TOPOSES ARE COHOMOLOGICALLY EQUIVALENT TO SPACES

By A. Joyal and I. Moerdijk[1]

This purpose of this paper is to prove that for every Grothendieck topos $\mathcal{E}$ there exist a space X and a covering $\varphi: X \to \mathcal{E}$ which induces an isomorphism in cohomology

$$H^n(\mathcal{E}, A) \xrightarrow{\sim} H^n(X, \varphi^* A) \qquad (n \geq 0)$$

for any abelian group A in $\mathcal{E}$. Moreover for $n = 1$ this is also true for nonabelian A. This implies, by a result of Artin and Mazur, that φ induces an isomorphism of etale homotopy groups.

1. Construction of the cover. Let $\mathcal{E}$ be a Grothendieck topos, and let G be an object of $\mathcal{E}$. $\mathrm{En}(G)$ is the space (in this paper 'space' means space in the sense of [JT], chapter IV, unless explicitly said otherwise) of infinite-to-one partial enumerations of G; in other words, $\mathrm{En}(G)$ is characterized by the property that for any map $f: \mathcal{F} \to \mathcal{E}$ of toposes, the points of the induced space $f^{\#}(\mathrm{En}(G))$ in $\mathcal{F}$ correspond to diagrams $\mathbf{N} \longleftarrow\!\!< U \longrightarrow\!\!\!\!\!\!\to f^* G$ in $\mathcal{F}$ with the property that for any $n \in \mathbf{N}$, $U - \{0, \ldots, n\} \to f^* G$ is still epi. We write $\mathcal{E}[\mathrm{En}(G)]$ for the category of sheaves in $\mathcal{E}$ on the space $\mathrm{En}(G)$, and $\varphi: \mathcal{E}(\mathrm{En}(G)] \to \mathcal{E}$ for the corresponding geometric morphism. The properties of the space $\mathrm{En}(G)$ and the map φ were extensively discussed in [JM]. For the present purpose, we recall the following basic facts. First of all, for a suitable object G of $\mathcal{E}$, $\mathcal{E}[\mathrm{En}(G)]$ is equivalent to the topos $\mathrm{Sh}(X_{\mathcal{E}})$ of sheaves on a space $X_{\mathcal{E}}$ in Sets, so that φ corresponds to a cover

(1) $$\varphi: \mathrm{Sh}(X_{\mathcal{E}}) \to \mathcal{E}.$$

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This geometric morphism is connected and locally connected; in particular, $\varphi^*:\mathscr{E} \to \mathrm{Sh}(X_{\mathscr{E}})$ has a left adjoint $\varphi_!$ such that for any $E \in \mathscr{E}$ and $S \in \mathrm{Sh}(X_{\mathscr{E}})$,

$$\varphi_!(\varphi^*(E) \times S) \cong E \times \varphi_!(S). \tag{2}$$

For any G in $\mathscr{E}$, there exists a surjective geometric morphism $p:\mathscr{B} \to \mathscr{E}$ where $\mathscr{B}$ is the category of sheaves on a complete Boolean algebra (Barr's theorem, [B]), such that p^*G is countable (cf. [JT]). $\mathscr{B}$ is a model of set theory, and the induced space $\mathrm{En}(p^*G) \cong p^{\#}(\mathrm{En}(G))$ in $\mathscr{B}$ has enough points, i.e. is an ordinary topological space, which can be described as follows: the points of $\mathrm{En}(p^{\#}(G))$ are functions $\alpha: U \to p^*G$ with $U \subset \mathbf{N}$ and $\alpha^{-1}(g)$ infinite for all $g \in p^*(G)$; the basic open sets are the sets of the form $V_u = \{\alpha \mid \forall i \in \mathrm{domain}(u)\colon i \in U \text{ and } \alpha(i) = u(i)\}$, where u ranges over all functions $u: K \to p^*G$ defined on a finite set $K \subset \mathbf{N}$. It is not difficult to prove that each basic open set V_u (in particular, the space itself, V_ϕ) is contractible ([JM]).

2. Relative Čech cohomology. In this section, let Y be a space in a topos $\mathscr{E}$. One can define the relative Čech cohomology groups of Y with coefficients in an abelian group object in $\mathscr{E}[Y]$, i.e. a sheaf (or in fact, just a presheaf) of abelian groups on Y in $\mathscr{E}$,

$$\check{H}^n_{\mathscr{E}}(Y, A).$$

These cohomology groups are group objects in $\mathscr{E}$. Their construction is completely parallel to the usual construction of the Čech cohomology groups of a topological space; indeed, the latter construction immediately translates to the context of a space in a topos $\mathscr{E}$, by viewing $\mathscr{E}$ as a universe for (constructive) set theory (cf. [BJ]).

More explicitly, let $S \in \mathscr{E}$ and let $\mathscr{U}: S \to \mathcal{O}(Y)$ be an open cover of Y indexed by S. Let A be a (pre)sheaf of abelian groups on Y in $\mathscr{E}$; so A is given by a map $A \to \mathcal{O}(Y)$ in $\mathscr{E}$ equipped with the structure of a (pre)sheaf. Let

$$\mathscr{U}_p: S_p = \underbrace{S \times \cdots \times S}_{p+1} \xrightarrow{\mathscr{U}^{p+1}} \mathcal{O}(Y)^{p+1} \xrightarrow{\wedge} \mathcal{O}(Y)$$

be the map in $\mathscr{E}$ obtained from $\mathscr{U}$ by intersection in Y, and let

$$C^p(\mathcal{U}, A) = \prod_{S_p} (A \underset{\mathcal{O}(Y)}{\times} S_p \to S_p) \tag{3}$$

where $\Pi_{S_p}: \mathscr{E}/S_p \to \mathscr{E}$ is the right adjoint of the functor $S_p{}^*: \mathscr{E} \to \mathscr{E}/S_p$ (cf. [J], p. 36). The $C^p(\mathcal{U}, A)$, $p \geq 0$, give a cochain complex $C^0(\mathcal{U}, A) \to C^1(\mathcal{U}, A) \to \cdots$ in the usual way, with the differential defined via alternating sums. The cohomology groups of this complex are denoted by $H^n_{\mathscr{E}}(\mathcal{U}, A)$. One may now take the colimit of these groups over the *internal* diagram in $\mathscr{E}$ of all open covers of $\mathcal{O}(Y)$ (so this involves internal covers of Y in $\mathscr{E}/E$ for arbitrary E!), and obtain the relative Čech cohomology groups

$$\check{H}^n_{\mathscr{E}}(Y, A) = \varinjlim_{\mathcal{U}} H^p_{\mathscr{E}}(\mathcal{U}, A) \qquad (p \geq 0). \tag{4}$$

Straightforward modifications of the standard argument show that these cohomology groups have the usual properties. For instance, if we write $\varphi: \mathscr{E}[Y] \to \mathscr{E}$ for the canonical geometric morphism and $e_E: \mathscr{E}/E \to \mathscr{E}$ for the geometric morphism given by $e_E^* = E^* = (X \mapsto X \times E \xrightarrow{\pi_2} E)$, then for any open cover $\mathcal{U}$ of $e_E^\#(Y)$ in $\mathscr{E}/E$,

$$H^0_{\mathscr{E}/E}(\mathcal{U}, e_E^*(A)) \cong e_E^* \varphi_* A, \tag{5}$$

where E is any object of $\mathscr{E}$; hence

$$H^0_{\mathscr{E}}(Y, A) \cong \varphi_* A. \tag{6}$$

And for an injective object I of the category $\underline{\text{Ab}}\,\mathscr{E}[Y]$ of abelian sheaves on Y in $\mathscr{E}$,

$$H^n_{\mathscr{E}/E}(\mathcal{U}, e_E^* I) = 0 \qquad (n > 0) \tag{7}$$

for any E in $\mathscr{E}$ and any open cover $\mathcal{U}$ of $e_E^\#(Y)$ in $\mathscr{E}/E$, so

$$\check{H}^n_{\mathscr{E}}(Y, I) = 0 \qquad (n > 0) \tag{8}$$

3. A relative Cartan-Leray spectral sequence. As before, let Y be a space in a topos $\mathscr{E}$, and let $\varphi: \mathscr{E}[Y] \to \mathscr{E}$ be the corresponding geometric morphism. $\mathscr{E}[Y]$ is a subtopos of the topos $\mathscr{E}^{\mathcal{O}(Y)^{op}}$ of presheaves on $\mathcal{O}(Y)$ in $\mathscr{E}$, and we write $i:\mathscr{E}[Y] \hookrightarrow \mathscr{E}^{\mathcal{O}(Y)^{op}}$ for the inclusion. The following is a relative version of SGA4, exp V, p. 24.

LEMMA 1. *For any abelian group A in $\mathscr{E}[Y]$, there exists a spectral sequence*

$$E_2^{p,q} = \check{H}^p_{\mathscr{E}}(Y, R^q i_*(A)) \Rightarrow R^{p+q}\varphi_*(A).$$

Proof. Let $0 \to A \to I^{\cdot}$ be an injective resolution of A in $\underline{\mathrm{Ab}}\,\mathscr{E}[Y]$. For an open cover $\mathcal{U}$ of Y in $\mathscr{E}$, one has a double complex of abelian groups $C^{p,q}(\mathcal{U}) = C^p(\mathcal{U}, I^q)$ (cf. (3)). By (5) and (7) above, the cohomology of the total complex is $H^n H^0(C^{**}(\mathcal{U})) = R^n\varphi_*(A)$, so we obtain a spectral sequence

$$(9) \qquad E_2^{p,q}(\mathcal{U}) = H^p H^q(C^{**}(\mathcal{U})) = H^p_{\mathscr{E}}(\mathcal{U}, R^q i_* A) \Rightarrow R^{p+q}\varphi_*(A)$$

in the standard way ([G]). The same applies to open covers of $e_E^\#(Y)$ in $\mathscr{E}/E$ for any object E of $\mathscr{E}$, so by taking the internal colimit in $\mathscr{E}$ over all open covers of Y, we obtain a spectral sequence as stated in the lemma.

Now let $\mathbf{B} \subset \mathcal{O}(Y)$ be a basis for Y in $\mathscr{E}$ which is closed under binary meets. Call $\mathbf{B}$ A-acyclic if for every morphism $B:E \to \mathbf{B}$ in $\mathscr{E}$,

$$(10) \qquad \check{H}^p_{\mathscr{E}/E}(B, A|B) = 0. \qquad (q > 0)$$

In (10), B stands for the open subspace of $e_E^\#(Y)$ determined by the given morphism $B:E \to \mathbf{B} \subset \mathcal{O}(Y)$, and $A|B \in \underline{\mathrm{Ab}}((\mathscr{E}/E)[B])$ is the sheaf induced by A.

$$(11) \qquad \begin{tikzcd} (\mathscr{E}/E)[B] \arrow[r, hook] & (\mathscr{E}/E)[e_E^\# Y] \arrow[r] \arrow[d] & \mathscr{E}[Y] \arrow[d, "\varphi"] \\ & (\mathscr{E}/E) \arrow[r, "e_E"] & \mathscr{E} \end{tikzcd}$$

LEMMA 2. *If $\mathbf{B}$ is an A-acyclic basis for Y as above, then $\check{H}^p_{\mathscr{E}}(Y, A) \cong R^p\varphi_* A$, for all $p \geq 0$.*

Proof. We show by induction on n that $E_2^{p,q} = 0$ for all p and all q with $0 < q < n$, in the spectral sequence of Lemma 1. Suppose this holds for n. Then (cf. [CE], p. 328) $\check{H}^i_{\mathscr{E}}(Y, A) = R^i\varphi_*A$ for $i < n$, and there is an exact sequence $0 \to \check{H}^n_{\mathscr{E}}(Y, A) \to R^i\varphi_*A \to E_2^{0,n} \to \check{H}^{n+1}_{\mathscr{E}}(Y, A) \to R^{n+1}\varphi_*A$. But $E_2^{0,n} = \check{H}^0_{\mathscr{E}}(Y, R^ni_*A) \rightarrowtail \varphi_*i^*R^ni_*(A) = \varphi_*R^ni^*i_*(A) = \varphi_*(0) = 0$ $(n > 0)$, so $\check{H}^n_{\mathscr{E}}(Y, A) \cong R^n\varphi_*A$. Applying this argument not to Y, but to any open subspace B (for any morphism $B: E \to \mathbf{B}$, cf the diagram (11)), our assumption on $\mathbf{B}$ gives that $R^ni_*(A) \mid \mathbf{B} = 0$, where $(-) \mid \mathbf{B}$ denotes the restriction functor $\mathscr{E}^{\mathcal{O}(X)^{op}} \to \mathscr{E}^{\mathbf{B}^{op}}$. Thus if in the spectral sequence (9) above, $\mathscr{U}$ is a cover consisting of basic opens from $\mathbf{B}$, then $E_2^{p,n}(\mathscr{U}) = H^p_E(\mathscr{U}, 0) = 0$ for all p. Since such covers consisting of basic opens are cofinal in the internal system of all covers, it follows by passing to the colimit that $E_2^{p,n} = 0$ (all p) in the spectral sequence of Lemma 1. So the inductive statement in the beginning of the proof holds for $n + 1$, and Lemma 2 is proved.

Remark. Let Y be a space in $\mathscr{E}$, as above. Recall (see [JT]) that an open $U \subset Y$ is called surjective if it holds in $\mathscr{E}$ that every cover of U is inhabited. If A is a sheaf on Y and $\{U_\alpha : \alpha \in \mathscr{A}\}$ is a family of opens, then $\Pi\{A(U_\alpha) | \alpha \in \mathscr{A}\} \cong \Pi\{A(U_\alpha) | \alpha \in \mathscr{A},\ U_\alpha \text{ surjective}\}$ (where Π is the internal product $\mathscr{E}/\mathscr{A} \to \mathscr{E}$, as in Section 2). This is analogous to the fact that for $\mathscr{E} = \text{Sets}$, $A(U) = \{*\}$ if the empty set covers U. Therefore in Lemma 2 it is enough to assume that $\mathbf{B}$ is closed under surjective binary meets (i.e. $B \wedge B' \in \mathbf{B}$ whenever B and $B' \in \mathbf{B}$ and $B \wedge B'$ is surjective), since by this isomorphism, surjective intersections are the only ones that need to be considered in the complexes $C^{p,q}(\mathscr{U})$.

4. The main theorem. Let $\mathscr{E}$ be a Grothendieck topos, and let $X_{\mathscr{E}}$ be the space constructed in Section 1. In the following theorem, $H^q(X_{\mathscr{E}}, -)$ denotes the sheaf cohomology of $X_{\mathscr{E}}$.

THEOREM. *The geometric morphism $\varphi: \mathrm{Sh}(X_{\mathscr{E}}) \to \mathscr{E}$ has the property that for any abelian group A in $\mathscr{E}$, $R^q\varphi_*(\varphi^*A) = 0$ for $q > 0$ (for $q = 0$, $R^q\varphi_*(\varphi^*A) \cong A$); consequently, φ induces an isomorphism*

$$H^q(\mathscr{E}, A) \xrightarrow{\sim} H^q(X_{\mathscr{E}}, \varphi^*A)$$

for each $q \geq 0$.

Proof. The second statement follows from the first by the Leray spectral sequence (SGA4, exp V, p. 35). The first statement is a special case (by construction of $X_{\mathscr{E}}$) of the general fact that for any object G in $\mathscr{E}$, the corresponding geometric morphism $\varphi:\mathscr{E}[\mathrm{En}(G)] \to \mathscr{E}$ induces isomorphisms $H^q(\mathscr{E}, A) \xrightarrow{\sim} H^q(\mathscr{E}[E(G)], \varphi^*A)$, for any abelian group A in $\mathscr{E}$ and any $q \geq 0$. Let $\mathbf{B}$ be the basis consisting of opens of the form V_u (u a finite partial function from $\mathbf{N}$ to G, cf. [JM]). $\mathrm{En}(G) = V_\phi \in \mathbf{B}$, and $V_u \wedge V_w$ is surjective iff u and w are compatible finite functions, and in that case $V_u \wedge V_w = V_{u \cup w}$, so $\mathbf{B}$ is closed under surjective finite meets (cf. the remark in Section 3).

We will show that for any injective object I of $\underline{\mathrm{Ab}}(\mathscr{E})$ and any $q > 0$

$$R^q\varphi_*(\varphi^*I) = 0. \tag{12}$$

This is enough, because φ is connected, i.e. $\varphi_*\varphi^* \cong \mathrm{id}$, and (12) says that φ^* maps injectives to φ_*-acyclic objects, so there is a spectral sequence ([G]) for the composition $\varphi^* \circ \varphi_*$, $E_2^{p,q} = (R^p\varphi_*)(R^q\varphi^*)A \Rightarrow R^{p+q}(\varphi_*\varphi^*)A$; φ^* is exact and $\varphi_*\varphi^* \cong \mathrm{id}$, so $E_2^{p,q} = 0$ for $q > 0$ and $E_2^{p,0} = R^p(\mathrm{id})(A) = 0$ for $p > 0$. Thus $R^p\varphi_*(\varphi^*A) = 0$ for $p > 0$.

To prove (12), let I be an injective in $\underline{\mathrm{Ab}}\,\mathscr{E}$, and let $\mathscr{U}$ be an open cover of $\mathrm{En}(G)$ by basic opens, say $\mathscr{U}:S \to \mathbf{B} \subset \mathcal{O}(Y)$ as in Section 2. Let us consider the nerve $N(\mathscr{U})$ of $\mathscr{U}$. This is the simplicial complex in $\mathscr{E}$ defined as follows: $S. = (S_p, p \geq 0)$ is a simplicial complex in $\mathscr{E}$, with as face $d_i:S_p \to S_{p-1}$ the projection $S^{p+1} \to S^p$ which deletes the i-th coordinate. The morphism $\mathscr{U}_p:S_p \to \mathbf{B} \subset \mathcal{O}(Y)$ can be viewed as an S_p-indexed sum of subobjects of the terminal object 1 of $\mathscr{E}[Y]$, and we write $\Sigma_{S_p} \mathscr{U}_p$ for their internal sum. Then

$$N_p(\mathscr{U}) = \varphi_!(\sum_{S_p} \mathscr{U}_p),$$

and the faces and degeneracies of $S.$ give $N.(\mathscr{U})$ the structure of a simplicial complex over $\mathscr{E}$. Moreover,

$$C^p(\mathscr{U}, \varphi^*I) \cong I^{N_p(\mathscr{U})}. \tag{13}$$

(cf. (2)), where the differentials on the left correspond to the differentials obtained on the right by alternating sums from the cofaces of the cosimplicial object $I^{N(\mathcal{U})}$. We claim that $C^p(\mathcal{U}, \varphi^*I)$ is an acyclic complex. Since I is injective, it suffices to prove that Free$(N\mathcal{U})$ is an acyclic chain complex in $\underline{\mathrm{Ab}}(\mathscr{E})$, where Free$(-)$ denotes the free abelian group functor. To this end, let $p: \mathscr{B} \to \mathscr{E}$ be a Boolean extension as at the end of Section 1, and consider the pullback square

$$\begin{tikzcd} \mathscr{B}[\mathrm{En}(p^*G)] \arrow[r, "p"] \arrow[d, "\psi"'] & \mathscr{E}[\mathrm{En}(G)] \arrow[d, "\varphi"] \\ \mathscr{B} \arrow[r, "p"] & \mathscr{E}. \end{tikzcd}$$

Since φ is locally connected so is ψ, and the Beck-Chevalley condition holds, i.e.

$$p^*\varphi_! \cong \psi_!\, p^*.$$

Consequently, if we write $\mathcal{U}'$ for the cover of $\mathrm{En}(p^*G)$ induced by $\mathcal{U}$ via pullback along p, we have $p^*(\mathrm{Free}(N\mathcal{U})) \cong \mathrm{Free}(N\mathcal{U}')$. But $\mathscr{B}$ is a model for set theory (with the axiom of choice), so we are now in a position to apply results from classical topology: the cover $\mathcal{U}'$ of $\mathrm{En}(p^*G)$ is a cover by basic opens, and $\mathrm{En}(p^*G)$ as well as each of its basic open subspaces are contractible, so the nerve $N(\mathcal{U}')$ of this cover is a contractible simplicial set, and $\mathrm{Free}(N\mathcal{U}')$ is an acyclic chain complex. Since $p^*(\mathrm{Free}\ N\mathcal{U}) = \mathrm{Free}(N\mathcal{U}')$ and p^* is faithful, it follows that $\mathrm{Free}(N\mathcal{U})$ is acyclic, as was to be shown.

Now apply this argument not just to $\mathrm{En}(G)$, but to any basic open $B \subset e_E^{\#}(\mathrm{En}(G))$ and any $E \in \mathscr{E}$ (cf (11), where $Y = \mathrm{En}(G)$ now). Then we conclude that **B** is an I-acyclic basis. (12) now follows by Lemma 2, since the whole space $\mathrm{En}(G)$ is a member of **B**. This completes the proof of the theorem.

5. Torsors. Let G be a group in a topos $\mathscr{E}$. A G-torsor in $\mathscr{E}$ (or principal G-bundle over $\mathscr{E}$) is an object T of $\mathscr{E}$ equipped with an action $\mu: G \times T \to T$ of G such that $T \to 1$ is epi and $(\mu, \pi_2): G \times T \to T \times T$ is an isomorphism. Recall ([Gi]) that $H^1(\mathscr{E}, G)$ is the pointed set of isomorphism classes of G-torsors (this is a group if G is abelian). For a space X and a sheaf of groups G on X, $H^1(X, G)$ stands for $H^1(\mathrm{Sh}(X), G)$.

THEOREM. *Let $\mathscr{E}$ be a topos, and let $\varphi : \mathrm{Sh}(X_{\mathscr{E}}) \to \mathscr{E}$ be the cover of Section* 1. *For any group G in $\mathscr{E}$, φ induces an isomorphism*

$$H^1(\mathscr{E}, G) \xrightarrow{\sim} H^1(X_{\mathscr{E}}, \varphi^*G)$$

Proof. The functor $\varphi^* : \mathscr{E} \to \mathrm{Sh}(X_{\mathscr{E}})$ is fully faithful, so it restricts to a fully faithful functor from the category of G-torsors in $\mathscr{E}$ to that of φ^*G-torsors in $\mathrm{Sh}(X_{\mathscr{E}})$. It thus suffices to show that this restriction of φ^* is essentially surjective. By [JM], there is a class $P \subset (X_{\mathscr{E}})^I$ of paths, such that $\mathscr{E}$ is equivalent to the full subcategory of $\mathrm{Sh}(X_{\mathscr{E}})$ consisting of those sheaves on $X_{\mathscr{E}}$ which are constant along the paths in P. Let T be a φ^*G-torsor in $\mathrm{Sh}(X_{\mathscr{E}})$. Then T is locally isomorphic to $\varphi^*(G)$, and $\varphi^*(G)$ is constant along all the paths in P. So T is locally constant along the paths in P, and hence constant along those paths (since the interval I is simply connected).

6. Etale homotopy. Let $\mathscr{E}$ be a locally connected topos, and let p be a point of $\mathscr{E}$. Artin and Mazur ([AM]) define the etale homotopy groups $\pi_n(\mathscr{E}, p)$ ($n \geq 0$), and prove a Whitehead theorem for toposes: a geometric morphism $(\mathscr{F}, q) \to (\mathscr{E}, p)$ of pointed locally connected toposes induces isomorphisms of etale homotopy groups iff it induces isomorphisms of cohomology groups with coefficients in a locally constant abelian group A in $\mathscr{E}$, as well as an isomorphism of the fundamental progroups $\pi_1(\mathscr{F}, q) \to \pi_1(\mathscr{E}, p)$. Our previous results give:

COROLLARY. *For any locally connected pointed topos $(\mathscr{E}, p)$ there exists a pointed space $(X_{\mathscr{E}}, q)$ and a cover $\varphi : (\mathrm{Sh}(X_{\mathscr{E}}), q) \to (\mathscr{E}, p)$ which induces isomorphisms in etale homotopy,*

$$\pi_n(X_{\mathscr{E}}, q) \xrightarrow{\sim} \pi_n(\mathscr{E}, p) \qquad (n \geq 0)$$

Proof. First of all, we need to modify the construction of the space $X_{\mathscr{E}}$ slightly, in order to lift the point p: if we replace the set $\mathbf{N}$ of natural numbers by an arbitrary infinite set S in the construction of Section 1 (and the space of infinite-to-one enumerations $\mathbf{N} \leftarrow\!\!\prec U \twoheadrightarrow G$ by that of infinite-to-one partial maps $\Delta(S) \leftarrow\!\!\prec U \twoheadrightarrow G$, where ΔS denotes the constant object of $\mathscr{E}$ corresponding to the set S), we obtain a cover (again called) $\varphi : X_{\mathscr{E}} \to \mathscr{E}$ with exactly the same properties as before. A straightforward classifying-topos argument shows that if we choose the cardinality of S sufficiently large (at least that of p^*G) then the given point p can be lifted to a point q of this (modified) space $X_{\mathscr{E}}$. $X_{\mathscr{E}}$ is locally connected since $\mathscr{E}$ is, and φ is a locally connected map. Now the result of Section 5 shows that φ induces an isomorphism in π_1 (since $H^1(\mathscr{E}, G) \cong \mathrm{Hom}(\pi_1(\mathscr{E}, p), G)$, cf [AM], Section 10). The corollary follows by the Whitehead theorem just quoted and the theorem of Section 4.

UNIVERSITÉ DU QUÉBEC À MONTREAL, CANADA
UNIVERSITY OF CHICAGO

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