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Toposes as Homotopy Groupoids

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The purpose of this paper is to deepen our understanding of toposes as generalized spaces. We prove that for any Grothendieck topos $\mathscr{E}$ there exists a space X and a collection P of paths in X so that $\mathscr{E}$ is equivalent to the category of sheaves on X which are constant along the paths in P. Furthermore, by taking homotopy classes of paths in P we obtain a continuous groupoid $G \rightrightarrows X$ so that the sheaves on X which are acted upon by G are exactly those sheaves which are constant along the paths in P. As a consequence, we thus deduce one of the main results proved in [JT], showing that any topos can be described as the category of equivariant sheaves for some continuous groupoid. Our construction has the following additional properties. The covering map $\varphi: X \to \mathscr{E}$ is connected and locally connected, and fiberwise contractible. This implies that φ induces a full embedding of $\mathscr{E}$ into the category of sheaves on X which preserves exponentials, in addition to first order logic. Moreover, it implies that φ induces an isomorphism between the (etale) cohomology groups of the topos $\mathscr{E}$ and the cohomology of the space X; this last aspect will be discussed in a different paper.

1. CONTRACTIBLE ENUMERATION SPACES

In this section we will consider the construction of a space $E(S)$ from an object S in some Grothendieck topos $\mathscr{E}$. The points of $E(S)$ will be the infinite-to-one partial enumerations of S, i.e., functions $\alpha: U \to S$ where $U \subseteq \mathbb{N}$ and $\alpha^{-1}(s)$ is infinite for every $s \in S$. If S is uncountable, there

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clearly cannot be any such points. Therefore we will define $E(S)$ as a space in the generalized sense of [JT, Sect. IV]. That is, $E(S)$ is defined in terms of its lattice of open sets. In describing the construction, we will abuse the notation and act as if $\mathscr{E}$ is the category of sets; this is justified as long as we use constructive and explicit arguments only, as is shown, e.g., in [BJ].

Let S be a "set." We define a space $E(S)$ by specifying a poset $\mathbb{P}$ equipped with a stable covering system (a Grothendieck topology): the elements of $\mathbb{P}$ are functions

$$u: K \to S,$$

where K is a finite subset $\{k_1, ..., k_n\}$ of $\mathbb{N}$, and where

$$u \leqslant v \quad \text{iff} \quad v \subseteq u,$$

i.e., $u \leqslant v$ if and only if, for all $n \in \mathbb{N}$, if $v(n)$ is defined so is $u(n)$, and $u(n) = v(n)$. $\mathbb{P}$ has a largest element ϕ, and "compatible" meets exist: $u \wedge v = u \cup v$ if the latter is a well-defined function. For a given $v \in \mathbb{P}$, there are covering families of v, one family

$$\mathscr{C}_{n,s}(v) = \{x \in \mathbb{P} \mid x \leqslant u \;\&\; \exists m \geqslant n\; x(m) = s\} \tag{1}$$

for each $n \in \mathbb{N}$ and each $s \in S$. This defines a presentation $(\mathbb{P}, \mathscr{C})$ of the space $E(S)$. We write $\mathcal{O}(E(S))$ for the lattice of opens of $E(S)$. So the elements of $\mathcal{O}(E(S))$ are subsets U of $\mathbb{P}$ such that (i) $u \leqslant v \in U$ implies $u \in U$, and (ii) if $\mathscr{C}_{n,s}(v) \subseteq U$ for some n and s, then $v \in U$. V_u will denote the basic open corresponding to $u \in \mathbb{P}$: $V_u = \{v \mid v \leqslant u\}$. Clearly the construction of $(\mathbb{P}, \mathscr{C})$ is preserved by inverse image functors, so we have the following stability lemma.

1.1. LEMMA. *For any geometric morphism $p: \mathscr{F} \to \mathscr{E}$ and any object S of $\mathscr{E}$, there is a natural isomorphism*

$$E(p^*S) \cong p^\# E(S)$$

of spaces in $\mathscr{F}$.

Here $p^\#$ denotes the functor from spaces in $\mathscr{E}$ to spaces in $\mathscr{F}$ given by pullback along p. More precisely, if X is a space in $\mathscr{E}$ and $\mathscr{E}[X]$ is the category of sheaves on X in $\mathscr{E}$, there is a canonical geometric morphism $\varphi_X: \mathscr{E}[X] \to \mathscr{E}$, and a pullback of toposes [JT, Sect. VI. 1]

$$\begin{tikzcd} \mathscr{F}[p^\# X] \arrow[r] \arrow[d, "\varphi_{p^\# X}"'] & \mathscr{E}[X] \arrow[d, "\varphi_X"] \\ \mathscr{F} \arrow[r, "p"] & \mathscr{E}. \end{tikzcd} \tag{2}$$

The next proposition says that the points of $E(S)$ are indeed the partial $\infty - 1$ enumerations. If $\mathbb{N} \supseteq U \xrightarrow{\alpha} S$ is an $\infty - 1$ enumeration, i.e., $\forall s\, \forall n\, \exists m \geqslant n\ \alpha(m) = s$, we use the notation

$$\mathbb{N} \xrightarrow{\infty - 1} S.$$

1.2. PROPOSITION. *Let S be an object of a topos $\mathscr{E}$. The $\mathscr{E}$-topos $\mathscr{E}[E(S)]$ classifies partial $\infty - 1$ enumerations of S; i.e., for any $\mathscr{E}$-topos $\mathscr{F} \to^{p} \mathscr{E}$ there is a natural equivalence between geometric morphisms $\mathscr{F} \to^{f} \mathscr{E}[E(S)]$ over $\mathscr{E}$ and partial $\infty - 1$ enumerations $\mathbb{N} \xrightarrow{\infty - 1} p^*S$ in $\mathscr{F}$.*

Proof. By Lemma 1.1 we have a pullback diagram of toposes

$$\begin{tikzcd} \mathscr{F}[E(p^*S)] \arrow[r] \arrow[d] & \mathscr{E}[E(S)] \arrow[d] \\ \mathscr{F} \arrow[r] & \mathscr{E}, \end{tikzcd}$$

so geometric morphisms $\mathscr{F} \to^{f} \mathscr{E}[E(S)]$ over $\mathscr{E}$ correspond to sections of $\mathscr{F}[E(p^*S)] \to \mathscr{F}$, i.e., to points of $E(p^*S)$ in $\mathscr{F}$. Therefore (replacing $\mathscr{F}$ by $\mathscr{E}$) it is enough to show that points of $E(S)$ are the same as $\infty - 1$ partial enumerations of S in $\mathscr{E}$. Using set-theoretic language, a point of $E(S)$ is described in terms of the presentation $(\mathbb{P}, \mathscr{C})$ by a subset $F \subseteq \mathbb{P}$ such that (1)–(3) hold:

(1) $\phi \in F;\ u \geqslant v \in F \Rightarrow u \in F$

(2) $u, v \in F \Rightarrow \exists w \in F\ (w \leqslant u \,\&\, w \leqslant v)$

(3) $v \in F \Rightarrow \exists x \in \mathscr{C}_{n,s}(v)\ x \in F$ (any n, s).

Given such an F, let $U = \{n \mid \exists v \in F\ n \in \mathrm{dom}(v)\}$, and let $\alpha\colon U \to S$ be defined by

$$\alpha(n) = s \qquad \text{iff} \quad \exists v \in F\ v(n) = s.$$

α is well-defined by (2) and $\forall s\, \forall n\, \exists m \geqslant n\ \alpha(m) = s$ by (1) and (3). Conversely, given an $\infty - 1$ partial enumeration α, one defines $F \subseteq \mathbb{P}$ satisfying (1)–(3) by $F = \{u \in \mathbb{P} \mid u \subseteq \alpha\}$. ∎

Let X be a space (in a topos $\mathscr{E}$), and let $X^I \to^{\varepsilon} X \times X$ be the map given by evaluation at the endpoints, where $I = [0, 1]$ in the unit interval (constructed as a generalized space in the topos $\mathscr{E}$). X is called *contractible* if $X \to 1$ is a surjection and ε has a section $H\colon X \times X \to X^I$; X is called *locally contractible* if it possesses a basis of contractible open subspaces. Notice that if X has a point x_0, $H(x_0, -)$ defines a contraction of X into x_0. Thus the existence of such a section H can be interpreted as a contraction of X into a "variable" base point, even though X may not have any points.

1.3. THEOREM. *Let S be an object of a topos $\mathscr{E}$, and let $E(S)$ be the space of $\infty - 1$ partial enumerations described above. $E(S)$ is contractible and locally contractible.*

Proof. We will explicitly describe a section $H: E(S) \times E(S) \to E(S)^I$ and it will be clear that H maps $V_u \times V_u$ into V_u^I for any basic open V_u ($u \in \mathbb{P}$). H will be defined in point-set terms. Or more precisely, we will define a natural transformation

$$H_Y : Cts(Y, E(S) \times E(S)) \to Cts(Y, E(S)^I)$$

which is natural in the parameter space Y. Given such a Y, a map of spaces $Y \to E(S) \times E(S)$ in $\mathscr{E}$ can be viewed as a pair of points (α, β) of $E(S)$ in $\mathscr{E}[Y]$ (here we identify S with the object $\varphi_Y^*(S)$ of $\mathscr{E}[Y]$, where $\varphi_Y : \mathscr{E}[Y] \to \mathscr{E}$ is the canonical geometric morphism). The map $H_Y(\alpha, \beta)$: $Y \to E(S)^I$, to be defined, is a path in the space $E(S)$ from α to β, constructed in $\mathscr{E}[Y]$ as follows. Work inside $\mathscr{E}[Y]$ and fix a sequence of rationals

$$0 < p_0 < p_1 < \cdots$$

converging to 1 ($p_i = 1 - 2^{-(i+1)}$ say). The easiest way is to describe the path

$$H_Y(\alpha, \beta): I \to E(S)$$

in point-set notation by

$$H_Y(\alpha, \beta)(0) = \alpha,$$

$$H_Y(\alpha, \beta)(t)(n) = \begin{cases} \alpha(n) & \text{if } t < p_n \\ \beta(n) & \text{if } t > p_n \\ \{\alpha(n) \mid \alpha(n) = \beta(n)\} & \text{if } t = p_n \end{cases} \qquad 0 < t < 1,$$

$$H_Y(\alpha, \beta)(1) = \beta.$$

In other words, $\gamma(t) = H_Y(\alpha, \beta)(t)$ is the enumeration α for $0 \leqslant t < p_0$; at $t = p_0$, $\gamma(t)(0)$ is defined only if $\alpha(0) = \beta(0)$, and for $p_0 < t$, $\gamma(t)(0) = \beta(0)$ while $\gamma(t)$ equals α at larger arguments; at $t = p_1$, $\gamma(t)(1)$ is defined only if $\alpha(1) = \beta(1)$, etc. This describes a well-defined map of spaces $I \to E(S)$ in $\mathscr{E}[Y]$. ∎

2. Spatial Covers Having Contractible Fibers

We will prove the following result.

2.1. Theorem. *For any topos $\mathscr{E}$ over the base topos $\mathscr{S}$ there exists a space $X_{\mathscr{E}}$ (in $\mathscr{S}$) and a connected, locally connected geometric morphism*

$$X_{\mathscr{E}} \xrightarrow{\varphi} \mathscr{E}. \tag{1}$$

Moreover, $X_{\mathscr{E}}$ is contractible as a space over $\mathscr{E}$. ($X_{\mathscr{E}}$ in (1) stands for the category $\mathscr{S}[X_{\mathscr{E}}]$ of sheaves on $X_{\mathscr{E}}$.)

Rephrasing what it means for a geometric morphism to be connected and locally connected, we obtain

2.2. Corollary. *For any topos $\mathscr{E}$ there exists a functor φ^* from $\mathscr{E}$ into the category of sheaves on a space $X_{\mathscr{E}}$ such that*

(i) *φ^* has both a left- and a right-adjoint (so φ^* preserves all limits and colimits).*

(ii) *φ^* is full and faithful.*

(iii) *φ^* preserves Π-functors; or equivalently, the induced functor*

$$\varphi^*/E\colon \mathscr{E}/E \to \text{sheaves}(X_{\mathscr{E}})/\varphi^*(E)$$

preserves exponentials, for every $E \in \mathscr{E}$.

In other words, $\mathscr{E}$ is fully embedded as a locally cartesian closed category into a category of sheaves on a space. Notice that this gives many embedding theorems for small categories $\mathbb{C}$, by equipping $\mathbb{C}$ with a suitable topology and applying the embedding φ^* above to the case $\mathscr{E} = Sh(\mathbb{C})$, and composing with the Yoneda embedding followed by sheafification:

$$\mathbb{C} \xhookrightarrow{y} \text{Sets}^{\mathbb{C}^{op}} \xrightarrow{\alpha} Sh(\mathbb{C}) = \mathscr{E} \xhookrightarrow{\varphi^*} Sh(X_{\mathscr{E}}).$$

For example, if $\mathbb{C}$ is a cartesian closed category, there exists a full embedding of $\mathbb{C}$ into a category of sheaves on a space which preserves finite limits, stable effective epimorphisms, and exponentials. We leave the statement of similar embedding theorems for regular categories, small pretoposes, small elementary toposes, etc., to the reader.

In the proof of 2.1, $X_{\mathscr{E}}$ will be the space defined by $X_{\mathscr{E}} = \mathscr{E}[E(S)]$, for a suitable object S of $\mathscr{E}$ (recall that $X_{\mathscr{E}}$ also stands for the topos of sheaves on $X_{\mathscr{E}}$), and $\varphi\colon X_{\mathscr{E}} \to \mathscr{E}$ will correspond to the canonical geometric morphism $\mathscr{E}[E(S)] \to \mathscr{E}$. So $E(S)$ can be thought of as the fiber of φ, explaining the title of the section. $E(S)$ is a contractible and locally contractible space by 1.3, and $E(S)$ is an open surjective space since every cover $\mathscr{C}_{n,s}(v)$ in the presentation $(\mathbb{P}, \mathscr{C})$ is inhabited [JT, Sect. VII.2]. It follows that $E(S)$ is connected and locally connected. (This is also easy to see directly, though: Since $\mathbb{P}$ has a terminal object, it is enough to show that for any $u \in \mathbb{P}$ and any cover R of u, R is inhabited and moreover any two $v, w \in R$ can be connected by a chain $v = v_0, v_1, ..., v_n = w$ in R such that $\forall i < n \exists x \in \mathbb{P}\ (x \leqslant v_i \;\&\; x \leqslant v_{i+1})$. But consider a cover $R = \mathscr{C}_{n,s}(u)$, and pick $v, w \in R$. Let $k > 0$ be so large that dom(v) and dom(w) are contained in $\{0, ..., k-1\}$. Then define a chain $v_0 = v$, $v_1 = u \cup (k, s)$, $v_2 = w$, and observe that $v \cup (k, s) \leqslant v_0$ and v_1, while $w \cup (k, s) \leqslant v_1, v_2$.)

We wish to emphasize that Theorem 2.1 expresses only a small portion of the nice properties of the geometric morphism $\varphi: X_{\mathscr{E}} \to \mathscr{E}$. For instance, contractibility of the fibers of φ also implies that the image of φ^* is closed under "locally isomorphic to," and that if $\varphi^*(S)$ is locally isomorphic to $\varphi^*(T)$ then S is locally isomorphic to T, for any objects S and T. It follows that φ induces an isomorphism between the fundamental groups of $X_{\mathscr{E}}$ and $\mathscr{E}$. More generally, it can be shown that φ induces isomorphisms in cohomology and in étale homotopy. Details of this will appear in a sequel to this paper [JM].

To prove Theorem 2.1, we use the following lemma, which is analogous to [JT, Sect. VII.3]. Here $S[U]$ is the object classifier,

$$\mathscr{S}[U] = \mathrm{Sets}^{\mathbb{F}},$$

where $\mathbb{F}$ is the category of finite sets.

2.3. Lemma. *Let $S[U]$ be the object classifier, with generic object U, and let $E(U)$ be the associated space in $\mathscr{S}[U]$ of partial $\infty - 1$ enumerations. Then $\mathscr{S}[U]\,[E(U)]$ is a spatial topos.*

Proof. $\mathscr{S}[U][E(U)]$ classifies equivalence relations on subsets of $\mathbb{N}$, with infinite equivalence classes. This corresponds to the geometric theory given by propositional letters $p_{n,m}$ $(n, m \geqslant 0)$, which express that n and m belong to the same equivalence class. So the axioms are

(1) $p_{n,m} \vdash p_{m,n}$

(2) $p_{n,m} \vdash p_{n,n}$

(3) $p_{n,m}, p_{m,k} \vdash p_{n,k}$

(4) $p_{n,m} \vdash \bigvee_{k>m} p_{n,k}$.

Any classifying topos for a propositional geometric theory is spatial. ∎

Now let $G \in \mathscr{E}$ be an object whose subobjects generate (such a G always exists, e.g., $\coprod_{C \in \mathbb{C}} \varepsilon(C)$, where $\mathbb{C}$ is a site for $\mathscr{E}$ and $\varepsilon: \mathbb{C} \to \mathscr{E}$ is the Yoneda embedding followed by sheafification). Then the geometric morphism

$$g: \mathscr{E} \to \mathscr{S}[U]$$

which classifies G (i.e., $g^*U \cong G$) is clearly spatial. Consider the pullback square (Lemma 1.1)

$$\begin{array}{ccc} \mathscr{E}[E(G)] & \xrightarrow{\bar{g}} & \mathscr{S}[U][E(U)] \\ \downarrow & & \downarrow \\ \mathscr{E} & \xrightarrow{g} & \mathscr{S}[U]. \end{array}$$

Since g is spatial, so is $\bar{g}$; hence since $\mathscr{S}[U][E(U)]$ is spatial by 2.3, so is $\mathscr{E}[E(G)]$. This means that there exists a space $X_{\mathscr{E}}$ (in the base $\mathscr{S}$) such that we have an equivalence of toposes sheaves $(X_{\mathscr{E}}) \simeq \mathscr{E}[E(G)]$. $X_{\mathscr{E}}$ thus comes equipped with a geometric morphism $\varphi: X_{\mathscr{E}} \to \mathscr{E}$, corresponding to $\varphi_{E(G)}: \mathscr{E}[E(G)] \to \mathscr{E}$ under the equivalence. That φ is connected and locally connected now follows from the fact that $E(G)$ is a connected and locally connected space in $\mathscr{E}$, as we pointed out above. This completes the proof of Theorem 2.1.

3. Sheaves Which Are Constant along Some Paths

Let X be a space. A *class of paths* is a subspace $P \subset X^I$, where I is the unit interval. Given a class of paths P, there are maps

$$P \xleftarrow{\pi} P \times I \xrightarrow{e} X,$$

where π is the projection and e is the restriction of the evaluation map $X^I \times I \to X$. A sheaf E on X is said to be *constant along the paths in* P if there exists a sheaf T on P and an isomorphism $\pi^*(T) \cong e^*(E)$. We write

$$Sh(X, P)$$

for the category of sheaves on X which are constant along the paths in P; it is a full subcategory of the category $Sh(X)$ of all sheaves on X. We will prove:

3.1. Theorem. *For any topos $\mathscr{E}$ there exists a space X and a class of paths $P \subset X^I$ such that $\mathscr{E}$ is equivalent to the category of sheaves on X which are constant along paths in P*:

$$\mathscr{E} \simeq Sh(X, P).$$

The space X for which we will prove this theorem is the space $X_{\mathscr{E}}$ constructed in the previous section, and P is defined by the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X_{\mathscr{E}}^{I} \arrow[d, "\varphi^{I}"] \\ \mathscr{E} \arrow[r] & \mathscr{E}^{I}, \end{tikzcd} \tag{1}$$

where $\mathscr{E} \to \mathscr{E}^{I}$ is the transposed of the projection $\mathscr{E} \times I \to \mathscr{E}$ (as before, $X_{\mathscr{E}}$ and I also denote the toposes of sheaves $\mathscr{S}[X_{\mathscr{E}}]$ and $\mathscr{S}[I]$). P is indeed a subspace of $X_{\mathscr{E}}^{I}$, by the following lemma.

3.2. LEMMA. *The "constant path" geometric morphism $\mathscr{E} \to \mathscr{E}^{I}$, transposed to the projection, is an inclusion of toposes.*

Proof. Suppose $\mathscr{E}$ classifies a geometric theory T, which we may assume to be single-sorted. Then $\mathscr{E}^{I}$ classifies the theory T' of sheaves of T-models on the unit interval. Let $M \to^{p} I$ be the generic model in $\mathscr{E}^{I}$ (so p is a local homeomorphism, and the codomain of p is the internal unit interval in $\mathscr{E}^{I}$). Add to T' the axiom that the canonical map $M \to \pi_0(M) \times I$ is an isomorphism. The resulting topos is a subtopos of $\mathscr{E}^{I}$, which is clearly equivalent to $\mathscr{E}$ itself. ∎

Another description of P is provided by the following lemma.

3.3. LEMMA. *Let $\mathscr{E}$ be a topos, let $\gamma\colon \mathscr{E} \to \mathscr{S}$ be the canonical geometric morphism into the base, let Y be a space in $\mathscr{E}$, and let K be an exponentiable space in $\mathscr{S}$ (e.g., K is compact regular, cf.* [JJ]). *Then the following diagram is a pullback of toposes*:

$$\begin{tikzcd} \mathscr{E}[Y^{\gamma^*(K)}] \arrow[r] \arrow[d] & \mathscr{E}[Y]^{K} \arrow[d, "\varphi^{K}"'] \\ \mathscr{E} \arrow[r] & \mathscr{E}^{K}, \end{tikzcd} \tag{2}$$

where $\varphi\colon \mathscr{E}[Y] \to \mathscr{E}$ is the canonical geometric morphism, $\mathscr{E} \to \mathscr{E}^{K}$ is transposed to the projection, and $Y^{\gamma^{\#}(K)}$ is the exponential of spaces in $\mathscr{E}$.

Proof. This is a completely elementary diagram chase argument, based on the (bicategorical) universal properties of exponentials and pullbacks. ∎

Proof of Theorem 3.1. Suppose F is a sheaf on $X_{\mathscr{E}}$ which is constant along the paths in P. By construction of $X_{\mathscr{E}}$, F corresponds to an object (also called F) of $\mathscr{E}[E(G)]$, i.e., F can be viewed as a sheaf on the enumeration space $E(G)$ inside $\mathscr{E}$. By 3.3

$$P = \mathscr{E}[E(G)^I]$$

so that if F is constant along the paths in P, then F is constant along *all* the paths in $E(G)$, when viewed as a sheaf on $E(G)$ inside $\mathscr{E}$, but $E(G)$ is a contractible space, i.e., evaluation at the endpoints $E(G)^I \to E(G) \times E(G)$ has a section, as proved in Section 1. It follows that the restriction map $E(G)^{\Delta} \to E(G)^{\partial\Delta}$ also has a section, where Δ is the standard 2-simplex and $\partial\Delta$ is its boundary (simply view Δ as the cone on $\partial\Delta$). To complete the proof, we now apply the following lemma to the case $Y = E(G)$. ∎

3.4. LEMMA. *Let Y be a connected, locally connected space (in a topos $\mathscr{E}$). Suppose Y is simply connected in the sense that the restriction map $Y^{\Delta} \to Y^{\partial\Delta}$ is a stable surjection. If F is a sheaf on Y which is constant along all paths in Y, then F is constant.*

Proof. Let F be a sheaf on Y, viewed as an étale space $F \to^{p} Y$, by the descent theorem [JT, M] it is enough to produce an equivalence relation $R \subset F \times F$ such that

$$\begin{array}{ccc} R & \xrightarrow{\pi_1} & F \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ Y \times Y & \xrightarrow{\pi_1} & Y \end{array} \tag{3}$$

is a pullback. Moreover, F is locally connected since Y is, so by writing F as the sum of its connected components, it suffices to consider the case where F is connected. We claim that in this case we may take $R = F \times F$ in (3), i.e., we claim that

$$1 \times p \colon F \times F \to F \times Y \tag{4}$$

is an isomorphism. To see this, first notice that since F is constant along paths, p has unique path-lifting, i.e., the map

$$\alpha \colon F^I \xrightarrow{\sim} F \times_Y Y^I \tag{5}$$

given by evaluation at 0 and composition with p, $\alpha = (ev_0, p^I)$, is an isomorphism. Now Y is connected and locally connected, so by a result of [MW], $Y^I \to^{\varepsilon} Y \times Y$ is an open surjection. The kernel pair of $Y^I \to^{\varepsilon} Y \times Y$ can be identified with

$$Y^{\partial\Delta} \overset{u}{\rightrightarrows} Y^I,$$

where u is the restriction along the composite of two edges of $\partial\Delta$ and v is the restriction to the other edge. Since open surjections are coequalizers of their kernel pairs [M2, Sect. 1.3], we obtain a stable coequalizer

$$Y^{\partial\Delta} \underset{v}{\overset{u}{\rightrightarrows}} Y^I \longrightarrow Y \times Y, \tag{6}$$

and hence by pullback a (stable) coequalizer

$$F \times_Y Y^{\partial\Delta} \underset{v'}{\overset{u'}{\rightrightarrows}} F \times_Y Y^I \xrightarrow{1 \times \varepsilon} F \times_Y (Y \times Y) \cong F \times Y, \tag{7}$$

where u' and v' are induced by u and v. Moreover, p has unique path-lifting in the sense that (5) is an isomorphism, and from this it follows that in the diagram

$$\begin{tikzcd} F^{\Delta} \arrow[r] \arrow[d, "\gamma"'] & F^{\partial\Delta} \arrow[d, "\beta"] \\ F \times_Y Y^{\Delta} \arrow[r] & F \times_Y Y^{\partial\Delta} \end{tikzcd} \tag{8}$$

first that γ is an isomorphism, and then by simple connectedness of Y that β is an isomorphism; here β and γ are given by evaluation at the vertex 0 and composition with p, like α is, and the horizontal maps in (8) are given by restriction. Now consider the diagram

$$\begin{tikzcd} F^{\partial\Delta} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\beta"'] & F^I \arrow[r] \arrow[d, "\alpha"'] & F \times F \arrow[d, "1 \times p"'] \\ F \times_Y Y^{\partial\Delta} \arrow[r, shift left] \arrow[r, shift right] & F \times_Y Y^I \arrow[r] & F \times Y. \end{tikzcd} \tag{9}$$

The bottom row is the coequalizer (7) and the top row is a coequalizer of the same type as (6). Since α and β are isomorphisms, so is $1 \times p$. This proves the lemma. ▮

4. Homotopy Groupoids

For any given topos $\mathscr{E}$, Joyal and Tierney have constructed a continuous groupoid G (i.e., a groupoid in the category of spaces) such that $\mathscr{E}$ is equivalent to the category of G-sheaves, or étale G-spaces (see [JT]). The purpose of this section is to show that such a groupoid can be realized as a groupoid of homotopy classes of paths in some space.

To begin with, let I be the unit interval as before, and define a space Δ

to be *totally disconnected* if the map $A \to A^I$, transposed to the projection, is an isomorphism of spaces. A *map* $Y \to^t X$ of spaces is said to be totally disconnected if Y is totally disconnected as a space in the topos $Sh(X)$ of sheaves over X—or equivalently, if

$$\begin{array}{ccc} Y & \xrightarrow{t} & X \\ \downarrow & & \downarrow \\ Y^I & \xrightarrow{t^I} & X^I \end{array} \tag{1}$$

is a pullback of spaces (this equivalence follows immediately from Lemma 3.3). The class of totally disconnected maps is stable under composition, and under pullback along an arbitrary map of spaces.

Next, we call a map c: $Y \to X$ *connected* if it is orthogonal to the class of totally disconnected maps, in the sense that for any solid commutative square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "c"'] & B \arrow[d, "t"] \\ X \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \tag{2}$$

with t totally disconnected, there exists a unique diagonal filling $X \to B$. The class of connected maps is closed under composition (but not under pullback).

It is not true in general that a given map can be factored as a connected map followed by a totally disconnected one, but if it can then such a factorization is unique (up to isomorphism), as is clear from the unique lifting property expressed by diagram (2). In particular, a map which is both connected and totally disconnected is an isomorphism.

A space X is called *path-connected* if the map $X \to 1$ and the evaluation at the endpoints ε: $X^I \to X \times X$ are both open surjections. A map $Y \to^f X$ is called path-connected if Y is path-connected as a space in $Sh(X)$; that is, f is an open surjection, as is the map $Y^I_{(X)} \to Y \times_X Y$ induced by ε: $Y^I \to Y \times Y$, where $Y^I_{(X)}$ is the path-space relative to X, constructed as the pullback

$$\begin{array}{ccc} Y^I_{(X)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y^I & \xrightarrow{f^I} & X^I. \end{array} \tag{3}$$

For example, every connected locally connected map is path-connected

[MW]. The class of path-connected maps is obviously stable under pullback.

4.1. LEMMA. (i) *Every path-connected map is connected.*

(ii) *If for a space X the map $X \to 1$ can be factored as $X \to A \to 1$ where $X \to A$ is path-connected and A is totally disconnected, then*

$$X^I \rightrightarrows X \to A$$

is a coequalizer (i.e., A is "the" space of path-components of X).

Proof. (i) By change of base, it is enough to prove that every path-connected space is connected. So let X be path-connected. We have to show that every map from X into a totally disconnected space A is constant. Since an open surjection is a coequalizer of its kernel pair [M2], we have a coequalizer $X \times X \rightrightarrows X \to 1$, and hence since $X^I \to X \times X$ is a surjection, a coequalizer

$$X^I \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} X \to 1, \tag{4}$$

ε_i being the evaluation at the endpoint i. But if $X \to^f A$ is a map into a totally disconnected space, then clearly $f \circ \varepsilon_0 = f \circ \varepsilon_1$, and therefore f is constant.

(ii) Let $X \to A \to 1$ be a factorization with $X \to A$ path-connected. Then $X^I_{(A)} \to X \times_A X$ is an open surjection, where $X^I_{(A)}$ is the relative exponential as in (3), so as in part (i) we have a coequalizer

$$X^I_{(A)} \rightrightarrows X \to A$$

(of the same type as (4), with A as the base now). But if A is moreover totally disconnected, then clearly $X^I_{(A)} = X^I$, and (ii) is proved. ▮

On the basis of this lemma, we introduce the following notation: *if $Y \to^f X$ is a map which can be factored as a totally disconnected map following a path-connected map, we say that $\pi_0(Y/X)$ exists*, and use it to denote the unique space for which there is a diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "p"'] & X \\ \pi_0(Y/X) \arrow[ur, "t"'] & \end{tikzcd} \tag{5}$$

with p path-connected and t totally disconnected; $\pi_0(Y/X)$ is the space of path-components of Y relative to X.

We also wish to point out that as a consequence of 4.1(i), the concept of connected map introduced in this section coincides with that of *SGA*4 when one restricts ones attention to locally connected maps. Thus, there is no conflict with our earlier use, as, e.g., in Theorem 2.1. We state this explicitly as a lemma.

4.2. LEMMA. *Let $Y \xrightarrow{f} X$ be a locally connected map of spaces. Then f is connected if and only if the pullback functor f^* from sheaves on X to sheaves on Y is fully faithful.*

Proof. ($\Leftarrow$) By [*MW*], $Y \xrightarrow{f} X$ is path-connected; so this direction follows from 4.1(i).

($\Rightarrow$) Suppose f is locally connected, and connected in the sense defined in this section. By taking the set of components (relative to X), we may factor f as $Y \overset{c}{\twoheadrightarrow} E \xrightarrow{p} X$ where c^* is fully faithful and p is étale, in the usual sense. c is also locally connected, so c must be connected by the direction ($\Leftarrow$) just proved. Since $p \circ c = f$ is also connected by assumption, p is connected as well as totally disconnected, hence an isomorphism. ∎

Let us now go back to the context of the previous section. Let X be a space, and let $P \subset X^I$ be a class of paths which is closed under the normal operations of composition, inversion, and reparametrization of paths, and moreover contains all the constant paths (i.e., $X \to X^I$ factors through P). Let $P \xrightarrow{\varepsilon} X \times X$ be the evaluation at the endpoints.

4.3. PROPOSITION. *If $\pi_0(P/X \times X)$ exists, then $\pi_0(P/X \times X) \to X \times X$ has the structure of a continuous groupoid, with operations induced from the usual operations of composition and inversion of paths in P. This groupoid is called the homotopy groupoid associated to P.*

Proof. Suppose $P \xrightarrow{q} \pi_0(P/X \times X) \xrightarrow{(\partial_0, \partial_1)} X \times X$ is a factorization of ε into a path-connected map q and a totally disconnected map (∂_0, ∂_1). For brevity, we write π_0 for $\pi_0(P, X \times X)$ in the remainder of the proof. Consider the solid diagram

$$\begin{tikzcd}
P \times P \arrow[r, "m"] \arrow[d, "q \times_X q"'] & P \arrow[d, "q"] \\
\pi_0 \times_X \pi_0 \arrow[r, dashed, "\hat{m}"] \arrow[dr] & \pi_0 \arrow[d] \\
& X \times X,
\end{tikzcd}$$

where m is composition of paths. Since $q \times_X q$ is path-connected (being the product (over X) of two path-connected maps), there exists a unique map $\bar{m}$ making the diagram commute. $\bar{m}$ defines an operation of composition on π_0. Inversion is similarly defined, and the unit is of course the composition $X \hookrightarrow P \to \pi_0$, $X \hookrightarrow P$ being the inclusion of constant paths into P. It is not hard to check that π_0 is a groupoid under these operations. For instance, the usual reparametrization shows that the two maps in $m(m \times 1)$ and $m(1 \times m)$ are homotopic, and hence that they induce one and the same map $\pi_0 \times_X \pi_0 \times_X \pi_0 \to \pi_0$, because homotopic maps into a totally disconnected space (over $X \times X$ in this case) must be equal. Therefore $\bar{m}$ is an associative operation

$$\begin{tikzcd}
P \times_X P \times_X P \arrow[r, shift left, "m(1 \times m)"] \arrow[r, shift right, "m(m \times 1)"'] \arrow[d] & P \arrow[d] \\
\pi_0 \times_X \pi_0 \times_X \pi_0 \arrow[r, dashed, "!"] & \pi_0 .
\end{tikzcd}$$

Notice that Lemma 4.1 (ii) expresses that π_0 is constructed by taking homotopy classes, relative to the endpoints, of paths in P.

Let $A \rightrightarrows^{\partial_0}_{\partial_1} X$ be any continuous groupoid. So X is the space of objects, A is the space of arrows, ∂_0 and ∂_1 are domain and codomain, and we are further given a composition operation $A \times_X A \to A$, a unit $X \to A$, and an inversion $A \to A$. An *action* of such a groupoid on a sheaf S over X, i.e., an étale map $S \to X$, is a map $S \times_X A \to S$ (pullback along ∂_1) satisfying the usual identities.

If $P \subset X^I$ is a class of paths in a space X, closed under composition and inversion and containing all the constant paths, it obviously still makes sense to speak of an action of P on a sheaf $S \to X$, even though composition in P is not strictly associative and there are no strict inverses or identities. Before stating the next lemma, we define a class $P \subset X^I$ to be closed under *restriction* if the composition map $X^I \times I^I \to X^I$ maps $P \times I^I$ into P.

4.4. LEMMA. *Let $P \subset X^I$ be a class of paths which is closed under composition, inversion, reparametrization, and contains all constant paths. Suppose, moreover, that the homotopy groupoid $\pi_0(P/X \times X) \to X \times X$ exists (cf. 4.3). Let $S \to X$ be a sheaf.*

(i) *There is a bijective correspondence between actions of P on S and actions of the associated homotopy groupoid on S.*

(ii) *If P is closed under restriction, then S can be equipped with a (necessarily unique) action by P if and only if S is constant along the paths in P (as defined in Section 3).*

Proof. (i) Let $Iso(X \times S, S \times X)$ be the subspace of isomorphisms contained in the exponential $(S\times X)^{(X\times S)}$ of sheaves over $X\times X$. As a space over $X\times X$, this is a subspace of a product of discrete spaces, hence totally disconnected. $Iso(X\times S, S\times X)\to X\times X$ defines a groupoid with X as space of objects. An action of P on S is a homomorphism (in the obvious sense of preserving the operations involved) of P into the groupoid $Iso(X\times S, S\times X)$. Such a homomorphism factors uniquely through $\pi_0=\pi_0(P/X\times X)$, since $P\to\pi_0$ is path-connected by assumption, hence connected by 4.1(ii).

(ii) If S is constant along paths in P, there is an obvious unique action of P on S. Conversely, suppose P acts on X. If $I\xrightarrow{\alpha} X$ is a path in P, then

$$(s, t)\mapsto(\alpha\circ\mu_t)\cdot s\colon S_{\alpha(0)}\to S_{\alpha(t)}$$

defines an isomorphism $S_{\alpha(0)}\times I \xrightarrow{\sim} \alpha^*(S)$, where $\mu_t\colon I\to I$ is the map $\mu_t(r)=tr$. Applying this to the generic path defined over P (viz. the evaluation $P\times I\to X$), we see that S is constant along paths in P. ∎

The result asserted in the opening lines of this section now follows from Theorem 3.2. In the following theorem, $X_{\mathscr{E}}$ and P are the same as Sections 2 and 3.

4.5. THEOREM. *Let $\mathscr{E}$ be a topos. There exists a space $X_{\mathscr{E}}$ and a class $P\subset X^I_{\mathscr{E}}$ of paths in $X_{\mathscr{E}}$ such that the homotopy groupoid associated to P exists, and $\mathscr{E}$ is equivalent to the category of sheaves on $X_{\mathscr{E}}$ equipped with an action of this homotopy groupoid.*

Proof. P and $X_{\mathscr{E}}$ are constructed as before. Consider the commutative diagram where the left-hand square is a pullback, and Δ is the diagonal.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & X^I_{\mathcal{E}} \arrow[r] \arrow[d] & X_{\mathcal{E}}\times X_{\mathcal{E}} \arrow[d] \\
\mathcal{E} \arrow[r] \arrow[rr, bend right, "\Delta"'] & \mathcal{E}^I \arrow[r] & \mathcal{E}\times\mathcal{E}
\end{tikzcd}$$

We get a factorization

$$P\to A\to X_{\mathscr{E}}\times X_{\mathscr{E}},$$

where $A=(\varphi\times\varphi)^{\#}(\Delta)$. We prove first that $A=\pi_0(P/X_{\mathscr{E}}\times X_{\mathscr{E}})$, i.e., that

$P \to A$ is path-connected and that $A \to X_{\mathscr{E}} \times X_{\mathscr{E}}$ is locally disconnected. For the latter, we have:

4.6. LEMMA. *For any topos $\mathscr{E}$ the diagonal $\mathscr{E} \to^{\Delta} \mathscr{E} \times \mathscr{E}$ is spatial and totally disconnected.*

Proof. If $\mathscr{E}$ classifies a geometric theory T, then $\mathscr{E} \times \mathscr{E}$ classifies the theory of a pair of models of T. Let $(M, N) \in \mathscr{E} \times \mathscr{E}$ be the generic such pair. As a topos over $\mathscr{E} \times \mathscr{E}$, $\mathscr{E} \to^{\Delta} \mathscr{E} \times \mathscr{E}$ classifies isomorphisms from M to N. So if $Iso_T(M, N)$ denotes the subspace of the function space N^M in $\mathscr{E} \times \mathscr{E}$ consisting of T-model isomorphisms, then $\mathscr{E} \simeq (\mathscr{E} \times \mathscr{E})[Iso_T(M, N)]$. So $\mathscr{E}$ is spatial over $\mathscr{E} \times \mathscr{E}$ (as was already pointed out in [JT]), and totally disconnected because N^M is clearly a totally disconnected space in $\mathscr{E} \times \mathscr{E}$, and hence so is any subspace of N^M. This proves the lemma.

Continuing with the proof of the theorem, we conclude that A is totally disconnected over $X_{\mathscr{E}} \times X_{\mathscr{E}}$, being the pullback of the totally disconnected map $\mathscr{E} \to \mathscr{E} \times \mathscr{E}$.

Next, $P \to A$ is path-connected. In fact, we claim that A is a deformation retract of P. To see this, notice that we have equivalences

$$\begin{array}{ccc}
\mathscr{E}[E(G)^I] & \xrightarrow{\simeq} & P \\
{\scriptstyle \varepsilon}\downarrow & & \downarrow \\
\mathscr{E}[E(G) \times E(G)] & \xrightarrow{\simeq} & A,
\end{array}$$

where ε is evaluation at the endpoints. But inside $\mathscr{E}$, $E(G)^I \to^{\varepsilon} E(G) \times E(G)$ has a section H, and $H \circ \varepsilon$ is homotopic to the identity because $E(G)$ is contractible. (Explicitly, the map K: $E(G)^I \times I \to E(G)^I$ defined in point-set notation by $K(\alpha, t)(s) = H(\alpha(s), H(\alpha(0), \alpha(1))(s))(t)$ defines a homotopy between $H \circ \varepsilon$ and the identity. It is easy to check from this explicit description of K that $E(G)^I \to E(G) \times E(G)$ is path-connected.)

The theorem now follows by applying Lemma 4.4 to Theorem 3.1, since P obviously has the required closure properties. ▮

In [JT], one constructs a continuous groipoid $X \times_{\mathscr{E}} X \rightrightarrows X$ from any open surjection from a space X onto a topos $\mathscr{E}$, such that $\mathscr{E}$ is equivalent to the category of sheaves on X equipped with an action of this groupoid. We should remark that our homotopy groupoid coincides with the groupoid constructed by this method from [JT] for the special case of the cover $X_{\mathscr{E}} \to \mathscr{E}$, as is clear from the construction.

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