NILPOTENT SYMMETRIC JACOBIAN MATRICES
AND THE JACOBIAN CONJECTURE II

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Abstract
It is shown that the Jacobian Conjecture holds for all polynomial maps $F : k^n \rightarrow k^n$ of the form $F = x + H$, such that $JH$ is nilpotent and symmetric, when $n \leq 4$. If $H$ is also homogeneous a similar result is proved for all $n \leq 5$.

Introduction
Let $F := (F_1, \ldots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map i.e. each $F_i$ is a polynomial in $n$ variables over $\mathbb{C}$. Denote by $JF := (\frac{\partial F_i}{\partial x_j})_{1 \leq i, j \leq n}$, the Jacobian matrix of $F$. Then the Jacobian Conjecture (which dates back to Keller [9], 1939) asserts that if $\det JF \in \mathbb{C}^\ast$, then $F$ is invertible. It was shown in [1] and [12] that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$, where $JH$ is homogeneous and nilpotent (these two conditions imply that $\det JF = 1$); in fact it is even shown that the case where $JH$ is nilpotent and $H$ is homogeneous of degree 3 is sufficient.

For $n = 3$ resp. $n = 4$ this so-called cubic homogeneous case was proved by Wright resp. Hubbers in [11] resp. [8]. For $n = 3$, the case $F = x + H$, where $H$ is not necessarily homogeneous, but of degree 3, was proved by Vistoli in [10]. On the other hand, if $H$ has degree $\geq 4$ not much is known; if for example $F$ is of the form $x + H$ where $H$ is homogeneous of degree $\geq 4$, then all cases $n \geq 3$ remain open. The aim of this paper is to study these type of problems under the additional hypothesis that $JH$ is symmetric. This is no loss of generality since it was recently shown by the authors in [3] that it suffices to prove the Jacobian Conjecture for all polynomial maps $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ of the form $F = x + H$ with $JH$ nilpotent, homogeneous of degree $\geq 2$ and symmetric.

For such maps the conjecture was proved for all $n \leq 4$ in [6]. The proof of this result is based on a remarkable theorem of Gordan and Noether, which asserts that if $n \leq 4$, then $h(f)$, the Hessian matrix of the homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, is singular iff $f$ is degenerate i.e. there exists a linear coordinate change $T$ such that $f(Tx) \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. However if $n = 5$ such a result does not hold: the polynomial $f = x_1^2x_3 + x_1x_2x_4 + x_2^2x_5$ has a singular Hessian but is not degenerate.
Nevertheless one of the main results of this paper (theorem 4.1) asserts that the Jacobian Conjecture holds for all polynomial maps $F : \mathbb{C}^5 \rightarrow \mathbb{C}^5$ of the form $F = x + H$ with $JH$ nilpotent, homogeneous and symmetric. To prove this result we first extend the 3 dimensional Gordan-Noether theorem to the case where $f$ needs not be homogeneous, but has the additional property that $\text{tr } h(f) = 0$ (proposition 3.2). Next we show, using a result of [4], that in case $n = 5$ and $f$ is homogeneous, the condition $h(f)$ is nilpotent implies that $f$ is degenerate. Then we are in the position to apply the main result of [2], to conclude the above mentioned 5-dimensional result.

Finally we also extend the 4-dimensional homogeneous result obtained in [6] to the case where $H$ needs not be homogeneous (theorem 5.1).

1 Preliminaries

The main aim of this section is to fix the notations, collect some results from [2] and [4] and to give some additional preliminaries which we will need in the sequel.

Throughout this paper $k$ denotes an algebraically closed field of characteristic zero and $k^{[n]} := k[x_1, \ldots, x_n]$ is the polynomial ring in $n$ variables over $k$. By $H = (H_1, \ldots, H_n) : k^n \rightarrow k^n$ we mean a polynomial map, i.e. each $H_i$ belongs to $k^{[n]}$. One easily verifies that $JH$ is symmetric iff there exists an $f \in k^{[n]}$ such that $H_i = f_{x_i}$, the partial derivative of $f$ with respect to $x_i$, for all $i$. In particular, $JH = h(f) := (\frac{\partial^2 f}{\partial x_i \partial x_j})$, the Hessian matrix of $f$. We may obviously assume that $f$ is reduced, i.e. does not contain terms of degree $\leq 1$. Our main interest is to study the Jacobian Conjecture for all polynomial maps of the form $F = x + H$, where $JH$ is nilpotent and symmetric. As already remarked above, this is sufficient for investigating the Jacobian Conjecture. Starting point is the main result of [2]. To explain it, we need to formulate the (homogeneous) symmetric dependence problem:

(Homogeneous) Symmetric Dependence Problem (H)SDP(n).

Let $f \in k^{[n]}$ be a (homogeneous) polynomial in $k^{[n]}$ of degree $d \geq 2$ such that $h(f)$ is nilpotent. Are the rows of $h(f)$ linearly dependent over $k$?

The following result can be found in [2].

**Proposition 1.1**

i) SDP(n) has an affirmative answer for all $n \leq 2$.

ii) If $n \leq 4$ and $f \in k^{[n]}$ is homogeneous, then $h(f)$ is singular implies that $f$ is degenerate. In particular HSDP(n) has an affirmative answer if $n \leq 4$.

Since $f$ is assumed to be reduced, it is shown in [2, 1.2] that the dependence of the rows of $h(f)$ is equivalent to the fact that the partials $f_{x_i}$, of $f$ are linearly dependent over $k$, which in turn is equivalent to $f$ being degenerate. The main result of [2] asserts the following.

**Proposition 1.2** Let $n \geq 2$ and $H \in k[x_1, \ldots, x_n]^n$ with $JH$ symmetric and nilpotent. Then
i) $x + H$ is invertible if SDP$(p)$ has an affirmative answer for all $p \leq n$.

ii) If $H$ is homogeneous, then $x + H$ is invertible if SDP$(p)$ has an affirmative answer for all $p \leq n - 2$ and HSDP$(p)$ for $p = n - 1$ and $p = n$.

The remainder of this paper is therefore devoted to showing that SDP$(p)$ has an affirmative answer for all $p \leq 4$ as well as HSDP$(5)$.

In order to investigate nilpotent Hessians we first recall our main results on singular Hessians obtained in [4]. To formulate them we need some preliminaries. First, let $f \in k[x]$. A polynomial $g \in k[y]$ is called equivalent to $f$ if there exists $T \in GL_n(k)$ such that $g = f \circ T$ i.e. $g(x) = f(Tx)$. It is well-known that

$$h(g) = T^t h(f)|_{T^t T}$$

(1)

So if $g$ is equivalent to $f$ and $\det h(f) = 0$, then $\det h(g) = 0$ as well. Furthermore, if $\det h(f) = 0$ there exists a nonzero polynomial $R(y_1, \ldots, y_n) \in k[y_1, \ldots, y_n]$ such that $R(f_{x_1}, \ldots, f_{x_n}) = 0$. We say that $R$ is a relation of $f$. Consequently (since $\det h(g) = 0$), also the partials of $g$ are algebraically dependent over $k$. This enables us to give the following definition: let $f \in k[x]$ with $\det h(f) = 0$. Then $s(f)$ is the maximal natural number, $0 \leq s \leq n - 1$ for which there exists a $g \in k[y]$ equivalent to $f$ which has a relation in $k[y_{s+1}, \ldots, y_n]$. In other words $n - s(f)$ is the least number of variables a relation of a with $f$ equivalent polynomial can have.

**Theorem 1.3** Let $f \in k[x]$ be reduced and satisfy $\det h(f) = 0$. Then

1) If $n = 3$ then either $f$ is degenerate or equivalent to a polynomial of the form $a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3$.

2) If $n = 4$ and $s(f) \geq 1$ then either $f$ is degenerate or equivalent to a polynomial of one of the following forms:

   i) $a_1(x_1, x_2) + a_2(x_1, x_2) x_3 + a_3(x_1, x_2)(x_4$ with $a_2$ and $a_3$ algebraically dependent over $k$.

   ii) $p(x_1, a) + b$, with $p(y_1, y_2) \in k[y_1, y_2]$ and $a, b \in A x_2 + A x_3 + A x_4$ where $A = k[x_1]$.

3) If $n = 5$ and $f$ is homogeneous, then either $f$ is degenerate or equivalent to a polynomial of the form $p(a)$, where $a = a_1 x_3 + a_2 x_4 + a_3 x_5$ with $a_i \in A = k[x_1, x_2]$ for all $i$ and $p(X) \in A[X]$.

2 Orthogonal equivalence of polynomials with singular Hessians

Theorem 1.3 gives a classification for small $n$ of reduced polynomials with singular Hessians up to equivalence. In this section we refine this result, namely we obtain a classification of such polynomials up to orthogonal equivalence: two polynomials $f$
and $g$ in $k^n$ are called orthogonally equivalent if there exists an orthogonal matrix $T \in O(n)$ i.e. $T \in M_n(k)$ with $T^TT = I_n$, such that $g = f \circ T$. The advantage of working with orthogonal equivalence is that it preserves the nilpotency of Hessians, i.e. $b(f)$ is nilpotent if $h(g)$ is nilpotent (which follows from (1)). The main result of this section is

**Theorem 2.1** Let $f \in k^n$ be reduced and satisfy $\det h(f) = 0$. Then

1) If $n = 3$, then either $f$ is degenerate or orthogonally equivalent to a polynomial of one of the following two forms:

$$a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3$$  \hspace{1cm} (2)

$$a_1(x_1 + ix_2) + a_2(x_1 + ix_2)x_2 + a_3(x_1 + ix_2)x_2$$  \hspace{1cm} (3)

2) If $n = 4$ and $s(f) \geq 1$, then either $f$ is degenerate or orthogonally equivalent to a polynomial of one of the following forms:

$$U := a_1(x_1, x_2) + a_2(x_1, x_2)x_3 + a_3(x_1, x_2)x_4$$  \hspace{1cm} (4)

with $a_2$ and $a_3$ algebraically dependent over $k$,

$$U|_{x_1 := x_1 + ix_3}$$  \hspace{1cm} (5)

with $a_2$ and $a_3$ algebraically dependent over $k$,

$$U|_{x_1 := x_1 + ix_3, x_2 := x_2 + ix_4}$$  \hspace{1cm} (6)

with $a_2$ and $a_3$ algebraically dependent over $k$,

$$p(x_1, a) + b$$  \hspace{1cm} (7)

with $p(y_1, y_2) \in k[y_1, y_2]$, $a, b \in Ax_2 + Ax_3 + Ax_4$ and $A = k[x_1]$,

$$p(x_1, a) + b|_{x_1 := x_1 + ix_2}$$  \hspace{1cm} (8)

3) If $n = 5$ and $f$ is homogeneous, then either $f$ is degenerate or orthogonally equivalent to a polynomial of one of the following forms:

$$p(x_1, x_2, a)$$  \hspace{1cm} (9)

with $a = a_1x_1 + a_2x_4 + a_3x_5$ and $a_i \in A := k[x_1, x_2]$ for all $i$ and $p(y_1, y_2, y_3) \in k[y_1, y_2, y_3]$,

$$p(x_1, x_2, a)|_{x_1 := x_1 + ix_3}$$  \hspace{1cm} (10)

$$p(x_1, x_2, a)|_{x_1 := x_1 + ix_3, x_2 := x_2 + ix_4}$$  \hspace{1cm} (11)

The proof of this result is based on theorem 1.3 and the following lemma
Lemma 2.2 Let $v_1, \ldots, v_r \in k^n$ be linearly independent over $k$. Then there exist an $s : 0 \leq s \leq r$, an $S \in GL_r(k)$ and an orthogonal matrix $T \in O(n)$ such that

$$S \begin{pmatrix} v_1^t \\ \vdots \\ v_r^t \end{pmatrix} T = \begin{pmatrix} I_r & iI_s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1^t + ie_{r+1}^t \\ \vdots \\ e_s^t + ie_{r+s}^t \\ e_{s+1}^t \\ \vdots \\ e_r^t \end{pmatrix},$$

where $e_i$ is the $i$-th standard basis vector in $k^n$ (if $s = 0$ read $S(v_1^t, \ldots, v_r^t)T = (e_1^t, \ldots, e_r^t)$).

Proof. Put $A := ((v_i, v_j))_{1 \leq i, j \leq r}$. Since $A$ is symmetric, there exist an $S \in GL_r(k)$ and an $s : 0 \leq s \leq r$ such that

$$S^tAS = J := \begin{pmatrix} 0_s & \ast \\ \ast & I_{r-s} \end{pmatrix}.$$

Put $(\vec{v}_1 \cdots \vec{v}_r) := (v_1 \cdots v_r)S$. Then one readily verifies (or see [2, lemma 1.3]) that $(\langle \vec{v}_i, \vec{v}_j \rangle)_{i,j} = J$. So replacing the $v_i$ by the $\vec{v}_i$, we may assume that $(\langle v_i, v_j \rangle)_{i,j} = J$. Now we distinguish two cases: $s = 0$ and $s \geq 1$.

- **Case 1**: $s = 0$.
  Then by the Gram-Schmidt theorem, there exists an orthogonal matrix $T \in GL_n(k)$ such that the $j$-th row $T_j$ of $T$ equals $v_j$ for all $j : 1 \leq j \leq r$. So $T_1v_1 = 1$ and $T_jv_i = 0$ for all $i : 1 \leq i \leq r$ and all $j \neq i$. In other words, $Tv_i = e_i$ for all $i : 1 \leq i \leq r$, i.e. $T$ is an orthogonal matrix satisfying $T(v_1 \cdots v_r) = (e_1 \cdots e_r)$.

- **Case 2**: $s \geq 1$.
  So $\langle v_1, v_j \rangle = 0$ for all $j : 1 \leq j \leq r$. Observe that $v_1$ is perpendicular to $kv_1 + \cdots + kv_r$, so $r \leq n - 1$. We may assume that $v_{11} = 1$. So $\langle v_1, v_1 \rangle = 1$. Hence if we put $u := i(e_1 - v_1)$, then $\langle e_1, u \rangle = 0$ and $\langle u, u \rangle = 1$. So by the Gram-Schmidt theorem there exists an orthogonal matrix $T \in GL_n(k)$ with $T_1 = e_1^t$ and $T_{r+1} = u^t$, where again $T_j$ is the $j$-th row of $T$. So $T_jv_1 = 0$ for all $j \neq 1$ and $T_ju = 0$ for all $j \neq r + 1$, which by the definition of $u$ implies that $T_jv_i = T_je_1 = 0$ for all $j \notin \{1, r+1\}$. Also $T_{r+1}v_1 = (v_1, u) = i(\langle v_1, e_1 \rangle - \langle v_1, v_1 \rangle) = i$. Summarizing $T(v_1 = (T_1v_1, \ldots, T_nv_1) = (e_1 + ie_{r+1})$.

Define $w_j := Tv_j$ for all $j$. Then $T(v_1 \cdots v_r) = (w_1 \cdots w_r) = (\langle v_1 + ie_{r+1}, w_2, \cdots, w_r \rangle).$. Since $T$ is orthogonal, we have that $\langle w_i, w_j \rangle = \langle v_i, v_j \rangle$ for all $i,j$. Now replace for each $j \geq 2$ $w_j$ by $w_j - c_jw_1$ for suitable $c_j \in k$ (which operation can be obtained by replacing $(w_1 \cdots w_r)$ by $(w_1 \cdots w_r)S$ for suitable $S \in GL_r(k)$) we may assume that the first component of $w_j$ equals zero. Since $\langle w_1, w_j \rangle = 0$ for all $j \geq 2$, it follows, using $w_1 = e_1 + ie_{r+1}$, that also the $(r + 1)$-th component of $w_j$ equals zero. Now consider the $r - 1$ vectors $w_2, \ldots, w_r$ in $k^{n-2} = ke_2 + \cdots + ke_r + ke_{r+2} + \cdots + ke_n$ and use induction on $n$ $\square$
Corollary 2.3 Let \( v_1, \ldots, v_r, v_{r+1}, \ldots, v_n \) be a \( k \)-basis of \( k^n \). Put \( V_i := \langle v_i, x \rangle \). Let \( f \) be of the form

\[
f = p \left( V_1, \ldots, V_r, \sum_{j=r+1}^{n} a_j(V_1, \ldots, V_r)V_j, \sum_{j=r+1}^{n} b_j(V_1, \ldots, V_r)V_j \right)
\]

Then \( f \) is orthogonally equivalent to a polynomial of the form

\[
q \left( X_0, \sum_{j=r+1}^{n} c_j(X_0)x_j, \sum_{j=r+1}^{n} d_j(X_0)x_j \right)
\]

where \( X_0 = (x_1 + ix_{r+1}, \ldots, x_s + ix_{r+s}, x_{s+1}, \ldots, x_r) \).

**Proof.** Choose \( T \) and \( S \) as in Lemma 2.2. Observe that

\[
f = \tilde{p} \left( S(V_1, \ldots, V_r), \sum_{j=r+1}^{n} \tilde{a}_j(S(V_1, \ldots, V_r))V_j, \sum_{j=r+1}^{n} \tilde{b}_j(S(V_1, \ldots, V_r))V_j \right)
\]

for suitable \( \tilde{p}, \tilde{a}_j \) and \( \tilde{b}_j \). Now we claim that \( f \circ T \) is of the desired form. Notice first that it follows from lemma 2.2 that

\[
E := S(V_1 \circ T, \ldots, V_r \circ T) = S(v_1^*Tx, \ldots, v_r^*Tx)
\]

Consequently,

\[
f \circ T = \tilde{p}(X_0, \sum_{j=r+1}^{n} \tilde{a}_j(X_0)W_j, \sum_{j=r+1}^{n} \tilde{b}_j(X_0)W_j)
\]

where \( W_j := V_j \circ T \) is a linear form in all \( x_i \) over \( k \). Finally observe that

\[
\sum_{j=r+1}^{n} \tilde{a}_j(X_0)W_j, \sum_{j=r+1}^{n} \tilde{b}_j(X_0)W_j \in k[X_0] + \sum_{j=r+1}^{n} k[X_0]x_j
\]

So we can write \( f \circ T \) in the desired form \( \square \)

**Proof of theorem 2.1.** In each of the cases in theorem 2.1 it follows from theorem 1.3 that there exists \( T \in Gl_n(k) \) such that \( f \circ T \) is of the form

\[
p \left( x_1, \ldots, x_r, \sum_{j=r+1}^{n} a_j(x_1, \ldots, x_r)x_j, \sum_{j=r+1}^{n} b_j(x_1, \ldots, x_r)x_j \right)
\]

for suitable \( r, p, a_j \) and \( b_j \). Hence \( f \) is of the form described in corollary 2.3, where \( v_i^* \) is the \( i \)-th row of \( T^{-1} \). Then apply this corollary \( \square \)
3 The symmetric Jacobian Conjecture in dimension 3

The main result of this section is

**Theorem 3.1** Let \( F = x + H : k^3 \to k^3 \) be a polynomial map with \( JH \) symmetric and nilpotent. Then \( F \) is invertible.

**Proof.** This is an immediate consequence of proposition 1.1 i), proposition 1.2 and proposition 3.2 below □

**Proposition 3.2** \( SDP(3) \) has an affirmative answer.

**Proof.** Let \( f \in k^{[3]} \) be reduced and assume that \( h(f) \) is nilpotent. Then by theorem 2.1 we may assume that \( f \) is either of the form (2) or of the form (3).

i) Suppose first that \( f \) is of the form (2). Since \( \text{tr} h(f) = 0 \) this gives \( a''_1(x_1) + a''_2(x_1)x_2 + a''_3(x_1)x_3 = 0 \). So \( \text{deg} a_i \leq 1 \) for all \( i \). Since \( f \) is reduced, this implies that \( f = c_1 x_1 x_2 + c_2 x_1 x_3 \) for some \( c_i \in k \). It follows that \( f_{x_2} \) and \( f_{x_3} \) are linearly dependent over \( k \), so \( f \) is degenerate.

ii) Now assume that \( f \) is of the form (3). Then a simple computation gives \( \text{tr} h(f) = \partial_2^2 f + \partial_3^2 f = 2i a'_2(x_1 + ix_2) \). Since \( \text{tr} h(f) = 0 \), this implies that \( a_2 \in k \) and hence that \( a_2 = 0 \), since \( f \) is reduced. Consequently, \( f = a_1(x_1 + ix_2) + a_3(x_1 + ix_2)x_3 \in k[x_1 + ix_2, x_3] \). So \( f \) is degenerate □

4 The homogeneous symmetric Jacobian Conjecture in dimension 5

The main result of this section is

**Theorem 4.1** Let \( F = x + H : k^5 \to k^5 \) be a polynomial map with \( JH \) symmetric, nilpotent and homogeneous of degree \( \geq 2 \). Then \( F \) is invertible.

**Proof.** By propositions 1.1 i) and 3.2, \( SDP(n) \) has an affirmative answer for all \( n \leq 3 \). Also \( HSDP(4) \) has an affirmative answer by proposition 1.1. Furthermore we will show in proposition 4.2 below that \( HSDP(5) \) has an affirmative answer. Then the desired result follows from proposition 1.2 ii) □

**Proposition 4.2** \( HSDP(5) \) has an affirmative answer.

**Proof.** Let \( f \in k^{[5]} \) be homogeneous and reduced and assume that \( h(f) \) is nilpotent. Then by theorem 2.1 we may assume that \( f \) is of the form (9), (10) or (11). We will show that in each of these cases \( f \) is degenerate.
i) First assume that $f$ is either of the form (9) or (10). Since $f$ is homogeneous it follows that all $a_i$ are homogeneous of the same degree, say $d$. If $d = 0$ then $f$ is trivially degenerate. So assume $d \geq 1$. Write $p = \gamma_r(y_1, y_2)y_3^d + \gamma_{r-1}(y_1, y_2)y_3^{d-1} + \cdots$ and $\partial_i$ instead of $\partial_{x_i}$. Then $g := \partial_5^{r-1}f$ is of the form

$$g = b_1(x_1 + cx_3, x_2) + b_2(x_1 + cx_3, x_2)x_3 + b_3(x_1 + cx_3, x_2)x_4 + b_4(x_1 + cx_3, x_2)x_5$$

with $c \in \{0, i\}$ and $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 2$. Since $tr h(f) = 0$ we have $\Delta f = 0$ where $\Delta = \partial_1^2 + \cdots + \partial_5^2$. Consequently, using that $\partial_5^{r-1}\gamma$ commutes with $\Delta$, we get that $\Delta \partial_5^{r-1}f = \partial_5^{r-1}\Delta f = 0$ i.e. $\Delta g = 0$. It then follows from the form of $g$ that $(\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + cx_3, x_2) = 0$ for all $j \geq 2$, since $x_j(\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + cx_3, x_2)$ is the leading term of $x_j$ of $\Delta f$, seen as polynomial over $x_1 + cx_3, x_2, \ldots, x_5$, for all $j \geq 2$.

If $c = 0$, this implies that $b_j(x_1, x_2)$ is of the form $a_j(x_1 + ix_3)^s + \mu_j(x_1 - ix_3)^s$ for some $\lambda_j, \mu_j \in k$ and $s \geq 1$. If $c = i$, then it follows from $\partial_i^2 b_j(x_1 + ix_3, x_2) = (\partial_1^2 + \partial_2^2 + \partial_3^2)b_j(x_1 + ix_3, x_2) = 0$ that each $b_j(x_1 + ix_3, x_2)$ is of the form $a_j(x_1 + ix_3)^s + \mu_j x_j(x_1 + ix_3)^{s-1}$ for some $a_j, \mu_j \in k$ and $s = 1$. In both cases, the polynomials $b_2, b_3, b_4$ belong to a 2-dimensional $k$-vectorspace and hence are linearly dependent over $k$. Since $b_j = r!a_3^{r-1}\gamma_r a_j$ for all $j \geq 2$, also the polynomials $a_2, a_3, a_4$ are linearly dependent over $k$. In case (9), it follows that $f_{x_3, f_{x_4}}$ are linearly dependent over $k$, so $f$ is degenerate. In case (10), first make the coordinate change which sends $x_1$ to $x_1 - ix_3$. Then the same argument shows that $f_{x_1, -ix_3}$ is degenerate and hence so is $f$.

ii) So it remains to show the case (11). We will show that $a_1$ and $a_2$ are linearly dependent over $k$, which will imply that $f$ is degenerate. Write again $p = \gamma_r(y_1, y_2)y_3^d + \cdots$. We distinguish two cases: $r \geq 2$ and $r = 1$.

First assume $r \geq 2$. Make the coordinate change $X_1 := x_1 + ix_3, X_2 := x_2 + ix_4, X_j := x_j$ for all $j \geq 3$. Put $U := a_1(X_1, X_2)X_3 + a_2(X_1, X_2)X_4 + a_3(X_1, X_2)X_5$. Then the condition $tr h(f) = 0$, i.e. $\Delta f = 0$, becomes

$$2i(\partial_{X_1}\partial_{X_3} + \partial_{X_2}\partial_{X_4}) + \partial_{X_4}^2 + \partial_{X_5}^2 + \partial_{X_5}^2)(\gamma_r(X_1, X_2)U + \cdots) = 0 \quad (12)$$

Applying $\partial_{X_3}^{r-1}$ to this equation gives

$$2i(\partial_{X_1}\partial_{X_3} + \partial_{X_2}\partial_{X_4} + \partial_{X_4}^2 + \partial_{X_5}^2 + \partial_{X_5}^2)(r!a_1^{r-1}U) = 0$$

So

$$\partial_{X_1}(\gamma_r a_1^r) + \partial_{X_2}(\gamma_r a_1^{r-1}a_2) = \partial_{X_1}\partial_{X_3}\gamma_r a_1^{r-1}U + \partial_{X_2}\partial_{X_4}\gamma_r a_1^{r-1}U = 0$$

Consequently there exists a homogeneous element $h_1 \in k[X_1, X_2]$ such that

$$\gamma_r a_1^r = \partial_{X_1}h_1 \quad \text{and} \quad \gamma_r a_1^{r-1}a_2 = -\partial_{X_1}h_1 \quad (13)$$
So if we put \( D = a_1 \partial X_1 + a_2 \partial X_2 \), then \( h_1 \in \ker D \). Similarly, applying \( \partial X_i^{-1} \) to the equation (12) gives \( \partial X_i (\gamma_r a_1 a_2^{s-1}) + \partial X_i (\gamma_r a_2^s) = 0 \). So there exists a homogeneous element \( h_2 \in k[X_1, X_2] \) such that
\[
\gamma_r a_1 a_2^{s-1} = \partial X_i h_2 \text{ and } \gamma_r a_2^s = -\partial X_i h_2
\]
(14)

So \( h_2 \in \ker D \).

Since \( a_1 \) and \( a_2 \) are homogeneous of the same degree, both \( h_1 \) and \( h_2 \) are also homogeneous of the same degree. Also \( \ker D = k[v] \) for some homogeneous element \( v \in k[X_1, X_2] \) (by [5, 1.2.25]). Consequently \( h_1 = c_1 v^s \) and \( h_2 = c_2 v^s \) for some \( c_j \in k \) and \( s \geq 1 \). It follows that \( h_1 \) and \( h_2 \) are linearly dependent over \( k \) and hence so are \( \partial X_i h_1 \) and \( \partial X_i h_2 \). Whence by (13) and (14) \( a_i^{-1} \) and \( a_i^s \) are linearly dependent over \( k \), which implies that \( a_1 \) and \( a_2 \) are linearly dependent over \( k \) (since \( r \geq 2! \)).

So it remains to consider the case \( r = 1 \), which follows immediately from the next lemma (which is a slightly generalized version of lemma 1.2 of [3]).

**Lemma 4.3** Let \( 0 \leq s \leq \frac{n}{2} \) and \( f \in k[x] \) of the form
\[
f = a_0(z) + a_1(z)x_{s+1} + a_2(z)x_{s+2} + \ldots + a_{n-s}(z)x_n
\]
where \( z \) is an abbreviation of \( x_1 + ix_{s+1}, x_2 + ix_{s+2}, \ldots, x_s + ix_{2s} \). Then \( h(f) \) is nilpotent iff \( J(a_1, \ldots, a_s) \) is nilpotent.

**Proof:** \( h(f) \) is nilpotent iff \( \det(\partial X_n f) = T^n \). Put \( q := \frac{1}{2} \sum_{i=1}^n x_i^2 \). Then \( h(Tq) = T h_n \). Let \( S := (x_1 - ix_{s+1}, x_2 - ix_{s+2}, \ldots, x_s - ix_{2s}, x_{s+1}, \ldots, x_n) \). Then \( f \circ S = a_0 + a_1 x_{s+1} + \ldots + a_{n-s} x_n \). Since \( \det S = 1 \) it follows from (1) in section 1 that \( M := h(Tq - f) \circ S \) satisfies \( M = T^n \) if \( h(f) \) is nilpotent. Now observe that
\[
q \circ S = \frac{1}{2} \sum_{j=1}^s (x_j^2 - 2ix_j x_{j+s}) + \frac{1}{2} \sum_{j=s+1}^n x_j^2
\]
\[
= \frac{1}{2} \sum_{j=1}^s (x_j^2 - 2ix_j x_{j+s}) + \frac{1}{2} \sum_{j=s+1}^n x_j^2,
\]
Then it follows that \( M \) is of the form
\[
M = \begin{pmatrix}
* & -iTI_n J(a_1, \ldots, a_s)^2 & * \\
-iTI_n J(a_1, \ldots, a_s) & 0 & 0 \\
0 & 0 & TI_n - 2s
\end{pmatrix}
\]
Finally observe that
\[
\det M = (-1)^s \cdot \det(iTI_n J(a_1, \ldots, a_s)) \cdot \det(iTI_n J(a_1, \ldots, a_s)^2) \cdot T^{n-2s}
\]
\[
= \det(TI_n - iJ(a_1, \ldots, a_s))^2 T^{n-2s}
\]
Consequently \( \det M = T^n \) iff \( \det(\partial X_n - iJ(a_1, \ldots, a_s)) = T^s \), which implies the desired result. \( \square \)
5 The symmetric Jacobian Conjecture in dimension 4

The main result of this section is

Theorem 5.1 Let $F = x + H : k^4 \to k^4$ be a polynomial map with $JH$ symmetric and nilpotent. Then $F$ is invertible.

Proof. This is an immediate consequence of propositions 1.2, 3.2, 1.1 and 5.2 below □

Proposition 5.2 SDP(4) has an affirmative answer.

The proof of this result is based on theorem 1.3 2). In order to use this result we will first show that the hypothesis $h(f)$ is nilpotent indeed implies that $s(f) \geq 1$. For the proof of this implication we need to recall some results obtained in [7], which we summarize in the next two propositions.

Proposition 5.3 Let $f \in k[n]$ be homogeneous and $R \in k[y_1, \ldots, y_n]$ such that $R(f_{x_1}, \ldots, f_{x_n}) = 0$. Put $h_i := R_{y_i}(f_{x_1}, \ldots, f_{x_n})$ and $D := \sum_{i=1}^n h_i \partial_{x_i}$. Then

i) $D^2(x_i) = 0$ for all $i$.

ii) Let $f = Ax_i^r + x_i^{r+1}(\ldots)$, where $0 \neq A \in K[x_2, \ldots, x_n]$. If $h_1 = 0$, then $A(h_2, \ldots, h_n) = 0$.

Proposition 5.4 Let $D = \sum_{i=1}^n h_i \partial_{x_i}$ be a homogeneous derivation on $k[n]$ such that $D^2(x_i) = 0$ for all $i$ and denote by $\mu$ the dimension of the rational map $h: \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$. If $\mu \leq 1$ then there exist at least two linearly independent linear relations between the $h_i$.

Now we are ready to prove

Proposition 5.5 Let $f \in k[4]$ be reduced and such that $h(f)$ is nilpotent. Then $s(f) \geq 1$, i.e. there exists a nonzero degenerate polynomial $R \in k[y_1, y_2, y_3, y_4]$ such that $R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$.

Proof. If $rk h(f) \leq 2$, then $rk J(f_{x_1}, f_{x_2}, f_{x_3}) \leq 2$. So by [5, proposition 1.2.9], $trdeg_k k[f_{x_1}, f_{x_2}, f_{x_3}] \leq 2$, which implies that there exists a nonzero polynomial $R \in k[y_1, y_2, y_3]$ with $R(f_{x_1}, f_{x_2}, f_{x_3}) = 0$. Clearly $R$ is degenerate in $k[y_1, y_2, y_3, y_4]$. So we may assume that $rk h(f) = 3$.

i) Let $d := \deg f$. Observe that $d \geq 2$ since $f$ is reduced. Since $det h(f) = 0$ there exists some nonzero polynomial $R \in k[y_1, y_2, y_3, y_4]$, say of degree $r$, such that $R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$. Let $\tilde{f}$ be the leading part of $f$ and $\tilde{R}$ the leading part of $R$. Then $\tilde{R}(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$. So it follows from proposition 1.1 ii) that $\tilde{f}$ is degenerate.
ii) Let \( S \in k[y_1, y_2, y_3, y_4, y_5] \) be homogeneous of degree \( r \) and \( S(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}, 1) = 0 \) for \( i \leq 4 \) and \( g_{x_6} = x_6^d + 1 \). Since \( S \) is homogeneous and \( S(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}, 1) = 0 \) it follows that \( S(g_{x_1}, g_{x_2}, g_{x_3}, g_{x_4}, g_{x_6}) = 0 \). Now we want to apply proposition 5.3 ii) to the polynomial \( g \) in proposition 5.3 and recall that \( z = (x_1, x_2, x_3, x_4, x_5) \) for all \( i \leq 4 \). Hence \( h = \alpha h \) for some \( \alpha \in k(x_1, x_2, x_3, x_4) \). Consequently the dimension of the rational map \( \tilde{h} \) is \( 0 \).

iii) Let \( M := h(f)^m \) where \( M \neq 0 \) and \( h(f)^{m+1} = 0 \). Choose a nonzero column \( \tilde{h} \) of \( M \). Since \( h(f)M = 0 \) it follows that \( h(f)\tilde{h} = 0 \). Furthermore \( \langle \tilde{h}, \tilde{h} \rangle = 0 \), for \( M^2 = 0 \). Since

\[
0 = \partial_{x_i} R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4})
\]

\[
= \sum_{j=1}^{4} R_{y_j}(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) f_{x_j} x_i
\]

\[
= \sum_{j=1}^{4} h_j f_{x_j} x_i
\]

for all \( 1 \leq i \leq 4 \), we get that \( h(f)\tilde{h} = 0 \). Since we already saw that \( h(f)\tilde{h} = 0 \), the hypothesis that \( rk h(f) = 3 \) implies that \( h = \alpha h \) for some \( \alpha \in k(x_1, x_2, x_3, x_4) \). Hence \( \langle \tilde{h}, \tilde{h} \rangle = 0 \) implies that \( h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0 \).

iv) The polynomial \( z_1^2 + z_2^2 + z_3^2 + z_4^2 \) is clearly homogeneous. Furthermore, substituting \( x_5 = 1 \) gives \( h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0 \) (by iii)). Hence \( z_1^2 + z_2^2 + z_3^2 + z_4^2 \) is homogeneous of degree \( r \) and \( S(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}, 1) = 0 \) for \( i \leq 4 \). Hence \( \langle h, h \rangle = 0 \) implies that \( h_1^2 + h_2^2 + h_3^2 + h_4^2 = 0 \).

Now define \( D = \sum_{i=1}^{6} z_i \partial_{x_i} \). Then by proposition 5.3 i) \( D(z_i) = 0 \) for all \( i \).

Now assume that \( R \) was taken of minimal degree, then it follows that \( \sum_{i=1}^{4} c_i R_{y_i} = 0 \), i.e. \( R \) is degenerate, which completes the proof \( \Box \)
Proof of proposition 5.2. According to proposition 5.5 we may assume that $f$ is of one of the forms (4)-(8) of theorem 2.1.

i) Let $f$ be of the form (4). Then

$$h(f) = \begin{pmatrix} h(a_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_3 + \begin{pmatrix} h(a_3) & 0 \\ 0 & 0 \end{pmatrix} x_4 + A$$

where $A$ is a $4 \times 4$ matrix which entries are polynomials in $x_1$ and $x_2$. Since $h(f)$ is nilpotent, so is $h(a_2)c_1 + h(a_3)c_2$ for each $c_1, c_2 \in k$ (look at the highest $x_3$-term of $h(f)|_{(x_1, x_2, c_1, c_2, x_3)}$). In particular both $h(a_2)$ and $h(a_3)$ are nilpotent.

Then it is well-known that the reduced parts of $a_2$ and $a_3$ are polynomials in $x_1 + ix_2$ or $x_1 - ix_2$ over $k$. Say the reduced part of $a_2$ is a nonzero polynomial in $x_1 + ix_2$. Consequently the reduced part of $a_3$ is also a polynomial in $x_1 + ix_2$, for otherwise $h(a_2) + h(a_3) = h(a_2 + a_3)$ cannot be nilpotent.

Write $a_2 = c_1 x_2 + g_1(x_1 + ix_2)$ and $a_3 = c_2 x_2 + g_2(x_1 + ix_2)$, with $c_1, c_2 \in k$. Since $a_2$ and $a_3$ are algebraically dependent over $k$, the same holds for $c_1 x_2 + g_1(x_1)$ and $c_2 x_2 + g_2(x_1)$ (make the coordinate change $x_1 \mapsto x_1 - ix_2$). If $c_1 \neq 0$ or $c_2 \neq 0$, it follows readily that $c_1 g_2 - c_2 g_1 \in k$ (make a coordinate change which sends one of the elements $c_1 x_2 + g_i(x_1)$ to $x_2$). Therefore $c_1 g_2 = c_2 g_1$, for $g_1(0) = g_2(0) = 0$ due to the reducedness of $f$. Hence $a_2$ and $a_3$ are linearly dependent over $k$ (since $a_2(0) = a_3(0) = 0$), which implies that $f$ is degenerate. So we may assume that $c_1 = c_2 = 0$. So both $a_2$ and $a_3$ belong to $k[x_1 + ix_2]$.

Finally $M_c := h(f)|_{(x_1, x_2, c, 0)}$ is nilpotent for all $c \in k$ and is of the form

$$M_c = \begin{pmatrix} h(a_1 + ca_2) & a_2' & a_3' \\ a_2' & ia_2' & 0 \\ a_3' & ia_3' & 0 \end{pmatrix}$$

An easy computation shows that the characteristic polynomial of a $4 \times 4$ matrix of the form

$$\begin{pmatrix} A & B \\ B^t & 0 \end{pmatrix}$$

is of the form $T^4 - (\text{tr} A)T^3 + (\det A)T^2 + \cdots$. Since $M_c$ is nilpotent this implies that $h(a_1 + ca_2)$ is nilpotent for all $c \in k$. Taking $c = 1$ (and using that $a_1$ has no terms of degree $\leq 1$, since $f$ is reduced) it follows as above from $a_2 \in k[x_1 + ix_2]$ that $a_1 \in k[x_1 + ix_2]$. Consequently $f \in k[x_1 + ix_2, x_3, x_4]$, i.e. $f$ is degenerate.

ii) Now assume that $f$ is of the form (5). Since $\text{tr} h(f) = 0$, it follows that $(\partial_1^2 + \partial_2^2 + \partial_3^2)f|_{x_1, x_2, x_3} = 0$. Looking at the coefficients of $x_3$ resp. $x_4$ we get that $(a_2)_{x_2 x_2} = 0$ resp. $(a_3)_{x_2 x_2} = 0$, i.e. $\deg_{x_2} a_i \leq 1$ for $i = 2, 3$. Suppose that $\deg_{x_2} a_2 = 1$ or $\deg_{x_2} a_3 = 1$. Since $a_2$ and $a_3$ are algebraically dependent over $k$, they are both polynomials in one polynomial, say $u$, with $u(0) = 0$, over $k$ (Gordan’s lemma). Hence $\deg_u u = 1$ and $\deg_u a_2, \deg_u a_3 \leq 1$. Since $f$ is
reduced, we have $a_2(0) = a_3(0) = 0$. So from $a(0) = 0$, it follows that $a_2 = c_2u$ and $a_3 = c_3u$ for some $c_i \in k$. Hence $a_2$ and $a_3$ are linearly dependent over $k$, whence $f$ is degenerate.

Now assume that $\deg_{x_2} a_2 = \deg_{x_2} a_3 = 0$, i.e. $a_2, a_3 \in k[x_1 + ix_3]$. We show that $a_2 \in k$, which implies that $a_2 = 0$ (since $f$ is reduced) and hence that $f \in k[x_1 + ix_3, x_2, x_4]$. So $f$ is degenerate. To see that $a_2 \in k$, observe that our assumption implies that $f$ is of the form

$$f = q(x_1 + ix_3, x_2, x_4) + a_2(x_1 + ix_3)x_3$$  \hfill (15)

So $M := h(f)|_{(x_1,x_2,0,x_3)}$ is of the form

$$M = \begin{pmatrix} q_{x_1x_1} & q_{x_2x_2} & q_{x_3x_1} + (a_2)x_1 & q_{x_3x_1} \\ * & * & * & * \\ q_{x_1x_1} + (a_2)x_1 & q_{x_2x_1} & -q_{x_1x_1} + 2i(a_2)x_1 & q_{x_3x_1} \\ * & * & * & * \end{pmatrix}$$

So if we substitute $T := i(a_2)x_1$ in the matrix $TI_4 - M$ we get a matrix which first and third row are linearly dependent over $k$. Consequently $i(a_2)x_1$ is a root of the characteristic polynomial $T^4$ of $M$. So $(a_2)x_1 = 0$ i.e. $a_2 \in k$, as desired.

iii) Now let $f$ be of the form (6). Then by lemma 4.3, $h(f)$ is nilpotent iff $J(a_2(x_1, x_2), a_3(x_1, x_2))$ is nilpotent. So by [5, 7.1.7] $a_2$ and $a_3$ are linearly dependent over $k$, which implies that $f$ is degenerate.

iv) Now let $f$ be of the form (7), with $a = a_1x_2 + a_2x_3 + a_3x_4$ and $b = b_1x_2 + b_2x_3 + b_3x_4$, where $a_i, b_j \in k[x_1]$ for all $i, j$. If $\deg_g p = 1$, then we can rewrite $f$ and “put the $a_i$’s in the $b_i$’s”, so that we may assume that $a_1 = a_2 = a_3 = 0 \in k$. Also if $\deg_g p \geq 2$, we get that $a_2, a_3, a_4 \in k$. To see for example that $a_3 \in k$, consider the coefficient of the highest $x_2$ power in $f$, say $c(x_1)$. Since $\text{tr } h(f) = 0$, it follows that $c''(x_1) = 0$ i.e. $\deg c(x_1) \leq 1$. Consequently, since $a_1(x_1)^2$ divides $c(x_1)$ (for $\deg p \geq 2$), we get that $a_1 \in k$. So $a_1 \in k$ for all $i$. Without loss of generality we may assume that $a_1 \neq 0$. Then $f$ is of the form

$$f = c_1(x_1, a_1x_2 + a_2x_3 + a_3x_4) + c_2(x_1)x_3 + c_3(x_1)x_4$$

where $a = a_1x_2 + a_2x_3 + a_3x_4$. So $f$ is of the form 2i) of theorem 1.3, since obviously $c_2(x_1, a) = c_2(x_1)$ and $c_3(x_1, a) = c_3(x_1)$ are algebraically dependent over $k$. So by the proof of theorem 2.1 $f$ is orthogonally equivalent to one the forms (4)-(6). For these cases we have already shown that $f$ is degenerate.

v) Finally assume that $f$ is of the form (8). The case $\deg_g p \leq 1$ and also the case $a_1, a_2, a_3 \in k$ follow by a similar argument as above. So we may assume that $\deg_g p \geq 2$ and that $\{a_1, a_2, a_3\}$ is not contained in $k$. We distinguish two subcases: $a_1 = 0$ and $a_1 \neq 0$. First assume $a_1 = 0$. Then $f$ is of the form
\[ f = q(x_1 + ix_2, x_3, x_4) + b_1(x_1 + ix_2)x_2 + a_2(x_1 + ix_2)x_3 + a_3(x_1 + ix_2)x_4, \] i.e. exactly of the form (15) with \( x_2 \) and \( x_3 \) interchanged. So by the argument given there we obtain \( b_1 = 0 \) and hence \( f \) is degenerate. Now assume that \( a_1 \neq 0 \). We will show that this case leads to a contradiction and hence cannot occur. Therefore put \( u := a_1(x_1 + ix_2)x_2 + a_2(x_1 + ix_2)x_3 + a_3(x_1 + ix_2)x_4. \) Then \( \partial_4^{-1}f = r!(\gamma a_3^{-1})(x_1 + ix_2)u. \) Since \( \text{tr}h(f) = 0 \) we have \( (\partial_1^2 + \ldots + \partial_4^2)f = 0 \) and hence \( (\partial_1^2 + \ldots + \partial_4^2)(\partial_4^{-1}f) = 0. \) Since \( \partial_4^{-1}f \) is linear in \( x_3 \) and \( x_4 \) and each polynomial in \( x_1 + ix_2, x_3 \) and \( x_4 \) belongs to \( \ker \partial_1^2 + \partial_2^2 \) we get that

\[ (\partial_1^2 + \partial_2^2)[(\gamma a_3^{-1})(x_1 + ix_2)x_2] = 0 \]

which implies that \( \gamma a_3^{-1}a_1 \in k \), as one easily verifies. Consequently \( a_3^{-1}a_1 \in k \). A similar argument gives that \( a_2^{-1}a_1 \in k \) (using \( \partial_3^{-1} \) instead of \( \partial_4^{-1} \)). Since \( a_1 \neq 0 \) and \( \{a_1, a_2, a_3\} \) is not contained in \( k \), it follows that \( a_2 = a_3 = 0. \) But then, using that \( \text{tr}h(f) = 0 \), now using \( (\partial_2 - i\partial_1)^{-1} \) instead of \( \partial_4^{-1} \), we obtain that \( \gamma a_1^{-1} \in k \), which implies that \( a_1 \in k \). So all \( a_i \) belong to \( k \), a contradiction. This completes the proof \( \square \)

References


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