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TOPOSES ARE COHOMOLOGICALLY EQUIVALENT TO SPACES

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This purpose of this paper is to prove that for every Grothendieck topos \( \mathcal{E} \) there exist a space \( X \) and a covering \( \varphi : X \to \mathcal{E} \) which induces an isomorphism in cohomology

\[
H^n(\mathcal{E}, A) \cong H^n(X, \varphi^*A) \quad (n \geq 0)
\]

for any abelian group \( A \) in \( \mathcal{E} \). Moreover for \( n = 1 \) this is also true for nonabelian \( A \). This implies, by a result of Artin and Mazur, that \( \varphi \) induces an isomorphism of etale homotopy groups.

1. Construction of the cover. Let \( \mathcal{E} \) be a Grothendieck topos, and let \( G \) be an object of \( \mathcal{E} \). \( \text{En}(G) \) is the space (in this paper ‘space’ means space in the sense of [JT], chapter IV, unless explicitly said otherwise) of infinite-to-one partial enumerations of \( G \); in other words, \( \text{En}(G) \) is characterized by the property that for any map \( f : \mathcal{T} \to \mathcal{E} \) of toposes, the points of the induced space \( f^*(\text{En}(G)) \) in \( \mathcal{T} \) correspond to diagrams \( \mathbb{N} \leftarrow U \to f^*G \) in \( \mathcal{T} \) with the property that for any \( n \in \mathbb{N}, U - \{0, \ldots, n\} \to f^*G \) is still epi. We write \( \mathcal{E}[\text{En}(G)] \) for the category of sheaves in \( \mathcal{E} \) on the space \( \text{En}(G) \), and \( \varphi : \mathcal{E}[\text{En}(G)] \to \mathcal{E} \) for the corresponding geometric morphism. The properties of the space \( \text{En}(G) \) and the map \( \varphi \) were extensively discussed in [IM]. For the present purpose, we recall the following basic facts. First of all, for a suitable object \( G \) of \( \mathcal{E} \), \( \mathcal{E}[\text{En}(G)] \) is equivalent to the topos \( \text{Sh}(X_\mathcal{E}) \) of sheaves on a space \( X_\mathcal{E} \) in \( \text{Sets} \), so that \( \varphi \) corresponds to a cover

\[
\varphi : \text{Sh}(X_\mathcal{E}) \to \mathcal{E}.
\]

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This geometric morphism is connected and locally connected; in particular, \( \varphi^* : \mathcal{E} \to \text{Sh}(X_\emptyset) \) has a left adjoint \( \varphi \), such that for any \( E \in \mathcal{E} \) and \( S \in \text{Sh}(X_\emptyset) \),

\[
(2) \quad \varphi_!(\varphi^*(E) \times S) \cong E \times \varphi_!(S).
\]

For any \( G \) in \( \mathcal{E} \), there exists a surjective geometric morphism \( p : \mathcal{B} \to \mathcal{E} \) where \( \mathcal{B} \) is the category of sheaves on a complete Boolean algebra (Barr’s theorem, [BJ]), such that \( p^*G \) is countable (cf. [JT]). \( \mathcal{B} \) is a model of set theory, and the induced space \( \text{En}(p^*G) \cong p^*(\text{En}(G)) \) in \( \mathcal{B} \) has enough points, i.e. is an ordinary topological space, which can be described as follows: the points of \( \text{En}(p^*(G)) \) are functions \( \alpha : U \to p^*G \) with \( U \subset \mathbb{N} \) and \( \alpha^{-1}(g) \) infinite for all \( g \in p^*(G) \); the basic open sets are the sets of the form \( V_u = \{ \alpha | \forall i \in \text{domain}(u) : i \in U \text{ and } \alpha(i) = u(i) \} \), where \( u \) ranges over all functions \( u : K \to p^*G \) defined on a finite set \( K \subset \mathbb{N} \). It is not difficult to prove that each basic open set \( V_u \) (in particular, the space itself, \( V_\emptyset \)) is contractible ([JM]).

2. Relative Čech cohomology. In this section, let \( Y \) be a space in a topos \( \mathcal{E} \). One can define the relative Čech cohomology groups of \( Y \) with coefficients in an abelian group object in \( \mathcal{E}[Y] \), i.e. a sheaf (or in fact, just a presheaf) of abelian groups on \( Y \) in \( \mathcal{E} \),

\[
\hat{H}^p_\mathcal{E}(Y, A).
\]

These cohomology groups are group objects in \( \mathcal{E} \). Their construction is completely parallel to the usual construction of the Čech cohomology groups of a topological space; indeed, the latter construction immediately translates to the context of a space in a topos \( \mathcal{E} \), by viewing \( \mathcal{E} \) as a universe for (constructive) set theory (cf. [BJ]).

More explicitly, let \( S \in \mathcal{E} \) and let \( \mathcal{U} : S \to \mathcal{O}(Y) \) be an open cover of \( Y \) indexed by \( S \). Let \( A \) be a (pre)sheaf of abelian groups on \( Y \) in \( \mathcal{E} \); so \( A \) is given by a map \( A \to \mathcal{O}(Y) \) in \( \mathcal{E} \) equipped with the structure of a (pre)sheaf. Let

\[
\mathcal{U}_p : S_p = S \times \cdots \times S \xrightarrow{\text{ag}\text{p+1}} \mathcal{O}(Y)^{p+1} \xrightarrow{\wedge} \mathcal{O}(Y)
\]

be the map in \( \mathcal{E} \) obtained from \( \mathcal{U} \) by intersection in \( Y \), and let
(3) \[ C^p(\mathcal{U}, A) = \prod_{S_p} (A \times_{S_p} S_p) \]

where \( \Pi_{S_p}: \mathcal{E}/S_p \to \mathcal{E} \) is the right adjoint of the functor \( S_p^*: \mathcal{E} \to \mathcal{E}/S_p \) (cf. [J], p. 36). The \( C^p(\mathcal{U}, A) \), \( p \geq 0 \), give a cochain complex \( C^0(\mathcal{U}, A) \to C^1(\mathcal{U}, A) \to \cdots \) in the usual way, with the differential defined via alternating sums. The cohomology groups of this complex are denoted by \( H^n_\mathcal{E}(\mathcal{U}, A) \). One may now take the colimit of these groups over the \textit{internal} diagram in \( \mathcal{E} \) of all open covers of \( \mathcal{O}(Y) \) (so this involves internal covers of \( Y \) in \( \mathcal{E}/E \) for arbitrary \( E \)), and obtain the relative Čech cohomology groups

(4) \[ \tilde{H}_\mathcal{E}(Y, A) = \lim_{\to \mathcal{U}} H^n_\mathcal{E}(\mathcal{U}, A) \quad (p \geq 0). \]

Straightforward modifications of the standard argument show that these cohomology groups have the usual properties. For instance, if we write \( \varphi: \mathcal{E}[Y] \to \mathcal{E} \) for the canonical geometric morphism and \( e_E: \mathcal{E}/E \to \mathcal{E} \) for the geometric morphism given by \( e_E^* = E^* = (X \mapsto X \times E \overset{\pi}{\to} E) \), then for any open cover \( \mathcal{U} \) of \( e_E^*(Y) \) in \( \mathcal{E}/E \),

(5) \[ H^n_{\mathcal{E}/E}(\mathcal{U}, e_E^*(A)) \cong e_E^*\varphi_*A, \]

where \( E \) is any object of \( \mathcal{E} \); hence

(6) \[ H^n_\mathcal{E}(Y, A) \cong \varphi_*A. \]

And for an injective object \( I \) of the category \( \text{Ab} \mathcal{E}[Y] \) of abelian sheaves on \( Y \) in \( \mathcal{E} \),

(7) \[ H^n_{\mathcal{E}/E}(\mathcal{U}, e_E^*I) = 0 \quad (n > 0) \]

for any \( E \) in \( \mathcal{E} \) and any open cover \( \mathcal{U} \) of \( e_E^*(Y) \) in \( \mathcal{E}/E \), so

(8) \[ \tilde{H}_\mathcal{E}(Y, I) = 0 \quad (n > 0) \]

3. A \textit{relative Cartan-Leray spectral sequence}. As before, let \( Y \) be a space in a topos \( \mathcal{E} \), and let \( \varphi: \mathcal{E}[Y] \to \mathcal{E} \) be the corresponding
geometric morphism. $\mathcal{E}[Y]$ is a subtopos of the topos $\mathcal{E}^{\mathcal{O}(Y)^{op}}$ of presheaves on $\mathcal{O}(Y)$ in $\mathcal{E}$, and we write $i: \mathcal{E}[Y] \hookrightarrow \mathcal{E}^{\mathcal{O}(Y)^{op}}$ for the inclusion. The following is a relative version of SGA4, exp V, p. 24.

**Lemma 1.** For any abelian group $A$ in $\mathcal{E}[Y]$, there exists a spectral sequence

$$E_2^{p,q} = \check{H}_q^p(Y, R^q i_*(A)) \Rightarrow R^{p+q} \varphi_*(A).$$

**Proof.** Let $0 \to A \to I$ be an injective resolution of $A$ in $\mathcal{A}b_{[\mathcal{E}][Y]}$. For an open cover $\mathcal{U}$ of $Y$ in $\mathcal{E}$, one has a double complex of abelian groups $C^{p,q}(\mathcal{U}) = C^p(\mathcal{U}, I^q)$ (cf. (3)). By (5) and (7) above, the cohomology of the total complex is $H^p H^q(C^{*,*}(\mathcal{U})) = R^p \varphi_*(A)$, so we obtain a spectral sequence

$$E_2^{p,q}(\mathcal{U}) = H^p H^q(C^{*,*}(\mathcal{U})) = H^p_\mathcal{E}(\mathcal{U}, R^q i_* A) \Rightarrow R^{p+q} \varphi_*(A)$$

in the standard way ([G]). The same applies to open covers of $e_E^*(Y)$ in $\mathcal{E}/E$ for any object $E$ of $\mathcal{E}$, so by taking the internal colimit in $\mathcal{E}$ over all open covers of $Y$, we obtain a spectral sequence as stated in the lemma.

Now let $B \subseteq \mathcal{O}(Y)$ be a basis for $Y$ in $\mathcal{E}$ which is closed under binary meets. Call $B$ $A$-acyclic if for every morphism $B : E \to B$ in $\mathcal{E}$,

$$\check{H}_q^p(B, A|B) = 0. \quad (q > 0)$$

In (10), $B$ stands for the open subspace of $e_E^*(Y)$ determined by the given morphism $B : E \to B \subseteq \mathcal{O}(Y)$, and $A|B \in \mathcal{A}b((\mathcal{E}/E)[B])$ is the sheaf induced by $A$.

$$\begin{array}{ccc}
(\mathcal{E}/E)[B] & \hookrightarrow & (\mathcal{E}/E)[e_E^* Y] \\
\downarrow & & \downarrow \varphi \\
(\mathcal{E}/E) & \to & \mathcal{E}
\end{array}$$

**Lemma 2.** If $B$ is an $A$-acyclic basis for $Y$ as above, then $\check{H}_q^p(Y, A) \cong R^p \varphi_* A$, for all $p \geq 0$. 


Proof. We show by induction on \( n \) that \( E^q_2 = 0 \) for all \( p \) and all \( q \) with \( 0 < q < n \), in the spectral sequence of Lemma 1. Suppose this holds for \( n \). Then (cf. [CE], p. 328) \( \tilde{H}^q(Y, A) = R^i\varphi_*A \) for \( i < n \), and there is an exact sequence \( 0 \to \tilde{H}^q(Y, A) \to R^i\varphi_*A \to E^{0,n}_2 \to \tilde{H}^{q+1}(Y, A) \to R^{n+1}\varphi_*A \). But \( E^{0,n}_2 = \tilde{H}^0(Y, R^ni_*A) \to \varphi^*R^ni_*A \) \( n \to 0 \), so \( H^q(Y, A) \equiv R^n\varphi_*A \). Applying this argument not to \( Y \), but to any open subspace \( B \) (for any morphism \( B,E \to B \), cf the diagram (11)), our assumption on \( B \) gives that \( R^n\varphi_i(A) \) \( B = 0 \), where \((-) \mid B \) denotes the restriction functor \( \mathcal{E}_Y \to \mathcal{E}_B \). Thus if in the spectral sequence (9) above, \( \mathcal{U} \) is a cover consisting of basic open sets from \( \mathcal{B} \), then \( E^{p,q}_2(\mathcal{U}) = H^q(\mathcal{U}, 0) = 0 \) for all \( p \). Since such covers consisting of basic opens are cofinal in the internal system of all covers, it follows by passing to the colimit that \( E^{p,q}_2 = 0 \) (all \( p \)) in the spectral sequence of Lemma 1. So the inductive statement in the beginning of the proof holds for \( n + 1 \), and Lemma 2 is proved.

Remark. Let \( Y \) be a space in \( \mathcal{E} \), as above. Recall (see [JT]) that an open \( U \subset Y \) is called surjective if it holds in \( \mathcal{E} \) that every cover of \( U \) is inhabited. If \( A \) is a sheaf on \( Y \) and \( \{ U_\alpha : \alpha \in \mathcal{A} \} \) is a family of opens, then \( \Pi\{ A(U_\alpha) | \alpha \in \mathcal{A} \} = \Pi\{ A(U_\alpha) | \alpha \in \mathcal{A}, U_\alpha \text{ surjective} \} \) (where \( \Pi \) is the internal product \( \mathcal{E}/\mathcal{A} \to \mathcal{E} \), as in Section 2). This is analogous to the fact that for \( \mathcal{E} = \text{Sets}, A(U) = \{ ^* \} \) if the empty set covers \( U \). Therefore in Lemma 2 it is enough to assume that \( \mathcal{B} \) is closed under surjective binary meets (i.e. \( B \cap B' \in \mathcal{B} \) whenever \( B \) and \( B' \in \mathcal{B} \) and \( B \cap B' \) is surjective), since by this isomorphism, surjective intersections are the only ones that need to be considered in the complexes \( C^{p,q}(\mathcal{U}) \).

4. The main theorem. Let \( \mathcal{E} \) be a Grothendieck topos, and let \( X_\mathcal{E} \) be the space constructed in Section 1. In the following theorem, \( H^q(X_\mathcal{E}, -) \) denotes the sheaf cohomology of \( X_\mathcal{E} \).

Theorem. The geometric morphism \( \varphi : \text{Sh}(X_\mathcal{E}) \to \mathcal{E} \) has the property that for any abelian group \( A \) in \( \mathcal{E} \), \( R^q\varphi_*(\varphi^*A) = 0 \) for \( q > 0 \) (for \( q = 0, R^q\varphi_*(\varphi^*A) \equiv A \)); consequently, \( \varphi \) induces an isomorphism

\[ H^q(\mathcal{E}, A) \to H^q(X_\mathcal{E}, \varphi^*A) \]

for each \( q \geq 0 \).
Proof. The second statement follows from the first by the Leray spectral sequence (SGA4, exp V, p. 35). The first statement is a special case (by construction of $X_\xi$) of the general fact that for any object $G$ in $\mathcal{C}$, the corresponding geometric morphism $\varphi : \mathcal{C}[\text{En}(G)] \to \mathcal{C}$ induces isomorphisms $H^q(\mathcal{C}, A) \to H^q(\mathcal{C}[E(G)], \varphi^*A)$, for any abelian group $A$ in $\mathcal{C}$ and any $q \geq 0$. Let $B$ be the basis consisting of opens of the form $V_u$ ($u$ a finite partial function from $\mathbb{N}$ to $G$, cf. [JM]). $\text{En}(G) = V_\varphi \in B$, and $V_u \cap V_w$ is surjective iff $u$ and $w$ are compatible finite functions, and in that case $V_u \cap V_w = V_{u\cup w}$, so $B$ is closed under surjective finite meets (cf. the remark in Section 3).

We will show that for any injective object $I$ of $\text{Ab}(\mathcal{C})$ and any $q > 0$

$$R^q\varphi_*(\varphi^*I) = 0. \tag{12}$$

This is enough, because $\varphi$ is connected, i.e. $\varphi_\ast \varphi^* \cong \text{id}$, and (12) says that $\varphi^*$ maps injectives to $\varphi_\ast$-acyclic objects, so there is a spectral sequence ([$G$]) for the composition $\varphi^* \circ \varphi_\ast$, $E_2^{p,q} = (R^p\varphi_\ast)(R^q\varphi^*)A \Rightarrow R^{p+q}(\varphi_\ast \varphi^*)A$; $\varphi^*$ is exact and $\varphi_\ast \varphi^* \cong \text{id}$, so $E_2^{p,q} = 0$ for $q > 0$ and $E_2^{0,0} = R^p(\text{id})(A) = 0$ for $p > 0$. Thus $R^q\varphi_*(\varphi^*A) = 0$ for $p > 0$.

To prove (12), let $I$ be an injective in $\text{Ab}(\mathcal{C})$, and let $\mathcal{U}$ be an open cover of $\text{En}(G)$ by basic opens, say $\mathcal{U} : S \to \mathcal{B} \subset \mathcal{O}(Y)$ as in Section 2. Let us consider the nerve $N(\mathcal{U})$ of $\mathcal{U}$. This is the simplicial complex in $\mathcal{C}$ defined as follows: $S_0 = (S_p, p \geq 0)$ is a simplicial complex in $\mathcal{C}$, with as face $d_i : S_p \to S_{p-1}$ the projection $S^{p+1} \to S^p$ which deletes the $i$-th coordinate. The morphism $\mathcal{U}_p : S_p \to \mathcal{B} \subset \mathcal{O}(Y)$ can be viewed as an $S_p$-indexed sum of subobjects of the terminal object 1 of $\mathcal{C}[Y]$, and we write $\Sigma_{S_p} \mathcal{U}_p$ for their internal sum. Then

$$N_p(\mathcal{U}) = \varphi_\ast(\Sigma_{S_p} \mathcal{U}_p),$$

and the faces and degeneracies of $S$. give $N(\mathcal{U})$ the structure of a simplicial complex over $\mathcal{C}$. Moreover,

$$C^p(\mathcal{U}, \varphi^*I) \cong I^{N_p(\mathcal{U})}. \tag{13}$$

(cf. (2)), where the differentials on the left correspond to the differentials obtained on the right by alternating sums from the cofaces of the co-
simplicial object $I^{N(U)}$. We claim that $C^p(U, \varphi^*I)$ is an acyclic complex. Since $I$ is injective, it suffices to prove that $\text{Free}(N(U))$ is an acyclic chain complex in $\text{Ab}(\mathfrak{C})$, where $\text{Free}(-)$ denotes the free abelian group functor. To this end, let $p: \mathcal{B} \to \mathfrak{C}$ be a Boolean extension as at the end of Section 1, and consider the pullback square

$$
\begin{array}{ccc}
\mathcal{B}[\text{En}(p^*G)] & \to & \mathfrak{C}[\text{En}(G)] \\
\downarrow \psi & & \downarrow \varphi \\
\mathcal{B} & \to & \mathfrak{C}.
\end{array}
$$

Since $\varphi$ is locally connected so is $\psi$, and the Beck-Chevalley condition holds, i.e.

$$p^*\varphi = \psi \circ p^*.$$

Consequently, if we write $U'$ for the cover of $\text{En}(p^*G)$ induced by $U$ via pullback along $p$, we have $p^*(\text{Free}(N(U))) = \text{Free}(N(U'))$. But $\mathcal{B}$ is a model for set theory (with the axiom of choice), so we are now in a position to apply results from classical topology: the cover $U'$ of $\text{En}(p^*G)$ is a cover by basic opens, and $\text{En}(p^*G)$ as well as each of its basic open subspaces are contractible, so the nerve $N(U')$ of this cover is a contractible simplicial set, and $\text{Free}(N(U'))$ is an acyclic chain complex. Since $p^*(\text{Free} N(U)) = \text{Free}(N(U'))$ and $p^*$ is faithful, it follows that $\text{Free}(N(U))$ is acyclic, as was to be shown.

Now apply this argument not just to $\text{En}(G)$, but to any basic open $B \subset e^*_p(\text{En}(G))$ and any $E \in \mathfrak{C}$ (cf (11), where $Y = \text{En}(G)$ now). Then we conclude that $\mathcal{B}$ is an $I$-acyclic basis. (12) now follows by Lemma 2, since the whole space $\text{En}(G)$ is a member of $\mathcal{B}$. This completes the proof of the theorem.

5. Torsors. Let $G$ be a group in a topos $\mathfrak{C}$. A $G$-torsor in $\mathfrak{C}$ (or principal $G$-bundle over $\mathfrak{C}$) is an object $T$ of $\mathfrak{C}$ equipped with an action $\mu: G \times T \to T$ of $G$ such that $T \to 1$ is epi and $(\mu, \pi_2): G \times T \to T \times T$ is an isomorphism. Recall ([Gi]) that $H^i(\mathfrak{C}, G)$ is the pointed set of isomorphism classes of $G$-torsors (this is a group if $G$ is abelian). For a space $X$ and a sheaf of groups $G$ on $X$, $H^i(X, G)$ stands for $H^i(\text{Sh}(X), G)$. 
Theorem. Let \( \mathcal{C} \) be a topos, and let \( \varphi : \text{Sh}(X_{\mathcal{C}}) \to \mathcal{C} \) be the cover of Section 1. For any group \( G \) in \( \mathcal{C} \), \( \varphi \) induces an isomorphism

\[
H^1(\mathcal{C}, G) \xrightarrow{\sim} H^1(X_{\mathcal{C}}, \varphi^*G)
\]

Proof. The functor \( \varphi^* : \mathcal{C} \to \text{Sh}(X_{\mathcal{C}}) \) is fully faithful, so it restricts to a fully faithful functor from the category of \( G \)-torsors in \( \mathcal{C} \) to that of \( \varphi^*G \)-torsors in \( \text{Sh}(X_{\mathcal{C}}) \). It thus suffices to show that this restriction of \( \varphi^* \) is essentially surjective. By [JM], there is a class \( P \subset (X_{\mathcal{C}})^I \) of paths, such that \( \mathcal{C} \) is equivalent to the full subcategory of \( \text{Sh}(X_{\mathcal{C}}) \) consisting of those sheaves on \( X_{\mathcal{C}} \) which are constant along the paths in \( P \). Let \( T \) be a \( \varphi^*G \)-torsor in \( \text{Sh}(X_{\mathcal{C}}) \). Then \( T \) is locally isomorphic to \( \varphi^*(G) \), and \( \varphi^*(G) \) is constant along all the paths in \( P \). So \( T \) is locally constant along the paths in \( P \), and hence constant along those paths (since the interval \( I \) is simply connected).

6. Etale homotopy. Let \( \mathcal{C} \) be a locally connected topos, and let \( p \) be a point of \( \mathcal{C} \). Artin and Mazur ([AM]) define the etale homotopy groups \( \pi_n(\mathcal{C}, p) \) \((n \geq 0)\), and prove a Whitehead theorem for toposes: a geometric morphism \((\mathcal{F}, q) \to (\mathcal{C}, p)\) of pointed locally connected toposes induces isomorphisms of etale homotopy groups iff it induces isomorphisms of cohomology groups with coefficients in a locally constant abelian group \( A \) in \( \mathcal{C} \), as well as an isomorphism of the fundamental progroups \( \pi_1(\mathcal{F}, q) \to \pi_1(\mathcal{C}, p) \). Our previous results give:

Corollary. For any locally connected pointed topos \((\mathcal{C}, p)\) there exists a pointed space \((X_{\mathcal{C}}, q)\) and a cover \( \varphi : (\text{Sh}(X_{\mathcal{C}}), q) \to (\mathcal{C}, p) \) which induces isomorphisms in etale homotopy,

\[
\pi_n(X_{\mathcal{C}}, q) \xrightarrow{\sim} \pi_n(\mathcal{C}, p) \quad (n \geq 0)
\]

Proof. First of all, we need to modify the construction of the space \( X_{\mathcal{C}} \) slightly, in order to lift the point \( p \): if we replace the set \( N \) of natural numbers by an arbitrary infinite set \( S \) in the construction of Section 1 (and the space of infinite-to-one enumerations \( N \leftarrow U \to G \) by that of infinite-to-one partial maps \( \Delta(S) \leftarrow U \to G \), where \( \Delta S \) denotes the constant object of \( \mathcal{C} \) corresponding to the set \( S \)), we obtain a cover (again called) \( \varphi : X_{\mathcal{C}} \to \mathcal{C} \) with exactly the same properties as before. A straightforward classifying-topos argument shows that if we
choose the cardinality of $S$ sufficiently large (at least that of $p^*G$) then the given point $p$ can be lifted to a point $q$ of this (modified) space $X_\mathcal{E}$. $X_\mathcal{E}$ is locally connected since $\mathcal{E}$ is, and $\varphi$ is a locally connected map. Now the result of Section 5 shows that $\varphi$ induces an isomorphism in $\pi_1$ (since $H^1(\mathcal{E}, G) \cong \text{Hom}(\pi_1(\mathcal{E}, p), G)$, cf [AM], Section 10). The corollary follows by the Whitehead theorem just quoted and the theorem of Section 4.

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REFERENCES

[JM] A. Joyal and I. Moerdijk, Toposes as homotopy groupoids, (to appear in _Advances in Math._).