ON THE MATHIEU CONJECTURE FOR $SU(2)$

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ABSTRACT. We study the Mathieu Conjecture for $SU(2)$ using the matrix elements of its unitary irreducible representations. We state a conjecture for the particular case $SU(2)$ implying the Mathieu Conjecture for $SU(2)$.

1. Introduction

Conjecture 1.1 (Mathieu [6]). Let $G$ be a compact connected Lie group and let $f$ be complex-valued $G$-finite function on $G$ such that $\int_G f^P(g) \, dg = 0$ for every $P \in \mathbb{N}_{>0}$. Then for any complex-valued $G$-finite function $h$ on $G$ we have $\int_G f^P(g) h(g) \, dg = 0$ for $P \gg 0$.

The Mathieu Conjecture 1.1 dates back to 1997 and is closely related to the Jacobian conjecture, since it actually implies the Jacobian conjecture, see [6]. See van den Essen [2], Smale [7] for more information on the history of the Jacobian conjecture. The Mathieu Conjecture 1.1 was proved for abelian compact groups by Duistermaat and Van der Kallen [1] in 1998. We study the Mathieu Conjecture 1.1 for the case $G = SU(2)$. Using explicit formulas for the Haar measure and known representation theoretic properties of $SU(2)$ we make the Mathieu Conjecture 1.1 more explicit. In particular, we use the fact that $SU(2)$-finite functions are finite linear combinations of matrix elements of finite dimensional irreducible representations of $SU(2)$ and that the matrix elements behave well under a subgroup $K \cong U(1)$ according to suitable characters. Note that the Mathieu Conjecture 1.1 is linear in the $G$-finite function $h$, but not in the $G$-finite function $f$. By the Peter-Weyl theorem, any $SU(2)$-finite function is the finite linear combination of matrix elements of irreducible representations. After recalling the necessary results on $SU(2)$ in Section 2, we show in Section 3 the validity of the Mathieu Conjecture 1.1 for $f$ a single matrix element or a sum of two matrix elements. For the sum of three matrix elements there is a partial result. These considerations lead to Conjecture 4.1 and Theorem 4.2 shows that this conjecture implies the Mathieu Conjecture 1.1 for $SU(2)$. Conjecture 4.1 describes the condition $\int_{SU(2)} f(g)^P \, dg = 0$ for all $P > 0$ in terms of a support condition on the characters of the abelian subgroup $U(1)$ of $SU(2)$ acting from the left and right on the individual matrix elements occurring in $f$.

We note that the Mathieu Conjecture 1.1 for bi-$K$-invariant functions is settled by Francoise et al. [3, Cor. 4.1], since the bi-$K$-invariant $SU(2)$-finite functions are the polynomials on $[-1, 1]$. 

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2. $SU(2)$

We briefly recall some required notions of $SU(2)$. Details can be found in e.g. [8], [9]. Let $k(\phi) = \left( e^{i\phi/2} \begin{array}{cc} 1 & 0 \\ 0 & e^{-i\phi/2} \end{array} \right)$ and $a(\theta) = \left( \begin{array}{cc} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{array} \right)$ be elements of $SU(2)$, then any element $g \in SU(2)$ can be expressed in terms of Euler angles $g = k(\phi)a(\theta)k(\psi)$ with $\phi \in [0,2\pi)$, $\theta \in (0,\pi)$, $\psi \in [-2\pi,2\pi)$. In terms of the Euler angles the Haar integral is, cf. [8, III, §3.9, (5)],

$$\int_{SU(2)} f(g) \, dg = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_{-2\pi}^{2\pi} F(\phi, \theta, \psi) \sin \theta \, d\psi \, d\theta \, d\phi,$$

(2.1)

where $F(\phi, \theta, \psi) = f(k(\phi)a(\theta)k(\psi))$. Denote the subgroup $K \cong U(1)$ generated by $k(\phi)$. For a function $f$ transforming by a non-trivial $K$-character under left- or right multiplication by $K$, we have $\int_{SU(2)} f(g) \, dg = 0$ by (2.1). The subgroup generated by $a(\theta)$ is the group $SO(2)$.

The finite-dimensional irreducible representations are labeled by the spin $\ell \in \frac{1}{2}\mathbb{N}$ and are of dimension $2\ell + 1$. The standard basis for the representation space is labeled as $\{ -\ell, -\ell + 1, \ldots, \ell \}$, and the corresponding matrix elements $t_{m,n}^\ell$ are $SU(2)$-finite functions, and any $SU(2)$-finite function is a finite linear combination of matrix elements of irreducible finite-dimensional representations. The matrix-elements $t_{m,n}^\ell$ behave well according to left and right action by $K$, cf. [8, III, §3.3, (3)]

$$t_{m,n}^\ell(k(\phi)g) = e^{-im\phi} t_{m,n}^\ell(g), \quad t_{m,n}^\ell(gk(\psi)) = e^{-in\psi} t_{m,n}^\ell(g).$$

(2.2)

In particular, $t_{0,0}^0(g) = 1$, and the algebra of bi-$K$-invariant $SU(2)$-finite functions consists of finite linear combinations of $t_{0,0}^\ell$, $\ell \in \mathbb{N}$. For $\ell \in \mathbb{N}$ we have $t_{0,0}^\ell(a(\theta)) = P_\ell(\cos \theta)$, cf. [8, III, §3.9, (5)] where $P_\ell$ is the Legendre polynomial in its standard normalisation $P_\ell(1) = 1$, [4, §4.5], [5, §1.8.3], which is real-valued on $[-1,1]$. The Legendre polynomials are orthogonal on $[-1,1]$ with respect to the uniform measure: $\int_{-1}^{1} P_\ell(x) P_m(x) \, dx = \delta_{\ell,m} 2/(2n+1)$. Moreover, the Schur orthogonality relations are, [8, III, §6.2, (1)]

$$\int_{SU(2)} t_{m,n}^{\ell_1}(g) t_{\ell,q}^{\ell_2}(g) \, dg = \frac{1}{2\ell_1 + 1} \delta_{\ell_1,\ell_2}\delta_{m,q}\delta_{n,q},$$

(2.3)

which in case $m = n = p = q = 0$ give the orthogonality for the Legendre polynomials.

3. The Mathieu Conjecture for $SU(2)$ for simple $f$

We start using some simple observations related to the condition in the Mathieu Conjecture \[14\] for $G = SU(2)$. Firstly, by the Schur orthogonality relations (2.3)

$$\int_{SU(2)} t_{m,n}^\ell(g) \, dg \neq 0 \iff \ell = 0. \quad (3.1)$$

Secondly, by the left and right $K$-behaviour of the matrix elements (2.2) and the Haar measure in Euler angles (2.1) we see

$$\int_{SU(2)} (t_{m_{1,n_1}}^{\ell_1})^{\alpha_1}(g) \cdots (t_{m_{k,n_k}}^{\ell_k})^{\alpha_k}(g) \, dg \neq 0 \implies \sum_{i=1}^k \alpha_i m_i = 0 = \sum_{i=1}^k \alpha_i n_i \quad (3.2)$$
for $\alpha_i \in \mathbb{N}$, $\ell_i \in \frac{1}{2}\mathbb{N}$ and $m_i, n_i \in \{-\ell_i, \ldots, \ell_i\}$.

**Lemma 3.1.** $\int_{SU(2)} (t_{m,n}^{\ell})^P (g) \, dg = 0$ for all integer $P > 0$ if and only if $m \neq 0$ or $n \neq 0$.

**Proof.** The implication $\Leftarrow$ follows from (3.2). To prove the other implication, we observe that for $\ell \in \mathbb{N}$

$$
\int_{SU(2)} (t_{0,0}^{\ell}(g))^2 \, dg = \frac{1}{2} \int_0^\pi (P_t(\cos \theta))^2 \sin \theta \, d\theta = \frac{1}{2} \int_{-1}^1 (P_t(x))^2 \, dx > 0. \quad \square
$$

Now we can verify the Mathieu Conjecture 1.1 in the case $f$ consists of one matrix element.

**Proposition 3.2.** The Mathieu Conjecture 1.1 is true for $G = SU(2)$ with $f$ a single matrix element $f = t_{m,n}^\ell$.

**Proof.** Since all non-negative powers of $f$ integrate to zero, Lemma 3.1 shows that $m \neq 0$ or $n \neq 0$, so in particular $\ell \neq 0$. Let $h = t_a^\ell \nu_{b,\alpha}$. We assume $m \neq 0$, the case $n \neq 0$ being similar.

By (3.2), we see that $Pm + a \neq 0$ implies $\int_{SU(2)} (f(g))^P h(g) \, dg = 0$, which is the case for $P > |a|/|m|$.

The same strategy can also be employed to deal with $f = A_1 t_{m_1,n_1}^{\ell_1} + A_2 t_{m_2,n_2}^{\ell_2}$, where $A_i \in \mathbb{C}$, assuming $A_1 \neq 0 \neq A_2$ and $(\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)$. Note

$$
\int_{SU(2)} (f(g))^P \, dg = \sum_{\alpha=0}^P \left( \begin{array}{c} P \\ \alpha \end{array} \right) A_1^\alpha A_2^{P-\alpha} \int_{SU(2)} (t_{m_1,n_1}^{\ell_1})^\alpha (g)(t_{m_2,n_2}^{\ell_2})^{P-\alpha} (g) \, dg. \quad (3.3)
$$

**Lemma 3.3.** Let $f$ be as above with at least one of $(m_1, m_2, n_1, n_2)$ non-zero, then

$$
\exists P > 0 : \quad \int_{SU(2)} (f(g))^P \, dg \neq 0 \quad \iff \quad \det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 0 \land m_1m_2 \leq 0 \land n_1n_2 \leq 0.
$$

**Remark 3.4.** Note that the condition in Lemma 3.3 means that $(0, 0)$ is on the line segment from $(m_1, n_1)$ to $(m_2, n_2)$.

**Proof.** $\Rightarrow$: Since at least one term in the right hand side of (3.3) has to be non-zero, (3.2) shows that $m_1\alpha + m_2(P-\alpha) = 0 = n_1\alpha + n_2(P-\alpha)$, which gives the result.

$\Leftarrow$: Note that $\dim \ker \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = 1$. Pick a solution $(\alpha, \beta) \in \mathbb{N}^2$ to $m_1\alpha + m_2\beta = 0 = n_1\alpha + n_2\beta$, and put $M = \alpha + \beta$. Then

$$
\int_{SU(2)} (f(g))^M \, dg = \left( \begin{array}{c} \alpha + \beta \\ \alpha \end{array} \right) A_1^\alpha A_2^\beta \int_{SU(2)} (t_{m_1,n_1}^{\ell_1})^\alpha (g)(t_{m_2,n_2}^{\ell_2})^\beta (g) \, dg, \quad (3.4)
$$

using (3.2), since for $\gamma \neq 0$

$$
\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \left( \begin{array}{c} \alpha + \gamma \\ \beta - \gamma \end{array} \right) = \gamma \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

since the kernel is one-dimensional. The integrand on the right hand side of (3.4) is a bi-$K$-invariant function, so that by (2.1) we can restrict to the integral over $g = a(\theta)$, $\theta \in [0, \pi]$. By [1, III, §3(3),(4)] the integrand in $a(\theta)$ is real-valued. In case the integral is non-zero
we are done. Otherwise, we put $P = 2M$, and then in the same way there is again at most one non-zero integral in the right hand side of (3.3), namely for $(2\alpha, 2\beta)$. The integral can be restricted to $SO(2)$ as before. Since this is the integral of a square, since the function $(t_{m_1, n_1}^{\ell_1})^\alpha(a(\theta)) (t_{m_2, n_2}^{\ell_2})^\beta(a(\theta))$ is real, the integral is non-zero. □

**Proposition 3.5.** The Mathieu Conjecture [1] is true for $G = SU(2)$ with $f$ a sum of two matrix element $f = A_1 t_{m_1, n_1}^{\ell_1} + A_2 t_{m_2, n_2}^{\ell_2}$, where $A_1 \neq 0 \neq A_2$ and $(\ell_1, m_1, n_1) \neq (\ell_2, m_2, n_2)$.

**Proof.** It suffices to take $h = t_{a,b}^\ell$ and to assume that $\int_{SU(2)} (f(g))^P \, dg = 0$ for all $P > 0$. We need to show that $\int_{SU(2)} (f(g))^P t_{a,b}^\ell(g) \, dg$ vanishes for sufficiently large $P$.

First assume that not all of $m_i$’s and $n_i$’s are zero, then by Lemma 3.3 we have $m_1 m_2 > 0$ or $n_1 n_2 > 0$ or $\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \neq 0$. Consider the last case, then by (3.2), (3.3) we see that $\int_{SU(2)} (f(g))^P t_{a,b}^\ell(g) \, dg$ can only be non-zero if

$$m_1 \alpha + m_2 \beta = -a, \quad n_1 \alpha + n_2 \beta = -b, \quad \alpha + \beta = P, \quad \alpha, \beta \in \mathbb{N}.$$ 

The first two equations have a unique solution $(\alpha_0, \beta_0) \in \mathbb{Q}^2$. In case $(\alpha_0, \beta_0) \in \mathbb{N}^2$, we see that for all $P > \alpha_0 + \beta_0$ the integral is zero. In case $m_1 m_2 > 0$, we consider $m_1 \alpha + m_2 \beta + a = 0$. In case $\text{sgn}(m_1) = \text{sgn}(a)$, we have no solution $(\alpha, \beta) \in \mathbb{N}^2$, so that integral is zero using (3.2), (3.3). In case $\text{sgn}(m_1) = -\text{sgn}(a)$, we see that the integral is zero for $P > |a|/\min(|m_1|, |m_2|)$. The case $n_1 n_2 > 0$ is dealt with analogously.

In case $m_1 = m_2 = n_1 = n_2 = 0$, $f$ is a bi-$K$-invariant function, and

$$\int_{SU(2)} (f(g))^P \, dg = \frac{1}{2} \int_{-1}^1 (A_1 P_{\ell_1}(x) + A_2 P_{\ell_2}(x))^P \, dx.$$ 

By Boyarchenko’s result, see [3, Cor. 4.1], there is no polynomial $f$ such that $\int_{-1}^1 (f(x))^P \, dx = 0$ for all $P > 0$, so the Mathieu Conjecture [1] is trivially valid in this case. □

The fact that at most one term in the binomial expansion leads to a non-zero integral is typical for $f$ a linear combination of two matrix elements. For a combination of three matrix-elements it gets more complicated.

**Proposition 3.6.** Let $f = \sum_{i=1}^3 A_i t_{m_i, n_i}^{\ell_i}$ with $A_i \neq 0$ for all $i$ and let $(\ell_i, m_i, n_i)$ be mutually different. Assume that $M = \begin{pmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$ has rank($M$) $\neq 2$. Then the Mathieu Conjecture [1] is valid for $f$.

**Proof.** The analogue of (3.3) is the trinomial expansion

$$\int_{SU(2)} f^P(g) \, dg = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = P} \left( A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} \right) \prod_{i=1}^3 \int_{SU(2)} \prod_{i=1}^3 (t_{m_i, n_i}^{\ell_i})^{\alpha_i}(g) \, dg.$$ 

(3.5)

As before, it suffices to consider the case $h = t_{a,b}^\ell$. We have to consider the cases rank($M$) $= 1$ and rank($M$) $= 3$. In the first case $m_i = m$ and $n_i = n$ for all $i$, and the integral in (3.5) is
zero if \( m \neq 0 \) or \( n \neq 0 \) by (3.2). In case \( m \neq 0 \) we see that

\[
\int_{SU(2)} f^P(g)t^\alpha_{a,b}(g) \, dg = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = P} \left( \frac{P}{\alpha_1, \alpha_2, \alpha_3} \right) \prod_{i=1}^{3} A^\alpha_i \int_{SU(2)} \prod_{i=1}^{3} (t^i_{m,n_i})^{\alpha_i} (g) t^\alpha_{a,b}(g) \, dg
\]

(3.6)
can only be non-zero if \( Pm + a = 0 \), so that for \( P > |a|/|m| \) the integral is zero. The case \( n \neq 0 \) is analogous. In case \( m = n = 0 \), we see that the condition in the Mathieu Conjecture is not valid using [3, Cor. 4.1] as in the proof of Proposition 3.5.

In case \( \text{rank}(M) = 3 \), \( M \) is invertible with \( M^{-1} \) having rational entries. In particular, for each \( P \in \mathbb{N} \) there is at most one term in the right hand side of (3.5) which can be non-zero, namely for \( \eta^P = \left( \frac{\alpha_1}{\alpha_2} \right) = M^{-1} \left( \frac{P}{0} \right) \) under the additional condition \( \eta^P \in \mathbb{N}^3 \). Assuming that this is the case, we see that, analogous to the proof of Proposition 3.5, \( \int_{SU(2)} f^P(g) \, dg \neq 0 \).

So we need to consider the case that \( \eta^P \notin \mathbb{N}^3 \) for all \( P > 0 \). Then the integral in (3.6) can only be non-zero in case

\[
M^{-1} \left( \begin{array}{c} P \\ -a \\ -b \end{array} \right) = \frac{1}{\det(M)} \left( \begin{array}{c} m_2n_3 - m_3n_2 \\ m_3n_1 - m_1n_3 \\ m_1n_2 - m_2n_1 \end{array} \right) - a \left( \begin{array}{c} n_2 - n_3 \\ n_3 - n_1 \\ n_1 - n_2 \end{array} \right) - b \left( \begin{array}{c} m_3 - m_2 \\ m_1 - m_3 \\ m_2 - m_1 \end{array} \right) \in \mathbb{N}^3
\]

Since \( \eta^P \) corresponds to the first term, i.e. \( a = b = 0 \), and \( \eta^P \notin \mathbb{N}^3 \) for all \( P > 0 \) we have \( \det(M)^{-1}(m_{n_i+1} - m_{i+1}n_i) < 0 \) for some \( i \in \{1, 2, 3\} \) with convention \( m_4 = m_1 \), \( n_4 = n_1 \). Then for \( P > (|a||n_i - n_{i+1}| + |b||m_{i+1} - m_i|)/|m_in_{i+1} - m_{i+1}n_i| \) the \( i \)-th coefficient is negative, so that the integral in (3.6) is zero.

Remark 3.7. In case \( \text{rank}(M) = 1 \) the convex hull \( C \) of \( \left( (m_i, n_i) \right)_{i=1}^{3} \) equals \( \{(m, n)\} \), and in case \( \text{rank}(M) = 3 \) we have \( (0, 0) \in C \) if and only if \( \exists \alpha^I \in \mathbb{Q}^{3}_{\geq 0} \) with \( M \alpha^I = (1, 0, 0)^t \).

From the proof of Proposition 3.5 we see that \( \int_{SU(2)} f(g)^P \, dg = 0 \) for all \( P > 0 \) precisely when \( (0, 0) \notin C \) in the cases \( \text{rank}(M) \neq 2 \). In case \( \text{rank}(M) = 2 \) the integral in (3.5) can have more than one non-zero term, and we have no control on possible cancellations. However, one expects that these cancellations cannot occur for all multiples of \( P \) as well. The techniques of Francoise et al. [3] might be useful in this regard considering it as polynomial identities in the \( A_i \)'s.

4. AN ALTERNATIVE CONJECTURE FOR THE MATHIEU CONJECTURE FOR \( SU(2) \)

Consider an arbitrary \( SU(2) \)-finite function \( f = \sum_{i=1}^{k} A_i t^\ell_{m_i, n_i} \) with \( A_i \neq 0 \) for every \( 1 \leq i \leq k \), then applying the multinomial finite theorem shows that if

\[
\int_{SU(2)} (f(g))^P \, dg = \sum_{\alpha_i \in \mathbb{N}, \sum_{i=1}^{k} \alpha_i = P} \left( \frac{P}{\alpha_1, \ldots, \alpha_k} \right) \prod_{i=1}^{k} A^\alpha_i \int_{SU(2)} \prod_{i=1}^{k} (t^\ell_{m_i, n_i})^{\alpha_i} (g) \, dg \neq 0
\]

(4.1)
for some $P > 0$, then for some $(\alpha_1, \ldots, \alpha_k)$ we have $\sum_{i=1}^k \frac{\alpha_i}{P} m_i = \sum_{i=1}^k \frac{\alpha_i}{P} n_i = 0$ by (3.2), so $(0, 0)$ is in the convex hull $C$ of $((m_i, n_i))_{i=1}^k$ over $\mathbb{Q}$.

**Conjecture 4.1.** For any $SU(2)$-finite function $f = \sum_{i=1}^k A_i t_{m_i, n_i}$, $A_i \neq 0$ for all $1 \leq i \leq k$, we have that $\int_{SU(2)} (f(g))^P \, dg = 0$ for all $P \in \mathbb{N}_{>0}$ if and only if $(0, 0)$ is not contained in the closed convex hull of $((m_i, n_i))_{i=1}^k$.

Lemma 3.1, Remarks 3.4, 3.7 support Conjecture 4.1.

**Theorem 4.2.** Assume Conjecture 4.1 holds, then the Mathieu Conjecture 1.1 for $SU(2)$ holds.

**Proof.** It suffices to show that $\int_{SU(2)} (f(g))^P t_{a,b}^c(g) \, dg = 0$ for $P$ sufficiently large assuming that $(0, 0)$ is not contained in the closed convex hull $C$ of $((m_i, n_i))_{i=1}^k$. Using (4.1) we see that $\int_{SU(2)} (f(g))^P t_{a,b}^c(g) \, dg$ can only be non-zero if $(-\frac{a}{P}, -\frac{b}{P}) \in C$. Since $(0, 0) \not\in C$, we see that for $P$ sufficiently large this is not the case and the integral is zero.

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**References**