A bounded transform approach to self-adjoint operators: Functional calculus and affiliated von Neumann algebras

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Abstract

Spectral theory and functional calculus for unbounded self-adjoint operators on a Hilbert space are usually treated through von Neumann’s Cayley transform. Based on ideas of Woronowicz, we redevelop this theory from the point of view of multiplier algebras and the so-called bounded transform (which establishes a bijective correspondence between closed operators and pure contractions). This also leads to a simple account of the affiliation relation between von Neumann algebras and self-adjoint operators.

1 Introductory overview

The theory of unbounded self-adjoint operators on a Hilbert space was initiated by von Neumann, partly motivated by mathematical problems of quantum mechanics [7]. The monograph by Schmüdgen [10] presents an excellent survey of the present state of the art.

Von Neumann’s approach was based on the Cayley transform and in its subsequent development the notion of a spectral measure played an important role, especially in defining a functional calculus. We consider this route a bit indirect and will avoid both by firstly invoking the bounded transform instead of the Cayley transform, i.e., the formal expressions

\[ S = T \sqrt{I + T^2}^{-1} ; \]
\[ T = S \sqrt{I - S^2}^{-1} , \]

make rigorous sense and provide a bijective correspondence between self-adjoint operators \( T \) and self-adjoint pure contractions \( S \) (i.e., \( \|Sx\| < \|x\| \) for each \( x \in \mathcal{H} \setminus \{0\} \)); cf. [3, 4, 10].
Note that the bounded transform $T \mapsto S$ is an operatorial version of the homeomorphism $\mathbb{R} \cong (-1, 1)$ given by the function $b : \mathbb{R} \to (-1, 1)$ and its inverse $u : (-1, 1) \to \mathbb{R}$, defined by
\begin{align}
b(x) &= \frac{x}{\sqrt{1 + x^2}}; \\
u(x) &= \frac{x}{\sqrt{1 - x^2}}.
\end{align}

Secondly, we replace spectral measures by simple arguments using multiplier algebras. Our approach is based on the work of Woronowicz [12, 13], whose functional calculus we adopt and to some extent complete, at least in the usual context of operators on a Hilbert space (Woronowicz’s work was mainly intended to deal with problems involving multiplier algebras and, even more generally, with operators on Hilbert $C^*$-modules [5]).

If $T$ is bounded (and, by standing assumption, self-adjoint), it is easy to prove the equality
\begin{equation}
C^*(T) = C^*(S),
\end{equation}
where $C^*(S)$ is the $C^*$-algebra generated within $B(H)$ by $S$ and the unit, etc. Furthermore, the spectral mapping theorem implies that the spectra of $S$ and $T$ are related by
\begin{align}
\sigma(T) &= \left\{ \mu(1 - \mu^2)^{-\frac{1}{2}} \mid \mu \in \sigma(S) \right\}; \\
\sigma(S) &= \left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} \mid \lambda \in \sigma(T) \right\},
\end{align}
preserving point spectra. As to the continuous functional calculus, for $S = S^* \in B(H)$ we have the familiar isomorphism $C(\sigma(S)) \cong C^*(S)$, written $g \mapsto g(S)$, given by the spectral theorem. Assuming $T = T^* \in B(H)$, the same applies to $T$. These calculi are related by
\begin{equation}
f(T) = (f \circ u)(S),
\end{equation}
where $f \in C(\sigma(T))$, so that $f \circ u \in C(\sigma(S))$. Self-adjointness is preserved, in that
\begin{equation}
f(T)^* = f^*(T),
\end{equation}
where $f^*(x) = \overline{f(x)}$. In particular, if $f$ is real-valued, then $f(T)$ is self-adjoint. At the level of von Neumann algebras, defining $W^*(S) = C^*(S)''$ and similarly for $T$, eq. (1.5) gives
\begin{equation}
\end{equation}

The functional calculus $f \mapsto f(T)$ may then be extended to bounded Borel functions $f$ on $\sigma(T)$, in which case it is still given by (1.8). We then have $f(T) \in W^*(T)$, whilst (1.9) remains valid; however, instead of the isometric property $\|f(T)\| = \|f\|_\infty$ for continuous $f$, we now have $\|f(T)\| \leq \|f\|_\infty$ (where $\| \cdot \|_\infty$ is the supremum-norm). See, e.g., [8].
Our aim is to generalize these results to the case where $T$ is unbounded. This indeed turns out to be possible, so that our main results are as follows. Throughout the remainder of this paper we assume that $T^* = T$ is possibly unbounded, with bounded transform $S$.

**Theorem 1.** The (point) spectra of $T$ and its bounded transform $S$ are related by

$$
s(T) = \left\{ \mu(1 - \mu^2)^{-\frac{1}{2}} : \mu \in \delta(T) \right\}; \quad (1.11)
$$

$$
s(S) = \left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(T) \right\}^-, \quad (1.12)
$$

where $^-$ denotes the closure in $\mathbb{R}$, and we abbreviate

$$
\delta(S) = \sigma(S) \cap (-1, 1). \quad (1.13)
$$

Note that $\delta(S) = \sigma(S)$ iff $T$ is bounded (in which case $\sigma(S)$ is a compact subset of $(-1, 1)$, since $\pm 1 \in \sigma(S)$ iff $T$ is unbounded). We define the following operator algebras within $B(\mathcal{H})$:

$$
C^*_b(S) = \left\{ g(S) : g \in C^*(\delta(S)) \right\}, \quad (1.14)
$$

where $\bullet$ is $b$, $c$, or $0$, so that we have defined $C^*_c(S)$, $C^*_b(S)$, and $C^*_0(S)$. Notice that $C(\sigma(S))$ consists of all $g \in C_b(\delta(S))$ for which \(\lim_{y \to \pm 1} g(y)\) exists, where this limit is 0 if and only if $g \in C_0(\delta(S))$. Hence we have the inclusions (of which the first set implies the second)

$$
C_c(\delta(S)) \subseteq C_0(\delta(S)) \subseteq C(\sigma(S)) \subseteq C_b(\delta(S)); \quad (1.15)
$$

$$
C^*_c(S) \subseteq C^*_0(S) \subseteq C^*(S) \subseteq C^*_b(S), \quad (1.16)
$$

with equalities iff $T$ is bounded. This means that $g(S)$ is defined for $g \in C_0(\delta(S))$, and hence * a fortiori also for $g \in C_c(\delta(S))$. Consequently, $f(T)$ may be defined by (1.8) whenever $f \in C_0(\sigma(T))$, including $f \in C_c(\sigma(T))$. To pass to the larger class $f \in C_b(\sigma(T))$, we define $C^*_0(S)\mathcal{H}$ as the linear span of all vectors of the form $g(S)\psi$, where $g \in C_0(\delta(S))$ and $\psi \in \mathcal{H}$. Then $C^*_0(S)\mathcal{H}$ is dense in $\mathcal{H}$ (Lemma 1). In the spirit of Woronowicz [5, 12], we then initially define $f(T)$ for $f \in C_b(\sigma(T))$ on the domain $C^*_0(S)\mathcal{H}$ by linear extension of the formula

$$
f_0(T)h(T)\psi = (fh)(T)\psi, \quad (1.17)
$$

where $h \in C_0(\sigma(T))$ and hence also $fh \in C_0(\sigma(T))$, since $C_b(\sigma(T))$ is the mutliplier algebra of $C_0(\sigma(T))$. Then $f_0(T)$ is bounded (Lemma 2), and we define $f(T)$ as its closure, i.e.,

$$
f(T) = f_0(T)^-. \quad (1.18)
$$

This also works for $f \in C(\sigma(T))$, in which case $f_0(T)$ may no longer be bounded, but remains closable (Lemma 3), so that we may once again define $f(T)$ as its closure, cf. (1.18). We have:
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Theorem 2. If \( f \in C(\sigma(T)) \) is real-valued, then \( f(T) \) is self-adjoint, i.e., \( f_0(T)^- = f_0(T)^* \); more generally, \( f(T)^* = f^*(T) \). Furthermore, the continuous functional calculus \( f \mapsto f(T) \) restricts to an isometric *-homomorphism from \( C_0(\sigma(T)) \) (with supremum-norm) to \( C^*(S) \).

See also Theorem 4. In addition, the map \( f \mapsto f(T) \) has the reassuring special cases

\[
\begin{align*}
1_{\sigma(T)}(T) &= I; \\
\text{id}(T) &= T; \\
(id - z)^{-1}(T) &= (T - z)^{-1}, \quad z \in \rho(T),
\end{align*}
\]

where \( 1_{\sigma(T)}(x) = 1 \) and \( \text{id}(x) = x \) (\( x \in \sigma(T) \)), and therefore does what it is supposed to do.

Finding the right analogue of (1.10) for unbounded \( T = T^* \) first requires a redefinition of \( W^*(T) \), which is standard [8]. If \( T \) is unbounded and \( R \in B(H) \), then we say that \( R \) and \( T \) commute, written \( TR \subset RT \), if \( R\psi \in D(T) \) and \( RT\psi = TR\psi \) for any \( \psi \in D(T) \). Let \( \{T\}' \) be the set of all bounded operators that commute with \( T \). If \( T^* = T \), then \( \{T\}' \) is a unital, strongly closed *-subalgebra of \( B(H) \), and hence a von Neumann algebra [8]. Its commutant

\[ W^*(T) = \{T\}'' \]

is a von Neumann algebra, too. If \( T \) is bounded, then \( W^*(T) \) is the von Neumann algebra generated by \( T \), which coincides with \( C^*(T)^'' \). As usual, we call a closed unbounded operator \( X \) affiliated to a von Neumann algebra \( A \subset B(H) \), written \( X \in A \)'', iff \( XR \subset RX \) for each \( R \in A' \). For example, if \( T^* = T \), then \( T\eta W^*(T) \), and if \( T \in A \), then \( W^*(T) \subseteq A \); in other words, \( W^*(T) \) is the smallest von Neumann algebra such that \( T \) is affiliated to it.

As a result of independent interest as well as a lemma for Theorem 4, we may then adapt [8, Lemma 5.2.8] to the bounded transform:

Theorem 3. Let \( A \subset B(H) \) be a von Neumann algebra. Then \( T \in A \) iff \( S \in A \).

Denoting the (Banach) space of (bounded) Borel functions on \( \sigma(T) \) (equipped with the supremum-norm) by \( B_b(\sigma(T)) \), we may still define \( f(T) \) by (1.8) and the usual Borel functional calculus for the bounded transform \( S \).

Theorem 4. The map \( f \mapsto f(T) \) is a norm-decreasing *-homomorphism from \( B_b(\sigma(T)) \) to

\[ W^*(T) = W^*(S). \]

More generally, if \( f \in B(\sigma(T)) \), then \( f(T) \) is affiliated with \( W^*(T) \).

The remainder of this paper simply consists of the proofs of these theorems.

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2 Proofs

This section contains all proofs. We will not repeat the theorems.

2.1 Proof of Theorem 1

The operator $\sqrt{1-S^2}$ is a bijection from $\mathcal{H}$ to $\mathcal{R}(\sqrt{1-S^2}) = \mathcal{D}(T)$. Let $\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T)$, so that $T - \lambda I$ is a bijection from $\mathcal{D}(T)$ to $\mathcal{H}$. Thus by composition we have a bijection $\mathcal{H} \to \mathcal{H}$; equivalently, $(T - \lambda I)(\sqrt{1-S^2})$ is invertible, which in turn is equivalent to invertibility of $S - \lambda \sqrt{1-S^2}$. Thus $\lambda \in \rho(T) \iff S - \lambda \sqrt{1-S^2}$ is a bijection, or, expressed contrapositively, $\lambda \in \sigma(T) \iff S - \lambda \sqrt{1-S^2}$ is not invertible in $B(\mathcal{H})$. This is the case iff $S - \lambda \sqrt{1-S^2}$ is not invertible in $C^*(S)$, which, using the Gelfand isomorphism $C^*(S) \cong C(\sigma(S))$, in turn is true iff the function $k_\lambda(x) = x - \lambda \sqrt{1-x^2}$ is not invertible in $C(\sigma(S))$, i.e., iff $0 \in \sigma(k_\lambda)$. Since in $C(X)$ we have $\sigma(f) = \mathcal{R}(f)$ (with $X$ a compact Hausdorff space), and $\sigma(S)$ is indeed compact and Hausdorff because $S$ is bounded, we obtain $\lambda \in \sigma(T)$ iff $0 \in \mathcal{R}(k_\lambda)$. If $\pm 1$ lie in $\sigma(S)$ they cannot give rise to this possibility, since $k_\lambda(\pm 1) = \pm 1$ for each $\lambda$. Hence we have $0 \in \mathcal{R}(k_\lambda)$ iff $\lambda = \mu(1 - \mu^2)^{-\frac{1}{2}}$ for some $\mu \in \sigma(S) \cap (-1, 1)$, which yields (1.11).

The same argument shows that $\mu \in \sigma(S) \cap (-1, 1)$ comes from $\lambda \in \sigma(T)$. But since $\sigma(S)$ is compact and hence closed in $[-1, 1]$ we obtain (1.12). \hfill \square

2.2 Proof of Theorem 2

This proof relies on three lemma’s.

Lemma 1. Let $C_c^*(S)\mathcal{H}$ be the linear span of all vectors of the form $g(S)\psi$, where $g \in C_c(\hat{\sigma}(S))$ and $\psi \in \mathcal{H}$. Then $C_c^*(S)\mathcal{H}$ is dense in $\mathcal{H}$.

Proof. Define $g_n : (-1, 1) \to [0, 1]$ by putting $g_n(x) = 0$ for $x \in (-1, \frac{1}{n} - 1] \cup [1 - \frac{1}{n}, 1)$, $g_n(x) = 1$ if $x \in \left[\frac{2}{n} - 1, 1 - \frac{2}{n}\right]$, and linear interpolation in between. The ensuing sequence converges pointwise to the unit $1$ on $(-1, 1)$. Restricting each $g_n$ to $\hat{\sigma}(S)$, the continuous functional calculus gives $g_n(S) \to 1_{\hat{\sigma}(S)}$ strongly. Therefore, for any $\psi \in \mathcal{H}$ we have a sequence $\psi_n = g_n(S)\psi$ in $C_c^*(S)\mathcal{H}$ such that $\psi_n \to \psi$. \hfill \square

Lemma 2. For $f \in C_0(\sigma(T))$, define an operator $f_0(T)$ on the domain $C_0^*(S)\mathcal{H}$ by (1.17). Then $f_0(T)$ is bounded, with bound

$$\|f(T)\| \leq \|f\|_\infty.$$  \hfill (2.24)

Proof. Let $\varepsilon > 0$. If $h \in C_0(\sigma(T))$, then $fh \in C_0(\sigma(T))$, so that we can find a compact subset $K \subset \sigma(T)$ such that $|h(x)f(x)| < \varepsilon$ for each $x \notin K$. Let $\tilde{h} = h \circ u$, cf. (1.4); then $\tilde{h} \in C_0(\hat{\sigma}(S))$ whenever $h \in C_0(\sigma(T))$; in fact, we have an isometric isomorphism

$$C_0(\sigma(T)) \overset{\cong}{\to} C_0(\hat{\sigma}(S)), \quad h \mapsto h \circ u.$$  \hfill (2.25)
Contractivity of the Borel functional calculus for bounded operators on $\mathcal{H}$ gives
\[
\|\overline{(1_K-\hat{f}h)}(S)\psi\| \leq \|\overline{(1_K-\hat{f}h)}(S)\|\|\psi\| \leq \|1_K-\hat{f}h\|_\infty \|\psi\| < \varepsilon \|\psi\|.
\]

Using also the homomorphism property of the Borel functional calculus, we then find
\[
\|\overline{(f\hat{h})}(S)\psi\| = \|\overline{(1_Kf\hat{h})}(S)\|\|\psi\| \\
\leq \|\overline{(1_Kf\hat{h})}(S)\| + \|\overline{(1_K-\hat{f}h)}(S)\| \\
= \|\overline{(1_Kf\hat{h})}(S)\| + \|\overline{(1_K-\hat{f}h)}(S)\| \\
< \|\overline{(1_Kf)}\|_\infty \|h(T)\psi\| + \varepsilon \|\psi\|.
\]

since $\|\overline{(1_Kf)}\|_\infty \leq \|\hat{f}\|_\infty = \|f\|_\infty$. Since the last expression above is independent of $K$, we may let $\varepsilon \to 0$, obtaining boundedness of $f(T)$ as well as $\|\psi\|$. \hfill \Box

The last claim in Theorem 2 now follows from the continuous functional calculus for $S$ and the isometric isomorphism (2.24). Although isometry may be lost if we go from $C_0(\sigma(T))$ to $C_0(\sigma(T))$, it easily follows from (1.17) - (1.18) that the map $f \mapsto f(T)$ at least defines a $^*$-homomorphism $C_0(\sigma(T)) \to B(\mathcal{H})$. This property will be used after Lemma 3 below.

**Lemma 3.** For $f \in C(\sigma(T))$, define an operator $f_0(T)$ on the domain $C^*_c(S)\mathcal{H}$ by (1.17). Then $f_0(T)$ is closable. Moreover, if $f$ is real-valued ($f^* = f$), then $f_0(T)$ is symmetric.

**Proof.** Suppose that $h_1(T)\psi_1$ and $h_2(T)\psi_2$ lie in $\mathcal{D}(f_0(T))$. Then we may compute:
\[
\langle h_2(T)\psi_2, f_0(T)h_1(T)\psi_1 \rangle = \langle \psi_2, \overline{f_0(T)}(fh_1)(T)\psi_1 \rangle = \langle \psi_2, \overline{(fh_1)}(T)\psi_1 \rangle; \quad (2.26)
\]
\[
\langle h_2(T)\psi_2, h_1(T)\psi_1 \rangle = \langle \psi_2, \overline{h_2(T)}h_1(T)\psi_1 \rangle = \langle \psi_2, \overline{(h_2)}h_1(T)\psi_1 \rangle. \quad (2.27)
\]

This implies that $\mathcal{D}(f_0(T)) \subseteq \mathcal{D}(f_0(T)^*)$ Since $\mathcal{D}(f_0(T))$ is dense, so is, $\mathcal{D}(f_0(T)^*)$, which implies that $f_0(T)$ is closable. The second claim is obvious from (2.26) - (2.27). \hfill \Box

**Proof.** To prove Theorem 2 we use a well-known result of Nelson [6]; see also [9] (this step was suggested to us by Nigel Higson). For convenience we recall this result (without proof):

**Lemma 4.** Let $\{U(t)\}_{t \in \mathbb{R}}$ be a strongly continuous unitary group of operators on a Hilbert space $\mathcal{H}$. Let $R: \mathcal{D}(R) \to \mathcal{H}$ be densely defined and symmetric. Assume that $\mathcal{D}(R)$ is invariant under $\{U(t)\}_{t \in \mathbb{R}}$, i.e. $U(t): \mathcal{D}(R) \to \mathcal{D}(R)$ for each $t$, and also that $\{U(t)\}_{t \in \mathbb{R}}$ is strongly differentiable on $\mathcal{D}(R)$. Then $-idU(t)/dt$ is essentially self-adjoint on $\mathcal{D}(R)$ and its closure is the self-adjoint generator of $\{U(t)\}_{t \in \mathbb{R}}$ (given by Stone’s Theorem). In particular, if $(dU(t)/dt)\psi = iRU(t)\psi$ for each $\psi \in \mathcal{D}(R)$, then $R$ is essentially self-adjoint.
Proofs

Set \( R = f_0(T) \) for \( f \in C(\sigma(T)) \), so that
\[
\mathcal{D}(R) = C^*_c(S)\mathcal{H}, \tag{2.28}
\]
and for each \( t \in \mathbb{R} \) define \( U(t) \) via the (bounded) function \( x \mapsto \exp(itf(x)) \) on \( \sigma(T) \), that is, for \( h \in C_c(\sigma(T)) \) and \( \psi \in \mathcal{H} \), we initially define
\[
U_0(t)h(T)\psi = (e^{itf}h)(T)\psi. \tag{2.29}
\]
Then \( U_0 \) bounded by Lemma [2] and we define \( U(t) \) as the closure of \( U_0(t) \). The remark before Lemma [3] then implies that \( t \mapsto U(t) \) defines a unitary representation of \( \mathbb{R} \) on \( \mathcal{H} \). Strong continuity of this representation follows from an \( \varepsilon/3 \) argument. First, for \( \varphi = h(T)\psi \), \( \|\psi\| = 1 \) for simplicity, eqs. (2.29) and (2.24) give
\[
\|U(t)\varphi - \varphi\| \leq \|e^{itf}h - h\|_\infty \leq \|h\|_\infty \|e^{itf} - 1\|_\infty, \tag{2.31}
\]
where \( K \) is the (compact) support of \( h \) in \( \sigma(T) \). Since the exponential function is uniformly convergent on any compact set, this gives \( \lim_{t \to 0} \|U(t)\varphi - \varphi\| = 0 \) for \( \varphi \) of the form (2.30); taking finite linear combinations thereof gives the same result for any \( \varphi \in C^*_c(S)\mathcal{H} \). Thus for any \( \varepsilon > 0 \) we can find \( \delta > 0 \) so that \( \|U(t)\varphi - \varphi\| < \varepsilon/3 \) whenever \( |t| < \delta \). For general \( \psi' \in \mathcal{H} \), we find \( \varphi \in C^*_c(S)\mathcal{H} \) such that \( \|\varphi - \psi'\| < \varepsilon/3 \), and estimate
\[
\|U(t)\psi' - \psi'\| \leq \|U(t)\psi' - U(t)\varphi\| + \|U(t)\varphi - \varphi\| + \|\varphi - \psi'\| \\
\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\]
since \( \|U(t)\psi' - U(t)\varphi\| = \|\psi' - \varphi\| \) by unitarity of \( U(t) \). Thus \( \lim_{t \to 0} \|U(t)\psi - \psi\| = 0 \) for any \( \psi \in \mathcal{H} \), so that the unitary representation \( t \mapsto U(t) \) is strongly continuous. Similarly,
\[
\left\| \frac{U(t+s)\varphi - U(t)\varphi}{s} - iRU(t)\varphi \right\| \leq \left\| \frac{e^{isf}h - h}{s} - ifh \right\|_\infty, \tag{2.32}
\]
assuming (2.30), so that by the same argument as in (2.31) we obtain
\[
\frac{dU(t)}{dt}\varphi = iRU(t)\varphi, \tag{2.33}
\]
initially for any \( \varphi \) of the form (2.30), and hence, taking finite sums, for any \( \varphi \in \mathcal{D}(R) \), cf. (2.28). The final part of Lemma [4] then shows that \( f_0(T) \) is essentially self-adjoint on its domain \( C^*_c(S)\mathcal{H} \). Its closure \( f(T) \) is therefore self-adjoint, and Theorem [2] is proved. \( \square \)
We now prove the examples (1.19) - (1.21), of which the first is trivial. Writing $T_0$ for the operator $\text{id}_0(T)$, the definition (1.17) gives

$$T_0\varphi = T\varphi$$

for $\varphi \in \mathcal{D}(T_0) = C_c^*(S)\mathcal{H}$. Let $\psi \in \mathcal{D}(T_0^{-})$, so that there is a sequence $(\varphi_n)$ in $\mathcal{D}(T_0)$ such that $\varphi_n \to \varphi$ and $(T_0\varphi_n)$ converges. Since $T$ is closed, it follows that $T_0\varphi_n = T\varphi_n \to T\varphi$, so that $\varphi \in \mathcal{D}(T)$. Hence $T_0^{-} \subset T$. Since both operators are self-adjoint, this implies $T_0^{-} = T$, which proves (1.20).

The proof of (1.21) is easier since $(T - z)^{-1}$ is bounded: writing $f(x) = (x - z)^{-1}$, where $z \notin \sigma(T)$ is fixed and $x \in \sigma(T)$, we have

$$f_0(T)h(T)\psi = (fh)(T)\psi = (T - z)^{-1}h(T)\psi,$$

and hence

$$f_0(T)\varphi = (T - z)^{-1}\varphi$$

for any $\varphi \in \mathcal{D}(f_0(T)) = C_c^*(S)\mathcal{H}$. So if $\varphi_n \to \varphi$ for $\varphi \in \mathcal{H}$ and $\varphi_n \in \mathcal{D}(f_0(T))$, boundedness and hence continuity of the resolvent implies

$$f(T)\varphi = \lim_{n \to \infty} f_0(T)\varphi_n = \lim_{n \to \infty} (T - z)^{-1}\varphi_n = (T - z)^{-1}\varphi.$$

### 2.3 Proof of Theorem 3

The first step consists in the observation that $T \eta A$ iff $TU \subset UT$ (or, equivalently, $UTU^* = T$) merely for each unitary $U \in A'$, which is well known [11].

The second step is to show that $TU \subset UT$ iff $SU = US$ for any unitary $U$. This is a simple computation. First suppose that $UTU^* = T$. Then:

$$U(1 + T^2)^{-1}U^* = (U(1 + T^2)U^*)^{-1} = ((U + UT^2)U^*)^{-1} = (UU^* + UTU^*)^{-1} = (1 + UTU^*UTU^*)^{-1} = (1 + T^2)^{-1}.$$

If $R$ is bounded and positive, then $UR = RU$ iff $U \in C^*(R)'$, and since $\sqrt{R} \in C^*(R)$ by the continuous functional calculus, we also have $U\sqrt{R} = \sqrt{RU}$. Consequently,

$$USU^* = U(T\sqrt{(1 + T^2)^{-1}})U^* = (UTU^*)\left(U\sqrt{(1 + T^2)^{-1}}U^*\right) = T\sqrt{(1 + T^2)^{-1}} = S.$$
Similarly, if $SU = US$, then
\[ UTU^* = US\sqrt{1 - S^2}^{-1} U^* = SU\sqrt{1 - S^2}^{-1} \] 
\[ = S \left( U \sqrt{1 - S^2} U^* \right)^{-1} = S \sqrt{1 - S^2}^{-1} = T. \]

Thirdly, as in the first step, $SU = US$ for any unitary $U \in A'$ if $S \in A'' = A$.

\[ \Box \]

2.4 Proof of Theorem 4

Eq. (2.23) in Theorem 3 follows from Theorem 3, taking $A = W^*(T)$, so that $T\eta A$, yields $S \in W^*(T)$, and hence $W^*(S) \subseteq W^*(T)$. On the other hand, taking $A = W^*(S)$, in which case $S \in A$, gives $T\eta W^*(S)$, and hence $W^*(T) \subseteq W^*(S)$.

Similar to (2.25), we have an isometric isomorphism
\[ B_b(\sigma(T)) \xrightarrow{\sim} B_b(\sigma(S)), \ h \mapsto h \circ u, \] (2.34)
so that the first claim of Theorem 4 follows from the Borel functional calculus for the bounded operator $S$ [8]. The proof of the last one is, \textit{mutatis mutandis}, practically the same as in [8, Theorem 5.3.8], so we omit the details; see [2].

As explained in [8, §5.3], there exists a Borel measure $\mu$ on $\sigma(T)$ such that the map $f \mapsto f(T)$ may also be seen as a so-called essential *-homomorphism from $B(\sigma(T)) \setminus N(\sigma(T))$ into the $^*$-algebra of normal operators affiliated with $W^*(T)$, where $N(\sigma(T))$ is the set of $\mu$-null functions on $\sigma(T)$. This remains true in our approach, with the same proof [2].

3 Epilogue

Let us finally note that although this paper was inspired by the work of Woronowicz, the $C^*$-algebraic affiliation relation he defines in [12] (as did, independently, also Baaj and Julg [1]) has not been used here. If we call his relation $\eta'$ to avoid confusion with the $W^*$-algebraic relation $\eta$ we do use, if $A \subseteq B(\mathcal{H})$ we have $T\eta'A \Rightarrow T \in A$ (and hence $T$ is bounded), cf. [12, Prop. 1.3]. Woronowicz does not define a $C^*$-algebraic counterpart of the von Neumann algebra $W^*(T)$, but it might be reasonable to define $C^*(T)$ as the smallest $C^*$-algebra $A$ in $B(\mathcal{H})$ such that $T\eta'A$. It follows from [12, Example 4] that this would give $C^*(T) = C_0^*(S)$, as defined in (1.4). This $C^*$-algebra contains $S$ (and hence $T$) if and only if $T$ is bounded, in which case $C_0^*(S) = C^*(S)$ and hence $C^*(T) = C^*(S)$, as in our approach, cf. (1.5). Also in general (i.e., if $T$ is possibly unbounded), the bicommutant $C^*(T)''$ coincides with $W^*(T)$ as defined in the usual way (1.22) this follows from $C_0^*(S)'' = C^*(S)'' = W^*(S)$ and (1.10).

Of course, we could also redefine $\eta'$, now calling it $\eta''$, by stipulating that $T\eta''A$ whenever $S \in A$, and redefine $C^*(T)$ accordingly (i.e., as the smallest $C^*$-algebra $A$ in $B(\mathcal{H})$ such that $T\eta''A$). This would give (1.5) even if $T$ is unbounded, though in a somewhat empty way.
References


