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# A bounded transform approach to self-adjoint operators: Functional calculus and affiliated von Neumann algebras

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## Abstract

Spectral theory and functional calculus for unbounded self-adjoint operators on a Hilbert space are usually treated through von Neumann's Cayley transform. Based on ideas of Woronowicz, we redevelop this theory from the point of view of multiplier algebras and the so-called bounded transform (which establishes a bijective correspondence between closed operators and pure contractions). This also leads to a simple account of the affiliation relation between von Neumann algebras and self-adjoint operators.

## 1 Introductory overview

The theory of unbounded self-adjoint operators on a Hilbert space was initiated by von Neumann, partly motivated by mathematical problems of quantum mechanics [7]. The monograph by Schmüdgen [10] presents an excellent survey of the present state of the art.

Von Neumann's approach was based on the Cayley transform and in its subsequent development the notion of a spectral measure played an important role, especially in defining a functional calculus. We consider this route a bit indirect and will avoid both by firstly invoking the *bounded transform* instead of the Cayley transform, i.e., the formal expressions

$$S = T\sqrt{I+T^2}^{-1}; \tag{1.1}$$

$$T = S\sqrt{I-S^2}^{-1}, \tag{1.2}$$

make rigorous sense and provide a bijective correspondence between self-adjoint operators  $T$  and self-adjoint pure contractions  $S$  (i.e.,  $\|Sx\| < \|x\|$  for each  $x \in \mathcal{H} \setminus \{0\}$ ); cf. [3, 4, 10].

Note that the bounded transform  $T \mapsto S$  is an operatorial version of the homeomorphism  $\mathbb{R} \cong (-1, 1)$  given by the function  $b : \mathbb{R} \rightarrow (-1, 1)$  and its inverse  $u : (-1, 1) \rightarrow \mathbb{R}$ , defined by

$$b(x) = \frac{x}{\sqrt{1+x^2}}; \quad (1.3)$$

$$u(x) = \frac{x}{\sqrt{1-x^2}}. \quad (1.4)$$

Secondly, we replace spectral measures by simple arguments using multiplier algebras. Our approach is based on the work of Woronowicz [12, 13], whose functional calculus we adopt and to some extent complete, at least in the usual context of operators on a Hilbert space (Woronowicz's work was mainly intended to deal with problems involving multiplier algebras and, even more generally, with operators on Hilbert  $C^*$ -modules [5]).

If  $T$  is bounded (and, by standing assumption, self-adjoint), it is easy to prove the equality

$$C^*(T) = C^*(S), \quad (1.5)$$

where  $C^*(S)$  is the  $C^*$ -algebra generated within  $B(\mathcal{H})$  by  $S$  and the unit, etc. Furthermore, the spectral mapping theorem implies that the spectra of  $S$  and  $T$  are related by

$$\sigma(T) = \left\{ \mu(1-\mu^2)^{-\frac{1}{2}} \mid \mu \in \sigma(S) \right\}; \quad (1.6)$$

$$\sigma(S) = \left\{ \lambda(1+\lambda^2)^{-\frac{1}{2}} \mid \lambda \in \sigma(T) \right\}, \quad (1.7)$$

preserving point spectra. As to the continuous functional calculus, for  $S = S^* \in B(\mathcal{H})$  we have the familiar isomorphism  $C(\sigma(S)) \xrightarrow{\cong} C^*(S)$ , written  $g \mapsto g(S)$ , given by the spectral theorem. Assuming  $T = T^* \in B(\mathcal{H})$ , the same applies to  $T$ . These calculi are related by

$$f(T) = (f \circ u)(S), \quad (1.8)$$

where  $f \in C(\sigma(T))$ , so that  $f \circ u \in C(\sigma(S))$ . Self-adjointness is preserved, in that

$$f(T)^* = f^*(T), \quad (1.9)$$

where  $f^*(x) = \overline{f(x)}$ . In particular, if  $f$  is real-valued, then  $f(T)$  is self-adjoint. At the level of von Neumann algebras, defining  $W^*(S) = C^*(S)''$  and similarly for  $T$ , eq. (1.5) gives

$$W^*(T) = W^*(S). \quad (1.10)$$

The functional calculus  $f \mapsto f(T)$  may then be extended to bounded Borel functions  $f$  on  $\sigma(T)$ , in which case it is still given by (1.8). We then have  $f(T) \in W^*(T)$ , whilst (1.9) remains valid; however, instead of the isometric property  $\|f(T)\| = \|f\|_\infty$  for continuous  $f$ , we now have  $\|f(T)\| \leq \|f\|_\infty$  (where  $\|\cdot\|_\infty$  is the supremum-norm). See, e.g., [8].

Our aim is to generalize these results to the case where  $T$  is unbounded. This indeed turns out to be possible, so that our main results are as follows. Throughout the remainder of this paper we assume that  $T^* = T$  is possibly unbounded, with bounded transform  $S$ .

**Theorem 1.** *The (point) spectra of  $T$  and its bounded transform  $S$  are related by*

$$\sigma(T) = \left\{ \mu(1 - \mu^2)^{-\frac{1}{2}} : \mu \in \tilde{\sigma}(S) \right\}; \quad (1.11)$$

$$\sigma(S) = \left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(T) \right\}^-, \quad (1.12)$$

where  $-$  denotes the closure in  $\mathbb{R}$ , and we abbreviate

$$\tilde{\sigma}(S) = \sigma(S) \cap (-1, 1). \quad (1.13)$$

Note that  $\tilde{\sigma}(S) = \sigma(S)$  iff  $T$  is bounded (in which case  $\sigma(S)$  is a compact subset of  $(-1, 1)$ , since  $\pm 1 \in \sigma(S)$  iff  $T$  is unbounded). We define the following operator algebras within  $B(\mathcal{H})$ :

$$C_{\bullet}^*(S) = \{g(S) : g \in C_{\bullet}(\tilde{\sigma}(S))\}, \quad (1.14)$$

where  $\bullet$  is  $b$ ,  $c$ , or  $0$ , so that we have defined  $C_c^*(S)$ ,  $C_0^*(S)$ , and  $C_b^*(S)$ . Notice that  $C(\sigma(S))$  consists of all  $g \in C_b(\tilde{\sigma}(S))$  for which  $\lim_{y \rightarrow \pm 1} g(y)$  exists, where this limit is 0 if and only if  $g \in C_0(\tilde{\sigma}(S))$ . Hence we have the inclusions (of which the first set implies the second)

$$C_c(\tilde{\sigma}(S)) \subseteq C_0(\tilde{\sigma}(S)) \subseteq C(\sigma(S)) \subseteq C_b(\tilde{\sigma}(S)); \quad (1.15)$$

$$C_c^*(S) \subseteq C_0^*(S) \subseteq C^*(S) \subseteq C_b^*(S), \quad (1.16)$$

with equalities iff  $T$  is bounded. This means that  $g(S)$  is defined for  $g \in C_0(\tilde{\sigma}(S))$ , and hence *a fortiori* also for  $g \in C_c(\tilde{\sigma}(S))$ . Consequently,  $f(T)$  may be defined by (1.8) whenever  $f \in C_0(\sigma(T))$ , including  $f \in C_c(\sigma(T))$ . To pass to the larger class  $f \in C_b(\sigma(T))$ , we define  $C_0^*(S)\mathcal{H}$  as the linear span of all vectors of the form  $g(S)\psi$ , where  $g \in C_0(\tilde{\sigma}(S))$  and  $\psi \in \mathcal{H}$ . Then  $C_0^*(S)\mathcal{H}$  is dense in  $\mathcal{H}$  (Lemma 1). In the spirit of Woronowicz [5, 12], we then initially define  $f(T)$  for  $f \in C_b(\sigma(T))$  on the domain  $C_0^*(S)\mathcal{H}$  by linear extension of the formula

$$f_0(T)h(T)\psi = (fh)(T)\psi, \quad (1.17)$$

where  $h \in C_0(\sigma(T))$  and hence also  $fh \in C_0(\sigma(T))$ , since  $C_b(\sigma(T))$  is the multiplier algebra of  $C_0(\sigma(T))$ . Then  $f_0(T)$  is bounded (Lemma 2), and we define  $f(T)$  as its closure, i.e.,

$$f(T) = f_0(T)^-. \quad (1.18)$$

This also works for  $f \in C(\sigma(T))$ , in which case  $f_0(T)$  may no longer be bounded, but remains closable (Lemma 3), so that we may once again define  $f(T)$  as its closure, cf. (1.18). We have:

**Theorem 2.** *If  $f \in C(\sigma(T))$  is real-valued, then  $f(T)$  is self-adjoint, i.e.,  $f_0(T)^- = f_0(T)^*$ ; more generally,  $f(T)^* = f^*(T)$ . Furthermore, the continuous functional calculus  $f \mapsto f(T)$  restricts to an isometric  $*$ -homomorphism from  $C_0(\sigma(T))$  (with supremum-norm) to  $C^*(S)$ .*

See also Theorem 4 . In addition, the map  $f \mapsto f(T)$  has the reassuring special cases

$$\mathbf{1}_{\sigma(T)}(T) = I; \quad (1.19)$$

$$\text{id}(T) = T; \quad (1.20)$$

$$(\text{id} - z)^{-1}(T) = (T - z)^{-1}, \quad z \in \rho(T), \quad (1.21)$$

where  $\mathbf{1}_{\sigma(T)}(x) = 1$  and  $\text{id}(x) = x$  ( $x \in \sigma(T)$ ), and therefore does what it is supposed to to.

Finding the right analogue of (1.10) for unbounded  $T = T^*$  first requires a redefinition of  $W^*(T)$ , which is standard [8]. If  $T$  is unbounded and  $R \in B(\mathcal{H})$ , then we say that  $R$  and  $T$  commute, written  $TR \subset RT$ , if  $R\psi \in \mathcal{D}(T)$  and  $RT\psi = TR\psi$  for any  $\psi \in \mathcal{D}(T)$ . Let  $\{T\}'$  be the set of all bounded operators that commute with  $T$ . If  $T^* = T$ , then  $\{T\}'$  is a unital, strongly closed  $*$ -subalgebra of  $B(\mathcal{H})$ , and hence a von Neumann algebra [8]. Its commutant

$$W^*(T) = \{T\}'' , \quad (1.22)$$

is a von Neumann algebra, too. If  $T$  is bounded, then  $W^*(T)$  is the von Neumann algebra generated by  $T$ , which coincides with  $C^*(T)''$ . As usual, we call a closed unbounded operator  $X$  affiliated to a von Neumann algebra  $A \subset B(H)$ , written  $X\eta A$ , iff  $XR \subset RX$  for each  $R \in A'$ . For example, if  $T^* = T$ , then  $T\eta W^*(T)$ , and if  $T\eta A$ , then  $W^*(T) \subseteq A$ ; in other words,  $W^*(T)$  is the smallest von Neumann algebra such that  $T$  is affiliated to it.

As a result of independent interest as well as a lemma for Theorem 4, we may then adapt [8, Lemma 5.2.8] to the bounded transform:

**Theorem 3.** *Let  $A \subset B(H)$  be a von Neumann algebra. Then  $T\eta A$  iff  $S \in A$ .*

Denoting the (Banach) space of (bounded) Borel functions on  $\sigma(T)$  (equipped with the supremum-norm) by  $\mathcal{B}_{(b)}(\sigma(T))$ , we may still define  $f(T)$  by (1.8) and the usual Borel functional calculus for the bounded transform  $S$ .

**Theorem 4.** *The map  $f \mapsto f(T)$  is a norm-decreasing  $*$ -homomorphism from  $\mathcal{B}_b(\sigma(T))$  to*

$$W^*(T) = W^*(S). \quad (1.23)$$

*More generally, if  $f \in \mathcal{B}(\sigma(T))$ , then  $f(T)$  is affiliated with  $W^*(T)$ .*

The remainder of this paper simply consists of the proofs of these theorems.

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## 2 Proofs

This section contains all proofs. We will not repeat the theorems.

### 2.1 Proof of Theorem 1

The operator  $\sqrt{1-S^2}$  is a bijection from  $\mathcal{H}$  to  $\mathcal{R}(\sqrt{1-S^2}) = \mathcal{D}(T)$ . Let  $\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T)$ , so that  $T - \lambda I$  is a bijection from  $\mathcal{D}(T)$  to  $\mathcal{H}$ . Thus by composition we have a bijection  $\mathcal{H} \rightarrow \mathcal{H}$ ; equivalently,  $(T - \lambda I)(\sqrt{1-S^2})$  is invertible, which in turn is equivalent to invertibility of  $S - \lambda\sqrt{1-S^2}$ . Thus  $\lambda \in \rho(T) \iff S - \lambda\sqrt{1-S^2}$  is a bijection, or, expressed contrapositively,  $\lambda \in \sigma(T) \iff S - \lambda\sqrt{1-S^2}$  is not invertible in  $B(\mathcal{H})$ . This is the case iff  $S - \lambda\sqrt{1-S^2}$  is not invertible in  $C^*(S)$ , which, using the Gelfand isomorphism  $C^*(S) \cong C(\sigma(S))$ , in turn is true iff the function  $k_\lambda(x) = x - \lambda\sqrt{1-x^2}$  is not invertible in  $C(\sigma(S))$ , i.e., iff  $0 \in \sigma(k_\lambda)$ . Since in  $C(X)$  we have  $\sigma(f) = \mathcal{R}(f)$  (with  $X$  a compact Hausdorff space), and  $\sigma(S)$  is indeed compact and Hausdorff because  $S$  is bounded, we obtain  $\lambda \in \sigma(T)$  iff  $0 \in \mathcal{R}(k_\lambda)$ . If  $\pm 1$  lie in  $\sigma(S)$  they cannot give rise to this possibility, since  $k_\lambda(\pm 1) = \pm 1$  for each  $\lambda$ . Hence we have  $0 \in \mathcal{R}(k_\lambda)$  iff  $\lambda = \mu(1 - \mu^2)^{-\frac{1}{2}}$  for some  $\mu \in \sigma(S) \cap (-1, 1)$ , which yields (1.11).

The same argument shows that  $\mu \in \sigma(S) \cap (-1, 1)$  comes from  $\lambda \in \sigma(T)$ . But since  $\sigma(S)$  is compact and hence closed in  $[-1, 1]$  we obtain (1.12).  $\square$

### 2.2 Proof of Theorem 2

This proof relies on three lemma's.

**Lemma 1.** *Let  $C_c^*(S)\mathcal{H}$  be the linear span of all vectors of the form  $g(S)\psi$ , where  $g \in C_c(\tilde{\sigma}(S))$  and  $\psi \in \mathcal{H}$ . Then  $C_c^*(S)\mathcal{H}$  is dense in  $H$ .*

*Proof.* Define  $g_n : (-1, 1) \rightarrow [0, 1]$  by putting  $g_n(x) = 0$  for  $x \in \left(-1, \frac{1}{n} - 1\right] \cup \left[1 - \frac{1}{n}, 1\right)$ ,  $g_n(x) = 1$  if  $x \in \left[\frac{2}{n} - 1, 1 - \frac{2}{n}\right]$ , and linear interpolation in between. The ensuing sequence converges pointwise to the unit  $\mathbf{1}$  on  $(-1, 1)$ . Restricting each  $g_n$  to  $\tilde{\sigma}(S)$ , the continuous functional calculus gives  $g_n(S) \rightarrow \mathbf{1}_{\tilde{\sigma}(S)}$  strongly. Therefore, for any  $\psi \in \mathcal{H}$  we have a sequence  $\psi_n = g_n(S)\psi$  in  $C_c^*(S)\mathcal{H}$  such that  $\psi_n \rightarrow \psi$ .  $\square$

**Lemma 2.** *For  $f \in C_b(\sigma(T))$ , define an operator  $f_0(T)$  on the domain  $C_0^*(S)\mathcal{H}$  by (1.17). Then  $f_0(T)$  is bounded, with bound*

$$\|f(T)\| \leq \|f\|_\infty. \quad (2.24)$$

*Proof.* Let  $\varepsilon > 0$ . If  $h \in C_0(\sigma(T))$ , then  $fh \in C_0(\sigma(T))$ , so that we can find a compact subset  $K \subset \sigma(T)$  such that  $|h(x)f(x)| < \varepsilon$  for each  $x \notin K$ . Let  $\tilde{h} = h \circ u$ , cf. (1.4); then  $\tilde{h} \in C_0(\tilde{\sigma}(S))$  whenever  $h \in C_0(\sigma(T))$ ; in fact, we have an isometric isomorphism

$$C_0(\sigma(T)) \xrightarrow{\cong} C_0(\tilde{\sigma}(S)), \quad h \mapsto h \circ u. \quad (2.25)$$

Contractivity of the Borel functional calculus for bounded operators on  $\mathcal{H}$  gives

$$\|(\widetilde{\mathbf{1}_{K^c}fh})(S)\psi\| \leq \|(\widetilde{\mathbf{1}_{K^c}fh})(S)\|\|\psi\| \leq \|\widetilde{\mathbf{1}_{K^c}fh}\|_\infty\|\psi\| < \varepsilon\|\psi\|.$$

Using also the homomorphism property of the Borel functional calculus, we then find

$$\begin{aligned} \|(fh)(T)\psi\| &= \|(\widetilde{fh})(S)\psi\| \\ &= \|(\widetilde{\mathbf{1}_Kfh})(S) + (\widetilde{fh} - \widetilde{\mathbf{1}_Kfh})(S)\psi\| \\ &\leq \|(\widetilde{\mathbf{1}_Kfh})(S)\psi\| + \|(\widetilde{\mathbf{1}_{K^c}fh})(S)\psi\| \\ &= \|(\widetilde{\mathbf{1}_Kf})(S)\widetilde{h}(S)\psi\| + \|(\widetilde{\mathbf{1}_{K^c}fh})(S)\psi\| \\ &< \|(\widetilde{\mathbf{1}_Kf})\|_\infty\|h(T)\psi\| + \varepsilon\|\psi\|, \\ &\leq \|f\|_\infty\|h(T)\psi\| + \varepsilon\|\psi\|, \end{aligned}$$

since  $\|(\widetilde{\mathbf{1}_Kf})\|_\infty \leq \|\widetilde{f}\|_\infty = \|f\|_\infty$ . Since the last expression above is independent of  $K$ , we may let  $\varepsilon \rightarrow 0$ , obtaining boundedness of  $f(T)$  as well as (2.24).  $\square$

The last claim in Theorem 2 now follows from the continuous functional calculus for  $S$  and the isometric isomorphism (2.25). Although isometry may be lost if we go from  $C_0(\sigma(T))$  to  $C_b(\sigma(T))$ , it easily follows from (1.17) - (1.18) that the map  $f \mapsto f(T)$  at least defines a \*-homomorphism  $C_b(\sigma(T)) \rightarrow B(H)$ . This property will be used after Lemma 4 below.

**Lemma 3.** *For  $f \in C(\sigma(T))$ , define an operator  $f_0(T)$  on the domain  $C_c^*(S)\mathcal{H}$  by (1.17). Then  $f_0(T)$  is closable. Moreover, if  $f$  is real-valued ( $f^* = f$ ), then  $f_0(T)$  is symmetric.*

*Proof.* Suppose that  $h_1(T)\psi_1$  and  $h_2(T)\psi_2$  lie in  $\mathcal{D}(f_0(T))$ . Then we may compute:

$$\langle h_2(T)\psi_2, f_0(T)h_1(T)\psi_1 \rangle = \langle \psi_2, \overline{h_2}(T)(fh_1)(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2}fh_1)(T)\psi_1 \rangle; \quad (2.26)$$

$$\langle (h_2\overline{f})(T)\psi_2, h_1(T)\psi_1 \rangle = \langle \psi_2, \overline{(h_2\overline{f})}h_1(T)\psi_1 \rangle = \langle \psi_2, (\overline{h_2}fh_1)(T)\psi_1 \rangle. \quad (2.27)$$

This implies that  $\mathcal{D}(f_0(T)) \subseteq \mathcal{D}(f_0(T)^*)$ . Since  $\mathcal{D}(f_0(T))$  is dense, so is  $\mathcal{D}(f_0(T)^*)$ , which implies that  $f_0(T)$  is closable. The second claim is obvious from (2.26) - (2.27).  $\square$

*Proof.* To prove Theorem 2 we use a well-known result of Nelson [6]; see also [9] (this step was suggested to us by Nigel Higson). For convenience we recall this result (without proof):

**Lemma 4.** *Let  $\{U(t)\}_{t \in \mathbb{R}}$  be a strongly continuous unitary group of operators on a Hilbert space  $\mathcal{H}$ . Let  $R : \mathcal{D}(R) \rightarrow \mathcal{H}$  be densely defined and symmetric. Assume that  $\mathcal{D}(R)$  is invariant under  $\{U(t)\}_{t \in \mathbb{R}}$ , i.e.  $U(t) : \mathcal{D}(R) \rightarrow \mathcal{D}(R)$  for each  $t$ , and also that  $\{U(t)\}_{t \in \mathbb{R}}$  is strongly differentiable on  $\mathcal{D}(R)$ . Then  $-idU(t)/dt$  is essentially self-adjoint on  $\mathcal{D}(R)$  and its closure is the self-adjoint generator of  $\{U(t)\}_{t \in \mathbb{R}}$  (given by Stone's Theorem). In particular, if  $(dU(t)/dt)\psi = iRU(t)\psi$  for each  $\psi \in \mathcal{D}(R)$ , then  $R$  is essentially self-adjoint.*

Set  $R = f_0(T)$  for  $f \in C(\sigma(T))$ , so that

$$\mathcal{D}(R) = C_c^*(S)\mathcal{H}, \quad (2.28)$$

and for each  $t \in \mathbb{R}$  define  $U(t)$  via the (bounded) function  $x \mapsto \exp(itf(x))$  on  $\sigma(T)$ , that is, for  $h \in C_c(\sigma(T))$  and  $\psi \in \mathcal{H}$ , we initially define

$$U_0(t)h(T)\psi = (e^{itf}h)(T)\psi. \quad (2.29)$$

Then  $U_0$  bounded by Lemma 2, and we define  $U(t)$  as the closure of  $U_0(t)$ . The remark before Lemma 3 then implies that  $t \mapsto U(t)$  defines a unitary representation of  $\mathbb{R}$  on  $\mathcal{H}$ . Strong continuity of this representation follows from an  $\varepsilon/3$  argument. First, for

$$\varphi = h(T)\psi, \quad (2.30)$$

assuming  $\|\psi\| = 1$  for simplicity, eqs. (2.29) and (2.24) give

$$\|U(t)\varphi - \varphi\| \leq \|e^{itf}h - h\|_\infty \leq \|h\|_\infty \|e^{itf} - \mathbf{1}\|_\infty^{(K)}, \quad (2.31)$$

where  $K$  is the (compact) support of  $h$  in  $\sigma(T)$ . Since the exponential function is uniformly convergent on any compact set, this gives  $\lim_{t \rightarrow 0} \|U(t)\varphi - \varphi\| = 0$  for  $\varphi$  of the form (2.30); taking finite linear combinations thereof gives the same result for any  $\varphi \in C_c^*(S)\mathcal{H}$ . Thus for any  $\varepsilon > 0$  we can find  $\delta > 0$  so that  $\|U(t)\varphi - \varphi\| < \varepsilon/3$  whenever  $|t| < \delta$ . For general  $\psi' \in \mathcal{H}$ , we find  $\varphi \in C_c^*(S)\mathcal{H}$  such that  $\|\varphi - \psi'\| < \varepsilon/3$ , and estimate

$$\begin{aligned} \|U(t)\psi' - \psi'\| &\leq \|U(t)\psi' - U(t)\varphi\| + \|U(t)\varphi - \varphi\| + \|\varphi - \psi'\| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

since  $\|U(t)\psi' - U(t)\varphi\| = \|\psi' - \varphi\|$  by unitarity of  $U(t)$ . Thus  $\lim_{t \rightarrow 0} \|U(t)\psi - \psi\| = 0$  for any  $\psi \in \mathcal{H}$ , so that the unitary representation  $t \mapsto U(t)$  is strongly continuous. Similarly,

$$\left\| \frac{U(t+s)\varphi - U(t)\varphi}{s} - iRU(t)\varphi \right\| \leq \left\| \frac{e^{isf}h - h}{s} - ifh \right\|_\infty, \quad (2.32)$$

assuming (2.30), so that by the same argument as in (2.31) we obtain

$$\frac{dU(t)}{dt}\varphi = iRU(t)\varphi, \quad (2.33)$$

initially for any  $\varphi$  of the form (2.30), and hence, taking finite sums, for any  $\varphi \in \mathcal{D}(R)$ , cf. (2.28). The final part of Lemma 4 then shows that  $f_0(T)$  is essentially self-adjoint on its domain  $C_c^*(S)\mathcal{H}$ . Its closure  $f(T)$  is therefore self-adjoint, and Theorem 2 is proved.  $\square$



We now prove the examples (1.19) - (1.21), of which the first is trivial. Writing  $T_0$  for the operator  $\text{id}_0(T)$ , the definition (1.17) gives

$$T_0\varphi = T\varphi$$

for  $\varphi \in \mathcal{D}(T_0) = C_c^*(S)\mathcal{H}$ . Let  $\psi \in \mathcal{D}(T_0^-)$ , so that there is a sequence  $(\varphi_n)$  in  $\mathcal{D}(T_0)$  such that  $\varphi_n \rightarrow \psi$  and  $(T_0\varphi_n)$  converges. Since  $T$  is closed, it follows that  $T_0\varphi_n = T\varphi_n \rightarrow T\psi$ , so that  $\psi \in \mathcal{D}(T)$ . Hence  $T_0^- \subset T$ . Since both operators are self-adjoint, this implies  $T_0^- = T$ , which proves (1.20).

The proof of (1.21) is easier since  $(T - z)^{-1}$  is bounded: writing

$$f(x) = (x - z)^{-1},$$

where  $z \notin \sigma(T)$  is fixed and  $x \in \sigma(T)$ , we have

$$f_0(T)h(T)\psi = (fh)(T)\psi = (T - z)^{-1}h(T)\psi,$$

and hence

$$f_0(T)\varphi = (T - z)^{-1}\varphi$$

for any  $\varphi \in \mathcal{D}(f_0(T)) = C_c^*(S)\mathcal{H}$ . So if  $\varphi_n \rightarrow \varphi$  for  $\varphi \in \mathcal{H}$  and  $\varphi_n \in \mathcal{D}(f_0(T))$ , boundedness and hence continuity of the resolvent implies

$$f(T)\varphi = \lim_{n \rightarrow \infty} f_0(T)\varphi_n = \lim_{n \rightarrow \infty} (T - z)^{-1}\varphi_n = (T - z)^{-1}\varphi.$$

### 2.3 Proof of Theorem 3

The first step consists in the observation that  $T\eta A$  iff  $TU \subset UT$  (or, equivalently,  $UTU^* = T$ ) merely for each unitary  $U \in A'$ , which is well known [11].

The second step is to show that  $TU \subset UT$  iff  $SU = US$  for any unitary  $U$ . This is a simple computation. First suppose that  $UTU^* = T$ . Then:

$$\begin{aligned} U(1 + T^2)^{-1}U^* &= (U(1 + T^2)U^*)^{-1} = ((U + UT^2)U^*)^{-1} \\ &= (UU^* + UT^2U^*)^{-1} = (1 + UTU^*UTU^*)^{-1} \\ &= (1 + T^2)^{-1}. \end{aligned}$$

If  $R$  is bounded and positive, then  $UR = RU$  iff  $U \in C^*(R)'$ , and since  $\sqrt{R} \in C^*(R)$  by the continuous functional calculus, we also have  $U\sqrt{R} = \sqrt{R}U$ . Consequently,

$$USU^* = U \left( T\sqrt{(1 + T^2)^{-1}} \right) U^* = (UTU^*) \left( U\sqrt{(1 + T^2)^{-1}}U^* \right) = T\sqrt{(1 + T^2)^{-1}} = S.$$

Similarly, if  $SU = US$ , then

$$UTU^* = US\sqrt{1-S^2}^{-1}U^* = SU\sqrt{1-S^2}^{-1}U^* = S\left(U\sqrt{1-S^2}U^*\right)^{-1} = S\sqrt{1-S^2}^{-1} = T.$$

Thirdly, as in the first step,  $SU = US$  for any unitary  $U \in A'$  iff  $S \in A'' = A$ .  $\square$

## 2.4 Proof of Theorem 4

Eq. (1.23) in Theorem 4 follows from Theorem 3: taking  $A = W^*(T)$ , so that  $T\eta A$ , yields  $S \in W^*(T)$ , and hence  $W^*(S) \subseteq W^*(T)$ . On the other hand, taking  $A = W^*(S)$ , in which case  $S \in A$ , gives  $T\eta W^*(S)$ , and hence  $W^*(T) \subseteq W^*(S)$ .

Similar to (2.25), we have an isometric isomorphism

$$\mathcal{B}_b(\sigma(T)) \xrightarrow{\cong} \mathcal{B}_b(\tilde{\sigma}(S)), \quad h \mapsto h \circ u, \quad (2.34)$$

so that the first claim of Theorem 4 follows from the Borel functional calculus for the bounded operator  $S$  [8]. The proof of the last one is, *mutatis mutandis*, practically the same as in [8, Theorem 5.3.8], so we omit the details; see [2].  $\square$

As explained in [8, §5.3], there exists a Borel measure  $\mu$  on  $\sigma(T)$  such that the map  $f \mapsto f(T)$  may also be seen as a so-called essential \*-homomorphism from  $\mathcal{B}(\sigma(T))/\mathcal{N}(\sigma(T))$  into the \*-algebra of normal operators affiliated with  $W^*(T)$ , where  $\mathcal{N}(\sigma(T))$  is the set of  $\mu$ -null functions on  $\sigma(T)$ . This remains true in our approach, with the same proof [2].

## 3 Epilogue

Let us finally note that although this paper was inspired by the work of Woronowicz, the  $C^*$ -algebraic affiliation relation he defines in [12] (as did, independently, also Baaq and Julg [1]) has not been used here. If we call his relation  $\eta'$  to avoid confusion with the  $W^*$ -algebraic relation  $\eta$  we do use, if  $A \subset B(\mathcal{H})$  we have  $T\eta' A \Rightarrow T \in A$  (and hence  $T$  is bounded), cf. [12, Prop. 1.3]. Woronowicz does not define a  $C^*$ -algebraic counterpart of the von Neumann algebra  $W^*(T)$ , but it might be reasonable to define  $C^*(T)$  as the smallest  $C^*$ -algebra  $A$  in  $B(\mathcal{H})$  such that  $T\eta' A$ . It follows from [12, Example 4] that this would give  $C^*(T) = C_0^*(S)$ , as defined in (1.14). This  $C^*$ -algebra contains  $S$  (and hence  $T$ ) if and only if  $T$  is bounded, in which case  $C_0^*(S) = C^*(S)$  and hence  $C^*(T) = C^*(S)$ , as in our approach, cf. (1.5). Also in general (i.e., if  $T$  is possibly unbounded), the bicommutant  $C^*(T)''$  coincides with  $W^*(T)$  as defined in the usual way (1.22) this follows from  $C_0^*(S)'' = C^*(S)'' = W^*(S)$  and (1.10).

Of course, we could also redefine  $\eta'$ , now calling it  $\eta''$ , by stipulating that  $T\eta'' A$  whenever  $S \in A$ , and redefine  $C^*(T)$  accordingly (i.e., as the smallest  $C^*$ -algebra  $A$  in  $B(\mathcal{H})$  such that  $T\eta'' A$ ). This would give (1.5) even if  $T$  is unbounded, though in a somewhat empty way.

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