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# Symmetric Jacobians\*

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## Abstract

This article is about polynomial maps with a certain symmetry and/or antisymmetry in their Jacobians, and whether the Jacobian Conjecture is satisfied for such maps, or whether it is sufficient to prove the Jacobian Conjecture for such maps.

For instance, we show that it suffices to prove the Jacobian conjecture for polynomial maps  $x + H$  over  $\mathbb{C}$  such that  $\mathcal{J}H$  satisfies all symmetries of the square, where  $H$  is homogeneous of arbitrary degree  $d \geq 3$ .

*Key words:* Jacobian conjecture, (anti)symmetric Jacobian matrix.

*MSC 2010:* 14R15, 14R10, 15B99, 15B57, 20F55.

## Introduction

Let  $F$  be a polynomial map over a field  $K$  of characteristic zero. The Jacobian Conjecture asserts that  $F$  has a polynomial inverse in case its Jacobian determinant  $\det \mathcal{J}F$  is a unit in  $K$ . It has been shown that in order to prove the Jacobian conjecture for all fields of characteristic zero, one can take an arbitrary such field  $K$  and a favorite integer  $d \geq 3$ , after which it suffices to prove the Jacobian Conjecture for polynomial maps over  $K$  of the form  $F = x + H$ , where  $x$  is the identity map and  $H$  is homogeneous of degree  $d$ .

We write  $\mathcal{J}F$  for the Jacobian of a polynomial map  $F$ ,  $\mathcal{H}f$  for the Hessian of a single polynomial  $f$ , and  $\nabla f$  for the gradient map of a single polynomial  $f$ . Notice that  $\mathcal{H}f = \mathcal{J}\nabla f$ . All results in this paper are about maps of the form  $F = x + H$ , with (anti)symmetry conditions on  $\mathcal{J}H$ .

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Most results are about two such maps, say  $F = x + H$  and  $\tilde{F} = \tilde{x} + \tilde{H}$ , each with their own (anti)symmetry conditions on  $\mathcal{J}H = \mathcal{J}_x H$  and  $\mathcal{J}_{\tilde{x}} \tilde{H}$  respectively, where the dimension of  $\tilde{F}$  is one, two or four times that of  $F$  (depending on the actual result), and  $\tilde{x}$  is the identity in the proper dimension. For each of these results, the reader may choose any of the following additional conditions when desired:

- $\mathcal{J}_x H$  and  $\mathcal{J}_{\tilde{x}} \tilde{H}$  are both singular or even nilpotent,
- for some fixed arbitrary subset  $S \subseteq \mathbb{N}$ ,  $H$  and  $\tilde{H}$  only have terms whose degrees are contained in  $S$ .

One can, e.g., assume that  $H$  and  $\tilde{H}$  only have terms of degree three, in which case the nilpotency of  $\mathcal{J}_x H$  and  $\mathcal{J}_{\tilde{x}} \tilde{H}$  already follows from the Keller condition. We have this Keller condition implicitly in case the actual result is the equivalence of the Jacobian conjecture for maps of the form  $F = x + H$  and that for maps of the form  $\tilde{F} = \tilde{x} + \tilde{H}$ , since the Jacobian conjecture is an assertion about Keller maps.

We associate vectors with column matrices and write  $M^t$  for the *transpose* of a matrix  $M$ . We write  $M^r$  for the *reverse* of a matrix  $M$ , i.e. if  $M$  has  $n$  rows, then the  $i$ -th row of  $M^r$  is equal to the  $(n + 1 - i)$ -th row of  $M$  for each  $i$ . Notice that the symmetries corresponding to the matrix operators  $M \mapsto M^t$  and  $M \mapsto M^r$  generate the whole dihedral symmetry group of the square. If  $f$  is a single polynomial, then we can view  $f$  as a polynomial map with only one component, and we have  $\mathcal{J}f = (\nabla f)^t$ .

Both  $x = x_1, x_2, \dots, x_n$  and  $y = y_1, y_2, \dots, y_n$  are  $n$ -tuples of variables, thus  $\mathbb{C}[x, y]$  is the coordinate ring of complex  $2n$ -space. By taking into account the order of variables in  $\mathbb{C}[x, y]$ ,  $(x, y)$  is the identity map of the above complex space. But  $(x, y)$  is also a vector of  $2n$  variables.

Notice that besides the matrix equality

$$M^r M' = (I_m^r M) M' = I_m^r (M M') = (M M')^r \quad (1)$$

where  $M$  has height  $m$  and  $I_m$  is the identity matrix of size  $m$ , we have the following equalities:

$$\mathcal{H}_X^r f := (\mathcal{H}_X f)^r = (\mathcal{J}_X (\nabla_X f))^r \quad (2)$$

$$\mathcal{J}_X^r F := (\mathcal{J}_X F)^r = \mathcal{J}_X (F^r) \quad (3)$$

$$\nabla_X^r f := (\nabla_X f)^r = \nabla_{X^r} f \quad (4)$$

We will use the above equalities in the rest of this paper, which is organized as follows. In section 1 and in theorem 2.5, we formulate results about the Jacobian conjecture for polynomial maps  $x + H$  such that  $\mathcal{J}H$  has certain (anti)symmetry properties with respect to the diagonal and/or the antidiagonal. In section 2, we formulate results about the Jacobian conjecture for polynomial maps  $x + H$  such that  $\mathcal{J}H$  has certain (anti)symmetry properties with respect to the center, possibly among other (anti)symmetry properties. The reason that theorem 2.5

contains results that belong to section 1, is that maps  $x + H$ , such that  $\mathcal{J}H$  has certain (anti)symmetry properties with respect to both the diagonal and the antidiagonal, belong to section 2 as well.

In section 3, we formulate results about the (linear) dependence problem (for Jacobians) for polynomial maps  $x + H$  such that  $\mathcal{J}H$  has certain (anti)symmetry properties. For the definition of this dependence problem, we refer to the beginning of section 3. At last, the conclusion follows.

## 1 Diagonally symmetric variants of the Jacobian conjecture

We define (anti)symmetry properties by pictures that visualize them.

**Definition 1.1.**  $\blacksquare(K, n)$  means that the Jacobian conjecture is satisfied for  $n$ -dimensional maps  $F = x + H$  over the field  $K$ , i.e.,  $F$  is invertible in case  $\det \mathcal{J}F \in K^*$ , such that the degree of each term of  $H$  is contained in a fixed set  $S \subseteq \mathbb{N}$ , and optionally  $\det \mathcal{J}H = 0$  or even  $(\mathcal{J}H)^n = 0$ .

So we do not assume that  $\det \mathcal{J}F = 1$  necessarily, except when we assume  $(\mathcal{J}H)^n = 0$ , in which case  $\det \mathcal{J}F = 1$  as a consequence, or  $1 \notin S$ , in which case  $\det \mathcal{J}F = 1$  follows from  $\det \mathcal{J}F \in K^*$ .

$\blacktriangleleft(K, n)$  and  $\blacktriangleright(K, n)$  mean that the Jacobian conjecture is satisfied for  $n$ -dimensional maps  $F = x + H$  over  $K$ , which have a symmetric Jacobian with respect to the diagonal and the anti-diagonal respectively, where  $H$  has the same partially chosen properties as in the definition of  $\blacksquare(K, n)$ .

$\blacktriangleleft(K, n)$  and  $\blacktriangleright(K, n)$  mean that the Jacobian conjecture is satisfied for  $n$ -dimensional maps  $F = x + H$  over  $K$ , for which  $\mathcal{J}H$  is anti-symmetric (i.e. applying the ‘symmetry’ negates the matrix) with respect to the diagonal and the anti-diagonal respectively, where  $H$  has the same partially chosen properties as in the definition of  $\blacksquare(K, n)$ .

In the definition of  $\blacktriangleleft(K, n)$ , the ‘symmetry’ is partially an antisymmetry, namely where colors on opposite sides of the diagonal do not match.

In the proofs of our results, we will use  $\blacksquare$ ,  $\blacktriangleleft$ ,  $\blacktriangleright$ ,  $\blacktriangleleft$ ,  $\blacktriangleright$  and  $\blacktriangleleft$  without parenthesized arguments to indicate the corresponding matrix (anti)symmetry. So we have, e.g.,  $\mathcal{J}H = \mathcal{J}H^t$  if  $\mathcal{J}H$  has symmetry  $\blacktriangleleft$ , and  $\mathcal{J}H = -\mathcal{J}H^t$  in case  $\mathcal{J}H$  has antisymmetry  $\blacktriangleleft$ .

If the (anti)symmetry separates several parts of the square, then entries on the edge of separation must satisfy both (anti)symmetry conditions. In this manner, the row and column in the middle of matrices with (anti)symmetry  $\blacktriangleleft$  must be zero when the dimension is odd, since they are equal to each other and to the opposites of each other.

For matrices with antisymmetry  $\blacktriangleleft(K, n)$ , the numbers on the diagonal must be equal to their opposites by definition of  $\blacktriangleleft$ , because the antisymmetry holds for the edge of the light and dark regions as well. So  $\blacktriangleleft$  is just the regular matrix antisymmetry with zeroes on the diagonal.

**Theorem 1.2** (Meng). *Assume  $K$  is a field of characteristic zero. Then  $\square(K, 2n)$  implies  $\square(K, n)$ .*

*Proof.* Assume  $F = x + H \in K[x]^n$  such that  $\mathcal{J}H$  has symmetry  $\square$ . Put  $f := y^t F = \sum_{i=1}^n y_i F_i$ . Then  $\mathcal{H}_{x,y} f(x, y^r)$  has the regular (Hessian) symmetry  $\square$ , whence

$$\mathcal{H}_{x,y}^r f(x, y^r) \stackrel{(2)}{=} \mathcal{J}_{x,y}^r \nabla_{x,y} f(x, y^r) \stackrel{(3)}{=} \mathcal{J}_{x,y} \nabla_{x,y}^r f(x, y^r) \stackrel{(4)}{=} \mathcal{J}_{x,y} \nabla_{y^r, x^r} f(x, y^r)$$

has symmetry  $\square$ . Since the first  $n$  components of  $\nabla_{y^r, x^r} f(x, y^r)$  are equal to  $\nabla_{y^r} (y^r)^t F = F$ , we see that  $\mathcal{J}_{x,y} \nabla_{y^r, x^r} f(x, y^r)$  is of the form

$$\mathcal{J}_{x,y} \nabla_{y^r, x^r} f(x, y^r) = \begin{pmatrix} \mathcal{J}F & \mathbf{0} \\ * & (((\mathcal{J}F)^r)^t)^r \end{pmatrix} = \begin{pmatrix} \mathcal{J}F & \mathbf{0} \\ * & \mathcal{J} \end{pmatrix}$$

where the zero submatrix right above appears since  $\deg_y f < 2$ , and the part  $y^t x$  of  $f$  does not affect  $*$  because  $\deg_x(y^t x) < 2$ . Thus  $*$  only depends on  $y^t H$ .

Hence  $\nabla_{y^r, x^r} f(x, y^r)$  is of the form  $\tilde{F} = (x, y) + \tilde{H}$  where  $\mathcal{J}_{x,y} \tilde{H}$  has symmetry  $\square$  and the properties to be chosen by the reader of  $\tilde{H}$  correspond to those of  $H$ . Thus if we assume  $\square(K, 2n)$ , then  $\nabla_{y^r, x^r} f(x, y^r)$  satisfies the Jacobian Conjecture. Since substituting  $y = 0$  in  $\nabla_{y^r, x^r} f(x, y^r)$  gives  $(F, 0)$ , we see that  $F$  satisfies the Jacobian conjecture as well. This gives the desired result.  $\square$

In [12], the author G. Meng constructs the map  $f := \sum_{i=1}^n y_i F_i$  in the above proof. The corresponding gradient map  $\nabla_{x,y} f$  has symmetry  $\square$ , but its linear part is  $(y, x)$  in case  $F$  has linear part  $x$ . In order to restore the linear part to  $(x, y)$ , we composed  $\nabla_{x,y} f$  with linear maps in the above proof, resulting  $\nabla_{y^r, x^r} f(x, y^r)$  with linear part  $(x, y)$  and symmetry  $\square$ . That is why the above theorem is considered to be due to Meng. The case that  $\mathcal{J}H$  is nilpotent of corollary 1.4 below was proved in [2].

**Theorem 1.3.**  $\square(\mathbb{C}, N)$  and  $\square(\mathbb{C}, N)$  are equivalent.

*Proof.* Notice that it suffices to show that polynomial maps  $H \in \mathbb{C}[X]^N$  with Jacobian symmetry  $\square$  can be transformed to polynomial maps  $\tilde{H} \in \mathbb{C}[X]^N$  with Jacobian symmetry  $\square$  by way of linear conjugation, and vice versa. We shall show that this is the case, where the conjugation map has the symmetric unitary Jacobian  $T := \frac{1}{2}\sqrt{2}(I_N + iI_N^r)$ .

( $\Rightarrow$ ): Let  $F = X + H \in \mathbb{C}[X]^N$  such that  $\mathcal{J}_X H$  has symmetry  $\square$ , where  $X = (x_1, x_2, \dots, x_N)$ . Since  $T$  is symmetric, we see that  $TH(TX)$  has the regular Jacobian symmetry  $\square$  as well. But  $T^{-1} = \frac{1}{2}\sqrt{2}(I_N - iI_N^r) = -iT^r$ , thus by (1),  $T^{-1}H(TX) = -i(TH(TX))^r$  has Jacobian symmetry  $\square$ . Since conjugations preserve the identity part  $X$  of  $F$ , we see that  $\tilde{F} := T^{-1}F(TX) = X + T^{-1}H(TX)$ , so  $\tilde{F} - X = T^{-1}H(TX)$  has Jacobian symmetry  $\square$ .

( $\Leftarrow$ ): Let  $F = X + H \in \mathbb{C}[X]^N$  such that  $\mathcal{J}_X H$  has symmetry  $\color{red}{\square}$ , where  $X = (x_1, x_2, \dots, x_N)$ . Then  $\mathcal{J}_X H^r$  has symmetry  $\color{red}{\square}$ . Since  $H^r = I_N^r H$  and  $I_N^r$  commutes with  $T$ , we obtain by (1) and by symmetry of  $T$  that

$$I_N^r T^{-1} H(TX) = T^{-1} H^r(TX) = -iT^r H^r(TX) = -i(TH^r(TX))^r$$

has Jacobian symmetry  $\color{red}{\square}$ . Thus  $T^{-1}H(TX)$  has Jacobian symmetry  $\color{red}{\square}$  on account of (1). Since conjugations preserve the identity part  $X$  of  $F$ , we see that  $\tilde{F} := T^{-1}F(TX) = X + T^{-1}H(TX)$ , so  $\tilde{F} - X = T^{-1}H(TX)$  has Jacobian symmetry  $\color{red}{\square}$ .  $\square$

**Corollary 1.4.**  $\color{red}{\square}(\mathbb{C}, 2n)$  implies  $\color{red}{\square}(\mathbb{C}, n)$ .

*Proof.* This follows from theorem 1.3 and theorem 1.2.  $\square$

**Theorem 1.5.** Assume  $K$  is a field of characteristic zero. Then  $\color{red}{\square}(K, 2n)$ ,  $\color{blue}{\square}(K, 2n)$  and  $\color{blue}{\square}(K, 2n+1)$  are equivalent.

*Proof.* The equivalence of  $\color{blue}{\square}(K, 2n)$  and  $\color{blue}{\square}(K, 2n+1)$  follows by stabilization: both the row and column in the middle of square matrices of odd dimension with (anti)symmetry  $\color{blue}{\square}$  are zero. Thus  $\color{red}{\square}(K, 2n) \Leftrightarrow \color{blue}{\square}(K, 2n)$  remains to be proved.

Notice that it suffices to show that polynomial maps  $H \in K[x, y]^{2n}$  with Jacobian symmetry  $\color{red}{\square}$  can be transformed to polynomial maps  $\tilde{H} \in K[x, y]^{2n}$  with Jacobian symmetry  $\color{blue}{\square}$  by way of linear conjugation, and vice versa. We shall show that this is the case, where the conjugation map has the Jacobian

$$T := \begin{pmatrix} I_n & I_n^r \\ -I_n^r & I_n \end{pmatrix}$$

Notice that  $T^t T = 2I_{2n}$  and  $T^t = I_{2n}^r T I_{2n}^r$ . Thus  $2T^{-1} = T^t = I_{2n}^r T I_{2n}^r$ .

( $\Rightarrow$ ): Let  $F = (x, y) + H \in K[x, y]^n$  such that  $\mathcal{J}_{x,y} H$  has symmetry  $\color{red}{\square}$ . Then  $H^r$  and hence also

$$T^t H^r(T(x, y)) \stackrel{(1)}{=} T^t I_{2n}^r H(T(x, y)) = 2T^{-1} I_{2n}^r H(T(x, y))$$

has the regular Jacobian symmetry  $\color{red}{\square}$ . Since negating the upper half of  $2T^{-1} = T^t$  has the same effect as reversing its columns, which is what  $I_{2n}^r$  does in the product  $2T^{-1} I_{2n}^r$ , we have that

$$2T^{-1} I_{2n}^r H(T(x, y)) = 2J_{2n} T^{-1} H(T(x, y))$$

where

$$J_{2n} := \begin{pmatrix} -I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix}$$

Combining the multiplication by  $2J_{2n}$  with the regular Jacobian symmetry  $\color{red}{\square}$ , we see that  $T^{-1}H(T(x, y))$  has Jacobian (anti)symmetry  $\color{blue}{\square}$ . Since conjugations preserve the identity part  $(x, y)$  of  $F$ , we see that  $\tilde{F} := T^{-1}F(T(x, y)) = (x, y) + T^{-1}H(T(x, y))$ , so  $\tilde{F} - (x, y) = T^{-1}H(T(x, y))$  has Jacobian (anti)symmetry  $\color{blue}{\square}$ .

( $\Leftarrow$ ): Let  $F = (x, y) + H \in K[x, y]^n$  such that  $\mathcal{J}_{x,y}H$  has (anti)symmetry  $\blacksquare$ . By negating the right half of  $\mathcal{J}_{x,y}H$ , we see that  $H(x, -y) = H(-J_{2n}(x, y))$  and hence also  $M^t H(-J_{2n}M(x, y))$  has Jacobian symmetry  $\blacksquare$  for each  $M \in \text{GL}_{2n}(K)$ . In particular

$$T^t(-J_{2n}^t)H((-J_{2n})^2T(x, y)) = -T^t J_{2n}H(T(x, y))$$

has Jacobian symmetry  $\blacksquare$ . Since negating the right half of  $T^t = 2T^{-1}$  has the same effect as reversing the order of its rows, we have that

$$T^t(-J_{2n}^t)H(T(x, y)) = (2T^{-1})^r H(T(x, y)) \stackrel{(1)}{=} \left(2T^{-1}H(T(x, y))\right)^r$$

Thus  $T^{-1}H(T(x, y))$  has Jacobian symmetry  $\blacksquare$ . Since conjugations preserve the identity part  $(x, y)$  of  $F$ , we see that  $\tilde{F} := T^{-1}F(T(x, y)) = (x, y) + T^{-1}H(T(x, y))$ , so  $\tilde{F} - (x, y) = T^{-1}H(T(x, y))$  has Jacobian symmetry  $\blacksquare$ .  $\square$

**Corollary 1.6** (Drużkowski). *Assume  $K$  is a field of characteristic zero. Then  $\blacksquare(K, 2n)$  implies  $\blacksquare(K, n)$ .*

*Proof.* This follows immediately from the above theorem and theorem 1.2.  $\square$

In fact, Drużkowski considers maps with (anti)symmetry  $\blacksquare$ , but linear part  $(-x, y) = J_{2n}(x, y)$  in [9]. Negating the first half of the map restores the linear part, and the (anti)symmetry becomes  $\blacksquare$ .

## Symmetry patterns that satisfy the Jacobian Conjecture

In some cases, the Jacobian Conjecture holds for polynomial maps  $F = x + H$  with certain (anti)symmetries of  $\mathcal{J}H$ , because  $F$  appears to be linear.

**Theorem 1.7.** *Assume  $F = x + H$  is a Keller map over  $\mathbb{R}$ , such that  $\mathcal{J}H$  has regular symmetry  $\blacksquare$ . If either  $H$  has no linear terms or  $\mathcal{J}H$  is nilpotent, then  $H$  is constant. In particular,  $F$  is invertible because  $F$  is translation in that case.*

*Proof.* The case that  $H$  has no linear terms follows from [4, Cor. 4.4], so assume that  $(\mathcal{J}H)^r = 0$  and  $(\mathcal{J}H)^{r-1} \neq 0$ . If  $r \geq 2$ , then

$$0 = (\mathcal{J}H)^{2r-2} = (\mathcal{J}H)^{r-1} \cdot (\mathcal{J}H)^{r-1} = (\mathcal{J}H)^{r-1} \cdot ((\mathcal{J}H)^{r-1})^t$$

Substituting generic reals in the variables in the rows of  $(\mathcal{J}H)^{r-1}$ , we obtain rows of real numbers that are isotropic (self-orthogonal), and hence zero. Contradiction, so  $r = 1$  and  $\mathcal{J}H = 0$ . Thus  $F = x + H$  is a translation.  $\square$

**Theorem 1.8.** *Assume  $K$  is a field of characteristic zero. Then  $\blacksquare(K, n)$  and  $\blacksquare(K, n)$  have affirmative answers. In particular, if the Jacobian of a polynomial map  $H$  over  $K$  has antisymmetry  $\blacksquare$  or  $\blacksquare$ , then  $\deg H \leq 1$ .*

*Proof.* Assume that  $H$  is a polynomial map over  $K$  with Jacobian antisymmetry  $\blacksquare$ . (The proof for Jacobian antisymmetry  $\blacktriangleright$  will be similar.) Then

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} H_k = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} H_j = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} H_i = -\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} H_k$$

and hence  $2\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} H_k = 0$ , for all  $i, j, k$ . So  $\deg H \leq 1$ .  $\square$

**Definition 1.9.**  $\boxtimes(K, n)$  means that the Jacobian conjecture is satisfied for polynomial maps  $F$  over  $K$  that have a symmetric Jacobian with respect to both the diagonal and the anti-diagonal, where  $H$  has the same partially chosen properties as in the definition of  $\blacksquare(K, n)$ .

In the definitions of  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$  and  $\boxtimes(K, n)$ , some ‘symmetries’ are antisymmetries, namely when colors on opposite sides of the symmetry axis do not match.

In the definitions of  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$  and  $\boxtimes(K, n)$ , the ‘symmetries’ are partially antisymmetries.

Notice that  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$  and  $\boxtimes(K, n)$  have affirmative answers as well as  $\blacksquare(K, n)$  and  $\blacktriangleright(K, n)$ , because the corresponding (anti)symmetries are stronger than at least one of those of  $\blacksquare$  and  $\blacktriangleright$  in theorem 1.8.

**Theorem 1.10.** *Assume  $K$  is a field of characteristic zero. Then  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$ ,  $\boxtimes(K, n)$  and  $\boxtimes(K, n)$  have affirmative answers. In particular, if the Jacobian of a polynomial map  $H$  over  $K$  has (anti)symmetry  $\boxtimes$ ,  $\boxtimes$ ,  $\boxtimes$  or  $\boxtimes$ , then  $\deg H \leq 1$ .*

*Proof.* Assume that  $H$  is a polynomial map over  $K$  with Jacobian (anti)symmetry  $\boxtimes$  or  $\boxtimes$ . (The proof for  $\boxtimes$  and  $\boxtimes$  will be similar). We show that  $\deg H \leq 1$ .

For that purpose, notice that above the anti-diagonal,  $\mathcal{J}H$  is anti-symmetric with respect to the diagonal. Now the assumption that  $x_1$  appears above the anti-diagonal in  $\mathcal{J}H$  implies that  $H_j$  has a term divisible by  $x_1 x_k$  for some  $j, k$  with  $j + k \leq n + 1$ , and leads to a contradiction in a similar manner as in the proof of theorem 1.8 (with  $i = 1$ ), because the argument in this proof does not get below the anti-diagonal of  $\mathcal{J}H$ , and stays in the part where  $\mathcal{J}H$  is anti-symmetric.

Thus there is no  $x_1$  above or on the anti-diagonal of  $\mathcal{J}H$ . By the (anti-)symmetry of the anti-diagonal, there is no  $x_1$  in  $\mathcal{J}H$ . Consequently, the first column of  $\mathcal{J}H$  is constant. By the (anti)symmetry conditions, all border entries of  $\mathcal{J}H$  are constant. Hence the entries of  $\mathcal{J}H$  that are not on the border do not contain  $x_1$  and neither  $x_n$ , and form a matrix with the same (anti)symmetry as  $\mathcal{J}H$  itself. So by induction on  $n$ , it follows that  $\deg H \leq 1$ . Hence  $F = x + H$  satisfies the Jacobian conjecture, as desired.  $\square$

In theorem 2.5 in the next section, we show that  $\boxtimes(K, n)$ ,  $\boxtimes(K, 2n - 1)$  and  $\boxtimes(K, 2n)$  are equivalent when  $K$  is a field of characteristic zero.



## 2 Centrally symmetric variants of the Jacobian conjecture

**Definition 2.1.**  $\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \blacksquare \\ \square \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \square \\ \blacksquare \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}(K, n)$  have horizontal and vertical (anti)symmetries in their definitions.

$\begin{smallmatrix} \times \\ \times \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \times \\ \square \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \square \\ \times \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \times \\ \times \end{smallmatrix}(K, n)$  have horizontal, vertical, and diagonal (anti)symmetries in their definitions.

$\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}(K, n)$  means that the Jacobian conjecture is satisfied for  $n$ -dimensional maps  $F = x + H$  as above that have Jacobians that are symmetric with respect to the center, i.e., entries  $(i, j)$  and  $(n + 1 - i, n + 1 - i)$  of  $\mathcal{J}H$  are equal for all  $i, j$ .

In the definition of  $\begin{smallmatrix} \bullet \\ \square \end{smallmatrix}(K, n)$ ,  $\begin{smallmatrix} \square \\ \bullet \end{smallmatrix}(K, n)$  and  $\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}(K, n)$ , the central point ‘symmetry’ is at least partially an antisymmetry.

Notice that for any matrix  $A \in \text{Mat}_n(K)$ , the matrix

$$M := \begin{pmatrix} +A & \pm A \\ \mp A & -A \end{pmatrix}$$

of size  $2n$  has some sort of tiling (anti)symmetry. By conjugating the map  $(x, y) \mapsto M(x, y)$  with  $(x, y^r)$ , we get a map with horizontal and vertical (anti)symmetries, since

$$\begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n^r \end{pmatrix} M \begin{pmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n^r \end{pmatrix} = \begin{pmatrix} +A & \pm((A^t)^r)^t \\ \mp A^r & -(((A^t)^r)^t)^r \end{pmatrix} = \begin{pmatrix} +A & \pm A \\ \mp A & -A \end{pmatrix}$$

Any square matrix  $M$  can be written as

$$M = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t)$$

which is the sum of a matrix with regular symmetry  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  and a matrix with regular antisymmetry  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ . In a similar manner, a square matrix  $M$  of size  $N$  with symmetry  $\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}$  can be written as

$$M = \frac{1}{2}(M + M^r) + \frac{1}{2}(M - M^r)$$

which is the sum of a matrix with symmetry  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  and a matrix with (anti)symmetry  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ , because the symmetry  $\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}$  means exactly that  $M^r = MI_N^r$ , and the left and right hand side of  $M \pm M^r = M \pm MI_N^r$  have a horizontal and a vertical (anti)symmetry axis respectively.

If  $F = X + H \in K[X]^N$  such that  $\mathcal{J}H$  has symmetry  $\begin{smallmatrix} \bullet \\ \bullet \end{smallmatrix}$ , then we can write

$$F = \frac{1}{2}(F + F^r) + \frac{1}{2}(F - F^r) \tag{5}$$

and  $\frac{1}{2}(F + F^r)$  and  $\frac{1}{2}(F - F^r)$  have Jacobian (anti)symmetries  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  and  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  respectively, because  $\mathcal{J}_X(F^r) = (\mathcal{J}_X F)^r$ .

**Theorem 2.2.** *Assume  $K$  is a field of characteristic zero. Then  $\square(K, n)$ ,  $\bullet(K, 2n - 1)$ ,  $\bullet(K, 2n)$ ,  $\boxplus(K, 2n - 1)$ ,  $\boxplus(K, 2n)$ ,  $\boxminus(K, 2n)$  and  $\boxminus(K, 2n + 1)$  are all equivalent.*

*Proof.* Since  $\square(K, n + 1)$  implies  $\square(K, n)$ , it suffices to prove the following.

$$\begin{aligned} \boxplus(K, 2n) &\iff \square(K, n) \\ \boxminus(K, 2n) &\iff \square(K, n) \quad \text{and} \\ \bullet(K, 2n) &\iff \square(K, n) \wedge \square(K, n) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \boxplus(K, 2n + 1) &\iff \square(K, n + 1) \\ \boxminus(K, 2n + 1) &\iff \square(K, n) \quad \text{and} \\ \bullet(K, 2n + 1) &\iff \square(K, n + 1) \wedge \square(K, n) \end{aligned} \tag{7}$$

We only prove (7), since (6) can be proved in a similar manner: you just ignore the  $(n + 1)$ -th row and column.

( $\Rightarrow$ ): Let  $(F, f) = (x, x_{n+1}) + (H, h)$  be a polynomial map in dimension  $n + 1$ , where  $f = x_{n+1} + h$  is a single polynomial and  $F = x + H$  is an  $n$ -tuple of polynomials. Let  $\tilde{F} = x + \tilde{H}$  be a polynomial map in dimension  $n$ . Then  $(F, f, \tilde{F}(y))$  is invertible or of Keller type, if and only if both  $(F, f)$  and  $\tilde{F}$  are invertible or of Keller type respectively.

By conjugating  $(F, f, \tilde{F}(y))$  with the linear map  $(x + y, x_{n+1}, x - y)$ , we obtain

$$\begin{aligned} \frac{1}{2} \begin{pmatrix} I_n & \mathbf{0} & I_n \\ \mathbf{0} & 2 & \mathbf{0} \\ I_n & \mathbf{0} & -I_n \end{pmatrix} \begin{pmatrix} F(x + y, x_{n+1}) \\ f(x + y, x_{n+1}) \\ \tilde{F}(x - y) \end{pmatrix} \\ = \frac{1}{2} \begin{pmatrix} F(x + y, x_{n+1}) + \tilde{F}(x - y) \\ 2f(x + y, x_{n+1}) \\ F(x + y, x_{n+1}) - \tilde{F}(x - y) \end{pmatrix} \end{aligned} \tag{8}$$

Now the Jacobian of the  $F$ -part and the  $\tilde{F}$ -part of the right hand side of (8), without the row and column in the middle, have tiling (anti)symmetries

$$\begin{pmatrix} +A & +A \\ +A & +A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} +A & -A \\ -A & +A \end{pmatrix}$$

respectively. A subsequent conjugation with  $(x, x_{n+1}, y^t)$  of the right hand side of (8) gives Jacobian symmetry  $\boxplus$  for the  $(F, f)$ -part, and Jacobian (anti)symmetry  $\boxminus$  for the  $\tilde{F}$ -part.

Since  $\bullet$  is a subsymmetry of both  $\boxplus$  and  $\boxminus$ , we have a map with symmetry  $\bullet$  in general, a map with symmetry  $\boxplus$  when  $\tilde{F}(y) = y$ , and a map with (anti)symmetry  $\boxminus$  when  $(F, f) = (x, x_{n+1})$ . This map is a conjugation of  $(F, f, \tilde{F}(y))$ , and the forward implications in (7) follow because conjugations preserve the linear part  $(x, x_{n+1}, y)$  of  $(F, f, \tilde{F}(y))$ .

( $\Leftarrow$ ): Take  $G \in K[x, x_{n+1}, y]^{2n+1}$  such that  $G - (x, x_{n+1}, y)$  has Jacobian symmetry  $\blacksquare$ . By (5), we can write  $G = \frac{1}{2}(G + G^r) + \frac{1}{2}(G - G^r)$ , where

$$G + G^r = \begin{pmatrix} F(x + y^r, x_{n+1}, x - y^r) \\ 2f(x + y^r, x_{n+1}, x - y^r) - x_{n+1} \\ F^r(x + y^r, x_{n+1}, x - y^r) \end{pmatrix}$$

has Jacobian symmetry  $\blacksquare$ , and

$$G - G^r = \begin{pmatrix} \tilde{F}(x + y^r, x_{n+1}, x - y^r) \\ x_{n+1} \\ -\tilde{F}^r(x + y^r, x_{n+1}, x - y^r) \end{pmatrix}$$

has Jacobian (anti)symmetry  $\blacksquare$ . From the (anti)symmetries  $\blacksquare$  and  $\blacksquare$ , we can derive that  $F_j, f \in K[x, x_{n+1}]$  and  $\tilde{F}_j \in K[y]$  for all  $j \leq n$ . By replacing  $\tilde{F}(x, x_{n+1}, y)$  by  $\tilde{F}(y)$  (and  $F(x, x_{n+1}, y)$  by  $F(x, x_{n+1})$ ), we obtain that  $G$  is of the form

$$G = \frac{1}{2}(G + G^r) + \frac{1}{2}(G - G^r) = \frac{1}{2} \begin{pmatrix} F(x + y^r, x_{n+1}) + \tilde{F}(x - y^r) \\ 2f(x + y^r, x_{n+1}) \\ F^r(x + y^r, x_{n+1}) - \tilde{F}^r(x - y^r) \end{pmatrix}$$

Hence the conjugation of  $G$  with the linear map  $(x, x_{n+1}, y^r)$  is equal to the right hand side of (8), which is the conjugation of  $(F, f, \tilde{F}(y))$  with the linear map  $(x + y, x_{n+1}, x - y)$ . This gives the last backward implication in (7). The other backward implications in (7) follow by taking  $\tilde{F}(y) = y$  and  $(F, f) = (x, x_{n+1})$  respectively.  $\square$

**Definition 2.3.** Define an ‘instance of  $\blacksquare(K, n)$ ’ as a Keller map  $F \in K[x]^n$  such that  $F - x$  has Jacobian (anti)symmetry  $\blacksquare$ , where  $\blacksquare$  is any (anti)symmetry of the square.

Notice that in both (6) and (7), the first and the second right hand side are satisfied, if and only if the last right hand side is satisfied. Hence we have proved the following as well.

**Theorem 2.4.** *Assume  $K$  is a field of characteristic zero. A map  $G = X + H$  is an (invertible) instance of  $\blacksquare(K, N)$ , if and only if  $X + \frac{1}{2}(H + H^r)$  is an (invertible) instance of  $\blacksquare(K, N)$  and  $X + \frac{1}{2}(H - H^r)$  is an (invertible) instance of  $\blacksquare(K, N)$ .*

When we combine horizontal and vertical (anti)symmetries with diagonal ones, we get the following.

**Theorem 2.5.** *Assume  $K$  is a field of characteristic zero. Then  $\blacksquare(K, n)$ ,  $\blacksquare(K, 2n - 1)$ ,  $\blacksquare(K, 2n)$ ,  $\blacksquare(K, 2n - 1)$ ,  $\blacksquare(K, 2n)$ ,  $\blacksquare(K, 2n)$  and  $\blacksquare(K, 2n + 1)$  are all equivalent.*

*Proof.* The proof is similar to that of the equivalence of  $\blacksquare(K, n)$ ,  $\bullet(K, 2n-1)$ ,  $\bullet(K, 2n)$ ,  $\blacksquare(K, 2n-1)$ ,  $\blacksquare(K, 2n)$ ,  $\blacksquare(K, 2n)$  and  $\blacksquare(K, 2n+1)$  in the proof of theorem 2.2.  $\square$

Notice that we have proved the following as well.

**Theorem 2.6.** *Assume  $K$  is a field of characteristic zero. Then a map  $G = X + H$  is an (invertible) instance of  $\blacksquare(K, N)$ , if and only if  $X + \frac{1}{2}(H + H^r)$  is an (invertible) instance of  $\blacksquare(K, N)$  and  $X + \frac{1}{2}(H - H^r)$  is an (invertible) instance of  $\blacksquare(K, N)$ .*

Now assume that  $F = x + H$  is power linear of *even* degree. Then the construction of an instance of  $\blacksquare(K, 2n)$  out of the instance  $F$  of  $\blacksquare(K, n)$  gives a map that is power linear of even degree again, say  $(x, y) + (B(x, y))^{*d}$ . Since  $d$  is even, we can assume that  $B$  has (anti)symmetry  $\blacksquare$  instead of  $\blacksquare$ . But that means that  $B^2 = 0$ .

In the general case, we can make  $(x, y) + (B(x, y))^{*d}$  out of  $F = x + (Ax)^{*d}$ , where

$$B := \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \otimes A = \begin{pmatrix} abA & -b^2A \\ a^2A & -abA \end{pmatrix}$$

and again we have  $B^2 = 0$  because the left factor of the Kronecker tensor product squares to zero as well. More precisely, if

$$T := \begin{pmatrix} a^{d-1}\sqrt{ab^d - a^db} & -b^{d-1}\sqrt{ab^d - a^db} \\ a^d & -b^d \end{pmatrix} \otimes I_n$$

then the determinant of the left factor of the Kronecker product just above is  $-(a^{d-1}\sqrt{ab^d - a^db})^d$ . By Cramer's rule,

$$T^{-1} = \left( \left( \frac{1}{a^{d-1}\sqrt{ab^d - a^db}} \right)^d \cdot \begin{pmatrix} b^d & -b^{d-1}\sqrt{ab^d - a^db} \\ a^d & -a^{d-1}\sqrt{ab^d - a^db} \end{pmatrix} \right) \otimes I_n$$

Since  $((Ax)^{*d}, 0) = (1, 0) \otimes (Ax)^{*d}$ , it follows from the mixed product property of the Kronecker product and the regular matrix product that

$$\begin{aligned} T^{-1}(F, y)|_{(x,y)=T(x,y)} &= (x, y) + \left( T^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \otimes (Ax)^{*d}|_{(x,y)=T(x,y)} \\ &= (x, y) + \left( \left( \frac{1}{a^{d-1}\sqrt{ab^d - a^db}} \right)^d \cdot \begin{pmatrix} b^d \\ a^d \end{pmatrix} \right) \otimes \\ &\quad \left( A(a^{d-1}\sqrt{ab^d - a^db}x - b^{d-1}\sqrt{ab^d - a^db}y) \right)^{*d} \\ &= (x, y) + \begin{pmatrix} b^d \\ a^d \end{pmatrix} \otimes \left( A(ax - by) \right)^d \\ &= (x, y) + \left( B(x, y) \right)^{*d} \end{aligned}$$

See also [8].

**Theorem 2.7.** *Assume  $K$  is a field of characteristic zero. Then  $\blacksquare(K, 2n) \Leftrightarrow \blacksquare(K, 2n+1)$  and  $\blacksquare(K, 2n) \Leftrightarrow \blacksquare(K, 2n+1)$  hold.*

*Proof.* Both equivalences follow by stabilization: the row and column in the middle of a matrix with (anti)symmetry  $\blacksquare$  or  $\blacksquare$  are both zero.  $\square$

The following theorem shows that complex polynomial maps can be seen as real polynomial maps with a certain Jacobian (anti)symmetry.

**Theorem 2.8.**  $\blacksquare(\mathbb{C}, n)$ ,  $\blacksquare(\mathbb{C}, 2n-1)$ ,  $\blacksquare(\mathbb{C}, 2n)$ ,  $\blacksquare(\mathbb{C}, 2n)$ ,  $\blacksquare(\mathbb{C}, 2n+1)$ ,  $\blacksquare(\mathbb{R}, 2n)$  and  $\blacksquare(\mathbb{R}, 2n+1)$  are equivalent.

*Proof.* The equivalence of  $\blacksquare(\mathbb{C}, n)$ ,  $\blacksquare(\mathbb{C}, 2n-1)$  and  $\blacksquare(\mathbb{C}, 2n)$  follows from theorem 2.2. The equivalence of  $\blacksquare(K, 2n)$  and  $\blacksquare(K, 2n+1)$  for  $K \in \{\mathbb{C}, \mathbb{R}\}$  follows from theorem 2.7. Since the implication  $\blacksquare(\mathbb{C}, 2n) \Rightarrow \blacksquare(\mathbb{C}, 2n)$  follows by conjugation with  $(x, iy)$  and the implication  $\blacksquare(\mathbb{C}, 2n) \Rightarrow \blacksquare(\mathbb{R}, 2n)$  is direct, the implication  $\blacksquare(\mathbb{R}, 2n) \Rightarrow \blacksquare(\mathbb{C}, n)$  remains to be proved.

Assume that  $\blacksquare(\mathbb{R}, 2n)$  holds. Let  $F$  be an instance of Keller type of  $\blacksquare(\mathbb{C}, n)$ . Then  $(F(x), \tilde{F}(y))$  is of Keller type as well, where the coefficients of the polynomial map  $\tilde{F}$  are the complex conjugates of those of  $F$ . Furthermore,  $F$  is invertible in case  $(F(x), \tilde{F}(y))$  is.

We prove that  $\blacksquare(\mathbb{C}, n)$  holds by showing that  $(F(x), \tilde{F}(y))$  can be transformed to an instance of  $\blacksquare(\mathbb{R}, 2n)$  by compositions with invertible linear maps. By [10, Prop. 1.1.7], the instance of  $\blacksquare(\mathbb{R}, 2n)$  satisfies the Jacobian conjecture over  $\mathbb{C}$  as well as over  $\mathbb{R}$ , which gives  $\blacksquare(\mathbb{C}, n)$ .

Notice that

$$\begin{aligned} \frac{1}{2} \begin{pmatrix} I_n & I_n \\ -iI_n & iI_n \end{pmatrix} \begin{pmatrix} F(x+iy) \\ \tilde{F}(x-iy) \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} F(x+iy) + \tilde{F}(x-iy) \\ -iF(x+iy) + i\tilde{F}(x-iy) \end{pmatrix} \\ &= \begin{pmatrix} \operatorname{Re} F(x+iy) \\ \operatorname{Im} F(x+iy) \end{pmatrix} \end{aligned} \quad (9)$$

if  $x$  and  $y$  are considered as real variables, and that (9) is a polynomial map with real coefficients by definition of  $\tilde{F}$ . Since the Jacobian of (9) is

$$\frac{1}{2} \begin{pmatrix} (\mathcal{J}F)|_{x=x+iy} + (\mathcal{J}\tilde{F})|_{x=x-iy} & i(\mathcal{J}F)|_{x=x+iy} - i(\mathcal{J}\tilde{F})|_{x=x-iy} \\ -i(\mathcal{J}F)|_{x=x+iy} + i(\mathcal{J}\tilde{F})|_{x=x-iy} & (\mathcal{J}F)|_{x=x+iy} + (\mathcal{J}\tilde{F})|_{x=x-iy} \end{pmatrix}$$

which has tiling (anti)symmetry

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

a conjugation of (9) with  $(x, y^t)$  gives a map with Jacobian (anti)symmetry  $\blacksquare$ . If we start with  $(F(x), \tilde{F}(y)) = (x, y)$ , then this map is equal to  $(x, y)$  as well, thus the compositions with linear maps add up to a linear conjugation. Hence the conjugation of (9) with  $(x, y^t)$  gives an instance of  $\blacksquare(\mathbb{R}, 2n)$ .  $\square$

## Symmetry patterns that satisfy the Jacobian Conjecture

**Theorem 2.9.** *Assume  $K$  is a field of characteristic zero. Then  $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}(K, N)$ ,  $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}(K, N)$ ,  $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}(K, N)$ , and  $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}(K, N)$  have affirmative answers.*

*Proof.* Let  $F = X + H$  be an instance of any of them. One can easily see that  $\mathcal{J}H \cdot H$  is a vector for which all of its  $N$  coordinates are sums of  $N$  terms that cancel out pairwise except maybe the term in the middle if  $N$  is odd. But that term can only be zero, since either the column in the middle of  $\mathcal{J}H$  is zero or the component in the middle of  $H$  is zero, depending on the actual (anti)symmetry. Therefore,

$$\mathcal{J}H \cdot H = 0 \tag{10}$$

Now it follows from ii)  $\Rightarrow$  i) in [3, Prop. 1.1] that  $G = x - H$  is the inverse of  $F$ .  $\square$

Maps  $x + H$  with inverse  $x - H$  are called quasi-translations. Over a field of characteristic zero, these are exactly the maps that satisfy (10). See also [3, Prop. 1.1]. Quasi-translations arise naturally with singular Hessians: if  $\det \mathcal{H}h = 0$ , then there exists a nonzero polynomial  $R$  such that  $R(\nabla h) = 0$ , and  $x + (\nabla R)(\nabla h)$  happens to be a quasi-translation on account of [1, Eq. (3)] and [3, Prop. 1.1].

## 3 Symmetric variants of the dependence problem

This section is about polynomial maps  $H$  instead of  $F = x + H$ , again with a certain (anti)symmetry in the Jacobian of the map  $H$ , but now the question is whether the (linear) dependence problem (for Jacobians) is satisfied for such maps. We say that  $H \in K[x]^n$  satisfies the dependence problem if

$$\lambda^t \mathcal{J}H = 0$$

for some nonzero  $\lambda \in K^n$ , which is equivalent to

$$\lambda_1 H_1 + \lambda_2 H_2 + \cdots + \lambda_n H_n \in K$$

when  $K$  is a field of characteristic zero. Notice that replacing  $H$  by any composition of  $H$  with invertible linear maps does not change whether  $H$  satisfies the dependence problem.

One may think that  $\det \mathcal{J}H = 0$  is a somewhat weak condition for getting linear dependence. Indeed, the dependence problem with  $\det \mathcal{J}H = 0$  is satisfied for arbitrary Jacobians only in dimension 1 and for homogeneous Jacobians only in dimensions 1 and 2. But the dependence problem with  $\det \mathcal{J}H = 0$  and additionally  $\mathcal{J}H$  symmetric with regular symmetry  $\begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$  reaches twice as far: it is satisfied for arbitrary Jacobians in dimensions 1 and 2, and for homogeneous Jacobians in all dimensions  $n \leq 4$ , see [5] and [6] respectively.

Corollary 3.5 below shows that the above results about symmetric Jacobians imply those about non-symmetric Jacobians. By theorem 3.8 below, we see that the dependence problem with  $\det \mathcal{J}H = 0$  and additionally  $\mathcal{J}H$  symmetric with symmetry  $\boxtimes$  is even satisfied for arbitrary Jacobians in dimensions  $n \leq 5$ , and for homogeneous Jacobians in all dimensions  $n \leq 9$ .

**Definition 3.1.**  $\square[K, n]$  means that the dependence problem is satisfied for  $n$ -dimensional maps  $H$  over the field  $K$ , such that the degree of each term of  $H$  is contained in a fixed set  $S \subseteq \mathbb{N}$ , and optionally  $\det \mathcal{J}H = 0$  or even  $(\mathcal{J}H)^n = 0$ .

$\boxtimes[K, n]$  and  $\boxminus[K, n]$  mean that the dependence problem is satisfied for  $n$ -dimensional maps  $H$  as above that have a symmetric Jacobian with respect to the diagonal and anti-diagonal respectively.

Et cetera. We replace the parenthesis of the symmetric variants of the Jacobian conjecture by square brackets all the time.

Notice that a horizontal symmetry axis in the Jacobian of  $H$  implies linear dependence over the base field  $K$  between the rows of  $\mathcal{J}H$  (in case the dimension is larger than one). Therefore, such (anti)symmetries will not be considered any further in this section. The same holds for (partial) antisymmetries that imply that the row in the middle of  $\mathcal{J}H$  is zero.

## Diagonally symmetric variants

**Definition 3.2.** Define an ‘instance of  $\square[K, n]$ ’ as a map  $H \in K[x]^n$  such that  $\mathcal{J}H$  has (anti)symmetry  $\square$ , where  $\square$  is any (anti)symmetry of the square.

**Theorem 3.3.** *Assume  $K$  is a field of characteristic zero. Then  $\boxtimes[K, 2n]$  implies  $\square[K, n]$ .*

*Proof.* If the set  $S$  of term degrees in the definition of  $\square[K, n]$  satisfies  $S \subseteq \{0, 1\}$ , then all that matters for  $\square[K, n]$  is whether  $\det \mathcal{J}H$  vanishes or not in the definition of  $\square[K, n]$ , and theorem 3.3 is trivially satisfied by definition. Thus assume that  $d \in S$  for some  $d \geq 2$ .

Suppose that  $H$  is an instance of  $\square[K, n]$ . With the proof of theorem 1.2, we obtain a map  $(H(x), G(x, y))$  which is an instance of  $\boxtimes[K, 2n]$ , and

$$\mathcal{J} \begin{pmatrix} H(x) \\ G(x, y) \end{pmatrix} = \begin{pmatrix} \mathcal{J}H & \mathbf{0} \\ * & (((\mathcal{J}H)^t)^r) \end{pmatrix} = \begin{pmatrix} \mathcal{J}H & \mathbf{0} \\ * & \mathcal{J}H \end{pmatrix}$$

Since the characteristic polynomial of the above matrix is the square of that of  $\mathcal{J}H$ , the nilpotency (or the vanishing of the determinant) of the Jacobian of  $(H(x), G(x, y))$  is completely determined by the nilpotency (or the vanishing of the determinant) of  $\mathcal{J}H$ . So we get another instance of  $\boxtimes[K, 2n]$  if we change  $G$  such that only the  $*$ -part of the above Jacobian changes. Hence we may change the terms without  $y$  in  $G$  by other terms without  $y$ , but we must not forget to preserve the symmetry  $\boxtimes$  of the  $*$ -part. We do this by replacing the part of  $G$  that has terms without  $y$  only by  $(x^r)^{*d} = (x_n^d, x_{n-1}^d, \dots, x_2^d, x_1^d)$ .

Now assume that  $\square[K, 2n]$  is satisfied. Then the components of  $(H, G)$  are linearly dependent over  $K$ , say that

$$\lambda^t H + \mu^t G \in K$$

where  $\lambda, \mu \in K^n$  are not both zero. If  $\mu = 0$ , then the components of  $H$  are linearly dependent over  $K$ , as desired, so it suffices to show that  $\mu = 0$ . Since  $H$  has no terms with  $y$ ,  $\lambda^t \mathcal{J}_y H = 0$  and we obtain that  $\mu^t \mathcal{J}_y G = 0$ , too. Thus  $\lambda^t \mathcal{J}_y H \lambda^r = \mu^t \mathcal{J}_y G \lambda^r = 0$ , and by the symmetry  $\square$  of  $\mathcal{J}_{x,y}(H, G)$  and by (1), we see that

$$\lambda^t \mathcal{J}_x H \mu^r = \lambda^t (I_{2n}^r \mathcal{J}_x^r H) I_{2n}^r \mu = (\mu^t I_{2n}^r (\mathcal{J}_x^r H)^t \lambda^r)^t = (\mu^t \mathcal{J}_y G \lambda^r)^t = 0$$

Hence by  $\lambda^t H + \mu^t G \in K$  and (1),

$$0 = (\lambda^t \mathcal{J}_x H \mu^r + \mu^t \mathcal{J}_x G \mu^r) \Big|_{y=0} = d \mu^t I_n^r \text{diag}(x^{*(d-1)}) \mu^r = d \sum_{i=1}^n \mu_{n+1-i}^2 x_i^{d-1}$$

which is a contradiction to  $\mu \neq 0$  because  $d \geq 2$ .  $\square$

**Theorem 3.4.**  $\square[\mathbb{C}, N]$  and  $\square[\mathbb{C}, N]$  are equivalent.

*Proof.* The proof is similar to that of theorem 1.3.  $\square$

**Corollary 3.5.**  $\square[\mathbb{C}, 2n]$  implies  $\square[\mathbb{C}, n]$ .

*Proof.* This follows from theorem 3.4 and theorem 3.3.  $\square$

Notice that there is no converse of theorems 1.2 and 3.3. But if we define  $\square(K, n)$  and  $\square[K, n]$  as  $\square(K, n)$  and  $\square[K, n]$  respectively with the extra condition that the upper right quadrant of the Jacobian is zero, then we do have a converse.

## Centrally symmetric variants

In a similar way as above,  $\square(K, N)$  and  $\square[K, N]$  are equivalent to  $\square(K, N)$  and  $\square[K, N]$  respectively. The proof is left as an exercise to the reader.

The following theorem is an analog of theorem 2.2, but one of the indexes of  $\square$  is different:  $2n + 1$  instead of  $2n - 1$ . This is because the  $\wedge$ 's become  $\vee$ 's. But first, we formulate a lemma.

**Lemma 3.6.** *Assume  $K$  is a field of characteristic zero. Then  $\square[K, n + 1]$  implies  $\square[K, n]$ .*

*Proof.* Assume  $H$  is an instance of  $\square[K, n]$ . If the set  $S$  in the definition of  $\square[K, n]$  is a subset of  $\{0\}$ , then  $H$  satisfies the dependence problem. Thus assume  $d \in S$  for some  $d \geq 1$ . If  $\det \mathcal{J}H = 0$  by definition of  $\square[K, n]$ , then there exists an  $i \leq n$  such that  $e_i^t$  is not contained in the row space (over  $K(x)$ ) of  $\mathcal{J}H$ . We take  $h = x_i^d$  in case  $\det \mathcal{J}H = 0$  by definition of  $\square[K, n]$  and  $h = x_{n+1}^d$  otherwise (because we need  $\frac{\partial}{\partial x_{n+1}} h = 0$  when  $\mathcal{J}H$  is nilpotent by definition of  $\square[K, n]$ ). Then  $(H, h)$  is an instance of  $\square[K, n + 1]$  and  $H$  satisfies the dependence problem in case  $(H, h)$  does, as desired.  $\square$



**Theorem 3.7.** *Assume  $K$  is a field of characteristic zero. Then  $\blacksquare[K, n]$ ,  $\bullet[K, 2n]$ , and  $\bullet[K, 2n + 1]$  are equivalent.*

*Proof.* By lemma 3.6, it suffices to prove that

$$\bullet[K, 2n] \iff \blacksquare[K, n] \vee \blacksquare[K, n]$$

and

$$\bullet[K, 2n + 1] \iff \blacksquare[K, n + 1] \vee \blacksquare[K, n]$$

This can be done in the same way as

$$\bullet(K, 2n) \iff \blacksquare(K, n) \wedge \blacksquare(K, n)$$

and

$$\bullet(K, 2n + 1) \iff \blacksquare(K, n + 1) \wedge \blacksquare(K, n)$$

in the proof of theorem 2.2. □

**Theorem 3.8.** *Assume  $K$  is a field of characteristic zero. Then  $\blacktriangle[K, n]$ ,  $\boxtimes[K, 2n]$  and  $\boxtimes[K, 2n + 1]$  are equivalent.*

*Proof.* The proof is similar to that of the previous theorem. □

**Theorem 3.9.**  $\blacksquare[\mathbb{C}, n]$ ,  $\bullet[\mathbb{C}, 2n]$ ,  $\bullet[\mathbb{C}, 2n + 1]$ ,  $\bullet[\mathbb{C}, 2n]$  and  $\bullet[\mathbb{R}, 2n]$  are equivalent.

*Proof.* The proof is similar to that of theorem 2.8. □

**Corollary 3.10.**  $\bullet[\mathbb{R}, 4n]$  implies  $\blacksquare[\mathbb{C}, n]$ .

*Proof.*  $\bullet[\mathbb{R}, 4n] \Leftrightarrow \blacksquare[\mathbb{R}, 2n] \Rightarrow \bullet[\mathbb{R}, 2n] \Leftrightarrow \blacksquare[\mathbb{C}, n]$ . □

## Planar singular Hessians and planar nilpotent Jacobians

If we assume that  $\det \mathcal{J}H = 0$  instead of that  $\mathcal{J}H$  is nilpotent, then we can transform (anti)symmetries more freely, because we do not need to conjugate. We can just compose with maps in  $\text{GL}_n(K)$ , where  $K$  is a field of characteristic zero. Or with maps in  $\text{GL}_n(A)$ , if we replace  $K$  by an integral domain  $A$ . Now if  $\mathcal{J}H$  has (anti)symmetry  $\blacktriangle$ , then its trace is zero. So if  $\mathcal{J}H$  has dimension 2, then  $\mathcal{J}H$  is nilpotent, if and only if  $\det \mathcal{J}H = 0$  and  $\mathcal{J}H$  has (anti)symmetry  $\blacktriangle$ . We can use this observation to prove the following (which can also be derived directly from [1, Th. 3.1], see [7, Cor. 5.1.2]).

**Theorem 3.11.** *Assume  $A$  is a unique factorization domain of characteristic zero and  $h \in A[x_1, x_2]$  such that  $\det \mathcal{H}h = 0$ . Then  $h$  is of the form*

$$g(ax_1 - bx_2) + (cx_1 - dx_2)$$

where  $g$  is an univariate polynomial over  $A$  and  $a, b, c, d \in A$ . Furthermore,  $g$  can be taken constant in case  $\text{rk } \mathcal{H}h = 0$ .

*Proof.* Let  $H := \nabla h$ . Since  $\mathcal{J}H$  has symmetry  $\blacksquare$ , we see that  $\mathcal{J}(-H_2, H_1)$  has (anti)symmetry  $\blacksquare$ , thus  $\text{tr } \mathcal{J}(-H_2, H_1) = 0$ . Since  $\det \mathcal{J}(-H_2, H_1) = 0$  as well, we get that  $\mathcal{J}(-H_2, H_1)$  is nilpotent. By [10, Th. 7.2.25] (see also [11]), we obtain that  $(-H_2, H_1)$  is of the form

$$\begin{pmatrix} -H_2 \\ H_1 \end{pmatrix} = \begin{pmatrix} bg'(ax_1 - bx_2) + d \\ ag'(ax_1 - bx_2) + c \end{pmatrix}$$

where  $g'$  is the derivative of an univariate polynomial  $g$  over  $A$ . Now one can easily see that  $H$  is the gradient map of  $g(ax_1 - bx_2) + (cx_1 - dx_2)$ . Hence  $h$  is of the desired form.  $\square$

## Conclusion

We have seen that besides the regular diagonal symmetry, there are many other interesting (anti)symmetry properties, over which a lot can be said in connection with the Jacobian conjecture.

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