \).
2.3. Notation

Syntactic notations:

\[ A, B, \ldots \text{ for formulas} \]
\[ M, N, \ldots \text{ for terms} \]
\[ a, b, \ldots \text{ for parameters} \]
\[ x, y, \ldots \text{ for variables} \]
\[ \Gamma, \Delta, \ldots \text{ for sets of formulas.} \]

Identity between syntactic objects is denoted by \( \equiv \).

\( \text{Par}(A) \) is the set of parameters that occur in \( A \).
\( \text{Par}(\Gamma) = \bigcup_{\alpha \in \Gamma} \text{Par}(A_\alpha) \).

\( A \) is a sentence if \( \text{Par}(A) = \emptyset \).

If \( \text{Par}(A) \subseteq \{ a \} \) or \( \text{Par}(\Gamma) \subseteq \{ a \} \) we write sometimes \( A(a) \) or \( \Gamma(a) \).

If \( \vec{N} \) corresponds to \( \vec{a} \), \( A(\vec{N}) \) denotes the result of substituting simultaneously \( N_i \) for \( a_i \) in \( A \).

\[ \Gamma(\vec{N}) = \{ A(\vec{N}) | A \in \Gamma \}. \]

Instead of giving the axioms and rules of \((CL)_I\) and \((CL)_C\) as usual in a Hilbert type system, we describe these theories in a natural deduction system à la Gentzen. See Prawitz [1965] for a discussion of those systems.

Every rule is called after the logical symbol (e.g. \( \land \)) which is introduced (\( \land I \)) or eliminated (\( \land E \)).

2.4. Definition

Let \( \Gamma \cup \{ A \} \) be a set of formulas of \((CL)\). We define the theory \((CL)_I\) by defining inductively \( \Gamma \vdash A : \)

\[ A \in \Gamma \Rightarrow \Gamma \vdash A \]
\[ \text{CL} \vdash A \Rightarrow \Gamma \vdash A, \text{ where } A \text{ is a formula of } \text{CL} \]

\( \land I ) \quad \Gamma \vdash A \text{ and } \Gamma \vdash B \Rightarrow \Gamma \vdash A \land B \)
\( \land E ) \quad \Gamma \vdash A \land B \Rightarrow \Gamma \vdash A \text{ and } \Gamma \vdash B \)
\( \lor I ) \quad \Gamma \vdash A \Rightarrow \Gamma \vdash A \lor B \)
\( \Gamma \vdash B \Rightarrow \Gamma \vdash A \lor B \)
\( \lor E ) \quad [ \Gamma \cup \{ A \} \vdash C, \Gamma \cup \{ B \} \vdash C \text{ and } \Gamma \vdash A \lor B ] \Rightarrow \Gamma \vdash C \)
\( \exists I ) \quad \Gamma \cup \{ A \} \vdash B \Rightarrow \Gamma \vdash A \exists B \)
\( \exists E ) \quad \Gamma \vdash A \text{ and } \Gamma \vdash A \exists B \Rightarrow \Gamma \vdash B \)
\( \forall I ) \quad \Gamma \vdash A(a) \Rightarrow \Gamma \vdash \forall x A(x), \text{ provided } a \notin \text{Par}(\Gamma) \)
\( \forall E ) \quad \Gamma \vdash \forall x A(x) \Rightarrow \Gamma \vdash A(N) \)
\( \exists I ) \quad \Gamma \vdash A(N) \Rightarrow \Gamma \vdash \exists x A(x) \)
\( \exists E ) \quad \Gamma \cup \{ A(a) \} \vdash B \text{ and } \Gamma \vdash \exists x A(x) \Rightarrow \Gamma \vdash B, \text{ provided } a \notin \text{Par}(\Gamma) \cup \text{Par}(B) \)
\( A ) \quad \Gamma \cup \{ A \} \vdash A \)

\( A \) is a theorem of \((CL)_I\) (notation \( \vdash A \)) if \( \emptyset \vdash A \).
To remember the deduction rules more easily we write them down in a more schematic way.

\[
\begin{align*}
\land I) & \quad \frac{A \quad B}{A \land B} & \land E) & \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \\
\lor I) & \quad \frac{A}{A \lor B} \quad \frac{B}{A \lor B} & \lor E) & \quad \frac{A \lor B \quad [A] \quad [B]}{C} \quad \frac{C}{C} \\
\exists I) & \quad \frac{[A]}{A \exists B} & \exists E) & \quad \frac{A \exists B \quad B}{A} \\
\forall I) & \quad \frac{A(a)}{\forall x \ A(x)} & \forall E) & \quad \frac{\forall x \ A(x)}{A(N)} \\
\exists I) & \quad \frac{A(N)}{\exists x A(x)} & \exists E) & \quad \frac{\exists x A(x) \quad [A(a)]}{B} \\
\land) & \quad \frac{A}{A} 
\end{align*}
\]

The meaning of these schema's is formalized in definition 2.4. See Práiwitz [1965] for a discussion. The rules \(\forall I\) and \(\exists E\) are subject to restrictions on the parameter \(a\), as in definition 2.4.

2.5. Definition

\((CL)_c\) is defined by adding to \((CL)\) the following extra deduction rule:

\[A_c) \quad \Gamma \cup \{\neg A\} \vdash A \Rightarrow \Gamma \vdash A.\]

In the schematic notation this is

\[
\frac{[-A]}{A_c}.
\]

To distinguish it from the intuitionistic system we write \(\Gamma \vdash_{c} A\) (resp. \(\vdash_{e} A\)) for the classical system (and \(\Gamma \vdash A\) (resp. \(\vdash A\)) for the intuitionistic system).

2.6. Lemma

If \(\Gamma(\vec{a}) \vdash_{*} A(\vec{a})\), then \(\Gamma(\vec{N}) \vdash_{*} A(\vec{N})\), where \(\vec{N}\) corresponds to \(\vec{a}\) and \(\vdash_{*}\) is \(\vdash\) or \(\vdash_{c}\).

Proof

Induction on the length of proof of \(\Gamma \vdash_{*} A\).
2.7. Lemma

\[ \vdash K \neq S \quad (\text{i.e. } \vdash \neg K = S). \]

Proof

Note that \( K \neq S \) stands for \( K = S \supset A \), i.e. for \( K = S \supset K = S \). A natural deduction of this is

\[ \frac{[K=S]}{K=S \supset K=S}. \]

From now on derivations are not shown in detail anymore.

2.8. Definition

We define the following axiom scheme of choice \((AC)\)

\[(AC) \quad \forall x \exists y A(x, y) \supset \exists z \forall x A(x, z).\]

2.9. Theorem (Scott)

\((CL)c + (AC)\) is inconsistent.

Proof

Since \( K \neq S \), \( \forall x \exists y x \neq y \). Hence by \((AC)\) \( \exists z \forall x z \neq x \), contradicting the fixed point theorem 1.8.

2.10. Corollary

\((AC) \quad \vdash \neg \exists x (x = K \lor x \neq K).\)

Proof

\( \forall x (x = K \lor x \neq K) \) implies (in \((CL)t) \( \forall x \exists y x \neq y \).

2.11. Corollary

Not \( \vdash (AC) \), i.e. not every instance of \((AC)\) is derivable.

Proof

\((CL)c\) has the canonical termmodel of \( CL \) as model (in which \( K \neq S \) by 1.7) and is therefore consistent.

Hence not \( \vdash (AC) \) and hence a fortiori not \( \vdash (AC). \)

2.12. Theorem

\((CL)c\), and hence \((CL)t\), is conservative over \( CL \).

Proof

If \( \vdash (M = N) \), then \( M = N \) is true in all models of \((CL)c\), hence in particular in the canonical term model of \( CL \) (with as domain the set of all \( CL \) terms up to provable equality) and hence \( CL \vdash M = N \).

Remark

Theorem 2.12. is a consequence of the following general fact. Let \( T \) be a set of quantifier free axioms (in a certain similarity type). Let \( T_{\text{prop}} \)
be the set of consequences of $T$ using proposition logic only. Let $T_{\text{pred}}$ be the set of all consequences of $T$ (in the full first other language).

Then $T_{\text{pred}}$ is conservative over $T_{\text{prop}}$. In the classical case this follows from the soundness of the predicate logic and the completeness of the proposition logic. In the intuitionistic case this follows from the normalization theorem for the intuitionistic predicate logic, see Prawitz [1965].

§ 3. COMBINATORY REALIZABILITY

In this § we discuss a combinatory version of Kleene’s ($\Gamma \vdash \cdot$)-realizability concept (Kleene [1952] p. 503).

3.1. Definition

Let $\Delta \cup \{ A \}$ be a set of formulas, $M$ a term (all of the language (CL)), where $M$ and $A$ may contain free parameters.

We define inductively $Mr_{\Delta} A$ ($M$ realizes-$\Delta \vdash A$).

$Mr_{\Delta} N_1 = N_2 \iff CL \vdash N_1 = N_2$.

$Mr_{\Delta} A \wedge B \iff (M)_0 r_{\Delta} A$ and $(M)_1 r_{\Delta} B$.

$Mr_{\Delta} A \vee B \iff [CL \vdash (M)_0 = K \text{ or } CL \vdash (M)_0 = K_A]$ and

$[CL \vdash (M)_0 = K \Rightarrow ((M)_1 r_{\Delta} A$ and $\Delta \vdash A)]$ and

$[CL \vdash (M)_0 = K_A \Rightarrow ((M)_1 r_{\Delta} B$ and $\Delta \vdash B)]$.

$Mr_{\Delta} A \supset B \iff$ for all $N$ $[M r_{\Delta} A$ and $\Delta \vdash A) \Rightarrow MN r_{\Delta} B]$.

$Mr_{\Delta} V x A(x) \iff$ for all $N$ $[M r_{\Delta} A(N)]$.

$Mr_{\Delta} \forall x A(x) \iff (M)_0 r_{\Delta} A((M)_1)$ and $\Delta \vdash A((M)_1)$.

In the above $N$ ranges over terms possibly with free parameters.

3.2. Definition

Let $\text{Par} (A) = \{ a_1, \ldots, a_n \} = \overrightarrow{a}$.

$Mrr_{\Delta} A (\overrightarrow{a})$ ($M$ strongly realizes-$\Delta \vdash A$) iff for all $\tilde{N}$ $M r_{\Delta} A (\tilde{N})$.

3.3. Definition

Let $\Gamma = \{ A_1, \ldots, A_n \}$ be a finite set of formulas.

\[ \bigwedge \forall \Gamma \supset A \equiv A_1 \supset (A_2 \supset \ldots (A_n \supset A) \ldots ) \text{ if } n \neq 0 \]

\[ \equiv A \text{ if } n = 0 \]

$Mrr_{\Delta} \Gamma \vdash A \iff Mrr_{\Delta} \bigwedge \forall \Gamma \supset A$.

3.4. Definition

$\Gamma \vdash A$ is (strongly) realizable if $\Gamma$ is finite and there is a term $M$ such that $M(\Gamma) r_{\Delta} \Gamma \vdash A$ for all $\Delta$.

3.5. Lemma

i) $Mrr_{\Delta} A \Rightarrow M \overrightarrow{a} r_{\Delta} A$, where $\{ \overrightarrow{a} \} = \text{Par} (A)$
ii) \([A \vdash A \Leftrightarrow A' \vdash A] \Rightarrow (r)r_A \) and \((r)r_{A'} \) are identical relations

iii) \([M(r)r_A \ A \text{ and } CL \vdash M = M'] \Rightarrow M'(r)r_A \ A\)

iv) \([M(r)r_A \ A(N) \text{ and } CL \vdash N = N'] \Rightarrow M(r)r_A \ A(N').

**Proof**

i) By definition. (The converse is false: take \(A \equiv (a = K \supset S = K)\)).

ii)–iv) Induction on the definition of \(r_A\).

### 3.6. Lemma

If \(\Gamma\) is finite and \(\Gamma \vdash A\), then \(\Gamma \vdash A\) is strongly realizable.

**Proof**

Induction on the length of proof of \(\Gamma \vdash A\).

**Case 1:** \(A \in \Gamma\). Let \(\Gamma = \{A_1, ..., A_n\}\) and \(A = A_i\). Let \(\overrightarrow{m} = m_1, ..., m_n\) and \(\overrightarrow{a} = \text{Par} (\Gamma \cup \{A_i\})\). Define \(M = \lambda \overrightarrow{m} \cdot \overrightarrow{a}\). Then \(Mrr_A \ \Gamma \vdash A\).

**Case 2:** \(CL \vdash A\). Then \(Krr_A \ \Gamma \vdash A\) by lemma 1.6.

**Case 3:** \(\Gamma \vdash A\) is the conclusion of a rule of inference. One has to show by examining the rules that the conclusion is strongly realizable assuming that the assumption(s) is (are) strongly realizable (induction hypothesis). We will show this only for the rules \(\land I), \land I), \lor I)\) and \(\exists E\), the other rules being left to the reader.

In the verification we will first assume that the only parameters occurring in the rule are the ones explicitly shown (simple proof).

In this proof \(\Gamma\) is always \(\{A_1, ..., A_p\}\) and \(\overrightarrow{M}_\Gamma = M_1, ..., M_n\) resp. \(\overrightarrow{m}_\Gamma = m_1, ..., m_p\) corresponding sequences of terms resp. parameters.

We write \(\overrightarrow{M}_\Gamma \ r_A \ \Gamma\) iff \(M_{1 \ r_A} \ A_1 \) and \(\Delta \vdash A, 1 < i < p\).

Then \(Mrr_A \Gamma \vdash A\) for all \(\overrightarrow{M}_\Gamma \ r_A \ \Gamma \Rightarrow \overrightarrow{M}_\Gamma \ r_A \ A\).

**Simple proof** (assuming there are no parameters): By the induction hypothesis, for some \(M_1, M_2\) we have \(M_{1 \ r_A} \Gamma \vdash A\) and \(M_{2 \ r_A} \Gamma \vdash B\).

Define \(M = \lambda \overrightarrow{m} \cdot [M_1 \overrightarrow{m}, M_2 \overrightarrow{m}]\), then \(Mrr_A \Gamma \vdash A \land B\).

**Detailed proof:** By the induction hypothesis, for some \(M_1, M_2\) we have \(M_{1 \ r_A} \Gamma \vdash A\) and \(M_{2 \ r_A} \Gamma \vdash B\).

Let \(\text{Par} (\Gamma) = \{a_0\} = \{a_{01}, ..., a_{0n_0}\}\)

\(\text{Par} (\Gamma \cup \{A\}) = \{a_1\}\)

\(\text{Par} (\Gamma \cup \{B\}) = \{a_2\}\)

\(\text{Par} (\Gamma \cup \{A, B\}) = \{a\}\).

Then \(\{a_0\} \subseteq \{a_1\} \cup \{a_2\} = \{a\}\).
Hence for all $\vec{N}_1, \vec{N}_2$ corresponding to $\vec{a}_1, \vec{a}_2$ we have

$$M_1 \vec{N}_1 r, A \quad \Gamma(\vec{N}_1) \supset A(\vec{N}_1)$$

and

$$M_2 \vec{N}_2 r, A \quad \Gamma(\vec{N}_2) \supset B(\vec{N}_2).$$

Let $n, m$ correspond to $\vec{a}, \Gamma$.

Let $n_1, n_2$ be the subsequences of $n$ corresponding to $\vec{a}_1, \vec{a}_2$.

Define $M = \lambda n \rightarrow [M_1 n_1 m r, M_2 n_2 m r]$. Claim $M r, A \quad \Gamma \vdash A \land B$.

Fix $\vec{N}$ corresponding to $\{\vec{a}\}$. Let $\vec{N}_0, \vec{N}_1$ and $\vec{N}_2$ be the subsequences corresponding to $a_0, a_1$ and $a_2$.

We have to show

$$M \vec{N} r, A \quad \Gamma(\vec{N}) \supset A(\vec{N}) \land B(\vec{N}).$$

Suppose $\vec{M} r, A \quad \Gamma(\vec{N})$.

Now $\Gamma(\vec{N}) \equiv \Gamma(\vec{N}_i) \equiv \Gamma(\vec{N}_0), i = 1, 2$ and $A(\vec{N}) \equiv A(\vec{N}_1), B(\vec{N}) \equiv B(\vec{N}_2)$.

Hence $M_1 \vec{N}_1 \vec{M} r, A(\vec{N}_1)$ and $M_2 \vec{N}_2 \vec{M} r, A(\vec{N}_2)$.

Therefore $M \vec{N} \vec{M} = [M_1 \vec{N}_1 \vec{M}, M_2 \vec{N}_2 \vec{M}] r, A(\vec{N}) \land B(\vec{N})$.

\[\begin{align*}
A & \quad \Gamma \vdash A, A \quad \text{i.e.} \quad \Gamma \vdash \neg A.
\end{align*}\]

By the induction hypothesis, for some $M_1, M_1 r, A \quad \Gamma$. Suppose $\vec{M} r, A \quad \Gamma$, then $M_1 \vec{M} r, A \equiv (S = K)$. But this is impossible since not $CL \vdash S = K$ by 1.7. Hence for no $\vec{M}, \vec{M} r, A$.

Therefore $M_1 r, A \quad \Gamma$.

\[\begin{align*}
A(a) & \quad \forall x A(x) \quad \Gamma \vdash A(a), \text{i.e.} \quad \Gamma \vdash \forall x A(x) \quad \text{where} \ a \notin \text{Par} (\Gamma).
\end{align*}\]

Simple proof (assuming $\text{Par} (\Gamma \cup \{A\}) = \{a\}$):

By the induction hypothesis, for some $M_1, M_1 r, A \quad (a)$.

Hence for all $\vec{N} M_1 \vec{N} r, A \quad \Gamma \vdash A(\vec{N})$.

(*)

Define $M = \lambda n m r \cdot M_1 n m r$. Claim $M r, A \quad \Gamma \vdash \forall x A(x)$.

Suppose $M r, A \quad \Gamma$. We have to show that $M \vec{M} r, A \quad \Gamma \vdash \forall x A(x)$ i.e. that for all $\vec{N}$ $M \vec{M} r, A \quad \Gamma$.

Since $M \vec{M} r, N = M_1 \vec{N} r, A \quad \Gamma$ this follows from (*).

Detailed proof

Let $\{\vec{a}\} = \text{Par} (\Gamma \cup \{\forall x A(x)\})$, then $a \notin \{\vec{a}\}$.

By the induction hypothesis, for some $M_1, M_1 r, A \quad (a)$.

Hence for all $\vec{N}$ (corresponding to $\vec{a}$), $\vec{N} : M_1 \vec{N} r, A \quad \Gamma(\vec{N}) \supset A(\vec{N}, N)$. 

By the induction hypothesis, for some $M_1, M_1 r, A \quad (a)$.
Define \( M = \lambda \overrightarrow{m_r}. n \cdot M_1 \overrightarrow{nnm_r} \), where \( n \) corresponds to \( \overrightarrow{N} \). As above it follows that \( Mrr_\alpha \Gamma \vdash \forall x A(x) \).

\[
\begin{align*}
\forall x A(x) \frac{[A(a)]}{B} & \quad \text{i.e. } \Gamma \vdash \forall x A(x) \quad \Gamma, A(a) \vdash B \\
\end{align*}
\]

where \( a \notin \text{Par}(\Gamma, B, \forall x A(x)) \).

**Simple proof** (Assuming \( a \) is the only parameter around).

By the induction hypothesis, for some \( M_1, M_2, M_1 rr_\alpha \Gamma \vdash \forall x A(x) \) and \( M_2 rr_\alpha \Gamma, A(a) \vdash B \) i.e. \( M_2 rr_\alpha \Gamma \supset (A(a) \supset B) \).

Define \( M = \lambda \overrightarrow{m_r}. (M_2(M_1 \overrightarrow{m_r})_1 \overrightarrow{m}(M_1 \overrightarrow{m_r})_0) \). Claim \( Mrr_\alpha \Gamma \vdash B \). Suppose \( \overrightarrow{Mrr_\alpha \Gamma} \). Then \( M_1 \overrightarrow{Mrr_\alpha \Gamma} \forall x A(x) \) and for all \( N, M_2 M_1 \overrightarrow{Mrr_\alpha \Gamma} A(N) \supset B \).

Thus \( (M_1 \overrightarrow{Mrr_\alpha \Gamma})_1 A((M_1 \overrightarrow{Mrr_\alpha \Gamma})_1) \) and \( M_2(M_1 \overrightarrow{Mrr_\alpha \Gamma})_1 \overrightarrow{Mrr_\alpha \Gamma} A((M_1 \overrightarrow{Mrr_\alpha \Gamma})_1) \supset B \).

Therefore \( M\overrightarrow{Mrr_\alpha \Gamma} = M_2(M_1 \overrightarrow{Mrr_\alpha \Gamma})_1 \overrightarrow{Mrr_\alpha \Gamma} \overrightarrow{0} rr_\alpha \Gamma \).

**Detailed proof**

Let \( \{\overrightarrow{a}\} = \text{Par}(\Gamma \cup \{\forall x A(x)\}) \)

\( \{\overrightarrow{b}\} = \text{Par}(\Gamma \cup \{B\}) - \{\overrightarrow{a}\} \), then \( a \notin \{\overrightarrow{a}, \overrightarrow{b}\} \).

Let \( \overrightarrow{N_a} \) and \( \overrightarrow{n_a} \) correspond to \( \overrightarrow{a} \) and let \( \overrightarrow{N_b} \) and \( \overrightarrow{n_b} \) correspond to \( \overrightarrow{b} \).

By the induction hypothesis, for some \( M_1, M_2, \)

\[
M_1 rr_\alpha \Gamma \supset (A(a) \supset B(\overrightarrow{a}, \overrightarrow{b})) \]

and

\[
M_2 rr_\alpha \Gamma \supset (A(a, a) \supset B(\overrightarrow{a}, \overrightarrow{b})).
\]

Define \( M = \lambda \overrightarrow{n_a} \overrightarrow{n_b} \overrightarrow{m_r}. M_2 \overrightarrow{n_a}(M_1 \overrightarrow{n_a} \overrightarrow{m_r})_1 \overrightarrow{n_b} \overrightarrow{m_r}(M_1 \overrightarrow{n_a} \overrightarrow{m_r})_0 \).

As above we see that \( Mrr_\alpha \Gamma \vdash B \).

3.7. **Corollary**

If \( \vdash A \), then there exists a term \( M \) such that for all \( A \) \( Mrr_\alpha A \).

**Proof**

Immediate by 3.6. and 3.4.i).

3.8. **Theorem**

\( (CL)_I \) is closed under the rule of choice, i.e. if \( (CL)_I \vdash \forall x \forall y A(x, y) \), then there exists a term \( M \) such that \( (CL)_I \vdash \forall x A(x, Mx) \).

**Proof**

Let \( (CL)_I \vdash \forall x \forall y A(x, y) \). Then, for some \( M_0, M_0 r_\phi \forall x \forall y A(x, y) \).

So \( M_0 r_\phi \forall y A(a, y) \). Therefore \( (M_0 a)_0 r_\phi A(a, (M_0 a)_1) \) and \( \vdash A(a, (M_0 a)_1) \).

Let \( M = \lambda a(M_0 a)_1 \). Then \( \vdash \forall x A(x, Mx) \).
3.9. **Theorem**

If \((AC) \vdash A\), then there exists a term \(M\) such that \(Mr_{(AC)} A\).

**Proof**

If \((AC) \vdash A\), then \(\Gamma \vdash A\), where \(\Gamma\) is a finite set of instances of \((AC)\).

Hence for some \(M_0\), \(M_0 r_{(AC)} \Gamma \vdash A\).

Now every closed instance of \((AC)\) is realized by

\[ M_1 = (\lambda a \cdot [\lambda b \cdot (ab)_0, \lambda b \cdot (ab)_1]). \]

If an instance of \((AC)\) contains the parameters \(\vec{a}\), then it is realized by \(\lambda a \cdot \vec{M}\).

Hence \(\vec{M} r_{(AC)} \Gamma\), where the elements of \(\vec{M}\) are of the form \(\lambda a \cdot M_1\).

Therefore \(M_0 \vec{M} r_{(AC)} A\). \(\blacksquare\)

3.10. **Corollary**

\((CL)_I + (AC)\) is conservative over \(CL\), and hence consistent.

**Proof**

Let \((AC) \vdash N_1 = N_2\). Then for some \(M\), \(M r_{(AC)} N_1 = N_2\).

Hence \(CL \vdash N_1 = N_2\). \(\blacksquare\)

3.11. **Definition**

1) A theory \(T\) is (strongly) disjunctive iff for all sentences (resp. formulas) \(A \lor B\) we have \(T \vdash A \lor B \Rightarrow T \vdash A\) or \(T \vdash B\).

2) A theory \(T\) is (strongly) existential iff for all sentences (resp. formulas) \(\exists x A(x)\) we have \(T \vdash \exists x A(x) \Rightarrow \) for some term \(M\) of \(T\), \(T \vdash A(M)\).

3.12. **Theorem**

The theories \((CL)_I\) and \((CL)_I + (AC)\) are strongly disjunctive and strongly existential.

**Proof**

Suppose \((CL)_I \vdash A \lor B\). Then, for some \(M\), \(M r_{(AC)} A \lor B\).

Hence \((M)_0 = K\) and \(\emptyset \vdash A\) or \((M)_0 = K_0\) and \(\emptyset \vdash B\).

Thus \((CL)_I \vdash A\) or \((CL)_I \vdash B\).

Similar for the existential property and similar for \((CL)_I + (AC)\). \(\blacksquare\)

**Remarks**

Theorems 3.8 and 3.12 (for \((CL)_I\)) can be proved also using the normalization theorem for intuitionistic logic, see Prawitz [1965]. Independently Statman [1972] gives an extension of the normalization theorem, from which theorems 3.10 and 3.12 (for \((CL)_I + (AC)\)) follow implicitly.
§ 4. Some other results

In this § we first obtain results analogous to those in § 3 about combinatory logic with extensionality and the term rule. In the case of the term rule we cannot work with realization by a combinator and use instead a concept analogous to Kleene’s.

4.1. Definition

Ext is the axiom $\forall x \forall y (\forall z (xz = yz) \supset x = y)$.

ext is the rule $\frac{Ma = Na}{M = N}$, where $a \notin \text{Par} (MN)$.

4.2. Lemma

$(CL) + Ext$, and hence $(CL)_I + Ext$, is conservative over $CL + ext$ and therefore consistent.

Proof

Like the proof of 2.12, using the well known fact that $CL + ext$ is consistent. (This follows for example from 4.5.).

4.3. Theorem

i) $(CL)_I + Ext + (AC)$ is conservative over $CL + ext$.

ii) $(CL)_I + Ext$ and $(CL)_I + Ext + (AC)$ are strongly disjunctive and strongly existential.

iii) $(CL)_I + Ext$ is closed under the rule of choice.

Proof

Replace in the definition of $Mr_A$, for the case that $A$ is $N_1 = N_2$, $CL \models N_1 = N_2$ by $CL + ext \models N_1 = N_2$. Then 3.7 and 3.9 becomes true for the new realization concept. Furthermore Ext is realizable in the new sense.

Hence we have the analogues of 3.8, 3.10 and 3.12.

4.4. Definition

$TR$ (term rule for $(CL)$) is the rule

\[
A(Z) \text{ for all closed } Z \\
\frac{\forall x A(x)}
\]

$tr$ (term rule for $CL$) is the rule

\[
MZ = NZ \text{ for all closed } Z \\
\frac{MZ = NZ}{M = N}
\]

$\omega$ (ω-rule for $CL$) is the rule

\[
MZ = NZ \text{ for all closed } Z \\
\frac{M = N}
\]

4.5. Lemma

$CL + tr + ext$ is equivalent with $CL + \omega$ and therefore consistent.
The equivalence is trivial. The consistency of $CL + \omega$ is proved in Barendregt [1971], § 2.2.

4.6. Theorem

i) $(CL)_c + TR + Ext$ is conservative over $CL + tr + ext$ and hence consistent.

ii) $(CL)_c + TR$ is conservative over $CL + tr$.

iii) $(CL)_c + TR + M = N (+ Ext)$ is conservative over $CL + tr + M = N (+ ext)$.

Proof

i) Let $\mathcal{M}$ be the strict term model of $CL + tr + ext$ (where strict refers to the fact that the domain consists of the closed terms up to provable equality). By the consistency of $CL + tr + ext$, $\mathcal{M} \models K^S$. Hence $\mathcal{M} \models ((CL)_c + TR + Ext).

ii), iii) Similar.

Remark

It is not known whether $CL + tr$ is conservative over $CL$ or whether $CL + tr + ext$ is conservative over $CL + ext$. For a partial result in this direction see Barendregt [1971], § 2.5.

4.7. Theorem

The various conservative extension result obtained can be summarized as follows:

\begin{center}
\begin{tabular}{c|c|c}
 & conservative & non conservative \\
 CL & $(CL)_L$ & $(CL)_L + AC$
\end{tabular}
\end{center}

1) $(CL)_c$

\begin{center}
\begin{tabular}{c|c|c}
 & conservative & non conservative \\
 CL + tr & $(CL)_L + TR$ & $(CL)_c + TR$
\end{tabular}
\end{center}

2) $(CL)_c$

The same holds for both cases if extensionality is included.
Proof
This follows by combining 2.11, 2.12, 3.10, 4.2, 4.3 and 4.6.

4.8. Theorem

(CL)_1 + TR + Ext and (CL)_2 + TR are disjunctive and existential.

Proof
(For (CL)_1 + TR + Ext, the proof for (CL)_2 + TR is similar).

Let T = (CL)_1 + TR + Ext.

We define inductively |A for sentences A.

|M = N iff CL + tr + ext |- M = N
|A \land B iff |A and |B
|A \lor B iff (|A and |-T A) or (|B and |-T B)
|A \supset B iff (|A and |-T A) |-B
|\forall xA(x) iff for all closed N |A(N).
|\exists xA(x) iff for some closed N |A(N) and |-T A(N).

If A is a formula, then |A iff |A*, where A* is the universal closure of A.

|\Gamma |- A iff |\Delta \Gamma \supset A (for finite \Gamma).

As in § 3 we can prove \Gamma |- T A |- |\Gamma |- A.
Hence |- T A |- |A.

Now suppose |- T A \lor B, where A, B are sentences.
Then |A \lor B. Hence |- T A or |- T B. Therefore T is disjunctive.
In the same way we see that T is existential.

In contrast to the case in arithmetic, where Peano + TR is complete, (CL)_c + TR + Ext is not a complete theory:

4.9. Theorem

Let \omega = (\lambda a \cdot aa) and \Omega = \omega \omega. Then \Omega K = K is an undecidable sentence in (CL)_c + TR + Ext.

Proof
Jacopini has proved (unpublished) that CL + \omega + \Omega = \lambda a \cdot a is consistent.
Hence by 4.5 and 4.6. iii) (CL)_c + TR + Ext + \Omega K = K is consistent.
Hence not (CL)_c + TR + Ext |- \Omega K \neq K.
Suppose now that (CL)_c + TR + Ext |- \Omega K = K.
Then by 4.5. and 4.6. CL + \omega |- \Omega K = K.
Hence by Barendregt [1971], 2.2.12 CL |- \Omega KK ... K = K, where K ... K is a sequence of K's.
This contradicts the Church-Rosser property of CL.
Hence not (CL)_c + TR + Ext |- \Omega K = K. ■
Open problems

1. Is $(CL)_I + TR + (AC)(+\text{Ext})$ consistent?
2. Is $(CL)_I + TR(+\text{Ext})$ conservative over $CL(+\text{ext})$? ¹)
3. Does the disjunctive and existential property for formulas hold for $(CL)_I + TR(+\text{Ext})$?

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References

Kleene, Stephen C., Introduction to metamathematics, Amsterdam (1952).
Prawitz, Dag, Natural deduction, a proof theoretic study, Stockholm (1965).

¹) Added in print (April 11, 1973): It follows from recent work of Mr. Plotkin, that this is not the case.