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HESSE AND THE JACOBIAN CONJECTURE

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Abstract

In this paper we give a survey of various recent results obtained by the authors in the study of the Jacobian Conjecture. It is shown that it suffices to investigate the conjecture for polynomial maps of the form $x + H$ with JH nilpotent and symmetric (and one may even assume that H is homogeneous of degree 3). Furthermore it is shown that for such maps the Jacobian Conjecture is true if $n \leq 4$ and if $n \leq 5$ when H is homogeneous (of arbitrary degree).

Introduction and history

Looking at the title, the reader may wonder what on earth Hesse has to do with the Jacobian Conjecture, since he died in 1874, i.e. 65 years before the Jacobian Conjecture was formulated by Keller in 1939?
The surprising answer will be: much more than one would expect!
Otto Hesse was born in 1811 in Königsberg, Germany, where he studied under Jacobi. He spent a while as a teacher of physics and chemistry, before he graduated from Königsberg in 1840. His main interest was in the study of algebraic functions, algebraic curves and the theory of invariants. It was in 1842, during an investigation of cubic and quadratic curves, that he introduced his famous Hesse matrix and Hessian determinant.
One of the questions he investigated is the following: for which polynomials $f \in \mathbb{C}^{[n]} := \mathbb{C}[x_1, \ldots, x_n]$, is the Hessian $h(f) := (\frac{\partial^2 f}{\partial x_i \partial x_j})$ singular at every point of $\mathbb{C}^n$, i.e. det $h(f) = 0$?
It is easy to verify that if $f \in \mathbb{C}[x_1, \ldots, x_{n-1}]$, then the last column and row of $h(f)$ consists of only zeroes, hence det $h(f) = 0$. More generally, if f is degenerate, i.e. if there exists a $T \in Gl_n(\mathbb{C})$ such that $f(Tx) \in \mathbb{C}[x_1, \ldots, x_{n-1}]$, then also det $h(f) = 0$. This follows readily from the formula

$$h(f \circ T) = T^t h(f)_{|Tx} T \tag{1}$$

In 1850, in volume 42 of Crelle's Journal and again in volume 56 of Crelle's Journal, Hesse stated that the converse is true for homogeneous polynomials, i.e. if $f \in \mathbb{C}^{[n]}$ is homogeneous and satisfies $\det h(f) = 0$, then f is degenerate or equivalently the partial derivatives $f_{x_1}, \ldots, f_{x_n}$ are linearly dependent over $\mathbb{C}$. In spite of the fact that Hesse's theorem appeared in most textbooks at his time, it turned out to be wrong. More precisely, in 1876 Gordan and Noether proved that Hesse's theorem is only correct if $n \leq 4$ and false for all $n \geq 5$. For example one easily verifies that

$$f = x_1^2 x_3 + x_1 x_2 x_4 + x_2^2 x_5 + x_6^3 + \ldots + x_n^3$$

has a singular Hessian, however the partial derivatives f_{x_i} are linearly independent over $\mathbb{C}$.

Now let's turn to the Jacobian Conjecture and explain how Hesse comes in. Recall that the Jacobian Conjecture asserts that a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ is invertible if det $JF \in \mathbb{C}^*$, where $JF = (\frac{\partial F_i}{\partial x_j})$ is the Jacobian matrix of F. In March 2002 the second author received a preprint from Tuck Washburn, in which he used the Gordan-Noether theorem to study the Jacobian Conjecture for so-called gradient mappings, i.e. mappings of the form $x + \nabla f = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n})$ with $f \in \mathbb{C}^{[n]}$. His main result asserted that for such maps the Jacobian Conjecture is true, in case f is homogeneous and $n \leq 4$. The arguments given in his preprint were not complete, however Washburn and the second author could overcome the difficulties, which resulted in the paper [11].

Reformulating the Jacobian Conjecture for gradient mappings, gives the following main result of [11]: let $F := x + H : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, where $H = (H_1, \ldots, H_n)$ is homogeneous of degree $d \geq 2$ and such that JH is nilpotent and symmetric. If $n \leq 4$, then F is invertible! Of course the interesting question to investigate next was: what happens in dimension 5 in case JH is homogeneous, nilpotent and symmetric? Since Hesse's theorem is false in dimension 5 our hope was to look for counterexamples.

In January 2003 the authors of this paper started to investigate the five dimensional case. This research has led to some surprising new discoveries, which will be described in this paper. For more details the reader is referred to the papers [2], [3], [4] and [5]. The most striking result improves the classical result of Bass, Connell, Wright and Yagzhev: it asserts that it suffices to investigate the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$ with JH homogeneous, nilpotent and symmetric (one may even assume that H is homogeneous of degree 3). Furthermore let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map of the form $x + H$ with JH nilpotent and symmetric. Then we proved the Jacobian Conjecture in the following cases:

i) $n \leq 4$ (H need not be homogeneous).

ii) $n \leq 5$ and H homogeneous (of arbitrary degree).

1 Dependence problems and the Jacobian Conjecture

Throughout this paper we have the following notations: k is an algebraically closed field of characteristic zero. So the equation $x^2 + 1 = 0$ has two solutions in k. We choose one and denote it by i. The polynomial ring in n variables over k is denoted by $k^{[n]}$ or $k[x]$ or $k[x_1, \ldots, x_n]$. Finally, the orthogonal group i.e. the set of all $n \times n$

matrices T with entries in k satisfying $T^tT = I_n$, is denoted by $\mathbb{O}(n)$.
In [1] Bass, Connell and Wright and in [19] Yagzhev showed that it sufffices to prove the Jacobian Conjecture for all polynomial maps of the form $F = x + H$, where $H = (H_1, \ldots, H_n)$ is homogeneous and its Jacobian matrix JH is nilpotent. In fact they even show that one may assume H to be of degree 3. So one becomes interested in studying nilpotent Jacobian matrices. This led various authors to the following problem (see [14, Conjecture 1], [15, Conjecture B], [16, Conjecture 11.3], [8, 7.1.7] and [6]).

Dependence Problem DP(n). Let $H = (H_1, \ldots, H_n) \in k[x]^n$ be such that JH is nilpotent. Are the rows of JH linearly dependent?

One easily verifies that in case $H_i(0) = 0$ for all i, the dependence of the rows of JH is equivalent to the linear dependence of the H_i over k. The importance of the Dependence Problem in connection with the Jacobian Conjecture comes from the following implication.

If $DP(p)$ has an affirmative answer for all $p \leq n$, then the Jacobian Conjecture holds for all polynomial maps $F : k^n \to k^n$ of the form $F = x + H$ with JH nilpotent.

A sketch of the proof of this implication is roughly as follows: we use induction on n and may obviously assume that $H(0) = 0$. So by $DP(n)$ there exist $c_i \in k$ not all zero such that $\sum c_i H_i = 0$. Take a $T \in Gl_n(k)$ which last row equals $(c_1 \ \cdots \ c_n)$. Then the last component of TH equals zero, so $THT^{-1} = (h_1, \ldots, h_{n-1}, 0)$ for some $h_i \in k^{[n]}$. Also $J(THT^{-1}) = TJH(T^{-1}x)T^{-1}$ is nilpotent and hence so is $J_{x_1,\ldots,x_{n-1}}(h_1, \ldots, h_{n-1})$. Viewing each h_i in $k(x_n)[x_1, \ldots, x_{n-1}]$ it follows from the induction hypothesis that $(x_1 + h_1, \ldots, x_{n-1} + h_{n-1})$ is invertible over $k(x_n)$. Then using 1.1.8 of [8] one can deduce that $T(x + H)T^{-1}$ is invertible over k and hence so is $x + H$.

So one can expect that $DP(n)$ has a negative answer in general. Indeed, only $DP(2)$ has an affirmative answer and for each $n \geq 3$ there exist counterexamples to $DP(n)$ ([8, 7.1.7]). On the other hand the homogeneous dependence problem is still open.

Homogeneous Dependence Problem HDP(n). Let $H = (H_1, \ldots, H_n) \in k[x]^n$ be homogeneous of degree ≥ 1 such that JH is nilpotent. Are the rows of JH linearly dependent over k ?

Apart from the case $n = 2$ mentioned above, affirmative answers to $HDP(n)$ are only known for $n = 3, d = 3$ (Wright, [18]) and $n = 4, d = 3$ (Hubbers, [13]). In 1993 C. Olech stated $HDP(n)$ for $k = \mathbb{R}$ as a conjecture (Conjecture B in [15]) and promised a bottle of Polish Vodka for either a proof or a counterexample.
The aim of this paper is to study both the homogeneous and inhomogeneous dependence problem under the additional hypothesis that JH is symmetric and secondly to investigate if for such H's the corresponding polynomial maps $F = x + H$ are invertible (they should be if the Jacobian Conjecture is true, since the nilpotence of JH implies that $JF = I + JH$ is invertible and hence that $\det JF \in k^*$). We have

(Homogeneous) Symmetric Dependence Problem (H)SDP(n). Let $H = (H_1, \ldots, H_n) : k^n \to k^n$ be a polynomial map such that JH is (homogeneous), nilpotent and symmetric. Are the rows of JH linearly dependent over k ?

Before we start looking at this problem, one might wonder: do there exist symmetric nilpotent Jacobian matrices other than the zero matrix?
Here is the first surprise: the answer is no if for example $k = \mathbb{R}$ or more generally if k is an ordered field. Namely we have

Proposition 1.1 *Let k be any ordered field. Then the zero matrix is the only nilpotent symmetric (Jacobian) matrix in $M_n(k[x_1, \ldots, x_n])$.*

Proof.

i) Let $M \in M_n(k)$ be such that M is symmetric and nilpotent. We show that $M = 0$. Namely suppose that M is non-zero. Then there exists $p \geq 2$ such that $A := M^{p-1} \neq 0$ and $M^p = 0$. Since $A \neq 0$ it has some non-zero row, say $(a_1, \ldots, a_n)$. Furthermore $A^2 = 0$ and A is symmetric. Consequently $a_1^2 + \ldots + a_n^2 = 0$, a contradiction since the left hand side is positive.

ii) Now let $J = J(x_1, \ldots, x_n) \in M_n(k[x_1, \ldots, x_n])$ be symmetric and nilpotent. Then for any n-tuple $(c_1, \ldots, c_n) \in k^n$ the matrix $J(c_1, \ldots, c_n) = 0$, according i). In particular for each i, j the polynomial J_{ij} vanishes on k^n. Since k is infinite (k is an ordered field and hence it has characteristic zero) it follows that $J_{ij} = 0$ for all i, j $\square$

However if for example k is an algebraically closed field then there do exist non-trivial symmetric, nilpotent Jacobian matrices. Before we give such an example, first observe that

Lemma 1.2 *Let $H = (H_1, \ldots, H_n) : k^n \to k^n$ be a polynomial map. Then JH is symmetric iff there exists $f \in k^{[n]}$ such that $H_i = f_{x_i}$ for all i iff $JH = h(f)$ for some $f \in k^{[n]}$.*

This is an immediate consequence of the classical Poincaré lemma which gives that $\partial_i H_j = \partial_j H_i$ for all i, j iff there exists $f \in k^{[n]}$ such that $H_i = f_{x_i}$ for all i. So symmetric Jacobian matrices are Hessian matrices.

Example 1.3 Let k be any field of characteristic zero containing elements $a_1, \ldots, a_n$, not all zero, such that $a_1^2 + \ldots + a_n^2 = 0$. Let $d \geq 2$ and $f := (a_1 x_1 + \ldots + a_n x_n)^d$. Then $h(f)$ is symmetric, nilpotent and non-zero.

Indeed, one readily verifies that $h(f) = d(d-1)(a_1 x_1 + \ldots + a_n x_n)^{d-2} (a_i a_j)_{i,j}$. So tr $h(f) = 0$ and rk $h(f) \leq 1$, which implies that $h(f)$ is nilpotent.

Now let's turn to the dependence problems for symmetric Jacobian matrices, i.e. to the problems $HSDP(n)$ and $SDP(n)$ and see how they are related to the Jacobian Conjecture. First we relate $HSDP(n)$ to the Gordan-Noether theorem, mentioned in the introduction. Recall that $f \in k^{[n]}$ is called degenerate if there exists $T \in Gl_n(k)$ such that $f \circ T = f(Tx) \in k[x_1, \ldots, x_{n-1}]$. Furthermore, if $f \in k^{[n]}$, then the polynomial $\tilde{f}$, obtained by subtracting from f all its monomials of degree ≤ 1, is called the **reduced part of** f and satisfies $h(f) = h(\tilde{f})$. We call f **reduced** if $f = \tilde{f}$. So in the study of Hessian matrices we may assume that f is reduced.

Proposition 1.4 *Let $f \in k^{[n]}$ be reduced. Then the following statements are equivalent*

i) The rows of $h(f)$ $(= J(f_{x_1}, \ldots, f_{x_n}))$ are linearly dependent over k.

ii) The columns of $h(f)$ are linearly dependent over k.

iii) $f_{x_1}, \ldots, f_{x_n}$ are linearly dependent over k.

iv) f is degenerate.

Proof. The equivalence of i), ii) and iii) is left to the reader. So assume iii). Then $\sum c_i f_{x_i} = 0$ for some c_i in k, not all zero. Choose $T \in Gl_n(k)$ with last column $(c_1, \ldots, c_n)$. Then the last component of the column $(J(f \circ T))^t = T^t (Jf)^t (Tx)$ is zero, whence $\partial_n (f \circ T) = 0$, which gives iv). For the converse implication, just reverse all implications □

Corollary 1.5 *$HSDP(n)$ has an affirmative answer for all $n \leq 4$.*

Proof. Let JH be homogeneous, symmetric and nilpotent. Then by 1.2 $JH = h(f)$ for some reduced homogeneous polynomial $f \in k^{[n]}$. Since $h(f)$ is nilpotent it follows in particular that $\det h(f) = 0$. So by the Gordan-Noether theorem f is degenerate, which by 1.4 implies that the rows of $h(f)$ are linearly dependent over k. □

Now we will investigate how the dependence problems are related to the Jacobian Conjecture. Above we showed that if $DP(p)$ holds for all $p \leq n$, then all maps $F : k^n \to k^n$ of the form $x + H$ with JH nilpotent are invertible. The main result of [2] shows that we can obtain a similar result in the symmetric case. More precisely we have

Theorem 1.6 *Let $n \geq 1$ and $H \in k[x]^n$ be such that JH is nilpotent and symmetric.*

i) If $SDP(p)$ holds for all $p \leq n$, then $x + H$ is invertible.

ii) If H is homogeneous and $SDP(p)$ holds for all $p \leq n - 2$ and $HSDP(p)$ for $p = n - 1$ and $p = n$, then $x + H$ is invertible.

Before we sketch the proof of this theorem let us first deduce the following result, which was obtained in [11].

Corollary 1.7 *Let $n \leq 4$ and $F = x + H : k^n \to k^n$ be a polynomial map such that H is homogeneous of degree $d \geq 2$. If $\det JF \in k^*$ and JF is symmetric, then F is invertible.*

Proof. Since H is homogeneous it is well-known that the condition $\det JF \in k^*$ implies that JH is nilpotent ([8, 6.2.11]). From the symmetry of JF it follows that JH is symmetric. Then the result follows from 1.6, 1.5 and the fact that $DP(2)$ and hence $SDP(2)$ have an affirmative answer □

To prove 1.6 we try to follow the argument in the proof of the implication "$DP(p)$ holds for all $p \leq n \Rightarrow x + H$ is invertible if JH is nilpotent" sketched above. Recall that in that proof the point was to replace H by THT^{-1}, where the last row of T consisted of the coefficients in the relation between the H_i. This implies that the last component of THT^{-1} equals zero. Since the map THT^{-1} also has a nilpotent Jacobian matrix, we obtained an induction situation etc.
Now turn to the case that JH is nilpotent and symmetric, so $H = (f_{x_1}, \ldots, f_{x_n})$ for some polynomial $f \in k^{[n]}$, and assume that $SDP(p)$ holds for all $p \leq n$. Then there exist $c_i \in k$, not all zero such that $\sum c_i f_{x_i} = 0$. Again we want to choose $T \in Gl_n(k)$ such that the last row equals $c = (c_1, \ldots, c_n)$. Then $J(THT^{-1})$ is nilpotent and the last component of THT^{-1} equals zero. However, in order to use induction on n, we must also have that $J(THT^{-1})$ is symmetric. This will be the case if we can choose T to be orthogonal, i.e. such that $T^tT = I_n$, since then $J(THT^{-1}) = J(THT^t) = TJH(T^tx)T^t = h(f \circ T^t)$ is symmetric. Observe that since the last component of THT^{-1} equals zero, this implies that $\partial_n(f \circ T^t) = 0$, i.e.

$$f \circ T^t \in k[x_1, \ldots, x_{n-1}] \tag{2}$$

However if T is any orthogonal matrix, then $rr^t = 1$ for any row r of T. So in case $c^tc(= c_1^2 + \ldots + c_n^2) = 0$ the row c cannot appear as (last) row of T. To overcome this difficulty we give a lemma which shows that also in case $cc^t = 0$ a result similar to (2) can be obtained. Let $\langle , \rangle$ denote the standard bilinear form on k^n, i.e. $\langle x, y\rangle = x^t y$ for all $x, y \in k^n$. Then we have

Lemma 1.8 *Let $c \in k^n$ be non-zero and $f \in k^{[n]}$ be reduced and satisfy $\langle c, (Jf)^t\rangle = 0$.*

i) If $\langle c, c\rangle \neq 0$ there exists $T \in \mathbb{O}(n)$ such that $f \circ T \in k[x_1, \ldots, x_{n-1}]$.

ii) If $\langle c, c\rangle = 0$ there exists $T \in \mathbb{O}(n)$ such that $f \circ T \in k[x_1, \ldots, x_{n-2}, x_{n-1} + ix_n]$.

Proof.

i) First assume that $\langle c, c\rangle \neq 0$. Replacing c by $c/\langle c, c\rangle^{\frac{1}{2}}$ (k is algebraically closed) we may assume that $\langle c, c\rangle = 1$. By the Gram-Schmidt theorem there exists an orthogonal $T \in Gl_n(k)$ which last column equals c. So it follows from the chain rule that $\partial_n(f \circ T)(x) = \sum c_j(\partial_j f)(Tx) = 0$, which gives i).

ii) Now let $\langle c, c\rangle = 0$. We may assume that $c_n = i$, whence $c_1^2 + \ldots + c_{n-1}^2 = 1$. So by the Gram-Schmidt theorem there exists an orthogonal matrix $S \in Gl_{n-1}(k)$ which last column equals $(c_1, \ldots, c_{n-1})$. Put

$$T = \begin{pmatrix} S & \\ & 1 \end{pmatrix}$$

Then the chain rule gives that

$$\partial_{n-1}(f \circ T)(x) = \sum_{j=1}^{n-1} c_j(\partial_j f)(Tx) = -i(\partial_n f)(Tx)$$

and $\partial_n(f \circ T) = (\partial_n f)(Tx)$. Consequently $(\partial_{n-1}+i\partial_n)(f \circ T) = 0$, which implies the desired result $\Box$

Proof of theorem 1.6 (started). Assume that $SDP(p)$ holds for all $p \leq n$ and let JH be nilpotent and symmetric. By 1.2 $H = (Jf)^t$ for some reduced $f \in k^{[n]}$ and by $SDP(n)$ there exists $0 \neq c \in k^n$ such that $\langle c, (Jf)^t \rangle = 0$.

i) If $\langle c, c \rangle \neq 0$ it follows from 1.8 i) that $g := f \circ T \in k[x_1, \ldots, x_{n-1}]$ for some $T \in \mathbb{O}(n)$. It suffices to prove that $x + (Jg)^t$ is invertible, since

$$x + (J(f \circ T))^t = T^t \circ (x + (Jf)^t) \circ T = T^{-1}(x + (Jf)^t)T$$

which implies that $x + (Jf)^t$ is invertible, too. Now observe that $h_{x_1,\ldots,x_{n-1}}(g)$ is nilpotent, since $h(g)$ $(= T^t h(f)_{|Tx} T)$ is nilpotent and $h(g)$ is of the form

$$\begin{pmatrix} h_{x_1,\ldots,x_{n-1}}(g) & 0 \\ 0 & 0 \end{pmatrix}$$

because g does not contain x_n. Then by the induction hypothesis $(x_1, \ldots, x_{n-1}) + (J_{n-1}g)^t$ is invertible over k. So $k[x_1+\partial_1(g), \ldots, x_{n-1}+\partial_{n-1}(g)] = k[x_1, \ldots, x_{n-1}]$. Since $\partial_n(f \circ T) = 0$, it follows that $k[x + J(f \circ T)^t] = k[x]$, i.e. $x + J(f \circ T)^t$ is invertible, as desired.

ii) So we may assume that $\langle c, c \rangle = 0$. Then by 1.8 ii), $g := f \circ T \in k[x_1, \ldots, x_{n-2}, x_{n-1} + ix_n]$ for some $T \in \mathbb{O}(n)$, or equivalently $(\partial_{n-1} + i\partial_n)g = 0$. This implies that the n-th column of $h(g)$ is equal to i times the $(n-1)$-th column of $h(g)$ and the same holds for the rows of $h(g)$ since it is a symmetric matrix. So $h(g)$ is of the form

$$\begin{pmatrix} A & u & iu \\ u^t & a & ia \\ iu^t & ia & -a \end{pmatrix}$$

where $A = h_{x_1,\ldots,x_{n-2}}(g)$, $a \in k[x]$ and $u \in k[x]^{n-2}$. Then by induction on r one easily verifies that the r-th power of such a matrix is again of the same form, with $A^r, \tilde{u}, \tilde{a}$ instead of A, u, a. Since $h(g)$ is nilpotent, it follows in particular that A is nilpotent as well, i.e. $h_{x_1,\ldots.x_{n-2}}(g)$ is nilpotent. Now put $R := k[x_{n-1} + ix_n]$. Then g can be viewed in $R[x_1, \ldots, x_{n-2}] \subset Q(R)[x_1, \ldots, x_{n-2}]$. Since $h_{x_1,\ldots,x_{n-2}}(g)$ is nilpotent it follows from the induction hypothesis that $G := (x_1+\partial_1(g), \ldots, x_{n-2}+\partial_{n-2}(g))$ is invertible over $Q(R)$. Also $\det J_{x_1,\ldots,x_{n-2}}G =$

1, since $h_{x_1,\dots,x_{n-2}}(g)$ is nilpotent. It then follows from [8, 1.1.8] that G is invertible over R. Hence, writing $g = u(x_1, \dots, x_{n-2}, x_{n-1} + ix_n)$ this gives

$$\begin{aligned} & k[x_1 + (\partial_1 u)(x_{**}, x_{n-1} + ix_n), \dots, \\ & x_{n-2} + (\partial_{n-2} u)(x_{**}, x_{n-1} + ix_n), x_{n-1} + ix_n] \\ & \quad = \quad k[x_1, \dots, x_{n-2}, x_{n-1} + ix_n] \end{aligned}$$

where $x_{**} = (x_1, \dots, x_{n-2})$ and $x_* = (x_1, \dots, x_{n-1})$. Making the substitutions $x_{n-1} \to x_{n-1} - ix_n$ and adding the extra k-algebra generator x_n to both k-algebras, we get

$$k[x_1 + (\partial_1 u)(x_*), \dots, x_{n-2} + (\partial_{n-2} u)(x_*), x_{n-1}, x_n] = k[x] \tag{3}$$

Finally let E_1 be the elementary map sending x_{n-1} to $x_{n-1} + ix_n$ and E_2 the elementary map sending x_n to $x_n - i(\partial_{n-1} u)(x_*)$. Then one readily verifies that $E_1 \circ (x + (Jg)^t) \circ E_1^{-1} \circ E_2 = (x_1 + (\partial_1 u)(x_*), \dots, x_{n-2} + (\partial_{n-2} u)(x_*), x_{n-1}, x_n)$. So by (3) this map is invertible and hence so is $x + (Jg)^t$. As before, this implies that $x + (Jf)^t$ is invertible, too □

2 The symmetric dependence problems

According to 1.6 we need to investigate both the homogeneous and the inhomogeneous symmetric dependence problems. For example to deduce that all polynomial maps $F : k^5 \to k^5$ of the form $x + H$ with JH nilpotent, homogeneous and symmetric are invertible, it suffices to show that both $SDP(3)$ and $HSDP(5)$ have an affirmative answer. Indeed, it turns out that both problems have an affirmative answer. This is proved in [5], based on results of [4]. In fact we have more generally

Theorem 2.1 *$SDP(p)$ has an affirmative answer for all $p \le 4$ and $HSDP(p)$ for all $p \le 5$.*

As a consequence of this result and 1.6 we get the following special case of the Jacobian Conjecture.

Theorem 2.2 *Let $F : k^n \to k^n$ be a polynomial map of the form $F = x + H$ with JH nilpotent and symmetric.*

i) If $n \le 4$, then F is invertible.

ii) If $n \le 5$ and H is homogeneous, then F is invertible.

To give an idea of the methods involved in the proof of 2.1 we first discuss the easiest case, namely we show that $SDP(3)$ has an affirmative answer. Then we sketch the proof of $HSDP(5)$. We refer to [4] and [5] for the complete proof and the more involved case concerning $SDP(4)$.
Starting point for our study of singular Hessians is the following observation: let

$f \in k^{[n]}$ with $\det h(f) = 0$. Since by 1.2 $h(f) = J(f_{x_1}, \ldots, f_{x_n})$, it follows from [8, 1.2.9] that the f_{x_i} are algebraically dependent over k. So there exists a non-zero polynomial $R \in k[y_1, \ldots, y_n]$ such that $R(f_{x_1}, \ldots, f_{x_n}) = 0$. We say that R is a **relation of** f. Suppose now that g is equivalent to f, i.e. $g = f \circ T$ for some $T \in Gl_n(k)$. Then, using the formula

$$h(f \circ T) = T^t h(f)_{|Tx} T$$

one deduces that $\det h(g) = 0$. So g has a relation as well. In fact one easily verifies that $\tilde{R} := R \circ (T^{-1})^t$ is a relation of g. Now it may happen that for suitable T, the relation $\tilde{R}$ of g contains less variables than n, for example when $\tilde{R} \in k[y_2, \ldots, y_n]$. This leads to the following definition

Definition 2.3 Let $f \in k^{[n]}$ with $\det h(f) = 0$. Then s_f is the maximum number $s : 0 \le s \le n-1$, for which there exists a g equivalent to f with a relation in $k[y_{s+1}, y_{s+2}, \ldots, y_n]$.

In other words, $n - s_f$ is the minimum number of variables that a relation of a polynomial equivalent to f can have. The number s_f plays an important role in the classification of f's with singular Hessian. To illustrate this, consider the following example

Example 2.4 If $f \in k^{[n]}$ is reduced, then $s_f = n-1$, if and only if f is degenerate.

Proof. If f is degenerate, then there is a $T \in Gl_n(k)$ such that $g = f \circ T \in k[x_1, \ldots, x_{n-1}]$. So $g_{x_n} = 0$ and y_n is a relation of g.
So assume that $s_f = n-1$. Then there is a $T \in Gl_n(k)$ such that $g = f \circ T$ satisfies $\tilde{R}(g_{x_n}) = 0$ for some $\tilde{R} \in k[y_n]$. So $g_{x_n} \in k$. Since f and hence g is reduced, it follows that $g_{x_n} = 0$, whence $g \in k[x_1, \ldots, x_{n-1}]$ □

One of the crucial problems in the understanding of the number s_f is the following:

$R_n(k)$. Let $f \in k^{[n]}$ with $\det h(f) = 0$. Is $s_f \ge 1$?

To obtain information on s_f we differentiate the relation $R(f_{x_1}, \ldots, f_{x_n})$
$= 0$ with respect to x_i. This gives

$$\sum R_{y_j}(f_{x_1}, \ldots, f_{x_n}) f_{x_j x_i} = 0$$

So if we define the derivation D by

$$D = D_R = \sum_{j=1}^{n} R_{y_j}(f_{x_1}, \ldots, f_{x_n}) \frac{\partial}{\partial x_j} \tag{4}$$

then $Df_{x_i} = 0$ for all i. Consequently, each coefficient $R_{y_j}(f_{x_1}, \ldots, f_{x_n})$ of D is contained in $\ker D$, whence

$$D^2(x_j) = 0 \text{ for all } 1 \le j \le n \tag{5}$$

Derivations having property (5) are the easiest type of locally nilpotent derivations. They appear in various papers such as [7], [10], [9] and [17], and without being mentioned explicitely they are also studied in [12].
A useful property of these derivations is that for small n they contain a non-zero linear form in their kernel. More precisely we have

Theorem 2.5 *Let D be a k-derivation on $k[x]$ satisfying (5).*

i) If $2 \leq n \leq 3$, then ker D contains a linear coordinate, however if $n = 7$ this need not be the case.

ii) If D is homogeneous and $3 \leq n \leq 4$, then ker D contains at least two independent linear coordinates, however if $n = 8$ this need not be the case.

iii) If D is as constructed in (4) and $n = 5$, then ker D contains two independent linear coordinates as well.

The proof of the affirmative statements can be found in [12]. The negative part of ii) is given in Exercise 6 of [8], p.164. Substituting $Q = 1$ in the example of that exercise gives the negative part of i).

Using these results we can give the following

Corollary 2.6 *Let $f \in k^{[n]}$ be such that det $h(f) = 0$.*

i) If $2 \leq n \leq 3$, then $s_f \geq 1$ and hence $R_n(k)$ has an affirmative answer.

ii) If f is homogeneous and $3 \leq n \leq 5$, then $s_f \geq 2$.

Proof.

i) Let R be a relation of f of minimal degree and D_R the corresponding derivation of (4). If $2 \leq n \leq 3$ it follows from 2.5 that D_R has a non-zero linear form $\sum \lambda_j x_j$ in its kernel. So $\sum_{j=1}^{n} \lambda_j R_{y_j}(f_{x_1}, \ldots, f_{x_n}) = 0$, i.e. $S := \sum \lambda_j R_{y_j}$ satisfies $S(f_{x_1}, \ldots, f_{x_n}) = 0$. By the minimal choice of $\deg R$ this implies that $S = 0$. So $JR \cdot \lambda = 0$. Now choose $T \in Gl_n(k)$ which first column equals λ. Then the first entry of $J(R \circ T) = JR(Ty)T$ equals zero. Consequently $\tilde{R} := R \circ T \in k[y_2, \ldots, y_n]$. Since $\tilde{R}$ is a relation of $f \circ (T^{-1})^t$ it follows that $s_f \geq 1$.

ii) The second part of the corollary follows from 2.5 by a similar argument □

In the proof of the statements that $SDP(3)$ and $HSDP(5)$ have affirmative answers, one more ingredient is needed. To describe it let $f \in k^{[n]}$ with det $h(f) = 0$ and assume that f is reduced and not degenerate. So by 2.4 $s := s_f \leq n-2$. Now let g be equivalent to f such that g has a relation $R \in k[y_{s+1}, \ldots, y_n]$, i.e. $R(g_{x_{s+1}}, \ldots, g_{x_n}) = 0$. Obviously we may assume that R is irreducible in $k[y_{s+1}, \ldots, y_n]$. Put $A :=$

$k[x_1, \ldots, x_s]$ and $K := Q(A)$, the quotient field of A. The g can be viewed as polynomial im $K[x_{s+1}, \ldots, x_n]$. As such, since $R(g_{x_{s+1}}, \ldots, g_{x_n}) = 0$, the partial derivatives of g are algebraically dependent over K, which by [8, 1.2.9] implies that the Hessian of g with respect to the $n-s$ variables $x_{s+1}, \ldots, x_n$, denoted $h_{x_{s+1},\ldots,x_n}(g)$, has rank $\leq (n-s)-1$. With these notations we have

Proposition 2.7 *If $R_{n-s}(K)$ has an affirmative answer, then rk $h_{x_{s+1},\ldots,x_n}(g) \leq (n-s)-2$.*

For the proof of this proposition we refer to [4]. Using the notations above we get

Corollary 2.8 *Let $n \geq 3$ and $s = n-2$. Then g is of the form $g = a_1 + a_2 x_{n-1} + a_3 x_n$, where all a_i belong to A and a_2 and a_3 are algebraically dependent over k.*

Proof. By 2.7 and the fact that $R_2(K)$ has an affirmative answer (2.6), it follows that rk $h_{x_{s+1},\ldots,x_n}(g) \leq (n-s)-2 = n-(n-2)-2 = 0$. So $h_{x_{s+1},\ldots,x_n}(g) = 0$, whence g is of the desired form. Finally, since $s = n-2$, it follows that $a_2 = g_{x_{n-1}}$ and $a_3 = g_{x_n}$ are algebraically dependent over k $\square$

Now we are able to show

$$SDP(3) \text{ has an affirmative answer.} \tag{6}$$

Proof. Let $f \in k^{[3]}$ and assume that $h(f)$ is nilpotent. Then in particular det $h(f) = 0$. So by 2.6 $s_f \geq 1$. If $s_f = 2 (= 3-1)$ then f is degenerate by 2.4 and we are done. So it remains the case that $s_f = 1 (= 3-2)$. Then by 2.8 it follows that f is equivalent to a polynomial of the form $a_1(x_1) + a_2(x_1)x_2 + a_3(x_1)x_3$. So $f = a_1(V_1) + a_2(V_1)V_2 + a_3(V_1)V_3$, with $V_i = \langle v_i, x \rangle$, $x = (x_1, x_2, x_3)$ and v_1, v_2, v_3 a k-basis of k^3.

i) Suppose that $\langle v_1, v_1 \rangle \neq 0$. Without loss of generality we may assume that $\langle v_1, v_1 \rangle = 1$. By the Gram-Schmidt theorem there exists $T \in \mathbb{O}(3)$ such that v_1 is the first column of T. Then

$$g := f \circ T = f(Tx) = a_1(\langle v_1, Tx \rangle) + a_2(\langle v_1, Tx \rangle)\langle v_2, Tx \rangle + a_3(\langle v_1, Tx \rangle)\langle v_3, Tx \rangle$$

Since $v_1^t T = e_1^t$ we get $\langle v_1, Tx \rangle = v_1^t T x = x_1$. So $g = a_1(x_1) + a_2(x_1)W_2 + a_3(x_1)W_3$, where $W_j = \langle v_j, Tx \rangle$. Observe that the W_j are linear in x_2 and x_3. So we can rewrite g in the form $g = b_1(x_1) + b_2(x_1)x_2 + b_3(x_1)x_3$, for some $b_j(x_1) \in k[x_1]$. Since $h(f)$ and hence $h(g)$ is nilpotent (for $T \in \mathbb{O}(3)$), it follows in particular that tr $h(g) = 0$. So $b_1''(x_1) + b_2''(x_1)x_2 + b_3''(x_1)x_3 = 0$. Since g is reduced this implies that $b_1 = 0, b_2 = c_1 x_1$ and $b_3 = c_2 x_1$ for some $c_i \in k$. So $g = c_1 x_1 x_2 + c_2 x_1 x_3 \in k[x_1, c_1 x_2 + c_2 x_3]$. Consequently, g is degenerate and hence so is f.

ii) Now assume that $\langle v_1, v_1 \rangle = 0$. Write $v_1 = (p_1, p_2, p_3)$. Without loss of generality we may assume that $p_1 = 1$. So $p_2^2 + p_3^2 = -1$. Let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -ip_2 & ip_3 \\ 0 & -ip_3 & -ip_2 \end{pmatrix}$$

Observe that $T \in \mathbb{O}(3)$ and that $v_1^t T = e_1 + ie_2$. So $\langle v_1, Tx \rangle = x_1 + ix_2$ and consequently

$$g := f \circ T = a_1(x_1 + ix_2) + a_2(x_1 + ix_2)W_2 + a_3(x_1 + ix_2)W_3$$

where $W_j = \langle v_j, Tx \rangle$. So we can rewrite g in the form $b_1(x_1 + ix_2) + b_2(x_1 + ix_2)x_2 + b_3(x_1 + ix_2)x_3$. Since tr $h(g) = 0$, a simple computation gives that $2ib_2''(x_1 + ix_2) = 0$, which implies that $b_2 = 0$, since g is reduced. Consequently $g \in k[x_1 + ix_2, x_3]$. So g is degenerate and hence so is f. $\square$

To conclude this section we outline the proof of the fact that $HSDP(5)$ has an affirmative answer. We refer to [5] for more details. So let $f \in k^{[5]}$ be homogeneous and such that $h(f)$ is nilpotent. The proof consists essentially of three steps.

- First we show that the singularity of the Hessian of f implies that either f is degenerate or f is equivalent to a polynomial of the form

$$p(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$$

  where each $a_i \in A := k[x_1, x_2]$ and $p(X) \in A[X]$.

- Then we deduce from this result that the singularity of the Hessian of f implies that either f is degenerate or is orthogonally equivalent to a polynomial of one of the following forms
  1) $p(a) := p(x_1, x_2, a_1(x_1, x_2)x_3 + a_2(x_1, x_2)x_4 + a_3(x_1, x_2)x_5)$.
  2) $p(a)_{|x_1 = x_1 + ix_3}$.
  3) $p(a)_{x_1 = x_1 + ix_3, x_2 = x_2 + ix_4}$.

- Finally we show that in each of these three cases the nilpotency of $h(f)$ implies that f is denegerate.

In fact in the cases 1) and 2) already the condition tr $h(f) = 0$ is sufficient to imply that f is degenerate. The same is true in case 3), when $\deg_X p(X) \geq 2$. However when in case 3) $\deg_X p(X) = 1$, then the condition tr $h(f) = 0$ is not sufficient to imply that f is degenerate. In fact, working out the nilpotency condition one finds the following

$$h(f) \text{ is nilpotent iff } J_{x_1, x_2}(a_1(x_1, x_2), a_2(x_1, x_2)) \text{ is nilpotent.} \tag{7}$$

Since this last condition gives that a_1 and a_2 are linearly dependent over k ([8, 7.1.7]), one easily deduces that f is degenerate, which completes the proof.

The equivalence (7) can be generalized as follows

Lemma 2.9 *Let $0 \leq s \leq \frac{n}{2}$ and $f \in k^{[n]}$ of the form*

$$f = a_0(z) + a_1(z)x_{s+1} + a_2(z)x_{s+2} + \ldots + a_{n-s}(z)x_n$$

where z is an abbreviation of $x_1 + ix_{s+1}, x_2 + ix_{s+2}, \ldots, x_s + ix_{2s}$. Then $h(f)$ is nilpotent iff $J(a_1, \ldots, a_s)$ is nilpotent.

Proof. $h(f)$ is nilpotent iff det $(TI_n - h(f)) = T^n$. Put $q := \frac{1}{2}\sum_{i=1}^n x_i^2$. Then $h(Tq) = TI_n$. Let $S := (x_1 - ix_{s+1}, x_2 - ix_{s+2}, \ldots, x_s - ix_{2s}, x_{s+1}, \ldots, x_n)$. Then $f \circ S = a_0 + a_1x_{s+1} + \ldots + a_{n-s}x_n$. Since det $S = 1$ it follows from (1) that $M := h((Tq - f) \circ S)$ satisfies det $M=$ det $(h(Tq) - h(f)) = T^n$ iff $h(f)$ is nilpotent. Now observe that

$$\begin{aligned} q \circ S &= \frac{1}{2}\sum_{j=1}^{s}(x_j^2 - 2ix_jx_{j+s} - x_{j+s}^2) + \frac{1}{2}\sum_{j=s+1}^{n} x_j^2 \\ &= \frac{1}{2}\sum_{j=1}^{s}(x_j^2 - 2ix_jx_{j+s}) + \frac{1}{2}\sum_{j=2s+1}^{n} x_j^2. \end{aligned}$$

Then it follows that M is of the form

$$M = \begin{pmatrix} * & -iTI_s - J(a_1, \ldots, a_s)^t & * \\ -iTI_s - J(a_1, \ldots, a_s) & 0 & 0 \\ * & 0 & TI_{n-2s} \end{pmatrix}$$

Finally observe that

$$\begin{aligned} \det M &= (-1)^s \det(iTI_s + J(a_1, \ldots, a_s)) \det(iTI_s + J(a_1, \ldots, a_s)^t) \cdot T^{n-2s} \\ &= \det(TI_s - iJ(a_1, \ldots, a_s))^2 T^{n-2s} \end{aligned}$$

Consequently det $M = T^n$ iff det $(TI_n - iJ(a_1, \ldots, a_s)) = T^s$, which implies the desired result $\square$

In fact this lemma formed the starting point of a rather surprising discovery, which will be described in the next section.

3 Reduction of the Jacobian Conjecture to the symmetric case

At first glance, studying polynomial maps of the form $x + H$ with JH nilpotent and the additional condition that JH is symmetric seems rather special. However, the main results of this section, 3.1 and 3.2, assert that it suffices to investigate the Jacobian Conjecture for all $n \geq 2$ and all polynomial maps of the form $F = x + H$ with JH nilpotent, symmetric and homogeneous (even of degree 3).
In order to formulate the main result and to honor Hesse, we introduce

Hessian Conjecture $HC(n)$. Let $f \in k^{[n]}$. If $h(f)$ is nilpotent, then $F := (x_1 + f_{x_1}, \ldots, x_n + f_{x_n})$ is invertible.

Since $h(f) = J(f_{x_1}, \ldots, f_{x_n})$, it follows that if the n-dimensional Jacobian Conjecture is true, then $HC(n)$ is true as well. The surprising point is now

Theorem 3.1 *The Jacobian Conjecture is equivalent to the Hessian Conjecture. More precisely, if $HC(2n)$ holds, then $x + H$ is invertible for every $H : k^n \to k^n$ with JH nilpotent.*

Proof. Let $H := (H_1, \ldots, H_n) \in k[x]^n$ with JH nilpotent. Then obviously also $J(-iH)$ is nilpotent. Introduce n new variables $x_{n+1}, \ldots, x_{2n}$ and put

$$f := (-i)(H_1(z)x_{n+1} + \cdots + H_n(z)x_{2n})$$

where $z := (x_1 + ix_{n+1}, \ldots, x_n + ix_{2n})$. Then it follows from 2.9, with $a_0 = 0$ and $a_j = (-i)H_j$ for all $1 \leq j \leq n$, that $h(f)$ is nilpotent. So the assumption $HC(2n)$ implies that $F := (x_1 + f_{x_1}, \ldots, x_{2n} + f_{x_{2n}})$ is invertible. Consequently $F \circ S$ is invertible, where

$$S = (x_1 - ix_{n+1}, \ldots, x_n - ix_{2n}, x_{n+1}, \ldots, x_{2n})$$

Writing x instead of $(x_1, \ldots, x_n)$, an easy calculation shows that

$$\begin{aligned} F \circ S \quad = \quad & (x_1 - ix_{n+1} - i\sum_j H_{jx_1}(x)x_{n+j}, \ldots, x_n - ix_{n+1} - i\sum_j H_{jx_n}(x)x_{n+j}, \\ & x_{n+1} + \sum_j H_{jx_1}(x)x_{n+j} - iH_1, \ldots, x_{2n} + \sum_j H_{jx_n}(x)x_{n+j} - iH_n) \end{aligned}$$

Hence $S^{-1} \circ F \circ S = (x_1 + H_1(x), \ldots, x_n + H_n(x), *, \ldots, *)$ is invertible, which implies that $x + H$ is invertible, as desired $\square$

So we get the following improvement of the classical result of [1] and [19].

Corollary 3.2 *It suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all F of the form $F = (x_1 + f_{x_1}, \ldots, x_n + f_{x_n})$, where $h(f)$ is nilpotent and f homogeneous of degree 4 (or equivalently for all $n \geq 2$ and all F of the form $F = x + H$, with JH nilpotent and symmetric and H homogeneous of degree 3).*

Proof. Follows immediately from 3.1 and corollary 2.2 of [1] $\square$

4 Final remarks

To conclude this paper we like to mention some open problems.

Problem 4.1 Does $HSDP(6)$ has an affirmative answer?

If $HSDP(6)$ does have an affirmative answer, then it follows from 1.6 ii) and 2.1 that the 6-dimensional Hessian Conjecture is true.
In connection with 4.1 it would be useful to have an affirmative answer to the following problem.

Problem 4.2 Let D be a homogeneous derivation on $k[x]$ of degree $d \geq 2$ and such that $D^2(x_j) = 0$ for all j. Does D have a non-zero linear form in its kernel?

We like to remark that this problem is a special case of the "classical" homogeneous dependence problem $HDP(n)$. To see this, just observe that the condition $D^2(x_j) = 0$ for all j implies that $F := \exp(D)$ is an automorphism of the form $F = x + H$, where $H_j = D(x_j)$ is homogeneous of degree $d \geq 2$. Consequently $\det JF = 1$ and hence by [8, 6.2.11] the matrix JH is nilpotent. Finally observe that D has a non-zero linear form in its kernel iff the H_j are linearly dependent over k.

References

[1] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture: Reduction of degree and formal Expansion of the Inverse, Bulletin of the AMS, 7(1982), 287-330.
[2] M. de Bondt and A. van den Essen, Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture, Report 0307, University of Nijmegen, 2003.
[3] M. de Bondt and A. van den Essen, A reduction of the Jacobian Conjecture to the symmetric case, Report 0308, University of Nijmegen, 2003, to appear in Proceedings of the AMS.
[4] M. de Bondt and A. van den Essen, Singular Hessians, Report 0317, University of Nijmegen, 2003.
[5] M. de Bondt and A. van den Essen, Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture, II, Report 0318, University of Nijmegen, 2003.
[6] A. Cima, A. Gasull and F. Mañosas, The discrete Markus-Yamabe problem, Nonlinear Analysis: Theory, Methods & Applications 35 (1999), no. 3, 343-354.
[7] J. Deveney and D. Finston, G_a actions on $\mathbb{C}^3$ and $\mathbb{C}^7$, Communications in Algebra, 22, no. 25 (1994), 6295-6302.
[8] A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture, Vol. 190, in Progress in Mathematics, Birkhäuser, 2000.
[9] A. van den Essen and E. Hubbers, $D_n(A)$ for polynomial automorphisms and stable tameness, J. of Algebra, 192 (1997), 460-475.
[10] A. van den Essen and T. Janssen, Kernels of elementary derivations, Report 9548, University of Nijmegen, 1995.
[11] A. van den Essen and S. Washburn, The Jacobian Conjecture for symmetric matrices, Report 0301, University of Nijmegen, 2003, to appear in Journal of Pure and Applied Algebra.
[12] P. Gordan and M. Noether, Über die algebraische Formen, deren Hesse'sche Determinante identisch verschwindet, Mathematische Annalen, 10 (1876), 547-568.
[13] E. Hubbers, The Jacobian Conjecture: Cubic homogeneous maps in Dimension Four, Master's thesis, University of Nijmegen, 1994.
[14] G. Meisters, Polymorphisms conjugate to Dilatations, pp. 67-88 in Automorphisms of Affine Spaces, Kluwer Academic Publishers (ed. A. van den Essen), 1995.
[15] C. Olech, On the Markus-Yamabe stability conjecture, pp. 127-137 in Proc. of the Intern. Meeting on Ordinary Differential Equations and their Applications, University of Florence, 1995.

[16] K. Rusek, Polynomial Automorphisms, preprint 456, Inst. of Math. Polish Acad. of Sciences, IMPAN, Warsaw, 1989.
[17] Z. Wang, Locally nilpotent derivations of polynomial rings, Ph.D. thesis, University of Ottawa, 1999.
[18] D. Wright, The Jacobian Conjecture: linear triangularization for cubics in dimension three, Linear and Multilinear Algebra, 34 (1993), 85-97.
[19] A, Yagzhev, On Keller's problem, Siberian Math. J., 21 (1980), 747-754

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