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NILPOTENT SYMMETRIC JACOBIAN MATRICES AND THE JACOBIAN CONJECTURE II

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Abstract

It is shown that the Jacobian Conjecture holds for all polynomial maps $F : k^n \to k^n$ of the form $F = x + H$, such that JH is nilpotent and symmetric, when $n \leq 4$. If H is also homogeneous a similar result is proved for all $n \leq 5$.

Introduction

Let $F := (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map i.e. each F_i is a polynomial in n variables over $\mathbb{C}$. Denote by $JF := (\frac{\partial F_i}{\partial x_j})_{1 \leq i,j \leq n}$, the Jacobian matrix of F. Then the Jacobian Conjecture (which dates back to Keller [9], 1939) asserts that if $\det JF \in \mathbb{C}^*$, then F is invertible. It was shown in [1] and [12] that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where JH is homogeneous and nilpotent (these two conditions imply that $\det JF = 1$); in fact it is even shown that the case where JH is nilpotent and H is homogeneous of degree 3 is sufficient.

For $n = 3$ resp. $n = 4$ this so-called cubic homogeneous case was proved by Wright resp. Hubbers in [11] resp. [8]. For $n = 3$, the case $F = x + H$, where H is not necessarily homogeneous, but of degree 3, was proved by Vistoli in [10]. On the other hand, if H has degree ≥ 4 not much is known; if for example F is of the form $x + H$ where H is homogeneous of degree ≥ 4, then all cases $n \geq 3$ remain open. The aim of this paper is to study these type of problems under the additional hypothesis that JH is symmetric. This is no loss of generality since it was recently shown by the authors in [3] that it suffices to prove the Jacobian Conjecture for all polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + H$ with JH nilpotent, homogeneous of degree ≥ 2 and symmetric.

For such maps the conjecture was proved for all $n \leq 4$ in [6]. The proof of this result is based on a remarkable theorem of Gordan and Noether, which asserts that if $n \leq 4$, then $h(f)$, the Hessian matrix of the homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, is singular iff f is degenerate i.e. there exists a linear coordinate change T such that $f(Tx) \in \mathbb{C}[x_1, \dots, x_{n-1}]$. However if $n = 5$ such a result does not hold: the polynomial $f = x_1^2 x_3 + x_1 x_2 x_4 + x_2^2 x_5$ has a singular Hessian but is not degenerate.

Nevertheless one of the main results of this paper (theorem 4.1) asserts that the Jacobian Conjecture holds for all polynomial maps $F : \mathbb{C}^5 \to \mathbb{C}^5$ of the form $F = x + H$ with JH nilpotent, homogeneous and symmetric. To prove this result we first extend the 3 dimensional Gordan-Noether theorem to the case where f needs not be homogeneous, but has the additional property that $\operatorname{tr} h(f) = 0$ (proposition 3.2). Next we show, using a result of [4], that in case $n = 5$ and f is homogeneous, the condition $h(f)$ is nilpotent implies that f is degenerate. Then we are in the position to apply the main result of [2], to conclude the above mentioned 5-dimensional result.

Finally we also extend the 4-dimensional homogeneous result obtained in [6] to the case where H needs not be homogeneous (theorem 5.1).

1 Preliminaries

The main aim of this section is to fix the notations, collect some results from [2] and [4] and to give some additional preliminaries which we will need in the sequel.

Throughout this paper k denotes an algebraically closed field of characteristic zero and $k^{[n]} := k[x_1, \ldots, x_n]$ is the polynomial ring in n variables over k. By $H = (H_1, \ldots, H_n) : k^n \to k^n$ we mean a polynomial map, i.e. each H_i belongs to $k^{[n]}$. One easily verifies that JH is symmetric iff there exists an $f \in k^{[n]}$ such that $H_i = f_{x_i}$, the partial derivative of f with respect to x_i, for all i. In particular, $JH = h(f) := (\frac{\partial^2 f}{\partial x_i \partial x_j})$, the Hessian matrix of f. We may obviously assume that f is reduced, i.e. does not contain terms of degree ≤ 1. Our main interest is to study the Jacobian Conjecture for all polynomial maps of the form $F = x + H$, where JH is nilpotent and symmetric. As already remarked above, this is sufficient for investigating the Jacobian Conjecture. Starting point is the main result of [2]. To explain it, we need to formulate the (homogeneous) symmetric dependence problem:

(Homogeneous) Symmetric Dependence Problem (H)SDP(n).
Let $f \in k^{[n]}$ be a (homogeneous) polynomial in $k^{[n]}$ of degree $d \geq 2$ such that $h(f)$ is nilpotent. Are the rows of $h(f)$ linearly dependent over k?

The following result can be found in [2].

Proposition 1.1

i) SDP(n) has an affirmative answer for all $n \leq 2$.

ii) If $n \leq 4$ and $f \in k^{[n]}$ is homogeneous, then $h(f)$ is singular implies that f is degenerate. In particular HSDP(n) has an affirmative answer if $n \leq 4$.

Since f is assumed to be reduced, it is shown in [2, 1.2] that the dependence of the rows of $h(f)$ is equivalent to the fact that the partials f_{x_i} of f are linearly dependent over k, which in turn is equivalent to f being degenerate. The main result of [2] asserts the following.

Proposition 1.2 *Let $n \geq 2$ and $H \in k[x_1, \ldots, x_n]^n$ with JH symmetric and nilpotent. Then* *i)* $x + H$ *is invertible if SDP(p) has an affirmative answer for all* $p \leq n$.

ii) *If* H *is homogeneous, then* $x + H$ *is invertible if SDP(p) has an affirmative answer for all* $p \leq n - 2$ *and HSDP(p) for* $p = n - 1$ *and* $p = n$.

The remainder of this paper is therefore devoted to showing that SDP(p) has an affirmative answer for all $p \leq 4$ as well as HSDP(5).
In order to investigate nilpotent Hessians we first recall our main results on singular Hessians obtained in [4]. To formulate them we need some preliminaries. First, let $f \in k^{[n]}$. A polynomial $g \in k^{[n]}$ is called equivalent to f if there exists $T \in Gl_n(k)$ such that $g = f \circ T$ i.e. $g(x) = f(Tx)$. It is well-known that

$$h(g) = T^t h(f)_{|Tx} T \tag{1}$$

So if g is equivalent to f and $\det h(f) = 0$, then $\det h(g) = 0$ as well. Furthermore, if $\det h(f) = 0$ there exists a nonzero polynomial $R(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ such that $R(f_{x_1}, \ldots, f_{x_n}) = 0$. We say that R is a relation of f. Consequently (since $\det h(g) = 0$), also the partials of g are algebraically dependent over k. This enables us to give the following definition: let $f \in k^{[n]}$ with $\det h(f) = 0$. Then $s(f)$ is the maximal natural number s, $0 \leq s \leq n-1$ for which there exists a $g \in k^{[n]}$ equivalent to f which has a relation in $k[y_{s+1}, \ldots, y_n]$. In other words $n - s(f)$ is the least number of variables a relation of a with f equivalent polynomial can have.

Theorem 1.3 *Let* $f \in k^{[n]}$ *be reduced and satisfy* $\det h(f) = 0$. *Then*

1) *If* $n = 3$ *then either* f *is degenerate or equivalent to a polynomial of the form* $a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3$.

2) *If* $n = 4$ *and* $s(f) \geq 1$ *then either* f *is degenerate or equivalent to a polynomial of one of the following forms:*

  i) $a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4$ *with* a_2 *and* a_3 *algebraically dependent over* k.

  ii) $p(x_1, a) + b$, *with* $p(y_1, y_2) \in k[y_1, y_2]$ *and* $a, b \in Ax_2 + Ax_3 + Ax_4$ *where* $A = k[x_1]$.

3) *If* $n = 5$ *and* f *is homogeneous, then either* f *is degenerate or equivalent to a polynomial of the form* $p(a)$, *where* $a = a_1x_3 + a_2x_4 + a_3x_5$ *with* $a_i \in A = k[x_1, x_2]$ *for all* i *and* $p(X) \in A[X]$.

2 Orthogonal equivalence of polynomials with singular Hessians

Theorem 1.3 gives a classification for small n of reduced polynomials with singular Hessians up to equivalence. In this section we refine this result, namely we obtain a classification of such polynomials up to orthogonal equivalence: two polynomials f

and g in $k^{[n]}$ are called orthogonally equivalent if there exists an orthogonal matrix $T \in O(n)$ i.e. $T \in M_n(k)$ with $T^tT = I_n$, such that $g = f \circ T$. The advantage of working with orthogonal equivalence is that it preserves the nilpotency of Hessians, i.e. $h(f)$ is nilpotent iff $h(g)$ is nilpotent (which follows from (1)). The main result of this section is

Theorem 2.1 *Let $f \in k^{[n]}$ be reduced and satisfy $\det h(f) = 0$. Then*

1) *If $n = 3$, then either f is degenerate or orthogonally equivalent to a polynomial of one of the following two forms:*

$$a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3 \tag{2}$$

$$a_1(x_1 + ix_2) + a_2(x_1 + ix_2)x_2 + a_3(x_1 + ix_2)x_2 \tag{3}$$

2) *If $n = 4$ and $s(f) \geq 1$, then either f is degenerate or orthogonally equivalent to a polynomial of one of the following forms:*

$$U := a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4 \tag{4}$$

with a_2 and a_3 algebraically dependent over k,

$$U_{|x_1:=x_1+ix_3} \tag{5}$$

with a_2 and a_3 algebraically dependent over k,

$$U_{|x_1:=x_1+ix_3, x_2:=x_2+ix_4} \tag{6}$$

with a_2 and a_3 algebraically dependent over k,

$$p(x_1, a) + b \tag{7}$$

with $p(y_1, y_2) \in k[y_1, y_2]$, $a, b \in Ax_2 + Ax_3 + Ax_4$ and $A = k[x_1]$,

$$(p(x_1, a) + b)_{|x_1:=x_1+ix_2} \tag{8}$$

3) *If $n = 5$ and f is homogeneous, then either f is degenerate or orthogonally equivalent to a polynomial of one of the following forms*

$$p(x_1, x_2, a) \tag{9}$$

with $a = a_1x_3 + a_2x_4 + a_3x_5$ and $a_i \in A := k[x_1, x_2]$ for all i and $p(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$,

$$p(x_1, x_2, a)_{|x_1:=x_1+ix_3} \tag{10}$$

$$p(x_1, x_2, a)_{|x_1:=x_1+ix_3, x_2:=x_2+ix_4} \tag{11}$$

The proof of this result is based on theorem 1.3 and the following lemma

Lemma 2.2 *Let $v_1, \ldots, v_r \in k^n$ be linearly independent over k. Then there exist an $s : 0 \leq s \leq r$, an $S \in Gl_r(k)$ and an orthogonal matrix $T \in O(n)$ such that*

$$S \begin{pmatrix} v_1^t \\ \vdots \\ v_r^t \end{pmatrix} T = \begin{pmatrix} I_r & \begin{matrix} iI_s \\ \emptyset \end{matrix} & \emptyset \end{pmatrix} = \begin{pmatrix} e_1^t + ie_{r+1}^t \\ \vdots \\ e_s^t + ie_{r+s}^t \\ e_{s+1}^t \\ \vdots \\ e_r^t \end{pmatrix}$$

where e_i is the i-th standard basis vector in k^n (if $s = 0$ read $S(v_1^t, \ldots, v_r^t)T = (e_1^t, \ldots, e_r^t)$).

Proof. Put $A := (\langle v_i, v_j \rangle)_{1 \leq i,j \leq r}$. Since A is symmetric, there exist an $S \in Gl_r(k)$ and an $s : 0 \leq s \leq r$ such that

$$S^t AS = J := \begin{pmatrix} 0_s & \\ & I_{r-s} \end{pmatrix}$$

Put $(\tilde{v_1} \ \cdots \ \tilde{v_r}) := (v_1 \ \cdots \ v_r) \cdot S$. Then one readily verifies (or see [2, lemma 1.3]) that $(\langle \tilde{v_i}, \tilde{v_j} \rangle)_{i,j} = J$. So replacing the v_i by the $\tilde{v_i}$, we may assume that $(\langle v_i, v_j \rangle)_{i,j} = J$. Now we distinguish two cases: $s = 0$ and $s \geq 1$.

- *Case 1: $s = 0$.*
  Then by the Gram-Schmidt theorem, there exists an orthogonal matrix $T \in Gl_n(k)$ such that the j-th row T_j of T equals v_j^t for all $j : 1 \leq j \leq r$. So $T_i v_i = 1$ and $T_j v_i = 0$ for all $i : 1 \leq i \leq r$ and all $j \neq i$. In other words, $Tv_i = e_i$ for all $i : 1 \leq i \leq r$, i.e. T is an orthogonal matrix satisfying $T(v_1 \ \cdots \ v_r) = (e_1 \ \cdots \ e_r)$.

- *Case 2: $s \geq 1$.*
  So $\langle v_1, v_j \rangle = 0$ for all $j : 1 \leq j \leq r$. Observe that v_1 is perpendicular to $kv_1 + \ldots + kv_r$, so $r \leq n - 1$. We may assume that $v_{11} = 1$. So $\langle v_1, e_1 \rangle = 1$. Hence if we put $u := i(e_1 - v_1)$, then $\langle e_1, u \rangle = 0$ and $\langle u, u \rangle = 1$. So by the Gram-Schmidt theorem there exists an orthogonal matrix $T \in Gl_n(k)$ with $T_1 = e_1^t$ and $T_{r+1} = u^t$, where again T_j is the j-th row of T. So $T_j e_1 = 0$ for all $j \neq 1$ and $T_j u = 0$ for all $j \neq r + 1$, which by the definition of u implies that $T_j v_1 = T_j e_1 = 0$ for all $j \notin \{1, r + 1\}$. Also $T_{r+1} v_1 = \langle v_1, u \rangle = i(\langle v_1, e_1 \rangle - \langle v_1, v_1 \rangle) = i$. Summarizing $Tv_1 = (T_1 v_1, \ldots, T_n v_1) = (e_1 + ie_{r+1})$.

  Define $w_j := Tv_j$ for all j. Then $T(v_1 \ \cdots \ v_r) = (w_1 \ \cdots \ w_r) = ((e_1 + ie_{r+1}) \ w_2 \ \cdots \ w_r)$. Since T is orthogonal, we have that $\langle w_i, w_j \rangle = \langle v_i, v_j \rangle$ for all i, j. Now replace for each $j \geq 2$ w_j by $w_j - c_j w_1$ for suitable $c_j \in k$ (which operation can be obtained by replacing $(w_1 \ \cdots \ w_r)$ by $(w_1 \ \cdots \ w_r)S$ for suitable $S \in Gl_r(k)$) we may assume that the first component of w_j equals zero. Since $\langle w_1, w_j \rangle = 0$ for all $j \geq 2$, it follows, using $w_1 = e_1 + ie_{r+1}$, that also the $(r+1)$-th component of w_j equals zero. Now consider the $r - 1$ vectors $w_2, \ldots, w_r$ in $k^{n-2} = ke_2 + \ldots + ke_r + ke_{r+2} + \ldots + ke_n$ and use induction on n $\square$

Corollary 2.3 *Let $v_1, \ldots, v_r, v_{r+1}, \ldots, v_n$ be a k-basis of k^n. Put $V_i := \langle v_i, x\rangle$. Let f be of the form*

$$f = p\left(V_1, \ldots, V_r, \sum_{j=r+1}^{n} a_j(V_1, \ldots, V_r)V_j, \sum_{j=r+1}^{n} b_j(V_1, \ldots, V_r)V_j\right)$$

Then f is orthogonally equivalent to a polynomial of the form

$$q\left(X_0, \sum_{j=r+1}^{n} c_j(X_0)x_j, \sum_{j=r+1}^{n} d_j(X_0)x_j\right)$$

where $X_0 = (x_1 + ix_{r+1}, \ldots, x_s + ix_{r+s}, x_{s+1}, \ldots, x_r)$.

Proof. Choose T and S as in Lemma 2.2. Observe that

$$f = \tilde{p}\left(S(V_1, \ldots, V_r), \sum_{j=r+1}^{n} \tilde{a_j}(S(V_1, \ldots, V_r))V_j, \sum_{j=r+1}^{n} \tilde{b_j}(S(V_1, \ldots, V_r))V_j\right)$$

for suitable $\tilde{p}$, $\tilde{a_j}$ and $\tilde{b_j}$. Now we claim that $f \circ T$ is of the desired form. Notice first that it follows from lemma 2.2 that

$$\begin{aligned} E &:= S(V_1 \circ T, \ldots, V_r \circ T) \\ &= S(v_1^t Tx, \ldots, v_r^t Tx) \\ &= X_0 \end{aligned}$$

Consequently,

$$f \circ T = \tilde{p}(X_0, \sum_{j=r+1}^{n} \tilde{a_j}(X_0)W_j, \sum_{j=r+1}^{n} \tilde{b_j}(X_0)W_j)$$

where $W_j := V_j \circ T$ is a linear form in all x_i over k. Finally observe that

$$\sum_{j=r+1}^{n} \tilde{a_j}(X_0)W_j, \sum_{j=r+1}^{n} \tilde{b_j}(X_0)W_j \in k[X_0] + \sum_{j=r+1}^{n} k[X_0]x_j$$

So we can write $f \circ T$ in the desired form □

Proof of theorem 2.1. In each of the cases in theorem 2.1 it follows from theorem 1.3 that there exists $T \in Gl_n(k)$ such that $f \circ T$ is of the form

$$p\left(x_1, \ldots, x_r, \sum_{j=r+1}^{n} a_j(x_1, \ldots, x_r)x_j, \sum_{j=r+1}^{n} b_j(x_1, \ldots, x_r)x_j\right)$$

for suitable r, p, a_j and b_j. Hence f is of the form described in corollary 2.3, where v_i^t is the i-th row of T^{-1}. Then apply this corollary □

3 The symmetric Jacobian Conjecture in dimension 3

The main result of this section is

Theorem 3.1 *Let $F = x + H : k^3 \to k^3$ be a polynomial map with JH symmetric and nilpotent. Then F is invertible.*

Proof. This is an immediate consequence of proposition 1.1 i), proposition 1.2 and proposition 3.2 below □

Proposition 3.2 *SDP(3) has an affirmative answer.*

Proof. Let $f \in k^{[3]}$ be reduced and assume that $h(f)$ is nilpotent. Then by theorem 2.1 we may assume that f is either of the form (2) or of the form (3).

i) Suppose first that f is of the form (2). Since $\operatorname{tr} h(f) = 0$ this gives $a_1''(x_1) + a_2''(x_1)x_2 + a_3''(x_1)x_3 = 0$. So $\deg a_i \leq 1$ for all i. Since f is reduced, this implies that $f = c_1x_1x_2 + c_2x_1x_3$ for some $c_i \in k$. It follows that f_{x_2} and f_{x_3} are linearly dependent over k, so f is degenerate.

ii) Now assume that f is of the form (3). Then a simple computation gives $\operatorname{tr} h(f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f = 2ia_2'(x_1 + ix_2)$. Since $\operatorname{tr} h(f) = 0$, this implies that $a_2 \in k$ and hence that $a_2 = 0$, since f is reduced. Consequently, $f = a_1(x_1 + ix_2) + a_3(x_1 + ix_2)x_3 \in k[x_1 + ix_2, x_3]$. So f is degenerate □

4 The homogeneous symmetric Jacobian Conjecture in dimension 5

The main result of this section is

Theorem 4.1 *Let $F = x + H : k^5 \to k^5$ be a polynomial map with JH symmetric, nilpotent and homogeneous of degree ≥ 2. Then F is invertible.*

Proof. By propositions 1.1 i) and 3.2, SDP(n) has an affirmative answer for all $n \leq 3$. Also HSDP(4) has an affirmative answer by proposition 1.1. Furthermore we will show in proposition 4.2 below that HSDP(5) has an affirmative answer. Then the desired result follows from proposition 1.2 ii) □

Proposition 4.2 *HSDP(5) has an affirmative answer.*

Proof. Let $f \in k^{[5]}$ be homogeneous and reduced and assume that $h(f)$ is nilpotent. Then by theorem 2.1 we may assume that f is of the form (9), (10) or (11). We will show that in each of these cases f is degenerate.

i) First assume that f is either of the form (9) or (10). Since f is homogeneous it follows that all a_i are homogeneous of the same degree, say d. If $d = 0$ then f is trivially degenerate. So assume $d \geq 1$. Write $p = \gamma_r(y_1, y_2)y_3^r + \gamma_{r-1}(y_1, y_2)y_3^{r-1} + \cdots$ and ∂_i instead of ∂_{x_i}. Then $g := \partial_5^{r-1} f$ is of the form

$$\begin{aligned} g &= b_1(x_1 + cx_3, x_2) + b_2(x_1 + cx_3, x_2)x_3 + \\ &\quad b_3(x_1 + cx_3, x_2)x_4 + b_4(x_1 + cx_3, x_2)x_5 \end{aligned}$$

with $c \in \{0, i\}$ and $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 2$. Since $\operatorname{tr} h(f) = 0$ we have $\Delta f = 0$ where $\Delta = \partial_1^2 + \ldots + \partial_5^2$. Consequently, using that ∂_5^{r-1} commutes with Δ, we get that $\Delta\partial_5^{r-1} f = \partial_5^{r-1}\Delta f = 0$ i.e. $\Delta g = 0$. It then follows from the form of g that $(\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + cx_3, x_2) = 0$ for all $j \geq 2$, since $x_j(\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + cx_3, x_2)$ is the leading term of x_j of Δf, seen as polynomial over $x_1 + cx_3, x_2, \ldots, x_5$, for all $j \geq 2$.

If $c = 0$, this implies that $b_j(x_1, x_2)$ is of the form $\lambda_j(x_1 + ix_2)^s + \mu_j(x_1 - ix_2)^s$ for some $\lambda_j, \mu_j \in k$ and $s \geq 1$. If $c = i$, then it follows from $\partial_2^2 b_j(x_1 + ix_3, x_2) = (\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + ix_3, x_2) = 0$ that each $b_j(x_1 + ix_3, x_2)$ is of the form $\lambda_j(x_1 + ix_3)^s + \mu_j x_2(x_1 + ix_3)^{s-1}$ for some $\lambda_j, \mu_j \in k$ and $s \geq 1$. In both cases, the polynomials b_2, b_3, b_4 belong to a 2-dimensional k-vectorspace and hence are linearly dependent over k. Since $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 2$, also the polynomials a_2, a_3, a_4 are linearly dependent over k. In case (9), it follows that $f_{x_3}, f_{x_4}, f_{x_5}$ are linearly dependent over k, so f is degenerate. In case (10), first make the coordinate change which sends x_1 to $x_1 - ix_3$. Then the same argument shows that $f_{|x_1 - ix_3}$ is degenerate and hence so is f.

ii) So it remains to show the case (11). We will show that a_1 and a_2 are linearly dependent over k, which will imply that f is degenerate. Write again $p = \gamma_r(y_1, y_2)y_3^r + \cdots$. We distinguish two cases: $r \geq 2$ and $r = 1$.

First assume $r \geq 2$. Make the coordinate change $X_1 := x_1 + ix_3, X_2 := x_2 + ix_4, X_j := x_j$ for all $j \geq 3$. Put $U := a_1(X_1, X_2)X_3 + a_2(X_1, X_2)X_4 + a_3(X_1, X_2)X_5$. Then the condition tr $h(f) = 0$, i.e. $\Delta f = 0$, becomes

$$(2i(\partial_{X_1}\partial_{X_3} + \partial_{X_2}\partial_{X_4}) + \partial_{X_3}^2 + \partial_{X_4}^2 + \partial_{X_5}^2)(\gamma_r(X_1, X_2)U^r + \cdots) = 0 \qquad (12)$$

Applying $\partial_{X_3}^{r-1}$ to this equation gives

$$2i(\partial_{X_1}\partial_{X_3} + \partial_{X_2}\partial_{X_4} + \partial_{X_3}^2 + \partial_{X_4}^2 + \partial_{X_5}^2)(r!\gamma_r a_1^{r-1}U) = 0$$

So

$$\partial_{X_1}(\gamma_r a_1^r) + \partial_{X_2}(\gamma_r a_1^{r-1}a_2) = \partial_{X_1}\partial_{X_3}\gamma_r a_1^{r-1}U + \partial_{X_2}\partial_{X_4}\gamma_r a_1^{r-1}U = 0$$

Consequently there exists a homogeneous element $h_1 \in k[X_1, X_2]$ such that

$$\gamma_r a_1^r = \partial_{X_2}h_1 \text{ and } \gamma_r a_1^{r-1}a_2 = -\partial_{X_1}h_1 \qquad (13)$$

So if we put $D = a_1\partial_{X_1} + a_2\partial_{X_2}$, then $h_1 \in \ker D$. Similarly, applying $\partial_{X_4}^{r-1}$ to the equation (12) gives $\partial_{X_1}(\gamma_r a_1 a_2^{r-1}) + \partial_{X_2}(\gamma_r a_2^r) = 0$. So there exists a homogeneous element $h_2 \in k[X_1, X_2]$ such that

$$\gamma_r a_1 a_2^{r-1} = \partial_{X_2} h_2 \text{ and } \gamma_r a_2^r = -\partial_{X_1} h_2 \tag{14}$$

So $h_2 \in \ker D$.

Since a_1 and a_2 are homogeneous of the same degree, both h_1 and h_2 are also homogeneous of the same degree. Also $\ker D = k[v]$ for some homogeneous element $v \in k[X_1, X_2]$ (by [5, 1.2.25]). Consequently $h_1 = c_1 v^s$ and $h_2 = c_2 v^s$ for some $c_j \in k$ and $s \geq 1$. It follows that h_1 and h_2 are linearly dependent over k and hence so are $\partial_{X_2} h_1$ and $\partial_{X_2} h_2$. Whence by (13) and (14) a_1^{r-1} and a_2^{r-1} are linearly dependent over k, which implies that a_1 and a_2 are linearly dependent over k (since $r \geq 2$!).

So it remains to consider the case $r = 1$, which follows immediately from the next lemma (which is a slightly generalized version of lemma 1.2 of [3]) □

Lemma 4.3 *Let $0 \leq s \leq \frac{n}{2}$ and $f \in k^{[n]}$ of the form*

$$f = a_0(z) + a_1(z)x_{s+1} + a_2(z)x_{s+2} + \ldots + a_{n-s}(z)x_n$$

where z is an abbreviation of $x_1 + ix_{s+1}, x_2 + ix_{s+2}, \ldots, x_s + ix_{2s}$. Then $h(f)$ is nilpotent iff $J(a_1, \ldots, a_s)$ is nilpotent.

Proof. $h(f)$ is nilpotent iff $\det(TI_n - h(f)) = T^n$. Put $q := \frac{1}{2}\sum_{i=1}^n x_i^2$. Then $h(Tq) = TI_n$. Let $S := (x_1 - ix_{s+1}, x_2 - ix_{s+2}, \ldots, x_s - ix_{2s}, x_{s+1}, \ldots, x_n)$. Then $f \circ S = a_0 + a_1 x_{s+1} + \ldots + a_{n-s}x_n$. Since $\det S = 1$ it follows from (1) in section 1 that $M := h(Tq - f) \circ S$ satisfies $\det M = T^n$ iff $h(f)$ is nilpotent. Now observe that

$$\begin{aligned} q \circ S &= \frac{1}{2}\sum_{j=1}^{s}(x_j^2 - 2ix_jx_{j+s} - x_{j+s}^2) + \frac{1}{2}\sum_{j=s+1}^{n} x_j^2 \\ &= \frac{1}{2}\sum_{j=1}^{s}(x_j^2 - 2ix_jx_{j+s}) + \frac{1}{2}\sum_{j=2s+1}^{n} x_j^2. \end{aligned}$$

Then it follows that M is of the form

$$M = \begin{pmatrix} * & -iTI_s - J(a_1, \ldots, a_s)^t & * \\ -iTI_s - J(a_1, \ldots, a_s) & 0 & 0 \\ * & 0 & TI_{n-2s} \end{pmatrix}$$

Finally observe that

$$\begin{aligned} \det M &= (-1)^s \cdot \det(iTI_s + J(a_1, \ldots, a_s)) \cdot \\ &\quad \det(iTI_s + J(a_1, \ldots, a_s)^t) \cdot T^{n-2s} \\ &= \det(TI_s - iJ(a_1, \ldots, a_s))^2 T^{n-2s} \end{aligned}$$

Consequently $\det M = T^n$ iff $\det(TI_n - iJ(a_1, \ldots, a_s)) = T^s$, which implies the desired result □

5 The symmetric Jacobian Conjecture in dimension 4

The main result of this section is

Theorem 5.1 *Let $F = x + H : k^4 \to k^4$ be a polynomial map with JH symmetric and nilpotent. Then F is invertible.*

Proof. This is an immediate consequence of propositions 1.2, 3.2, 1.1 and 5.2 below
$\square$

Proposition 5.2 *SDP(4) has an affirmative answer.*

The proof of this result is based on theorem 1.3 2). In order to use this result we will first show that the hypothesis $h(f)$ is nilpotent indeed implies that $s(f) \geq 1$. For the proof of this implication we need to recall some results obtained in [7], which we summarize in the next two propositions.

Proposition 5.3 *Let $f \in k^{[n]}$ be homogeneous and $R \in k[y_1, \ldots, y_n]$ such that $R(f_{x_1}, \ldots, f_{x_n}) = 0$. Put $h_i := R_{y_i}(f_{x_1}, \ldots, f_{x_n})$ and $D := \sum_{i=1}^{n} h_i \partial_{x_i}$. Then*

i) $D^2(x_i) = 0$ for all i.

ii) Let $f = Ax_1^r + x_1^{r+1}(\ldots)$, where $0 \neq A \in K[x_2, \ldots, x_n]$. If $h_1 = 0$, then $A(h_2, \ldots, h_n) = 0$.

Proposition 5.4 *Let $D = \sum_{i=1}^{n} h_i \partial_{x_i}$ be a homogeneous derivation on $k^{[n]}$ such that $D^2(x_i) = 0$ for all i and denote by μ the dimension of the rational map $h : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$. If $\mu \leq 1$ then there exist at least two linearly independent linear relations between the h_i.*

Now we are ready to prove

Proposition 5.5 *Let $f \in k^{[4]}$ be reduced and such that $h(f)$ is nilpotent. Then $s(f) \geq 1$, i.e. there exists a nonzero degenerate polynomial $R \in k[y_1, y_2, y_3, y_4]$ such that $R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$.*

Proof. If $\operatorname{rk} h(f) \leq 2$, then $\operatorname{rk} J(f_{x_1}, f_{x_2}, f_{x_3}) \leq 2$. So by [5, proposition 1.2.9], $\operatorname{trdeg}_k k(f_{x_1}, f_{x_2}, f_{x_3}) \leq 2$, which implies that there exists a nonzero polynomial $R \in k[y_1, y_2, y_3]$ with $R(f_{x_1}, f_{x_2}, f_{x_3}) = 0$. Clearly R is degenerate in $k[y_1, y_2, y_3, y_4]$. So we may assume that $\operatorname{rk} h(f) = 3$.

i) Let $d := \deg f$. Observe that $d \geq 2$ since f is reduced. Since $\det h(f) = 0$ there exists some nonzero polynomial $R \in k[y_1, y_2, y_3, y_4]$, say of degree r, such that $R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$. Let $\bar{f}$ be the leading part of f and $\bar{R}$ the leading part of R. Then $\bar{R}(\bar{f}_{x_1}, \bar{f}_{x_2}, \bar{f}_{x_3}, \bar{f}_{x_4}) = 0$. So it follows from proposition 1.1 ii) that $\bar{f}$ is degenerate.

ii) Put $S := y_6^r R(\frac{y}{y_6})$. Then $S \in k[y_1, y_2, y_3, y_4, y_6]$ is homogeneous of degree r and $S(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}, 1) = 0$. Put $g := x_5^d f(\frac{x}{x_5}) + x_5^{d-1} x_6$. Then $g_{x_i} = x_5^{d-1} f_{x_i}(\frac{x}{x_5})$ for all $i \le 4$ and $g_{x_6} = x_5^{d-1} \cdot 1$. Since S is homogeneous and $S(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}, 1) = 0$ it follows that $S(g_{x_1}, g_{x_2}, g_{x_3}, g_{x_4}, g_{x_6}) = 0$. Now we want to apply proposition 5.3 ii) to the polynomial $g \in k^{[6]}$ and the relation $S \in k[y_1, \ldots, y_6]$ which does not contain y_5. Put $z_i := S_{y_i}(g_{x_1}, g_{x_2}, g_{x_3}, g_{x_4}, g_{x_6})$ for all $i : 1 \le i \le 6$. Observe that $z_5 = 0$ and that $g = \bar{f}(x_1, x_2, x_3, x_4) + (\ldots)x_5$ (since $d \ge 2$). So taking $A := \bar{f}$ in proposition 5.3 we get that $\bar{f}(z_1, z_2, z_3, z_4) = 0$.

iii) Let $M := h(f)^m$ where $M \neq 0$ and $h(f)^{m+1} = 0$. Choose a nonzero column $\tilde{h}$ of M. Since $h(f)M = 0$ it follows that $h(f)\tilde{h} = 0$. Furthermore $\langle \tilde{h}, \tilde{h} \rangle = 0$, for $M^2 = 0$. Since

$$\begin{aligned} 0 &= \partial_{x_i} R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) \\ &= \sum_{j=1}^{4} R_{y_j}(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) f_{x_j x_i} \\ &= \sum_{j=1}^{4} h_j f_{x_j x_i} \end{aligned}$$

for all $1 \le i \le 4$, we get that $h(f)h = 0$ Since we already saw that $h(f)\tilde{h} = 0$, the hypothesis that $\operatorname{rk} h(f) = 3$ implies that $h = \alpha \tilde{h}$ for some $\alpha \in k(x_1, x_2, x_3, x_4)$. Hence $\langle \tilde{h}, \tilde{h} \rangle = 0$ implies that $h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0$.

iv) The polynomial $z_1^2 + z_2^2 + z_3^2 + z_4^2$ is clearly homogeneous. Furthermore, substituting $x_5 = 1$ gives $h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0$ (by iii)). Hence $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$, which is an irreducible non-degenerate relation between the polynomials z_1, z_2, z_3, z_4. Since we also found a degenerate relation between the z_i in ii), namely $\bar{f}(z_1, z_2, z_3, z_4) = 0$, it follows that $\operatorname{trdeg}_k k(z_1, z_2, z_3, z_4) \le 2$. Consequently the dimension of the rational map $z : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ defined by $z(x) = (z_1, z_2, z_3, z_4, 0)$ is at most 1.

Now define $D = \sum_{i=1}^{6} z_i \partial_{x_i}$. Then by proposition 5.3 i) $D(z_i) = 0$ for all i. Observe that $z_i \in k[x_1, \ldots, x_5]$ and recall that $z_5 = 0$. So also $\tilde{D}(z_i) = 0$ for all $i \le 4$, where $\tilde{D}$ is the derivation $\sum_{i=1}^{4} z_i \partial_{x_i}$ on $k[x_1, \ldots, x_5]$. Then it follows from proposition 5.4 that besides the relation $z_5 = 0$ there is another linear relation between $z_1, \ldots, z_5$. So z_1, z_2, z_3, z_4 are linearly dependent over k. Taking $x_5 = 1$ it follows that h_1, h_2, h_3, h_4 are linearly dependent over k. Consequently there exist $c_i \in k$, not all zero with

$$\sum_{i=1}^{4} c_i R_{y_i}(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0 \text{ i.e. } \left(\sum_{i=1}^{4} c_i R_{y_i} \right) (f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$$

Now assume that R was taken of minimal degree, then it follows that $\sum_{i=1}^{4} c_i R_{y_i} = 0$, i.e. R is degenerate, which completes the proof $\square$

Proof of proposition 5.2. According to proposition 5.5 we may assume that f is of one of the forms (4)-(8) of theorem 2.1.

i) Let f be of the form (4). Then

$$h(f) = \begin{pmatrix} h(a_2) & 0 \\ 0 & 0 \end{pmatrix} x_3 + \begin{pmatrix} h(a_3) & 0 \\ 0 & 0 \end{pmatrix} x_4 + A$$

where A is a 4×4 matrix which entries are polynomials in x_1 and x_2. Since $h(f)$ is nilpotent, so is $h(a_2)c_1 + h(a_3)c_2$ for each $c_1, c_2 \in k$ (look at the highest x_3-term of $h(f)_{|(x_1,x_2,c_1x_3,c_2x_3)}$). In particular both $h(a_2)$ and $h(a_3)$ are nilpotent. Then it is well-known that the reduced parts of a_2 and a_3 are polynomials in $x_1 + ix_2$ or $x_1 - ix_2$ over k. Say the reduced part of a_2 is a nonzero polynomial in $x_1 + ix_2$. Consequently the reduced part of a_3 is also a polynomial in $x_1 + ix_2$, for otherwise $h(a_2) + h(a_3) = h(a_2 + a_3)$ cannot be nilpotent.

Write $a_2 = c_1x_2 + g_1(x_1 + ix_2)$ and $a_3 = c_2x_2 + g_2(x_1 + ix_2)$, with $c_1, c_2 \in k$. Since a_2 and a_3 are algebraically dependent over k, the same holds for $c_1x_2 + g_1(x_1)$ and $c_2x_2 + g_2(x_1)$ (make the coordinate change $x_1 \mapsto x_1 - ix_2$). If $c_1 \neq 0$ or $c_2 \neq 0$, it follows readily that $c_1g_2 - c_2g_1 \in k$ (make a coordinate change which sends one of the elements $c_ix_2 + g_i(x_1)$ to x_2). Therefore $c_1g_2 = c_2g_1$, for $g_1(0) = g_2(0) = 0$ due to the reducedness of f. Hence a_2 and a_3 are linearly dependent over k (since $a_2(0) = a_3(0) = 0$), which implies that f is degenerate. So we may assume that $c_1 = c_2 = 0$. So both a_2 and a_3 belong to $k[x_1 + ix_2]$.

Finally $M_c := h(f)_{|(x_1,x_2,c,0)}$ is nilpotent for all $c \in k$ and is of the form

$$M_c = \begin{pmatrix} h(a_1 + ca_2) & \begin{matrix} a_2' & a_3' \\ ia_2' & ia_3' \end{matrix} \\ \begin{matrix} a_2' & ia_2' \\ a_3' & ia_3' \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

An easy computation shows that the characteristic polynomial of a 4×4 matrix of the form

$$\begin{pmatrix} A & B \\ B^t & 0 \end{pmatrix} \text{ where } B = \begin{pmatrix} p & p \\ ip & iq \end{pmatrix}$$

is of the form $T^4 - (\operatorname{tr} A)T^3 + (\det A)T^2 + \cdots$. Since M_c is nilpotent this implies that $h(a_1 + ca_2)$ is nilpotent for all $c \in k$. Taking $c = 1$ (and using that a_1 has no terms of degree ≤ 1, since f is reduced) it follows as above from $a_2 \in k[x_1 + ix_2]$ that also $a_1 \in k[x_1 + ix_2]$. Consequently $f \in k[x_1 + ix_2, x_3, x_4]$, i.e. f is degenerate.

ii) Now assume that f is of the form (5). Since $\operatorname{tr} h(f) = 0$, it follows that $(\partial_1^2 + \partial_2^2 + \partial_3^2)(f)_{|x_1 - ix_3} = 0$. Looking at the coefficients of x_3 resp. x_4 we get that $(a_2)_{x_2x_2} = 0$ resp. $(a_3)_{x_2x_2} = 0$, i.e. $\deg_{x_2} a_i \leq 1$ for $i = 2, 3$. Suppose that $\deg_{x_2} a_2 = 1$ or $\deg_{x_2} a_3 = 1$. Since a_2 and a_3 are algebraically dependent over k, they are both polynomials in one polynomial, say u, with $u(0) = 0$, over k (Gordan's lemma). Hence $\deg_{x_2} u = 1$ and $\deg_u a_2$, $\deg_u a_3 \leq 1$. Since f is reduced, we have $a_2(0) = a_3(0) = 0$. So from $u(0) = 0$, it follows that $a_2 = c_2 u$ and $a_3 = c_3 u$ for some $c_i \in k$. Hence a_2 and a_3 are linearly dependent over k, whence f is degenerate.

Now assume that $\deg_{x_2} a_2 = \deg_{x_2} a_3 = 0$, i.e. $a_2, a_3 \in k[x_1 + ix_3]$. We show that $a_2 \in k$, which implies that $a_2 = 0$ (since f is reduced) and hence that $f \in k[x_1 + ix_3, x_2, x_4]$. So f is degenerate. To see that $a_2 \in k$, observe that our assumption implies that f is of the form

$$f = q(x_1 + ix_3, x_2, x_4) + a_2(x_1 + ix_3)x_3 \tag{15}$$

So $M := h(f)_{|(x_1,x_2,0,x_3)}$ is of the form

$$M = \begin{pmatrix} q_{x_1x_1} & q_{x_2x_1} & iq_{x_1x_1} + (a_2)_{x_1} & q_{x_3x_1} \\ * & * & * & * \\ iq_{x_1x_1} + (a_2)_{x_1} & iq_{x_2x_1} & -q_{x_1x_1} + 2i(a_2)_{x_1} & iq_{x_3x_1} \\ * & * & * & * \end{pmatrix}$$

So if we substitute $T := i(a_2)_{x_1}$ in the matrix $TI_4 - M$ we get a matrix which first and third row are linearly dependent over k. Consequently $i(a_2)_{x_1}$ is a root of the characteristic polynomial T^4 of M. So $(a_2)_{x_1} = 0$ i.e. $a_2 \in k$, as desired.

iii) Now let f be of the form (6). Then by lemma 4.3, $h(f)$ is nilpotent iff $J(a_2(x_1, x_2), a_3(x_1, x_2))$ is nilpotent. So by [5, 7.1.7] a_2 and a_3 are linearly dependent over k, which implies that f is degenerate.

iv) Now let f be of the form (7), with $a = a_1x_2 + a_2x_3 + a_3x_4$ and $b = b_1x_2 + b_2x_3 + b_3x_4$, where $a_i, b_j \in k[x_1]$ for all i, j. If $\deg_{y_2} p = 1$, then we can rewrite f and "put the a_i's in the b_i's", so that we may assume that $a_1 = a_2 = a_3 = 0 \in k$. Also if $\deg_{y_2} p \geq 2$, we get that $a_2, a_3, a_4 \in k$. To see for example that $a_1 \in k$, consider the coefficient of the highest x_2 power in f, say $c(x_1)$. Since $\operatorname{tr} h(f) = 0$, it follows that $c''(x_1) = 0$ i.e. $\deg c(x_1) \leq 1$. Consequently, since $a_1(x_1)^2$ divides $c(x_1)$ (for $\deg p \geq 2$), we get that $a_1 \in k$. So $a_i \in k$ for all i. Without loss of generality we may assume that $a_1 \neq 0$. Then f is of the form

$$\begin{aligned} f &= c_1(x_1, a_1x_2 + a_2x_3 + a_3x_4) + c_2(x_1)x_3 + c_3(x_1)x_4 \\ &= c_1(x_1, a) + c_2(x_1, a)x_3 + c_3(x_1, a)x_4 \end{aligned}$$

where $a = a_1x_2 + a_2x_3 + a_3x_4$. So f is of the form 2i) of theorem 1.3, since obviously $c_2(x_1, a) = c_2(x_1)$ and $c_3(x_1, a) = c_3(x_1)$ are algebraically dependent over k. So by the proof of theorem 2.1 f is orthogonally equivalent to one the forms (4)-(6). For these cases we have already shown that f is degenerate.

v) Finally assume that f is of the form (8). The case $\deg_{y_2} p \leq 1$ and also the case $a_1, a_2, a_3 \in k$ follow by a similar argument as above. So we may assume that $\deg_{y_2} p \geq 2$ and that $\{a_1, a_2, a_3\}$ is not contained in k. We distinguish two subcases: $a_1 = 0$ and $a_1 \neq 0$. First assume $a_1 = 0$. Then f is of the form

$f = q(x_1 + ix_2, x_3, x_4) + b_1(x_1 + ix_2)x_2$, i.e. exactly of the form (15) with x_2 and x_3 interchanged. So by the argument given there we obtain $b_1 = 0$ and hence f is degenerate. Now assume that $a_1 \neq 0$. We will show that this case leads to a contradiction and hence cannot occur. Therefore put $u := a_1(x_1 + ix_2)x_2 + a_2(x_1 + ix_2)x_3 + a_3(x_1 + ix_2)x_4$. Then $\partial_4^{r-1} f = r!(\gamma a_3^{r-1})(x_1 + ix_2)u$. Since $\operatorname{tr} h(f) = 0$ we have $(\partial_1^2 + \ldots + \partial_4^2)f = 0$ and hence $(\partial_1^2 + \ldots + \partial_4^2)(\partial_4^{r-1} f) = 0$. Since $\partial_4^{r-1} f$ is linear in x_3 and x_4 and each polynomial in $x_1 + ix_2, x_3$ and x_4 belongs to ker $\partial_1^2 + \partial_2^2$ we get that

$$(\partial_1^2 + \partial_2^2)[(\gamma a_3^{r-1} a_1)(x_1 + ix_2)x_2] = 0$$

which implies that $\gamma a_3^{r-1} a_1 \in k$, as one easily verifies. Consequently $a_3^{r-1} a_1 \in k$. A similar argument gives that $a_2^{r-1} a_1 \in k$ (using ∂_3^{r-1} instead of ∂_4^{r-1}). Since $a_1 \neq 0$ and $\{a_1, a_2, a_3\}$ is not contained in k, it follows that $a_2 = a_3 = 0$. But then, again using that $\operatorname{tr} h(f) = 0$, now using $(\partial_2 - i\partial_1)^{r-1}$ instead of ∂_4^{r-1}, we obtain that $\gamma a_1^r \in k$, which implies that $a_1 \in k$. So all a_i belong to k, a contradiction. This completes the proof $\Box$

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