Type Theories, Toposes and Constructive Set Theory: Predicative Aspects of AST

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Dedicated to Anne S. Troelstra on the occasion of his 60th birthday.

1 Introduction

We are honoured to have the opportunity to dedicate this paper to Anne S. Troelstra. Anne supervised Moerdijk's PhD thesis, and was closely involved (as an "opponent") with Palmgren's thesis. We were both privileged to get to know Anne as a very scholarly mathematician, who generously gave his time and ideas to younger mathematicians. Many of the ideas in this paper go back to what he implicitly or explicitly taught us about intuitionistic logical systems, predicative and constructive set theories and type theories. Here, we study the relation between these theories and "Algebraic Set Theory" (AST), developed in the context of categorical logic and topos theory [10, 19, 5].

In categorical logic an important tool is the correlation between type theories and categorical structures; for example, between simply typed lambda calculus and cartesian closed categories, or between higher order intuitionistic logic and elementary toposes. The axioms for a topos can be seen as type-theoretic in nature, and have a type construction for the "power set". Much of ordinary (constructive) mathematics can be formalised inside intuitionistic type theory, or, what is the same, in a topos. The scope of formalisation is even larger for intuitionistic set theory (IZF, [7]), which allows transfinite iterations of the power set operation. For many particular toposes, it is possible to model IZF inside them. In fact, using an auxiliary notion of "small map", it is possible to extend the axioms for a topos, and provide a general theory for building models of set theory out of toposes ("algebraic set theory", [10, 19, 5]).

Topos theory and IZF are essentially impredicative in nature. However, the study of constructive logical systems, such as Martin-Löf type theory, Aczel–Myhill set theory (CZF) and Feferman's systems for explicit mathematics, has shown (or indicates) that large parts of mathematics can be formalised without the use of an "impredicative" notion of power set. Instead, these systems use generalized inductive definitions. These predicative systems are closely related to each other. For example, Aczel showed how CZF can be interpreted in Martin-Löf type theory [13].

Thus the question naturally arises as to what would be a useful notion of "predicative topos". On the one hand, such a notion should serve as a categorical counterpart for constructive predicative type theories such as Martin-Löf's. On the other hand, it should allow such crucial constructions as that of the topos of sheaves and of presheaves, and of constructions of toposes by glueing and realizability. Furthermore, it is desirable that these constructions can be performed "internally", so as to provide extensions of an arbitrary "predicative topos" by sheaves, presheaves, etc., just as in the case of ordinary toposes. In addition, given such a notion of "predicative topos", the question arises whether the algebraic set theory based on small maps in a topos can be adapted to predicative toposes so as to give sheaf and other categorical models for CZF.

One of the purposes of this paper is to present one possible such notion of "predicative topos", and show that for this notion, all these desiderata can in fact be proved. This is the notion of what we call a *stratified pseudotopos.* (The definitions of stratified pseudotopos vs topos bear some formal resemblance to those of stratified pseudomanifold vs manifold, whence our choice of terminology.) The basic ingredients of a stratified pseudotopos are, on the one hand, the structure of a pretopos with dependent products and W-types, as discussed in our earlier [15]; and on the other hand, a filtration or stratification of the pretopos by classes of small maps, similar to but different from the classes discussed in [10]. One of our main results, then, is that the sheaves on an internal site in a stratified pseudotopos again form such a pseudotopos. Another main result is that any stratified pseudotopos provides a model of CZF. The combination of these two results provides many different kinds of models of CZF. For instance, one can combine CZF with choice sequences, in order to model an extension of CZF validating Brouwer's continuity principles, or with various omniscience principles and countable choice in order to model a set theory for predicative classical mathematics (cf. Remark 12.8). Since forcing extensions can be seen as sheaf models, these results also provide a general framework for proving independence results for CZF by (forcing) methods similar to those used for ZF.

Our method for obtaining CZF models from stratified pseudotoposes is related to Aczel's original interpretation of CZF into type theory. A complication arises in our case, however, because Aczel's construction uses the fact that Martin-Löf type theory satisfies a general choice principle for types — or in categorical terms, it uses the existence of "enough" projectives. As is well-known, the property of having enough projectives is in general destroyed by sheaf constructions. Because of this, we modify Aczel's model construction and employ instead a choice principle to be called the Axiom of Multiple Choice (AMC). This principle is weaker than the existence of enough projectives, and related to the categorical Collection Axiom introduced in [10]. It suffices for many constructions where one would perhaps natural be inclined to use these projectives, and, crucially, it is preserved under the construction of sheaves.

We briefly outline the plan of this paper. In Section 2, we recall the basic properties of the categories with which we shall work. These are pretoposes with dependent products and W-types. In Section 3, we discuss a variation on the axioms for small maps presented in [10], suitable for such pretoposes with dependent products and W-types. One of the main new axioms for such a class of small maps is the Axiom of Multiple Choice. This axiom, which is introduced and discussed in Section 4, is based on the notion of a so-called "collection map", also defined in Section 4. In Section 5, we discuss some general properties of the extensional — or "Mostowski" — collapse of W-types. This Mostowski collapse has particularly good, "universal", properties in the special case of a W-type associated to a collection map, as we will explain in Section 6. Using these general properties, it is now quite straightforward to prove that the Mostowski collapse of a universal small map is a model for Aczel's constructive set theory CZF; this is done in Section 7. The next three sections are concerned with the stability under internal sheaf constructions of the categorical properties introduced in Section 3 and 4. In Section 8, AMC is identified as a sufficient condition for the existence of the (internal) associated sheaf functor. In Section 9 and 10 the class of "point-wise small" natural transformations between sheaves is shown to satisfy the axioms for small maps and AMC. Up to this point in the paper, we have only assumed that the underlying category is a pretopos with dependent products and W-types, and equipped with a single class of small maps. In the next section, however, we will introduce the notion of a stratified pseudotopos, which is, roughly speaking, such a pretopos with dependent products and W-types, which is moreover filtered by an entire sequence of classes of small maps. In the last section, we explain how theories in a standard version of type theory with universes naturally give rise to stratified pseudotoposes.

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2 Preliminaries

The basic categorical context in which we shall work is that of a pretopos with dependent products and W-types. We briefly recall the relevant definitions here.

A *pretopos* is a category with finite limits, finite sums which are stable and disjoint, and where for each equivalence relation $R \rightrightarrows X$ there exists a stable quotient X/R, i.e. an object which fits into a diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & X/R \end{tikzcd}$$

which is exact and remains so after pulling back along any map into X/R. The "internal logic" of a pretopos is an intuitionistic first order logic (coherent logic with sums and quotients). We will often exploit this fact, and work inside a pretopos $\mathcal{E}$ as if it is a universe of sets with this logic.

A pretopos $\mathcal{E}$ is said to have *dependent products* if, for any map $\alpha : Y \longrightarrow X$, the pullback functor $\alpha^* : \mathcal{E}/X \longrightarrow \mathcal{E}/Y$ has a right adjoint. This condition is equivalent to $\mathcal{E}$ being locally cartesian closed.

In a pretopos $\mathcal{E}$ with dependent products, any map $f : B \longrightarrow A$ gives rise to a "polynomial functor" $P_f : \mathcal{E} \longrightarrow \mathcal{E}$, defined by

$$P_f(X) = \sum_{a \in A} X^{B_a} \tag{1}$$

where $B_a = f^{-1}(a)$. The *W-type of signature* f is the initial algebra for this endofunctor, and is denoted $W(f)$ (if it exists). One can think of $W(f)$ as the free algebra with a B_a-ary operation for each $a \in A$. For $a \in A$ and a morphism $t : B_a \longrightarrow W(f)$, the value under this operation is denoted

$$\mathrm{sup}_a(t).$$

A pretopos with dependent products $\mathcal{E}$ is said to *have W-types* if for any map f this initial algebra $W(f)$ exists. This implies that the same is true for any slice pretopos $\mathcal{E}/X$. Note that if such a pretopos $\mathcal{E}$ has W-types, it has in particular a natural number object $\mathbb{N}$, the W-type of the coproduct inclusion $1 \hookrightarrow 1 + 1$. We refer to [15] for an extensive discussion of W-types in pretoposes.

3 Small Maps

Consider a pretopos $\mathcal{E}$ with dependent products and W-types, and a class $\mathcal{S} \subseteq \mathcal{E}$ of arrows in $\mathcal{E}$. We will discuss a variation of the axioms in [10] which intuitively express that maps in $\mathcal{S}$ should be thought of as maps all of whose fibers are "small" in some sense. The axioms for $\mathcal{S}$ naturally fall apart in three groups. The first axioms S1-3 should naturally be required of any class of maps determined by the properties of the fibers, and state that $\mathcal{S}$ is a stack (cf. Remark 3.2 below). On any such class one can impose fullness and representability conditions. These conditions (together with S1-3) apply in a much more general context, e.g. $\mathcal{E}$ could be a regular category with (stable) sums. If $\mathcal{E}$ has more structure, stable under slicing (i.e. each $\mathcal{E}/X$ has the same structure and change-of-base preserves the structure), it is natural to require that the category $\mathcal{S}/X \subseteq \mathcal{E}/X$, of "small" maps into X, is closed under this structure. In the case of a pretopos $\mathcal{E}$ as above, this is expressed in Axioms F1–5 below.

Definition 3.1 A class of maps $\mathcal{S} \subseteq \mathcal{E}$ is said to be *stable* if it satisfies the following three axioms S1–3:

(S1) (Pullback stability) In a pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd} \tag{2}$$

f' belongs to $\mathcal{S}$ whenever f does.

(S2) (Descent) If in a pullback square (2), the map p is epi, then f belongs to $\mathcal{S}$ whenever f' does.

(S3) (Sum) If two maps $Y \longrightarrow X$ and $Y' \longrightarrow X'$ both belong to $\mathcal{S}$ then so does their sum $Y + Y' \longrightarrow X + X'$.

Any map $\pi : E \longrightarrow U$ in $\mathcal{E}$ determines a stable class $\mathcal{S}(\pi)$, consisting of those maps $f : Y \longrightarrow X$ which fit into a double pullback diagram

$$\begin{tikzcd}
Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[r] \arrow[d, "f'"'] & E \arrow[d, "\pi"'] \\
X & X' \arrow[l, two heads, "p"'] \arrow[r] & U
\end{tikzcd} \tag{3}$$

where p is epi as indicated. In the internal logic of $\mathcal{E}$, one can think of $\mathcal{S}(\pi)$ as the class of those maps $f : Y \longrightarrow X$ for which every fiber is isomorphic to some fiber of π:

$$f \in \mathcal{S}(\pi) \text{ iff } \forall x \in X \, \exists u \in U \, \exists \text{ iso } f^{-1}(x) \cong \pi^{-1}(u).$$

A class $\mathcal{S}$ of the form $\mathcal{S} = \mathcal{S}(\pi)$ is said to be *representable*, and we often refer to π as the *universal small map* in this context. Note, however, that given a representable class $\mathcal{S}$, there can be many different maps π for which $\mathcal{S} = \mathcal{S}(\pi)$. For example, for any pullback diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
U' \arrow[r, two heads] & U
\end{tikzcd}$$

with an epi on the bottom, one has $\mathcal{S}(\pi) = \mathcal{S}(\pi')$.

Remark 3.2 (For readers familiar with stacks; see e.g. [4].) For any object X of $\mathcal{E}$, let $\mathcal{S}_X$ be the full subcategory of $\mathcal{E}/X$ whose objects belong to $\mathcal{S}$. Then the axioms S1-3 express that $\mathcal{S}_{(-)}$ is a *stack* on $\mathcal{E}$. Any map $\pi : E \longrightarrow U$ determines a full internal subcategory **Full**(π) in $\mathcal{E}$ (see [9]), and hence a (representable) sheaf of categories **Full**(π). The stack completion of this sheaf is precisely the stack $\mathcal{S}(\pi)$ just described.

A stable class $\mathcal{S} \subseteq \mathcal{E}$ is said to be a *locally full subcategory* if

(S4) For any two composable arrows f and g with $f \in \mathcal{S}$, the map g belongs to $\mathcal{S}$ iff the composite fg does.

In this paper, we will only consider classes $\mathcal{S}$ which are locally full subcategories.

In the particular context we are interested in, our ambient category $\mathcal{E}$ is a pretopos with dependent products and W-types, as is each of its slices $\mathcal{E}/X$. We require the same for each of the subcategories $\mathcal{S}_X \subseteq \mathcal{E}/X$:

Definition 3.3 A collection of maps $\mathcal{S} \subseteq \mathcal{E}$ is said to be a *(representable) class of small maps* if $\mathcal{S}$ is a stable, (representable) locally full subcategory, and if for every object $X \in \mathcal{E}$, the category $\mathcal{S}_X$ is a pretopos with dependent products and W-types, and the inclusion $\mathcal{S}_X \hookrightarrow \mathcal{E}/X$ preserves this structure.

In this context, we refer to an object X of $\mathcal{E}$ for which $X \longrightarrow 1$ belongs to $\mathcal{S}$ as a *small object.* More generally, we refer to a small map $Y \longrightarrow X$ (i.e. a map in $\mathcal{S}$) as a *small object over X.*

Remark 3.4 The last condition in the previous definition is equivalent to the following five properties:

(F1) $1_X \in \mathcal{S}$ for every object X of $\mathcal{E}$.

(F2) $0 \longrightarrow X$ is in $\mathcal{S}$, and if $Y \longrightarrow X$ and $Z \longrightarrow X$ are in $\mathcal{S}$ then so is $Y + Z \longrightarrow X$.

(F3) For an exact diagram in $\mathcal{E}/X$,

$$\begin{tikzcd}
R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r] \arrow[d] & Y/R \arrow[dl] \\
& X &
\end{tikzcd} \tag{4}$$

if $R \longrightarrow X$ and $Y \longrightarrow X$ belong to $\mathcal{S}$ then so does $Y/R \longrightarrow X$.

(F4) For any $Y \longrightarrow X$ and $Z \longrightarrow X$ in $\mathcal{S}$, their exponent $(Z^Y)_X \longrightarrow X$ in $\mathcal{E}/X$ belongs to $\mathcal{S}$.

(F5) For a commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\
& X &
\end{tikzcd}$$

with all maps in $\mathcal{S}$, the W-type $W_X(f)$ taken in $\mathcal{E}/X$ (which is a map in $\mathcal{E}$ with codomain X) belongs to $\mathcal{S}$.

Indeed, (F1) states that the terminal object of $\mathcal{E}/X$ belongs to $\mathcal{S}_X$. Since $\mathcal{S}$ is closed under pullback and composition (S1,4), it follows that $\mathcal{S}_X \subseteq \mathcal{E}/X$ is closed under all finite limits. (F2) states that $\mathcal{S}_X \subseteq \mathcal{E}/X$ is closed under finite sums, and (F3) similarly concerns quotients of equivalence relations, (F4) exponentials and (F5) W-types.

Remark 3.5 (Slicing) The structure of a pretopos $\mathcal{E}$ with dependent products and W-types, equipped with a (representable) class of small maps $\mathcal{S}$, is stable under slicing: for any object X of $\mathcal{E}$, the slice $\mathcal{E}/X$ is again such a pretopos equipped with a class of small maps $\mathcal{S}/X$.

Remark 3.6 ($\mathcal{S}$-separated objects) Let $\mathcal{S}$ be a class of small maps. Call a map $Y \longrightarrow X$ ($\mathcal{S}$-)*separated,* or an ($\mathcal{S}$-)*separated* object over X, if its diagonal $Y \longrightarrow Y \times_X Y$ belongs to $\mathcal{S}$. If $Y \longrightarrow X$ is small then it is also separated. If, in a diagram

$$\begin{tikzcd}
B \arrow[rr, "\alpha"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

f is separated and g is small, then $\mathrm{Im}(\alpha)$ is a small object over X. (These remarks are immediate consequences of the axioms for a class of small maps.)

Remark 3.7 ($\mathcal{S}$-separation) Recall from [10] that, for a representable class of small maps, each object X of $\mathcal{E}$ has a corresponding power object $\mathcal{P}_s(X)$, representing small subobjects of X. Thus, arrows $I \longrightarrow \mathcal{P}_s(X)$ correspond to subobjects $A \rightarrowtail I \times X$ for which $A \longrightarrow I$ belongs to $\mathcal{S}$. These power objects are closed under the propositional connectives. Moreover, if $f : Y \longrightarrow X$ is small then there is a pullback operation

$$f^{\#} : \mathcal{P}_s(X) \longrightarrow \mathcal{P}_s(Y).$$

If X and Y are separated objects, this operation has both adjoints $\forall_f$ and $\exists_f$. Let us call a first order formula φ of the language of $\mathcal{E}$ an $\mathcal{S}$-*bounded* formula if it contains quantifiers over "small sets" (i.e. along small maps) only. Then each power object $\mathcal{P}_s(X)$ of a separated object X satisfies "$\mathcal{S}$-separation", in the sense that if $A \subseteq X$ is small and $\varphi(x)$ is $\mathcal{S}$-bounded then $\{x \in A \ : \ \varphi(x)\} \subseteq X$ is small (i.e. belongs to $\mathcal{P}_s(X)$).

4 The Axiom of Multiple Choice

The axioms we are about to discuss will be expressed using quasi-pullbacks. Recall that a commutative square

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
A \arrow[r] & X
\end{tikzcd} \tag{5}$$

is called a *quasi-pullback* whenever the canonical map $C \longrightarrow A \times_X B$ is an epi. Thus if g is epi, then so is f. Note also that juxtaposing two quasi-pullbacks with a common edge yields a quasi-pullback.

We recall from [10] that a class of small maps $\mathcal{S}$ is said to satisfy the *collection axiom* (CA) if

(CA) For any map $f : A \longrightarrow X$ in $\mathcal{S}$ and any epi $C \longrightarrow A$ there exists a quasi-pullback of the form

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C \arrow[r, two heads] & A \arrow[d] \\
Y \arrow[rr, two heads] & & X,
\end{tikzcd} \tag{6}$$

where $Y \longrightarrow X$ is epi as indicated and $B \longrightarrow Y$ belongs again to $\mathcal{S}$.

In the internal logic of the ambient pretopos $\mathcal{E}$, this axiom can be stated as a schema

(CA) $\forall$ small $A[\forall a \in A\, \exists c \in C\, \varphi(a,c) \Rightarrow \exists$ small $B\ (\exists f : B \longrightarrow C)\forall a \in A\, \exists b \in B\, \varphi(a, fb)]$.

Here φ is a formula of the internal language of $\mathcal{E}$, and the quantifiers "$\forall$ small A", "$\exists$ small B" range over small objects in (slices of) $\mathcal{E}$. These make sense in the usual interpretation — in fact, one can replace these quantifiers by quantification over the "universal small object" U. We will often use the internal language of $\mathcal{E}$ in a more informal way, and apply the collection axiom in the following form:

(CA) (informal) For any small set A and any surjection $C \longrightarrow A$ there exists a surjection $B \longrightarrow A$ from a small set which factors through C.

This collection axiom was partly inspired by the axiom with the same name of (intuitionistic) Zermelo-Fraenkel set theory. (In fact, one of the goals in [10] was to construct models of IZF from classes of small maps.)

For our present purposes, related to Aczel's constructive set theory CZF, we need a strengthening of this categorical collection axiom, which we call the *axiom of multiple choice*, and which we will now explain.

First, we introduce the notion of a collection map. Informally, a map $g : D \longrightarrow C$ in $\mathcal{E}$ is a *collection map* if (it holds in the internal logic of $\mathcal{E}$ that) for any surjection $E \twoheadrightarrow D_c$ on some fiber $D_c = g^{-1}(c)$ of g, there is another fiber $D_{c'}$ for which there is a surjection $D_{c'} \longrightarrow D_c$ which factors through $E \longrightarrow D_c$ by some map $D_{c'} \longrightarrow E$. Diagrammatically, we can express this by asking that for any map $c : T \longrightarrow C$ and any epi $E \longrightarrow T \times_C D$ there is a diagram of the form

$$\begin{tikzcd}
D \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & D \times_C T' \arrow[l] \arrow[r] \arrow[d] & E \arrow[r, two heads] \arrow[d, phantom, "\text{q.p.b.}"] & T \times_C D \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & D \arrow[d] \\
C & T' \arrow[l] \arrow[rr, two heads] & & T \arrow[r] & C
\end{tikzcd} \tag{7}$$

where the middle square is a quasi-pullback, involving the given map $E \twoheadrightarrow T \times_C D$ and with an epi on the bottom, while the other two squares are pullbacks.

More generally, if $D \longrightarrow C$ is a map over an object A of $\mathcal{E}$, we say that $D \longrightarrow C$ is a *collection map over* A if it is a collection map in the pretopos $\mathcal{E}/A$.

Example 4.1 The collection axiom (CA) stated above is equivalent to the axiom that the universal small map $E \longrightarrow U$ is a collection map. This is easily seen, using the fact that in a pretopos, pasting a pullback to a quasi-pullback yields a quasi-pullback.

Remark 4.2 We observe some basic properties of collection maps:

(i) If $D \longrightarrow C$ is a collection map over A then for any $A' \longrightarrow A$, the pullback $D \times_A A' \longrightarrow C \times_A A'$ is a collection map over A'.

(ii) If $D \longrightarrow C$ is a collection map over A then for any map $A \longrightarrow B$ the map $D \longrightarrow C$ is also a collection map over B.

(iii) In a pullback square over A of the form

$$\begin{tikzcd}
D' \arrow[r] \arrow[d] & D \arrow[d] \\
C' \arrow[r] & C
\end{tikzcd}$$

$D' \longrightarrow C'$ is a collection map over A iff $D \longrightarrow C$ is, *provided* the map $C' \longrightarrow C$ is epi.

(iv) For a family of maps $D_i \longrightarrow C_i$ over A_i, the sum $\sum D_i \longrightarrow \sum C_i$ is a collection map over $\sum A_i$ iff each $D_i \longrightarrow C_i$ is a collection map over A_i.

We now introduce a strengthening of the Collection Axiom called the *Axiom of Multiple Choice*. In the internal language of $\mathcal{E}$, where we think of objects of $\mathcal{E}$ as "sets", this axiom can informally be expressed as follows

> (AMC) ("internal") For any small set B there exists a collection map $D \longrightarrow C$ where D and C are small, and C is inhabited, together with a map $D \longrightarrow B$ making $D \longrightarrow B \times C$ into a surjection.

Diagrammatically, the small set B (in any slice of $\mathcal{E}$!) is the fiber of a small map $B \longrightarrow A$, and the interpretation of "there exists" takes place on a cover $A' \twoheadrightarrow A$, so that (AMC) takes the following form:

> (AMC) ("diagrammatic") For any map $B \longrightarrow A$ in $\mathcal{S}$, there exists an epi $A' \twoheadrightarrow A$ and a quasi-pullback of the form
>
> $$\begin{tikzcd}
D \arrow[rr] \arrow[d] & & B \arrow[d] \\
C \arrow[r, two heads] & A' \arrow[r, two heads] & A
\end{tikzcd} \tag{8}$$
>
> where $D \longrightarrow C$ is a small collection map over A' and $C \longrightarrow A'$ is a small epi.

We will use both versions of (AMC) in the sequel.

Proposition 4.3 *AMC implies the Collection Axiom.*

Proof. By way of example we will give both an informal and a diagrammatic proof.

To show that the informal version of (CA) follows from the informal version of (AMC), take any small "set" A and any epi $E \twoheadrightarrow A$. By (AMC) there exists a surjection $(f, g) : D \longrightarrow A \times C$ where $g : D \longrightarrow C$ is a collection map, and C is inhabited. For any $x \in C$, we get $f : g^{-1}(x) \twoheadrightarrow A$. Applying the fact that g is a collection map to the cover $E \times_A g^{-1}(x) \twoheadrightarrow$

$g^{-1}(x)$, we find another $y \in C$ and a surjection $t : g^{-1}(y) \longrightarrow g^{-1}(x)$ so that $f \circ t$ factors through $E \longrightarrow A$. Thus $B = g^{-1}(y)$ shows that (CA) holds.

A possible diagrammatic proof would go as follows. First, for a given small map $A \longrightarrow X$, (AMC) gives a quasi-pullback of the form

$$\begin{tikzcd}
D \arrow[rr] \arrow[d] & & A \arrow[d] \\
C \arrow[r, two heads] & X' \arrow[r, two heads] & X
\end{tikzcd} \tag{9}$$

where $D \longrightarrow C$ is small collection map over X'. Applying the description of a collection map (7) to the cover $E \times_A D \twoheadrightarrow D$, we find a diagram of the shape

$$\begin{tikzcd}
D \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & D \times_C T' \arrow[l] \arrow[r] \arrow[d] & E \times_A D \arrow[r, two heads] \arrow[d, phantom, "\text{q.p.b.}"] & D \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{q.p.b.}"] & & A \arrow[d] \\
C & T' \arrow[l] \arrow[rr, two heads] & & C \arrow[r, two heads] & X' \arrow[r, two heads] & X
\end{tikzcd} \tag{10}$$

Since $D \longrightarrow C$ is small, so is $D \times_C T' \longrightarrow T'$. Hence the composition of the two right hand quasi-pullbacks yields a quasi-pullback witnessing (CA). ■

Remark 4.4 , about the difference between (AMC) and (CA): For a small set A and an epi $E \twoheadrightarrow A$, (CA) gives an epi $B \twoheadrightarrow A$ from a small B which factors through E. This B depends in general on the cover E. (AMC) bounds the choice of the B's needed to a small family, which does not depend on E, namely the fibers of the collection map $D \longrightarrow C$.

The (AMC) is related to the axiom of choice in the following way. In many examples (e.g. realizability categories, or categories defined from Martin-Löf type theories), the class $\mathcal{S}$ of small maps in $\mathcal{E}$ is defined from a class of *choice maps* $\mathcal{A}$ in $\mathcal{E}$ (see [10, p. 97]). More precisely, each map in $\mathcal{A}$ is internally projective, and for any map $f : B \longrightarrow A$ in $\mathcal{S}$ there exists a quasi-pullback of the form

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\
C \arrow[r] & A
\end{tikzcd}$$

with $g \in \mathcal{A}$. In this situation, (AMC) is always satisfied, by the proposition below. We remark, however, that (AMC) is much more flexible then any condition having to do with the existence of (internal) projectives. Indeed, (AMC) is always stable under the sheaf-construction (see Section 9), while the existence of projectives rarely is.

Proposition 4.5 *A map $D \longrightarrow C$ is a collection map over C if, and only if, it is a choice map.*

Proof. Working informally inside $\mathcal{E}$, this is quite clear: By definition the map $D \longrightarrow C$ is a choice map precisely when its fibers are projective. Then the proposition asserts the obvious fact that, internally, a "set" P is projective iff $P \longrightarrow 1$ is a collection map.

By way of illustration, we also give a diagrammatic proof of the implication ($\Leftarrow$). Suppose $D \longrightarrow C$ is a choice map. Recall from [10, p. 96] that this means that D is internally projective in $\mathcal{E}/C$, so that given any $T \longrightarrow C$ and any epi $E \twoheadrightarrow T \times_C D$, there will exist an epi $T' \twoheadrightarrow T$ and a map $T' \times_C D \longrightarrow E$ which fits into a commutative square of the form

$$\begin{tikzcd}
T' \times_C D \arrow[r] \arrow[d] & E \arrow[d] \\
T \times_C D \arrow[r, "1"] & T \times_C D
\end{tikzcd}$$

From this square we obtain a diagram

$$\begin{tikzcd}
D \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & T' \times_C D \arrow[l] \arrow[r] \arrow[d] \arrow[drr, phantom, "\text{p.b.}"] & E \arrow[r, two heads] & T \times_C D \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & D \arrow[d] \\
C & T' \arrow[l] \arrow[rr, two heads] & & T \arrow[r] & C
\end{tikzcd} \tag{11}$$

which is more than required for (AMC). ∎

We wish to apply the (AMC) to a small map $B \longrightarrow A$ where the object A is also small. Informally in $\mathcal{E}$, this means that we are given a small family $\{B_a : a \in A\}$ of small sets. The informal version of (AMC) applied to each member of this family gives for each $a \in A$ a collection map $D_x \longrightarrow C_x$ with a surjection $D_x \longrightarrow C_x \times B_a$, etc. Lacking the axiom of choice, we cannot choose one x for each a. But, applying collection (CA) to the quantifier combination $\forall a \in A \, \exists x \, (\cdots)$, we find that a *small* family of such collection maps suffice. This gives a quasi-pullback

$$\begin{tikzcd}
D \arrow[rr] \arrow[d] & & B \arrow[d] \\
C \arrow[r, two heads] & X \arrow[r, two heads] & A
\end{tikzcd}$$

where $D \longrightarrow C$ is a collection map over X while $C \longrightarrow X$ and $X \longrightarrow A$ are both small surjections. Using Remark 4.2.(ii), we obtain an informal proof of the following assertion:

Proposition 4.6 *(Preliminary formulation) Suppose (AMC) holds. Then, internally in $\mathcal{E}$, it holds that for any small map $B \longrightarrow A$ into a small object A there exists a quasi-pullback*

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r, two heads] & A
\end{tikzcd}$$

where all objects and maps are small and $D \longrightarrow C$ is a collection map over A.

Taking the existential quantifier in this proposition outside $\mathcal{E}$, we can rephrase it as follows.

Proposition 4.6 *(AMC) Let $B \longrightarrow A$ be a small map into a small object A. Then there exists a $T \twoheadrightarrow 1$ and a quasi-pullback of the form*

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & B \times T \arrow[d] \\
C \arrow[r, two heads] & A \times T
\end{tikzcd}$$

where $D \longrightarrow C$ is a small collection map over $A \times T$, and $C \longrightarrow T$ is small.

Proof. Although this proposition has already been proved informally, we also give a diagrammatic proof. (AMC) gives a diagram

$$\begin{tikzcd}
D_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{q.p.b.}"] & A' \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\
C_0 \arrow[r, two heads] & A' \arrow[r, two heads] & A
\end{tikzcd} \tag{12}$$

where $D_0 \longrightarrow C_0$ is a small collection map over A' and $C_0 \longrightarrow A'$ is small. The collection axiom gives a quasi-pullback of the form

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & A' \arrow[r, two heads] & A \arrow[d] \\
T \arrow[rr, two heads] & & 1
\end{tikzcd} \tag{13}$$

where $V \longrightarrow T$ is small. Pulling back the collection map $D_0 \longrightarrow C_0$ over A' along $V \longrightarrow A'$ yields a map $D \longrightarrow C$ which fits into a pullback square

$$\begin{tikzcd}
D = D_0 \times_{A'} V \arrow[r] \arrow[d] & D_0 \arrow[d] \\
C = C_0 \times_{A'} V \arrow[r, two heads] & C_0
\end{tikzcd} \tag{14}$$

with a small collection map over V on the left. Juxtaposing (14) with the outer quasi-pullback of (12) on the right gives the upper quasi-pullback in

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r, two heads] \arrow[d] & A \arrow[d] \\
T \arrow[r, two heads] & 1
\end{tikzcd} \tag{15}$$

The bottom is a quasi-pullback because $C \longrightarrow V$ is epi and V fits into the quasi-pullback (13). It follows, using the pasting lemma for quasi-pullbacks, that

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \times T \arrow[d] \\ C \arrow[r, two heads] & A \times T \end{tikzcd} \tag{16}$$

is a quasi-pullback with an epi on the bottom, and on the left a small collection map over V, hence a fortiori over $A \times T$. Finally, it is easy to check that $C \longrightarrow T$ is small. ∎

5 Mostowski Collapse of W-types

We continue to work in a pretopos $\mathcal{E}$, with dependent products and W-types, which is equipped with a class of small maps $\mathcal{S}$.

Consider the W-type $W(f)$ associated to a small map $f : B \longrightarrow A$ in $\mathcal{E}$. By the universal property of $W(f)$, one can define a map

$$E : W(f) \times W(f) \longrightarrow \mathcal{P}_s(1)$$

by "double induction". The E here stands for "extensional equality". If $x = \mathrm{sup}_a(t)$ and $y = \mathrm{sup}_{a'}(t')$, then

$$\begin{array}{rcl} E(x,y) & \text{iff} & \forall b \in f^{-1}(a)\, \exists b' \in f^{-1}(a')\, E(t(b), t'(b')) \\ & \text{and} & \forall b' \in f^{-1}(a')\, \exists b \in f^{-1}(a)\, E(t(b), t'(b')). \end{array}$$

Note that the value $E(x,y)$ indeed lies in $\mathcal{P}_s(1)$, because the inductive definition only involves quantifiers ranging over the fibers of $f : B \longrightarrow A$, and these are assumed to be small. We also define two other relations $W(f) \times W(f) \longrightarrow \mathcal{P}_s(1)$ by induction, M for membership and I for inclusion, by

$$\begin{array}{rcl} M(y,x) & \text{iff} & \exists b \in f^{-1}(a)\, E(y, t(b)), \\ I(y,x) & \text{iff} & \forall b' \in f^{-1}(a')\, \exists b \in f^{-1}(a)\, E(t(b), t'(b')). \end{array}$$

Notice that these definitions are intertwined, as follows

$$I(x,y) \quad \text{iff} \quad \forall u \in W(f)\, (M(u,x) \longrightarrow M(u,y)) \tag{17}$$

$$E(x,y) \quad \text{iff} \quad I(x,y) \text{ and } I(y,x). \tag{18}$$

We will also write $M, I, E \subseteq W(f) \times W(f)$ for the corresponding subobjects. We observe that the map $E \rightarrowtail W(f) \times W(f)$ is small by construction, and is easily seen to be an equivalence relation, by induction on elements of $W(f)$. Thus we can form the quotient, which we denote by

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] & W(f) \arrow[r, "\theta_f"] & V(f) \end{tikzcd} \tag{19}$$

and call the *Mostowski* (or extensional) *collapse* of $W(f)$.

Note that I and M pass down to well-defined relations on $V(f)$, denoted by $\subseteq$ and ε. Thus, in the internal logic of $\mathcal{E}$,

$$\forall x, y \in W(f) : (I(x,y) \Leftrightarrow \theta_f(x) \subseteq \theta_f(y)) \,\&\, (M(x,y) \Leftrightarrow \theta_f(x) \,\varepsilon\, \theta_f(y)). \tag{20}$$

Lemma 5.1 *(i) The diagonal* $\Delta : V(f) \longrightarrow V(f) \times V(f)$ *is small, i.e.* $V(f)$ *is* $\mathcal{S}$-*separated.*

(ii) The composition $\varepsilon \rightarrowtail V(f) \times V(f) \xrightarrow{\pi_2} V(f)$ *is small.*

(iii) If $f : B \longrightarrow A$ *and* A *are both small then so is* $V(f)$*.*

Proof. (i): By construction of $V(f)$ there is a pullback

$$\begin{tikzcd}
E \arrow[r, two heads] \arrow[d, tail] & V(f) \arrow[d, "\Delta"] \\
W(f) \times W(f) \arrow[r, two heads] & V(f) \times V(f)
\end{tikzcd}$$

Since $E \longrightarrow W(f) \times W(f)$ is small, so is Δ by axiom (S2).

(ii): We argue in the internal logic of $\mathcal{E}$. Take any $v \in V(f)$, and write $v = \theta_f(x)$, where $x = \sup_a(t)$ is some element of $W(f)$. We need to show that $\{w \in V(f) : w \,\varepsilon\, v\}$ is small. But this set is the image of the map

$$\begin{tikzcd}
f^{-1}(a) \arrow[r, tail, "t"] & W(f) \arrow[r, "\theta_f"] & V(f)
\end{tikzcd}$$

and this image is small by part (i) and Remark 3.6.

(iii): By Remark 3.4, (F5) applied with $X = 1$ gives that $W(f)$ is small. But then so is $V(f)$, by (F3) and the fact that $E \rightarrowtail W(f) \times W(f)$ is small. ∎

By part (ii) of this lemma, any $v \in V(f)$ defines a small subobject ("subset")

$$\mathrm{Ext}(v) = \{w \in V(f) : w \,\varepsilon\, v\}$$

of $V(f)$, the "externalization" of v (from elements of $V(f)$ to objects of $\mathcal{E}$).

Lemma 5.2 *("extensionality") The map*

$$\mathrm{Ext} : V(f) \longrightarrow \mathcal{P}_s(V(f))$$

is injective.

Proof. This is just another way of phrasing (17) – (18) above. ∎

We next consider the functoriality of the construction of $V(f)$ out of a small map f. To this end, we first introduce some terminology. A map $\iota : V(f) \longrightarrow V(g)$ is called a *transitive embedding* if

$$w \,\varepsilon\, \iota(v) \Leftrightarrow (\exists u \,\varepsilon\, v)\, w = \iota(u).$$

It follows easily by induction that each transitive embedding is mono. Recall from [15] that $W(f)$ is contravariant, in the sense that any commutative triangle

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{21}$$

induces a map $\alpha^* : W(f) \longrightarrow W(g)$. This map is mono whenever α is epi. Moreover, $W(f)$ is covariant in the sense that any pullback square

$$\begin{tikzcd} D \arrow[r, "\tilde{\alpha}"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "\alpha"] & A \end{tikzcd} \tag{22}$$

induces a map $\alpha_! : W(g) \longrightarrow W(f)$. This map is epi whenever α is.

Proposition 5.3 *(i) If α in (21) is epi, then $\alpha^* : W(f) \longrightarrow W(g)$ induces a transitive embedding (again denoted)*

$$\alpha^* : V(f) \rightarrowtail V(g)$$

such that $\theta_g \circ \alpha^* = \alpha^* \circ \theta_f$.

(ii) For any pullback square (22), the map $\alpha_! : W(g) \longrightarrow W(f)$ induces a transitive embedding

$$\alpha_! : V(g) \rightarrowtail V(f)$$

such that $\theta_f \circ \alpha_! = \alpha_! \circ \theta_g$. If α is epi, then $\alpha_! : V(g) \longrightarrow V(f)$ is an isomorphism.

(iii) The map $\alpha_! : V(g) \longrightarrow V(f)$ does not depend on the maps α and $\tilde{\alpha}$ making up the pullback square (22).

Proof. (i): Recall that $\alpha^* : W(f) \longrightarrow W(g)$ is defined "inductively" by

$$\alpha^*(\sup_a t) = \sup_a(\alpha^* \circ t \circ \alpha_a), \tag{23}$$

where $a \in A$, $t : B_a \longrightarrow W(f)$ and $\alpha_a : C_a \longrightarrow B_a$ is the restriction of α. If α is epi then so is α_a for each a, and one shows easily by induction on $x, y \in W(f)$ that

$$x \sim y \Longleftrightarrow \alpha^*(x) \sim \alpha^*(y)$$

where $\sim$ is the equivalence relation E on $W(f)$ (respectively on $W(g)$). Then by using (23) and that α_a is epi, again, one shows

$$M(w, \alpha^*(\sup_a t)) \Leftrightarrow \exists u\, (M(u, \sup_a t) \wedge w \sim \alpha^*(u)).$$

Thus α^* factors by a transitive embedding which makes the square

$$\begin{tikzcd}
W(g) \arrow[r, tail, "\alpha^*"] \arrow[d, two heads, "\theta_g"'] & W(f) \arrow[d, two heads, "\theta_f"] \\
V(g) \arrow[r, tail, "\alpha^*"] & V(f)
\end{tikzcd}$$

commute.

(ii): Recall that $\alpha_! : W(g) \longrightarrow W(f)$ is defined, again inductively, by

$$\alpha_!(\sup_c(t)) = \sup_{\alpha(c)}(\alpha_! \circ t \circ \tilde{\alpha}_c^{-1}), \tag{24}$$

where $c \in C$, $t : D_c \longrightarrow W(g)$ and $\tilde{\alpha}_c : D_c \xrightarrow{\sim} B_{\alpha(c)}$ is the restriction of $\tilde{\alpha}$. Again, it is easy to show by induction that for any $x, y \in W(f)$,

$$x \sim y \Longleftrightarrow \alpha_!(x) \sim \alpha_!(y).$$

Using (24), and that each $\tilde{\alpha}_c^{-1}$ is iso, one shows that $\alpha_!$ factors by a transitive embedding $\alpha_! : V(g) \rightarrowtail V(f)$, analogous to the case of α^* in part (i). If α is epi then so is $\alpha_! : W(g) \longrightarrow W(f)$, and hence $\alpha_! : V(g) \longrightarrow V(f)$ must be epi as well. Then $\alpha_! : V(g) \xrightarrow{\sim} V(f)$ is an isomorphism.

(iii): Suppose both squares in

$$\begin{tikzcd}
D \arrow[r, shift left, "\tilde{\alpha}"] \arrow[r, shift right, "\tilde{\beta}"'] \arrow[d, "g"'] & B \arrow[d, "f"] \\
C \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & A
\end{tikzcd}$$

are pullbacks. Then for any $c \in C$, we have an isomorphism

$$\gamma_c = \tilde{\beta}_c \circ \tilde{\alpha}_c^{-1} : B_{\alpha(c)} \xrightarrow{\sim} B_{\beta(c)}.$$

Now, if $t : g^{-1}(c) \longrightarrow W(g)$ is any map such that

$$\theta_f \circ \alpha_! \circ t = \theta_f \circ \beta_! \circ t : g^{-1}(c) \longrightarrow W(g) \rightrightarrows W(f) \longrightarrow V(f),$$

i.e. $\forall d \in g^{-1}(c)\, E(\alpha_! t(d), \beta_! t(d))$, then, using the inductive definition of E and the isomorphism γ_c, one sees that $E(\alpha_!(\sup_c t), \beta_!(\sup_c t))$, i.e.

$$\theta_f \alpha_!(\sup_c(t)) = \theta_f \beta_!(\sup_c(t)).$$

By induction, it follows that

$$\theta_f \circ \alpha_! = \theta_f \circ \beta_! : W(g) \longrightarrow V(f),$$

and hence $\alpha_! = \beta_!$ as maps $V(g) \longrightarrow V(f)$. ∎

Now consider the "universal small map" $\pi : E \longrightarrow U$, and write $V = V(\pi)$. (In fact, we will see that V only depends on $\mathcal{S}$ and not on the representing map π.)

Corollary 5.4 *(i) For any small map f there is a canonical transitive embedding $i_f : V(f) \rightarrowtail V$ (by which we can identify $V(f)$ with a subobject of V).*

(ii) The maps $\alpha^* : V(f) \rightarrowtail V(g)$ and $\alpha_! : V(g) \rightarrowtail V(f)$, in the preceeding proposition respect this embedding, i.e.

$$i_g \circ \alpha^* = i_f, \qquad i_f \circ \alpha_! = i_g.$$

Proof. We give an explicit description of i_f, proving (i), and then leave the verification of (ii) to the reader.

For the given small map f, there exists a double pullback

$$\begin{tikzcd}
Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[r, "\tilde{c}"] \arrow[d] & E \arrow[d, "\pi"] \\
X & X' \arrow[l, two heads, "p"'] \arrow[r, "c"'] & U
\end{tikzcd} \tag{25}$$

and hence by Proposition 5.3 (ii) a transitive embedding

$$i_f = c_! \circ (p_!)^{-1} : V(f) \longrightarrow V.$$

We claim that this embedding does not depend on the choice of the double pullback (25). Indeed, it does not depend on c and p by 5.3 (iii). And if there is another double pullback with $Y'' \longrightarrow X''$ in the middle, one can construct a "common refinement" $Y' \times_Y Y'' \longrightarrow X' \times_X X''$, and use 5.3 (iii) again to show that i_f does not depend on $Y' \longrightarrow X'$ either. ∎

Corollary 5.5 *(i) When the maps f and g fit into a quasi-pullback of the form*

$$\begin{tikzcd}
\cdot \arrow[r, two heads] \arrow[d, "g"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, two heads] & \cdot
\end{tikzcd}$$

then $V(f) \subseteq V(g)$.

(ii) In particular, if (AMC) holds then, internally in $\mathcal{E}$, it holds that for any (small) map f between small objects there is a collection map g between small objects such that $V(f) \subseteq V(g)$.

Proof. (i) Decomposing the quasi-pullback as a pullback and a triangle

$$\begin{tikzcd}
\cdot \arrow[r, two heads] \arrow[dr, "g"'] & \cdot \arrow[r, two heads] \arrow[d, "f'"] & \cdot \arrow[d, "f"] \\
 & \cdot \arrow[r, two heads] & \cdot
\end{tikzcd}$$

we obtain a mono $V(f) \cong V(f') \rightarrowtail V(g)$ by Proposition 5.3 (i), (ii). By Corollary 5.4, $V(f) \subseteq V(g)$ as subobjects of V. Part (ii) follows by using Proposition 4.6. ∎

6 Mostowski Collapse for Collection Maps

We will briefly discuss some properties of objects of the form $V(f)$ for the special case where f is a (small) collection map. To begin with, observe that any small map $f : B \longrightarrow A$ defines a functor

$$\mathcal{P}_f : \mathcal{E} \longrightarrow \mathcal{E}$$

by

$$\mathcal{P}_f(X) = \{S \subseteq X \mid \exists a \in A \, \exists t : B_a \longrightarrow X : S = \mathrm{Im}(t)\}.$$

In other words, $\mathcal{P}_f(X)$ is the family of those subsets of X which can be enumerated by some fiber of f. This object can be constructed in any pretopos with dependent products, as a quotient of $\Sigma_{a \in A} X^{B_a}$. If X is separated, then by 3.7 $\mathcal{P}_f(X)$ is a subobject of $\mathcal{P}_s(X)$, the object of small subsets of X. In particular, for the universal small map $\pi : E \longrightarrow U$ we have $\mathcal{P}_\pi(X) = \mathcal{P}_s(X)$ whenever X is separated.

This construction is clearly a covariant functor of X. We denote the effect of a map $\varphi : X \longrightarrow Y$ in $\mathcal{E}$ by

$$\varphi_! : \mathcal{P}_f(X) \longrightarrow \mathcal{P}_f(Y).$$

It is useful to observe the following property of collection maps, an immediate consequence of the definition.

Lemma 6.1 *For any collection map $f : B \longrightarrow A$, the map $\varphi_! : \mathcal{P}_f(X) \longrightarrow \mathcal{P}_f(Y)$ is epi whenever $\varphi : X \longrightarrow Y$ is.* ∎

The functor $\mathcal{P}_f$ is closely related to the polynomial functor P_f of (1). Indeed, by taking images of functions $B_a \longrightarrow X$, one obtains an epimorphism

$$\mathrm{Im} : P_f(X) \longrightarrow \mathcal{P}_f(X), \tag{26}$$

natural in X.

As to any endofunctor, there is a category of $\mathcal{P}_f$-algebras associated to $\mathcal{P}_f$. Its objects are pairs $(X, \alpha : \mathcal{P}_f(X) \longrightarrow X)$, and its arrows $\varphi : (X, \alpha) \longrightarrow (Y, \beta)$ are maps $\varphi : X \longrightarrow Y$ in $\mathcal{E}$ such that $\beta \circ \varphi_! = \varphi \circ \alpha$.

Before discussing the next two propositions, we observe that the map Ext of Lemma 5.2 factors through $\mathcal{P}_f$, i.e. defines a map

$$\text{Ext} : V(f) \longrightarrow \mathcal{P}_f(V(f)). \tag{27}$$

Proposition 6.2 *If f is a small collection map, then $V(f)$ has the structure of a $\mathcal{P}_f$-algebra, denoted* Int : $\mathcal{P}_f(V(f)) \longrightarrow V(f)$. *This structure map fits into a commutative square*

$$\begin{tikzcd}
P_f(W(f)) \arrow[r, two heads] \arrow[d, "\text{sup}"] & \mathcal{P}_f(V(f)) \arrow[d, "\text{Int}"] \\
W(f) \arrow[r, two heads, "\theta_f"] & V(f)
\end{tikzcd} \tag{28}$$

where the map on top is the composite $(\theta_f)_! \circ \text{Im} : P_f(W(f)) \longrightarrow \mathcal{P}_f(W(f)) \longrightarrow \mathcal{P}_f(V(f))$.

Proof. Let $S \in \mathcal{P}_f(V(f))$, and choose $\sigma : B_a \longrightarrow V(f)$ such that S is the image of σ. Consider the pullback

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & W(f) \arrow[d, "\theta_f"] \\
B_a \arrow[r, "\sigma"] & V(f)
\end{tikzcd}$$

Since $f : B \longrightarrow A$ is a collection map, there exists an $a' \in A$ and a surjection $p : B_{a'} \longrightarrow B_a$ which factors through X. Thus σp factors through $W(f)$, and we find a map $\tau : B_{a'} \longrightarrow W(f)$ such that

$$\theta_f \circ \tau = \sigma \circ p.$$

Let $x = \text{sup}_{a'}(\tau) \in W(f)$, and $y = \theta_f(x) \in V(f)$. We claim that y only depends on S, and not on a, σ, etc. Indeed, the small subset $\text{Ext}(y) \subseteq V(f)$ is the image of $\theta_f \circ \tau = \sigma \circ p$, hence since p is epi, $\text{Ext}(y) = \text{Im}(\sigma) = S$. So by the extensionality lemma 5.2, y only depends on S, and we can define $\text{Int}(S)$ to be this y. In this way, we obtain a map $\text{Int} : \mathcal{P}_f(V(f)) \longrightarrow V(f)$, which by construction makes the diagram commute. ∎

We remark that in this proof, we construct $\text{Int}(S)$ as the y for which $\text{Ext}(y) = S$. So $\text{Ext} \circ \text{Int} = \text{Id}$. Since Ext is mono (5.2), also $\text{Int} \circ \text{Ext} = \text{Id}$. Thus Int and Ext are mutually inverse, as stated in the following proposition.

Proposition 6.3 *Let f be a small collection map. Then $(V(f), \text{Int})$ is the initial $\mathcal{P}_f$-algebra. In particular,* Int *is an isomorphism, inverse to* Ext.

Proof. Let X be any $\mathcal{P}_f$-algebra, say with structure map $\mu : \mathcal{P}_f(X) \longrightarrow X$. By precomposing with $\mathrm{Im} : P_f(X) \longrightarrow \mathcal{P}_f(X)$, we can define a P_f-algebra structure on X. Since $W(f)$ is the free P_f-algebra, there is a unique map λ making the diagram

$$\begin{tikzcd}
P_f(W(f)) \arrow[r, "P_f(\lambda)"] \arrow[dd, "\sup"] & P_f(X) \arrow[d, "\mathrm{Im}"] \\
 & \mathcal{P}_f(X) \arrow[d, "\mu"] \\
W(f) \arrow[r, "\lambda"] & X
\end{tikzcd} \tag{29}$$

commute. In other words,

$$\lambda(\sup_a(t)) = \mu(\{\lambda(tb) : b \in f^{-1}(a)\}).$$

From this last expression, one readily shows by induction on elements of $W(f)$ that λ respects the equivalence relation E, hence factors as a map $\overline{\lambda} : V(f) \longrightarrow X$ with

$$\overline{\lambda} \circ \theta_f = \lambda. \tag{30}$$

Then $\overline{\lambda}$ is a homomorphism of $\mathcal{P}_f$-algebras. Indeed,

$$\begin{aligned}
\overline{\lambda} \circ \mathrm{Int} \circ (\theta_f)_! \circ \mathrm{Im} &= \overline{\lambda} \circ \theta_f \circ \sup && (\text{ cf. } (28)) \\
&= \lambda \circ \sup \\
&= \mu \circ \mathrm{Im} \circ P_f(\lambda) && (\text{cf. } (29)) \\
&= \mu \circ \lambda_! \circ \mathrm{Im} && (\text{naturality of Im}) \\
&= \mu \circ \overline{\lambda}_! \circ (\theta_f)_! \circ \mathrm{Im}. && (\text{by } (30))
\end{aligned}$$

Since $(\theta_f)_! \circ \mathrm{Im}$ in (28) is epi, cf. Lemma 6.1, we conclude that $\overline{\lambda} \circ \mathrm{Int} = \mu \circ \overline{\lambda}_!$, as required. The uniqueness of $\overline{\lambda}$ follows readily from that of λ, and we omit the details. ∎

Corollary 6.4 *For any small collection map f, one has (in the internal logic of $\mathcal{E}$):*

$$v \,\varepsilon\, \mathrm{Int}(S) \textit{ iff } v \in S$$

for any $v \in V$ and $S \in \mathcal{P}_f(V(f))$.

Proof. For any two $v, w \in V(f)$, we have

$$v \in \mathrm{Ext}(w) \text{ iff } v \,\varepsilon\, w$$

by definition of Ext. Applying this to $w = \mathrm{Int}(S)$ gives the stated equivalence. ∎

Corollary 6.5 *For any collection map f, the object $V(f)$ only depends (up to isomorphism) on the representable class $\mathcal{S}(f)$ of maps. In particular, for the universal small map π, the object $V = V(\pi)$ only depends on the class $\mathcal{S}$ of small maps.*

Proof. Clearly $\mathcal{P}_f$ depends only on $\mathcal{S}(f)$, so this follows from Proposition 6.3. ■

Remark 6.6 For a transitive embedding $\eta : V(f) \longrightarrow V(g)$ the square

$$\begin{tikzcd}
V(f) \arrow[r, "\mathrm{Ext}"] \arrow[d, "\eta"'] & \mathcal{P}_s(V(f)) \arrow[d, "\mathrm{Im}(\eta)"] \\
V(g) \arrow[r, "\mathrm{Ext}"] & \mathcal{P}_s(V(g)).
\end{tikzcd}$$

commutes (as is immediate from the definition). If f and g are collection maps, then we observed that Ext factors through $\mathcal{P}_f(V(f))$ and is inverse to the algebra structure Int : $\mathcal{P}_f(V(f)) \longrightarrow V(f)$, and similarly for $V(g)$. It follows that the transitive embeddings given by Proposition 5.3 are compatible with the algebra structure of Proposition 6.3.

Remark 6.7 We can immediately relate Proposition 6.3 to the theory of ZF-algebras developed in [10]. Indeed, the functor $\mathcal{P}_\pi$ is part of a monad $(\mathcal{P}_\pi, \cup, \{\cdot\})$, with union and singleton as multiplication and unit. By Theorem A5, p. 106 in [10], it follows from Proposition 6.3 that $V = V(\pi)$ is the free successor algebra for this monad, with as algebra structure $\mathcal{P}_\pi(V) \longrightarrow V$ the union map $S \mapsto \mathrm{Int}\{w : \exists v \in S\, w \,\varepsilon\, v\}$, and as successor map $V \longrightarrow V$ the singleton map $v \mapsto \mathrm{Int}(\{v\})$. (Thus V is the free ZF-algebra, if we define ZF-algebras here using $\mathcal{P}_\pi$ rather than $\mathcal{P}_s$. Note that, unlike the treatment in [10], we do not assume here that the quotient of a small object is again small. So $\mathcal{P}_s$ is not a covariant functor and we have to use $\mathcal{P}_\pi$ instead.)

7 Categorical models of CZF

In this section we consider Aczel's constructive set theory, CZF. The main references for models of CZF are Aczel [1, 2] and the exposition in Troelstra and van Dalen [22]. Our main result will be that if $\mathcal{E}$ is a pretopos with dependent products and W-types, and $\mathcal{S}$ is a class of small maps satisfying (AMC), then $V = V(\pi)$ is a model of CZF. (In fact, it is also a model of the regular extension axiom [2], cf. Theorem 7.1 below.) This result is analogous to the result for IZF in [10]. In relation to Section 9, it is relevant to observe here that we do not need all W-types to exist in $\mathcal{E}$ for this result to hold, but only the W-types $W(f)$ for small maps f.

We begin by listing Aczel's axioms. The formulae φ in 1 – 10 below are arbitrary unless otherwise stated.

1. Extensionality: $v = w \leftrightarrow \forall x\, (x \,\epsilon\, v \leftrightarrow x \,\epsilon\, w)$

2. Pairing: $\forall v, w\, \exists x\, \forall y\, (y \in x \leftrightarrow (y = v \vee y = w))$
3. Union: $\forall v\, \exists x\, \forall y\, (y \in x \leftrightarrow \exists w\, (y \in w \in x))$
4. Set induction: $\forall x\, (\forall y \in x\, \varphi(y) \longrightarrow \varphi(x)) \longrightarrow \forall x\, \varphi(x)$
5. Δ_0-separation: $\exists v\, \forall x\, (x \in v \leftrightarrow \varphi(x) \wedge x \in w)$ for *bounded* formulas φ, not containing v as a free variable
6. Infinity: $\exists z\, \forall x\, [x \in z \leftrightarrow x = \emptyset \vee (\exists y \in z)\, x = y \cup \{y\}]$. (In view of the extensionality axiom the equalities and the constant $\emptyset$ can be eliminated from the Infinity axiom.)
7. Exponentials: $\forall x, y\, \exists a\, \forall v\, (v \in a \leftrightarrow v$ is a function from x to $y)$
8. Strong collection: $\forall x \in v\, \exists y\, \varphi(x, y) \longrightarrow \exists w\, [(\forall x \in v\, \exists y \in w\, \varphi(x, y)) \wedge (\forall y \in w\, \exists x \in v\, \varphi(x, y))]$
9. Subset collection: $\forall a, b\, \exists c\, \forall u\, \Big[\forall x \in a\, \exists y \in b\, \varphi(x, y, u) \longrightarrow \exists d \in c\, [(\forall x \in a\, \exists y \in d\, \varphi(x, y, u)) \wedge$ $(\forall x \in d\, \exists y \in a\, \varphi(x, y, u))]\Big]$.
10. Regular extension axiom (REA): $\forall x\, \exists r\, (x \in r \wedge r$ regular$)$

The last axiom uses the notion of a *regular set.* A relation $R \subseteq b \times A$ is called a *multi-valued function* if $\forall x \in b\, \exists y \in A\, \langle x, y\rangle \in R$. We recall from [2] that a nonvoid set A is *regular* if it is transitive and if for every $b \in A$ and every multi-valued function $R \subseteq b \times A$, there is a bounding set $d \in A$ so that

$$\forall x \in b\, \exists y \in d\, \langle x, y\rangle \in R \text{ and } \forall y \in d\, \exists x \in b\, \langle x, y\rangle \in R.$$

Aczel's original theory CZF is axiomatized by 1-6,8,9. In [1, Prop. 2.1, 2.2] and [3, Prop. 4.5] it is shown that in the presence of the axioms 1-6:

REA and Strong Collection $\Rightarrow$ Subset Collection $\Rightarrow$ exponentials.

So CZF+REA can be axiomatized by 1-6,8,10. We denote by CZF^- the theory axiomatized by 1-8, which is similar to Myhill's original set theory. By CZF^+ we denote the theory given by 1-6,8,10. Thus $\mathrm{CZF}^- \subseteq \mathrm{CZF} \subseteq \mathrm{CZF}^+$.

Theorem 7.1 *Let $\mathcal{E}$ be a pretopos with dependent products and W-types $W(f)$, for all (small) maps f, and let $\mathcal{S}$ be the class of small maps in $\mathcal{E}$.*

(i) If $\mathcal{S}$ satisfies (CA) then $V = V(\pi)$ is a model of CZF^-.

(ii) If $\mathcal{S}$ satisfies (AMC) then V is a model of CZF^+.

Proof. Write $V = V(\pi)$, and recall the mutually inverse isomorphisms

$$\mathrm{Int} : \mathcal{P}_s(V) \longrightarrow V, \qquad \mathrm{Ext} : V \longrightarrow \mathcal{P}_s(V).$$

(Note that $\mathcal{P}_s(V) = \mathcal{P}_\pi(V)$ since V is separated, cf. Lemma 5.1.(i).) For a small subset $A \subseteq V$ and an element $v \in V$ we have

$$x \in A \text{ iff } x \,\epsilon\, \mathrm{Int}(A) \qquad \text{and} \qquad x \in \mathrm{Ext}(v) \text{ iff } x \,\epsilon\, v.$$

(So we could also write $\mathrm{Ext}(v) = \{w \in V : w \,\epsilon\, v\}$, $\mathrm{Int}(A) = \{a \in V : a \in A\}$; however, these are different kinds of braces: the first ones are braces of $\mathcal{E}$ coming with any $\mathcal{P}_s(X)$; the second ones are internal to the structure $(V, \epsilon\,)$. To avoid confusion, we will *only* use $\{\quad\}$ in the first sense.)

We now check the axioms 1-8.

1. (Extensionality) Since $=$ on $\mathcal{P}_s(V)$ is extensional in $\in$, the $=$ on V is extensional for ϵ (as already observed in Lemma 5.2).

2. (Pairing) Take $v, w \in V$. Since $1+1$ is small we have $\{v, w\} \in \mathcal{P}_s(V)$ (this uses Remark 3.6 and the fact that V is separated). Then $x = \mathrm{Int}(\{v, w\})$ is the required pair.

3. (Union) Take $v \in V$, with $A = \mathrm{Ext}(v)$, and let $B = \{\langle u, w\rangle : u \,\epsilon\, w \in A\}$. Then by Lemma 5.1.(ii), $\pi_2 : B \longrightarrow A$ is small (the fiber is $\mathrm{Ext}(w)$). So B is small since A is. Thus the union $\cup v$ can be constructed as $\mathrm{Int}(\mathrm{Im}(\pi_1 : B \longrightarrow V))$.

4. (Set induction) This is just the initiality of V (Proposition 6.3) in disguise. Indeed, if φ satisfies $\forall x\, (\forall y \,\epsilon\, x\, \varphi(y) \longrightarrow \varphi(x))$, then $A = \{x \in V : \varphi(x)\}$ has the property that for any small $S \subseteq A$, also $\mathrm{Int}(S) \in A$. Thus (A, Int) is a sub-$\mathcal{P}_s$-algebra of V. Since V is initial, $A = V$.

5. (Δ_0-separation) First observe that any bounded formula of CZF corresponds to an $\mathcal{S}$-bounded formula in the language of $\mathcal{E}$ (cf. 3.7), which speaks about the object V and its subobject $\epsilon \rightarrowtail V \times V$ (the image of $M \rightarrowtail W \times W$). For example, a bounded quantifier of the form $\forall y \,\epsilon\, x\, \psi(y)$ corresponds to quantification along the small map $\pi_2 : B \longrightarrow V$ where $B = \{(y, x) : y \,\epsilon\, x \wedge \psi(y)\} \subseteq V \times V$.

For any such bounded formula φ of CZF, we now need to construct a set $w \in V$ with $\forall x \in V\, (x \,\epsilon\, w \leftrightarrow \varphi(x) \wedge x \,\epsilon\, v)$. By bounded comprehension for $\mathcal{P}_s(V)$ (3.7), there is a small subset $A \subseteq V$ such that $\forall x \in V\, (x \in A \text{ iff } \varphi(x) \text{ and } x \in \mathrm{Ext}(w))$. Thus we can take $w = \mathrm{Int}(A)$.

6. (Infinity) $\mathcal{E}$ has a small natural numbers object $\mathbb{N} = W(1 \hookrightarrow 1+1)$. Define $f : \mathbb{N} \longrightarrow \mathcal{P}_s(V)$ by induction, by $f(0) = \emptyset$, and $f(n+1) = \{f(n)\} \cup f(n)$. Then $\mathrm{Int}(\mathrm{Im}(\mathrm{Int} \circ f))$ is the standard set $\omega \in V$ of natural numbers.

7. (Exponentials) Take $x, y \in V$, and consider the small subsets $X = \mathrm{Ext}(x)$ and $Y = \mathrm{Ext}(y)$ of V. Their exponential Y^X is a small object in $\mathcal{E}$. Define

$$g : Y^X \longrightarrow \mathcal{P}_s(V)$$

by $g(f) = \{\langle x, f(x)\rangle : x \in X\}$, and let $a = \mathrm{Int}(\mathrm{Im}(\mathrm{Int} \circ g))$.

8. (Strong collection) Suppose $\forall x \,\epsilon\, v\, \exists y\, \varphi(x, y)$. Take $v \in V$ and write $A = \mathrm{Ext}(v) \subseteq V$. Let $B = \{\langle x, y\rangle : \varphi(x, y) \wedge x \in A\}$. Then $\pi_1 : B \longrightarrow A$ is epi by assumption. Since A is small, the collection axiom (CA) for $\mathcal{S}$ gives the existence of a small object S and a map $\lambda : S \twoheadrightarrow A$ factoring through B, say $\lambda = \pi_1 \mu$ for $\mu : S \longrightarrow B$. Let C be the image of $\pi_2 \circ \mu$. Then C is a small subset of V (since V is separated), and $w = \mathrm{Int}(C)$ verifies the conclusion of the collection axiom.

This completes the proof of part (i) of the theorem, and it remains to discuss the validity of REA. We note that, by the usual construction of the transitive closure, any set $x \in V$

belongs to some transitive $x' \in V$, so that we may restrict our attention to transitive x. Validity of REA in the presence of (AMC) thus follows from the following two lemmas. ∎

Before stating the first lemma, we introduce some notation. For any (small) map $f : B \longrightarrow A$ between small objects, the subobject $V(f) \subseteq V$ is small, and we write

$$v(f) = \mathrm{Int}(V(f)).$$

Lemma 7.2 *(AMC) For every $x \in V$ there exists a collection map g between small objects such that $x \mathrel{\epsilon} v(g)$.*

Proof. It suffices to prove that if x is transitive then there exists such a small collection map g with $x \subseteq v(g)$, i.e. $\mathrm{Ext}(x) \subseteq V(g)$. Consider the small map

$$\mu(x) : M(x) \longrightarrow \mathrm{Ext}(x)$$

whose fiber over $y \mathrel{\epsilon} x$ is $\mathrm{Ext}(y)$. Let $g : B \longrightarrow A$ be a small collection map for which there is a quasi-pullback of the form

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "g"'] & \cdot \arrow[d, "\mu(x)"] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

Such a g exists (internally), by (AMC). We now show that

$$\forall y \mathrel{\epsilon} x : y \in V(g)$$

by induction on the elements y of the transitive set x. Suppose $y \mathrel{\epsilon} x$ and $\forall z \mathrel{\epsilon} y : z \in V(g)$. Then $\mathrm{Ext}(y) \subseteq V(g)$. Since $\mathrm{Ext}(y)$ is a fiber of $\mu(x)$, clearly $\mathrm{Ext}(y) \in \mathcal{P}_g(V(g))$, so by the algebra structure Int on $V(g)$,

$$y = \mathrm{Int}(\mathrm{Ext}(y)) \in V(g). \blacksquare$$

Lemma 7.3 *If $g : D \longrightarrow A$ is a collection map between small objects then $v(g) \in V$ is regular.*

Proof. It is easy to see that any set of the form $v(g)$ is transitive.

To see that $V(g)$ is regular, take any $b \in V(g)$ and suppose that $B = \mathrm{Ext}(b)$ has the property that

$$\forall x \in B \, \exists y \in V(g) \, \varphi(x, y).$$

We need to find a $c \in V(g)$ such that $C = \mathrm{Ext}(c)$ has the property

$$\forall x \in B \, \exists y \in C \, \varphi(x, y).$$

Since $B \in \mathcal{P}_g(V(g))$, there is a map $t : D_a \longrightarrow V(g)$ from a fiber D_a of g with B as image. Thus

$$\forall d \in D_a \, \exists y \in V(g) \, \varphi(t(d), y).$$

Since g is a collection map, there is a another fiber $D_{a'}$, a surjection $p : D_{a'} \twoheadrightarrow D_a$, and a map $s : D_{a'} \longrightarrow V(g)$ such that

$$\forall d \in D_a \, \varphi(t(d), s(d)).$$

Thus, we can take $C = \mathrm{Im}(s)$ to obtain $c = \mathrm{Int}(C)$ with the required property. ∎

8 Sheaves on Collection Sites

For the moment, let $\mathcal{E}$ be a pretopos with dependent products and W-types, and let $\mathcal{S}$ be a class of small maps. We work inside $\mathcal{E}$. Before we turn to small maps between sheaves in the next section, we need to explain how the Axiom of Multiple Choice enables us to derive some standard properties of internal sheaves. First, let us establish some conventions concerning (internal) sites. We take a site $\mathbb{C}$ to be given by a category with finite limits, and for each $C \in \mathbb{C}$ a family $\mathrm{Cov}(C)$ of covers of C. So each $S \in \mathrm{Cov}(C)$ is a family of arrows with codomain C. These are supposed to satisfy the usual conditions: trivial covers, and stability under pullback and under composition. Write $\underline{\mathrm{Cov}}(C) = \{(S, f) : f \in S \in \mathrm{Cov}(C)\}$. Then there is a commutative square

$$\begin{tikzcd}
\underline{\mathrm{Cov}} \arrow[r] \arrow[d] & \mathbb{C}_1 \arrow[d, "\mathrm{cod}"] \\
\mathrm{Cov} \arrow[r] & \mathbb{C}_0 .
\end{tikzcd} \tag{31}$$

A site $\mathbb{C}$ is said to have *small covers* if $\underline{\mathrm{Cov}} \to \mathrm{Cov}$ is a small map. A *collection site* is one in which $\underline{\mathrm{Cov}} \to \mathrm{Cov}$ is a small collection map over $\mathbb{C}_0$. A *small site* is one in which $\underline{\mathrm{Cov}} \to \mathrm{Cov} \to \mathbb{C}_0$ are small maps and $\mathbb{C}_0, \mathbb{C}_1$ are small objects. Thus, for a small site, all maps in the square (31) will be small.

Lemma 8.1 *For a collection site $\mathbb{C}$, the inclusion of presheaves into sheaves has a left adjoint, the associated sheaf functor*

$$\mathbf{a} : \mathrm{Psh}_{\mathcal{E}}(\mathbb{C}) \to \mathrm{Sh}_{\mathcal{E}}(\mathbb{C}).$$

Proof. The usual construction of $\mathbf{a}$ by the "double plus" construction applies, but some care is needed when working in a pretopos with dependent products, and one has to use repeatedly that $\mathbb{C}$ is a collection site. By way of illustration we discuss a few aspects.

Let X be a presheaf and let S be a cover of C, say $S = \{\alpha_i : C_i \to C \mid i \in I\}$. A compatible family of elements of x over S is an assignment

$$i \mapsto x_i \in X(C_i)$$

such that $x_i \cdot \pi_1 = x_j \cdot \pi_2$ for any $i, j \in I$ and the corresponding pullback

$$\begin{tikzcd}
C_i \times_C C_j \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_j \arrow[d, "\alpha_j"] \\
C_i \arrow[r, "\alpha_i"'] & C.
\end{tikzcd}$$

If $R = \{\beta_k : D_k \longrightarrow C\}_k$ is another cover, we say that R *refines* S if $\forall k \exists i (\exists \gamma : D_k \longrightarrow C_i)\ (\alpha_i \gamma = \beta_k)$. If R refines S then x induces a compatible family $x|R$ over R, by

$$(x|R)_k = x_i \cdot \gamma$$

(for given k this does not depend on the index i and the arrow γ, by the compatibility of x).

If x is a compatible family over S and y one over T, call them equivalent, $(S, x) \sim (T, y)$, if there exists a common refinement R of S and T such that $x|R = y|R$. Denote by $X^+(C)$ the "set" of equivalence classes (this construction can be done in internally in $\mathcal{E}$). If x is a compatible family over S and $\beta : D \longrightarrow C$ then one obtains a compatible family $\beta^*(x)$ over the pullback $\beta^*(S)$ in the obvious way:

$$\begin{tikzcd}
D \times_C C_i \arrow[r, "\beta_i"] \arrow[d, "\overline{\alpha}_i"'] & C_i \arrow[d, "\alpha_i"] \\
D \arrow[r, "\beta"'] & C,
\end{tikzcd} \qquad \beta^*(x)_i := x \cdot \beta_i.$$

This gives X^+ the structure of a presheaf. One now shows, as usual, that X^+ is separated, and that X^+ is a sheaf whenever X itself is separated. We only discuss the first property:

X^+ is separated: Suppose (S, x) and (T, y) represent elements of $X^+(C)$ which agree on a cover $U = \{\alpha_i : C_i \longrightarrow C \mid i \in I\}$ of C. Thus for each i there exists $R \in \mathrm{Cov}(C_i)$ such that R refines $\alpha_i^*(S)$ and $\alpha_i^*(T)$ and $\alpha_i^*(x)|R = \alpha_i^*(y)|R$. Since $\mathbb{C}$ is a collection site, there is a reindexing of the cover U, say $V = \{\alpha_j : C_j \longrightarrow C\}_{j \in J}$, such that R is given by a function R_j of $j \in J$. Now by composition, one obtains a cover from these $R_j = \{\beta_k : D_{j,k} \longrightarrow C_j\}_{k \in K_j}$, namely

$$\{\alpha_j \circ \beta_k : D_{j,k} \longrightarrow C_j \longrightarrow C\}_{j \in J, k \in K_j},$$

which refines S and T, and x and y agree on this cover. So $(S, x) \sim (T, y)$. ∎

Proposition 8.2 *For a collection site $\mathbb{C}$, the category $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ internal sheaves is a pretopos with dependent products and W-types of small maps.*

Proof. The finite limits and dependent products are those of $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$. The sums and quotients are constructed from those of $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ using the associated sheaf functor, from the previous lemma. Proposition 5.7 of [15] shows that $\mathrm{Sh}(\mathbb{C})$ is closed under the formation of W-types. An examination of its proof reveals that W-types $W(f)$ exists in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ for any small map f. ∎

Using (AMC), we can replace each site with small covers by a collection site, and deduce that the sheaves again form a pretopos:

Lemma 8.3 (AMC) *For every site $\mathbb{C}$ with small covers there is an equivalent collection site. It is obtained from $\mathbb{C}$ by taking the same underlying category and by reindexing the covers by a small collection map.*

Proof. Take a quasi-pullback

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[d] & \underline{\mathrm{Cov}} \arrow[d] \\
A \arrow[r, two heads] & \mathrm{Cov}
\end{tikzcd}$$

where $B \to A$ is a small collection map over Cov, and hence a fortiori over $\mathbb{C}_0$. Then A "reindexes" the covers of each object $C \in \mathbb{C}$, and for each $a \in A$, B_a reindexes the elements of the particular cover with new index a, so $(\mathbb{C}, B \to A, B \to \mathbb{C}_1, A \to \mathbb{C}_0)$ is really the same site with all families reindexed, and will hence define the same category of sheaves. ∎

Corollary 8.4 *Let $\mathcal{E}$ be a pretopos with dependent products and W-types, equipped with a class of small maps $\mathcal{S}$. If $\mathcal{S}$ satisfies (AMC), then for any internal site $\mathbb{C}$ in $\mathcal{E}$ with small covers, the category of internal sheaves on $\mathbb{C}$ is a pretopos with dependent products and W-types of small maps.*

We do not know whether the condition that $\mathcal{S}$ satisfies (AMC) can be dropped here.

9 Small Maps Between Sheaves

We will now turn to the construction of a class $\overline{\mathcal{S}}$ of small maps between sheaves, from the given class $\mathcal{S}$ of small maps in $\mathcal{E}$, and then show that $\overline{\mathcal{S}}$ inherits the basic properties of $\mathcal{S}$. We fix a *small* collection site $\mathbb{C}$, internal to $\mathcal{E}$. Furthermore, we assume $\mathbb{C}$ is subcanonical and write C for the representable presheaf $\mathbb{C}(-, C)$. (The assumption that $\mathbb{C}$ is subcanonical is not essential though, and one can replace everywhere the occurrence of C by the associated sheaves $\mathbb{C}(-, C)^{++}$ of representable presheaves.) Throughout this section we work with internal presheaves and sheaves over $\mathbb{C}$.

Definition 9.1 Let Y and X be sheaves. A map $f : Y \to X$ is a *small map of sheaves* iff for each $C \in \mathbb{C}$, the map $f_C : Y(C) \to X(C)$ is small. This defines the class of small maps $\overline{\mathcal{S}}$.

Remark 9.2 (i) The definition of course makes sense for presheaves as well.

(ii) If $f : Y \to X$ is a small map between sheaves, then

$$\{(\beta, y) \mid \beta : D \to C,\ y \in Y(D) \text{ and } f(y) = x \cdot \beta\}$$

is small for all C and $x \in X(C)$, since a small sum of small sets is small.

(iii) $Y \to 1$ is small iff $Y(C)$ is small for each C.

Our aim is to prove that $\overline{\mathcal{S}}$ inherits properties from $\mathcal{S}$. Certain properties are immediately verified by considering the sheaves and their maps "pointwise" or rather "objectwise", i.e. considering properties of $Y(C) \longrightarrow X(C)$ for each object C. For the rest we need a couple of lemmas and definitions. If S is a cover of C, write

$$|S| = \sum_{\alpha \in S} \mathrm{dom}(\alpha),$$

with an associated canonical map of sheaves $\pi_S : |S| \longrightarrow C$. Thus an arrow $x : |S| \longrightarrow \mathcal{A}$ between sheaves is the same as an indexed family $\{x_\alpha \in \mathcal{A}(\mathrm{dom}(\alpha))\}_{\alpha \in S}$.

Lemma 9.3 *Let $C \in \mathbb{C}$ and suppose $\mathcal{A}$ is a sheaf. If $f : \mathcal{A} \longrightarrow C$ is epi, then there exists a cover $S \in \mathrm{Cov}(C)$ and a map x making the diagram*

$$\begin{tikzcd}
{|S|} \arrow[r, "x"] \arrow[dr, "\pi_S"'] & \mathcal{A} \arrow[d, "f"] \\
 & C
\end{tikzcd}$$

commute.

Proof. Since f is epi, there exists a cover $T \in \mathrm{Cov}(C)$ such that for each $\alpha \in S$ there exists an $x \in \mathcal{A}(\mathrm{dom}(\alpha))$ with $f(x) = \alpha$. Since $\underline{\mathrm{Cov}} \longrightarrow \mathrm{Cov}$ is a collection map over $\mathbb{C}_0$, there is another cover S of C for which there exists a choice function for the quantifier combination "for each α there exists an x" above, i.e. a function x on S with $x_\alpha \in \mathcal{A}(\mathrm{dom}(\alpha))$ and $f(x_\alpha) = \alpha$ for all $\alpha \in S$. This is exactly a map $x : |S| \longrightarrow \mathcal{A}$ as required. ∎

Next we need to verify that sheafification preserves smallness in the following sense.

Lemma 9.4 *If $P \longrightarrow X$ is a small map from a presheaf into a sheaf, then $P^+ \longrightarrow X$ is again small, and hence so is the map $\mathbf{a}(P) = P^{++} \longrightarrow X$ from the associated sheaf.*

Proof. The result follows easily from the explicit description of P^+ given in Lemma 8.1. ∎

The following result is often useful for checking that a particular map is small.

Lemma 9.5 *Let $f : Y \longrightarrow X$ be a map of sheaves. Suppose that $x \in X(C)$ and that $S = \{C_i \xrightarrow{\alpha_i} C\}_i$ is a cover. If for all i and all $\beta : D \longrightarrow C_i$, the set $f_D^{-1}(x \cdot \alpha_i \beta)$ is small, then $f_C^{-1}(x)$ is small.*

Proof. The assumption implies that the set

$$B = \prod_{i \in I} f_{C_i}^{-1}(x \cdot \alpha_i)$$

is small. By the sheaf property, $f_C^{-1}(x)$ is isomorphic to $\{g \in B : g \text{ compatible}\}$. Now "$g$ compatible" is expressed by the formula

$$(\forall i, j \in I)(\forall D \in \mathbb{C}_0)(\forall \beta : D \longrightarrow C_i)(\forall \gamma : D \longrightarrow C_j)\, g(i) \cdot \beta = g(j) \cdot \gamma,$$

which is $\mathcal{S}$-bounded since the site is small and each $f_D^{-1}(x \cdot \alpha_i \beta)$ is small. Hence $f_C^{-1}(x)$ is small. ∎

The main theorem now reads as follows. We emphasize that it does not assert the existence of the W-type $W(f)$ in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ for arbitrary maps f, only for small maps.

Theorem 9.6 *Let $\mathbb{C}$ be a small collection site. Let $\overline{\mathcal{S}}$ be the class of small maps between sheaves in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ obtained from a given class $\mathcal{S}$ of small maps in $\mathcal{E}$ (cf. Definition 9.1). Then:*

(i) $\overline{\mathcal{S}}$ is a stable class of small maps in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$.

(ii) If $\mathcal{S}$ is representable, then so is $\overline{\mathcal{S}}$.

(iii) If $\mathcal{S}$ satisfies (AMC), then so does $\overline{\mathcal{S}}$.

Proof. (i): We check the conditions (S1-4), (F1-5) for small maps are satisfied by $\overline{\mathcal{S}}$ whenever they are satisfied by $\mathcal{S}$. (S1), (S4) and (F1) can easily be checked by "pointwise" reasoning. To check (S2) consider a pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, two heads, "p"'] & X \end{tikzcd}$$

with g small and p epi. Let $C \in \mathbb{C}$ and $x \in X(C)$. Since p is epi there is a cover $\{C_i \xrightarrow{\alpha_i} C\}_{i \in I}$, and $x_i \in X(C_i)$ so that $p(x_i) = x \cdot \alpha_i$ for all $i \in I$. For $\beta : D \longrightarrow C_i$ we have $p(x_i \cdot \beta) = p(x_i) \cdot \beta = x \cdot \alpha_i \beta$. Now since

$$g_C^{-1}(x_i \cdot \beta) \cong f_C^{-1}(p(x_i \cdot \beta)),$$

and $g_C^{-1}(x_i \cdot \beta)$ is small, $f_C^{-1}(x \cdot \alpha_i \beta)$ is small. Thus $f_C^{-1}(x)$ is small by Lemma 9.5.

Condition (S3) is checked by a similar use of Lemma 9.5.

Conditions (F2) and (F3) are verified by first making the constructions in presheaves, and then applying Lemma 9.4. By inspecting the arguments in [15] one easily sees that (F4) and (F5) holds for $\overline{\mathcal{S}}$.

(ii): We continue to work with a small site, and discuss the representability axiom for small maps. In fact, one can deduce the representability of the class $\overline{\mathcal{S}}$ in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ from that of $\mathcal{S}$ in $\mathcal{E}$, exactly as in [10, p. 91]. To see this, suppose $\mathcal{S} = \mathcal{S}(\pi)$ where $\pi : E \longrightarrow U$ is in $\mathcal{E}$. Define a sheaf (a sum of representables)

$$\underline{U} = \sum_C \sum_f \sum_R C \tag{32}$$

as follows: C ranges over all objects of $\mathbb{C}$. For a given C, f in (32) ranges over all small families

$$f = \{f_i : D_i \longrightarrow C\}_{i \in I}$$

of arrows into C (not necessarily covers). Write

$$|f| = \sum_{i \in I} D_i$$

for such a family. Next, given C and f, R in (32) ranges over small families of arrows

$$R = \left\{E_k \longrightarrow |f| \times_C |f| = \sum_{i,j} D_i \times_C D_j\right\}_k,$$

with $|R| = \sum E_k$ as before, such that the image of $|R|$ defines an equivalence relation on $|f|$ (contained in $|f| \times_C |f| \subseteq |f| \times |f|$).

Thus, for each such C, R, f we have a coequalizer, mapping into C:

$$\begin{tikzcd}
{|R|} \arrow[r, shift left] \arrow[r, shift right] & {|f|} \arrow[r] & {|f|/|R| =: |f, R|} \arrow[r, "\pi_{f,R}"'] & C,
\end{tikzcd}$$

and we can define a sheaf $\underline{E}$ over $\underline{U}$ by letting $\underline{E} = \sum_C \sum_f \sum_R |f, R|$ and $\underline{\pi} : \underline{E} \longrightarrow \underline{U}$ be the map given by $\pi_{f,R}$ as above on each summand indexed by C, f, R.

Since each $|f, R|$ is a small sheaf, $\underline{\pi}$ is a small map of sheaves. We claim it is universal. To see this, take a small map $g : Y \longrightarrow X$ of sheaves, and consider the "canonical" quasi-pullback

$$\begin{tikzcd}
\sum_C \sum_x \sum_{\gamma,y} D \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
\sum_{C,x} C \arrow[r, two heads] & X
\end{tikzcd}$$

where C, x range over all $C \in \mathbb{C}$ and $x \in X(C)$, and for given C, x, the sum $\sum_{\gamma,y}$ ranges over the small set

$$g_{C,x} = \{(\gamma : D \longrightarrow C, y) \mid y \in Y(D), g(y) = x \cdot \gamma\}.$$

For a given C and $x \in X(C)$, also consider R given by

$$\{E \xrightarrow{(\alpha,\beta)} D \times_C D' \mid (D \xrightarrow{\gamma} C, y) \text{ and } (D' \xrightarrow{\gamma'} C, y') \text{ belong to } g_{C,x} \text{ and } y \cdot \alpha = y' \cdot \beta\}.$$

Then with the notation $|\cdot|$ for the sum, we get a coequalizer

$$\begin{tikzcd}
{|R|} \arrow[r, two heads] & {|g_{C,x}| \times_{Y_x} |g_{C,x}|} \arrow[r, shift left] \arrow[r, shift right] & {|g_{C,x}|} \arrow[r, two heads] & Y_x
\end{tikzcd}$$

where $Y_x = C \times_X Y$. This now gives a double pullback

$$\begin{tikzcd}
\underline{E} \arrow[d] & Y_x \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\
\underline{U} & C \arrow[l] \arrow[r, "x"] & X
\end{tikzcd}$$

where $C \longrightarrow \underline{U}$ is the inclusion of the summand given by $g_{C,x}$ and R as above. Taking the sum over all C and x gives a double pullback as required for the representability axiom. This proves that the class $\overline{\mathcal{S}}$ is representable if $\mathcal{S}$ is.

(iii): This will be proved in the next section. ∎

10 The Axiom of Multiple Choice for Sheaves

This section is devoted to the proof of Theorem 9.6.(iii). First, we make some preparatory remarks. Throughout this section we assume $\mathbb{C}$ is a small collection site.

Remark 10.1 Suppose that $\mathcal{T} \longrightarrow \mathcal{R}$ is a map between sheaves. Then this map is a collection map (internally in $\mathcal{E}$) iff for any $D \in \mathbb{C}$ and $\alpha : D \longrightarrow \mathcal{R}$, and for any epi $p : \mathcal{X} \twoheadrightarrow \mathcal{T}_\alpha = \mathcal{T} \times_{\mathcal{R}} D$, there exists a cover $\{\gamma_k : D_k \longrightarrow D\}_k$ of D, and for each index k a map $\beta_k : D_k \longrightarrow \mathcal{R}$ and a factorisation

$$\begin{tikzcd}
 & \mathcal{X}_{\gamma_k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{X} \arrow[d] \\
\mathcal{T}_{\beta_k} \arrow[ur] \arrow[r, two heads] & \mathcal{T}_{\alpha\gamma_k} \arrow[r] & \mathcal{T}_\alpha .
\end{tikzcd}$$

Since $\mathbb{C}$ is a collection site this property can equivalently be phrased as: there exists a cover R of D and maps $\beta : |R| \longrightarrow \mathcal{R}$ and $x : |\mathcal{T}_\beta| \longrightarrow \mathcal{X}$, for which

$$\begin{tikzcd}
\mathcal{T} \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{T}_\beta \arrow[l] \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[r, two heads, "p"] \arrow[d, phantom, "\text{q.p.b.}"] & \mathcal{T}_\beta \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{T} \arrow[d] \\
\mathcal{R} & {|R|} \arrow[l, "\beta"'] \arrow[rr, two heads, "\pi_R"'] & & D \arrow[r, "\alpha"'] & \mathcal{R}
\end{tikzcd} \tag{33}$$

commutes, and $p \circ x : \mathcal{T}_\beta \longrightarrow \mathcal{T}_\alpha$ fits into a quasi-pullback, as indicated.

We will now prove the main result of this section, thus verifying (AMC).

Proposition 10.2 *For any small map $\mathcal{B} \longrightarrow \mathcal{A}$ between sheaves, there exists a quasi-pullback*

$$\begin{tikzcd}
\mathcal{T} \arrow[rr] \arrow[d] & & \mathcal{B} \arrow[d] \\
\mathcal{R} \arrow[r, two heads] & \mathcal{A}' \arrow[r, two heads] & \mathcal{A}
\end{tikzcd} \tag{34}$$

where $\mathcal{T} \longrightarrow \mathcal{R}$ is a small collection map over $\mathcal{A}'$, and $\mathcal{R} \longrightarrow \mathcal{A}'$ is small as well.

Proof. Consider the canonical epi $\mathcal{A}' \twoheadrightarrow \mathcal{A}$ where $\mathcal{A}$ is the sum of representables $\mathbb{C}(-,C)$ indexed by all pairs (a,C) where $a \in \mathcal{A}(C)$ and $C \in \mathbb{C}_0$. By pulling back $\mathcal{B} \longrightarrow \mathcal{A}$ along $\mathcal{A}' \twoheadrightarrow \mathcal{A}$, and by using Remark 4.2.(iv), we find that it is enough to prove the proposition in the case where $\mathcal{A}$ is a representable sheaf C. Next, by replacing the small site $\mathbb{C}$ by $\mathbb{C}/C$, we find that it is in fact enough to assume that $\mathcal{A} = 1$, and to construct a quasi-pullback of the form

$$\begin{tikzcd}
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{R} \arrow[r, two heads] & 1
\end{tikzcd}$$

where $\mathcal{T} \longrightarrow \mathcal{R}$ is a collection map between small sheaves. To this end, take any small sheaf $\mathcal{B}$ in the ambient category $\mathcal{E}$, and use (AMC) in $\mathcal{E}$ to obtain a quasi-pullback of small sets

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \sum_C (\mathcal{B}/C)_0 \arrow[d] \\
L \arrow[r, two heads] & \sum_C 1 = \mathbb{C}_0
\end{tikzcd} \tag{35}$$

where $M \longrightarrow L$ is a small collection map over $\mathbb{C}_0$. Here we have written $(\mathcal{B}/C)_0 = \{(\beta, b) \,|\, \beta : D \longrightarrow C, b \in \mathcal{B}(D)\}$.

For a given $\ell \in L$, define $\mathrm{Cov}(\ell)$ to be the collection of M_ℓ-indexed families

$$F = \{(S_i, E_i \xrightarrow{\alpha_i} C, |S_i| \xrightarrow{b_i} \mathcal{B}) : i \in M_\ell\} \tag{36}$$

where $C = \pi(\ell)$, $\alpha_i : E_i \longrightarrow C$ is an arrow in $\mathbb{C}$, S_i is a cover of E_i, and the canonical map

$$|F| := \sum_{i \in M_\ell} |S_i| \longrightarrow C \times \mathcal{B} \tag{37}$$

is epi.

We emphasize that $\mathrm{Cov}(\ell)$ is a small set. This can be seen by writing out the condition that this last map (37) is epi, explicitly in terms of the site: it will only involve quantifiers over small sets because the site $\mathbb{C}$ and the sheaf $\mathcal{B}$ are assumed to be small.

Now define

$$\begin{aligned}
\mathcal{R} &= \sum_{\ell \in L} \sum_{F \in \mathrm{Cov}(\ell)} \pi(l), \\
\mathcal{T} &= \sum_{\ell \in L} \sum_{F \in \mathrm{Cov}(\ell)} |F|,
\end{aligned}$$

and notice that these sheaves are small.

Lemma 10.3 *There is a quasi-pullback of small sheaves*

$$\begin{tikzcd}
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{R} \arrow[r, two heads] & 1
\end{tikzcd} \tag{38}$$

with $\mathcal{R} \twoheadrightarrow 1$ *epi.*

Proof. The map $\mathcal{T} \longrightarrow \mathcal{R}$ is the canonical map, obtained as the sum of the maps $|F| = \sum_{i \in M_\ell} |S_i| \longrightarrow C$, for $F = \{(S_i, E_i \longrightarrow C, b_i)\}_i$ as in (36). The square (38) is the "sum" of squares

$$\begin{tikzcd}
{|F|} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
C \arrow[r, two heads] & 1
\end{tikzcd}$$

ranging over all ℓ and $F \in \mathrm{Cov}(\ell)$, where $C = \pi(\ell)$. Each of these squares is a quasi-pullback (simply by the assumption that (37) is epi), hence so is the sum (38).

It remains to show that $\mathcal{R} \longrightarrow 1$ is epi. In fact, we will show that for each $C \in \mathbb{C}$, there exists an $\ell \in L$ and an $F \in \mathrm{Cov}(\ell)$ so that $\pi(\ell) = C$. Now surely there is an $\ell \in L$ with $\pi(\ell) = C$ because $L \longrightarrow \mathbb{C}_0$ is epi (cf. (35)). The diagram (35) also provides for this ℓ a surjection

$$\rho_\ell : M_\ell \longrightarrow (\mathcal{B}/C)_0.$$

Write

$$\rho_\ell(i) = (E_i \xrightarrow{\alpha_i} C, b_i),$$

for each $i \in M_\ell$, and let $F \in \mathrm{Cov}(\ell)$ be the family which assigns to each i the trivial cover of this E_i,

$$F = \{(\{1_{E_i}\}, E_i \xrightarrow{\alpha_i} C, b_i)\}_{i \in M_\ell}.$$

Then F indeed belongs to $\mathrm{Cov}(\ell)$, in fact $|F| \longrightarrow C \times \mathcal{B}$ is already an epi of presheaves. This proves the lemma. ∎

Lemma 10.4 *The map* $\mathcal{T} \longrightarrow \mathcal{R}$ *is a collection map of sheaves.*

Proof. We use the description of the collection maps of sheaves given in Remark 10.1. Take any map $\alpha : D \longrightarrow \mathcal{T}$ and an epi

$$p : \mathcal{X} \twoheadrightarrow \mathcal{T}_\alpha = \mathcal{T} \times_{\mathcal{R}} D.$$

By moving to a cover of D, we may assume that α maps into a summand C of $\mathcal{R}$ indexed by ℓ and F. Write this F as

$$F = \{(S_i, E_i \xrightarrow{\alpha_i} C, |S_i| \xrightarrow{b_i} \mathcal{B})\}_{i \in M_\ell}$$

as before. Thus $C = \pi(\ell)$, and

$$\mathcal{T}_\alpha = |F| = \sum_{i \in M_\ell} |S_i|,$$

so that $p : \mathcal{X} \longrightarrow \mathcal{T}_\alpha$ is a sum of epis

$$p_i : \mathcal{X}_i \twoheadrightarrow |S_i| \qquad (i \in M_\ell).$$

Since $\mathbb{C}$ is a collection site by assumption, we find for each $(\gamma : D \longrightarrow E_i) \in S_i$ a cover R of D and a factorization

$$\begin{tikzcd}
{|R|} \arrow[r, "x"] \arrow[d, "\pi_R"'] & \mathcal{X} \arrow[d] & \\
D \arrow[r, tail] & {|S_i|} \arrow[r, "\pi_{S_i}"'] & E_i
\end{tikzcd}$$

Since $\mathbb{C}$ is a collection site (again), there is a reindexing T of the cover S_i so that R and x are given as a function of $\gamma \in T$, say R_γ, x_γ. Now $R = \{\gamma \circ \delta \mid \gamma \in T, \delta \in R\}$ is also a cover of E_i (by composition of covers), and it is a refinement of S_i; in fact there are epis

$$|R| \twoheadrightarrow |T| \twoheadrightarrow |S_i| \twoheadrightarrow |E_i|.$$

Also, the $\{x_\gamma : \gamma \in T\}$ paste together to a map x as in

$$\begin{tikzcd}
{|R|} \arrow[r, "x"] \arrow[d] & \mathcal{X}_i \arrow[d] \\
{|T|} \arrow[r, two heads] & {|S_i|}
\end{tikzcd} \tag{39}$$

So we have proved:

(*) For each $i \in M_\ell$ there exists a cover R of E_i, refining S_i by an epi $|R| \longrightarrow |S_i|$, for which there is a commutative square (39).

Since $M \longrightarrow L$ is a collection map over $\mathbb{C}_0$, we can find another $\ell' \in L$, with $\pi(C') = C$ also, and an epi $\tau : M_{\ell'} \twoheadrightarrow M_\ell$, so that R and x in (*) are given by functions $R_{i'}$ and $x_{i'}$ depending on $i' \in M_{\ell'}$. So for each i' there is a cover $R_{i'}$ of $E_{\tau(i')}$, and $x_{i'}$ fits into a square

$$\begin{tikzcd}
{|R_{i'}|} \arrow[r, "x_{i'}"] \arrow[d] & \mathcal{X}_{\tau(i')} \arrow[d] \\
{|T|} \arrow[r, two heads] & {|S_{\tau(i')}|}
\end{tikzcd} \tag{40}$$

Now define

$$F' = \{(R_{i'}, E_{\tau(i')}, b_{\tau(i')})\}_{i' \in M_{\ell'}}.$$

Then F' belongs to $\mathrm{Cov}(\ell')$, so (ℓ, F') is the index of a summand C of $\mathcal{R}$. Thus $\alpha : D \longrightarrow C$ (the map we started out with) defines a map $\alpha' : D \longrightarrow \mathcal{R}$ also and we obtain a diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{T}_{\alpha'} \arrow[l] \arrow[r, two heads, "\tilde{\tau}"'] \arrow[d] & \mathcal{T}_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{T} \arrow[d] \\
\mathcal{R} & D \arrow[l] \arrow[r, two heads, "1"'] & D \arrow[r, "\alpha"'] & \mathcal{R}
\end{tikzcd} \tag{41}$$

on top of which is the following triangle

$$\begin{tikzcd}
 & \sum_{i\in M_\ell} \mathcal{X}_i \arrow[d] \\
\sum_{i'\in M_{\ell'}} |R_{i'}| \arrow[ur, "\tilde{x}"] \arrow[r, two heads, "\tilde{\tau}"'] & \sum_{i'\in M_\ell} |R_i|
\end{tikzcd}$$

here $\tilde{\tau} : \mathcal{T}_{\alpha'} \longrightarrow \mathcal{T}_\alpha$ sends $|R_{i'}|$ to $|S_{\tau(i')}|$ and $\tilde{x}$ sends $|R'_i|$ into $\mathcal{X}_{\tau(i')}$ as in (40). Since τ is surjective and each $|R_{i'}| \longrightarrow |S_{\tau(i')}|$ is epi, so is the map $\tilde{\tau}$. Thus the middle square in (41) is a quasi-pullback, and the proof is complete. ∎

By these lemmas we have proved Proposition 10.2, and hence completed the proof of Theorem 9.6.

11 Stratifications

In this section we introduce the notion of stratified pseudotopos, which is a predicative analogue of elementary topos, in that it enjoys closure under the internal sheaves (Theorem 11.2). Stratified pseudotoposes also arise naturally from Martin-Löf type theory (Section 12).

Let $\mathcal{E}$ be a pretopos with dependent products and W-types. A *filtration* of $\mathcal{E}$ is a sequence of subcategories

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots$$

of $\mathcal{E}$ with the property that $\mathcal{E} = \cup_{n\geq 0} \mathcal{S}_n$. We will consider such filtrations where each $\mathcal{S}_n$ is a class of small maps, satisfying (AMC) as described in previous sections. If $\mathcal{S} \subseteq \mathcal{S}'$ are two classes of small maps, we say that $\mathcal{S}$ is *properly contained* in $\mathcal{S}'$ (notation: $\mathcal{S} \leq \mathcal{S}'$) if there exists a representing map $\pi : E \longrightarrow U$ for $\mathcal{S}$ (i.e. $\mathcal{S} = \mathcal{S}(\pi)$) for which $U \longrightarrow 1$ belongs to $\mathcal{S}'$, i.e. U is an $\mathcal{S}'$-small object.

Definition 11.1 A *(representable) stratification* of $\mathcal{E}$ is a filtration of $\mathcal{E}$ by (representable) classes of small maps $(\mathcal{S}_n)$ such that

$$\mathcal{S}_0 \leq \mathcal{S}_1 \leq \mathcal{S}_2 \leq \cdots \qquad \text{and} \qquad \mathcal{E} = \cup_n \mathcal{S}_n.$$

A *pseudotopos* is a pretopos $\mathcal{E}$ with dependent products and W-types for which such a stratification exists. A pair $(\mathcal{E}, (\mathcal{S}_n)_n)$ consisting of a pseudotopos and an explicitly given stratification will be referred to as a *stratified pseudotopos.*

Note that this notion of a stratified pseudotopos is stable under slicing, for if $(\mathcal{S}_n)$ is a stratification of $\mathcal{E}$ then $(\mathcal{S}_n/X)$ is one of $\mathcal{E}/X$.

Theorem 11.2 *If $\mathcal{E}$ is a stratified pseudotopos, and $\mathbb{C}$ is a collection site in $\mathcal{E}$, then* $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$ *is a stratified pseudotopos.*

Proof. Let $(\mathcal{S}_n)$ be a stratification of $\mathcal{E}$. By reindexing this stratification, if necessary, we may assume that $\mathbb{C}$ is a $\mathcal{S}_0$-small collection site. Let $\overline{\mathcal{S}}_n$ be the class of $\mathcal{S}_n$-small sheaf maps. According to Theorem 9.6, each $\overline{\mathcal{S}_n}$ is a stable representable class of small maps in $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$, which satisfies AMC. By the construction of $\overline{\mathcal{S}}$ from $\mathcal{S}$ it is clear that $\mathcal{S}_n \leq \mathcal{S}_{n+1}$ implies $\overline{\mathcal{S}}_n \subseteq \overline{\mathcal{S}}_{n+1}$. To see that the inclusion is proper, we need only to inspect the construction of the representing map $\underline{\pi} : \underline{E} \longrightarrow \underline{U}$ for $\overline{\mathcal{S}_n}$. The sheaf $\underline{U}$ is constructed in (32) by twice taking sums over the collection all $\mathcal{S}_n$-small families of arrows, and then sheafifying. Thus by using Lemma 9.4 one sees that $\underline{U}(D)$ is in $\mathcal{S}_{n+1}$. Hence $\underline{U} \longrightarrow 1 \in \mathcal{S}_{n+1}$, proving that the inclusion is proper. Also $\cup_n \overline{\mathcal{S}}_n = \mathrm{Sh}_{\mathcal{E}}(\mathbb{C})$, since each sheaf is given by a map in $\mathcal{E}$ and $\cup_n \mathcal{S}_n = \mathcal{E}$. ∎

Remark 11.3 Suppose $\mathcal{S} \leq \mathcal{S}'$ and $\pi' : E' \longrightarrow U'$ is a universal $\mathcal{S}'$-small map, so that $\mathcal{S}' = \mathcal{S}(\pi)$. Then there exists a double pullback

$$\begin{tikzcd}
U \arrow[d] & V \arrow[l] \arrow[r] \arrow[d] & E' \arrow[d] \\
1 & T \arrow[l, two heads] \arrow[r] & U'
\end{tikzcd} \tag{42}$$

In many examples, related to type theory, 1 is projective, so the epi $T \twoheadrightarrow 1$ splits and we obtain a pullback of the form

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & E' \arrow[d] \\
1 \arrow[r] & U'
\end{tikzcd} \tag{43}$$

12 Relation to Type Theory

In this section we give a predicative, constructive example of a stratified pseudotopos by building such a category inside one of Martin-Löf's type theories (cf. Theorem 12.7). This pseudotopos, **Sets**, plays a similar fundamental role as the ordinary category of sets.

We recall some background from our previous paper [15]. In Martin-Löf type theory [13] the category of sets, denoted **Sets**, is naturally defined to be the category of types (or presets) with equivalence relations and functions preserving these equivalences. The basic type theory of Martin-Löf consists of rules for Σ- and Π-types, disjoint sum-type $(+)$, natural numbers $\mathbb{N}$, the canonical finite sets $\mathbb{N}_k = \{0, \ldots, k-1\}$, and the (intensional) identity type. (See Troelstra [21] for a discussion about the relation between these basic axioms.) We will consider an extension $\mathbf{ML}_{<\omega}\mathbf{W}$ of this basic theory with W-types (see [13, 15]) and an infinite, cumulative sequence of universes. For each external natural number n there is a family of types $\mathbb{T}_n(x)$ $(x \in \mathbb{U}_n)$ called a *universe.* The type $\mathbb{U}_n$ is to be thought of as a collection of *codes* for types, and $\mathbb{T}_n(a)$ is the type corresponding to the code $a \in \mathbb{U}_n$. That the sequence is cumulative means that for each type A there is some external index n and some $a \in \mathbb{U}_n$ such that $A = \mathbb{T}_n(a)$. Moreover we have an embedding function $\mathsf{t}_n : \mathbb{U}_n \longrightarrow \mathbb{U}_{n+1}$ and a constant $\mathsf{u}_n \in \mathbb{U}_{n+1}$ satisfying the equations

$$\mathbb{T}_{n+1}(\mathsf{t}_n(a)) = \mathbb{T}_n(a), \qquad \mathbb{T}_{n+1}(\mathsf{u}_n) = \mathbb{U}_n.$$

In the first universe $\mathbb{U}_0, \mathbb{T}_0$ there are codes for the basic types $\mathbb{N}$ and $\mathbb{N}_k$, $k = 0, 1, 2, \ldots$, i.e. there are constants $\mathsf{n}, \mathsf{n}_k \in \mathbb{U}_0$ with $\mathbb{T}_0(\mathsf{n}) = \mathbb{N}$ and $\mathbb{T}_0(\mathsf{n}_k) = \mathbb{N}_k$. Moreover, in each universe $\mathbb{U}_n, \mathbb{T}_n$ there are codes for type constructions Σ, Π, $\mathbb{W}$ and Id, that is for $a \in \mathbb{U}_n$ and $b(x) \in \mathbb{U}_n$ $(x \in \mathbb{T}_n(a))$, we have

$$\hat{\Sigma}(a, b) \in \mathbb{U}_n, \qquad \mathbb{T}_n(\hat{\Sigma}(a, b)) = (\Sigma x \in \mathbb{T}_n(a))\, \mathbb{T}_n(b(x)),$$

and similarly for Π and $\mathbb{W}$. For identity types we have for each $a \in \mathbb{U}_n$ and $b, c \in \mathbb{T}_n(a)$ a code $\hat{\mathrm{Id}}(a, b, c) \in \mathbb{U}_n$ and its decoding as identity type

$$\mathbb{T}_n(\hat{\mathrm{Id}}(a, b, c)) = \mathrm{Id}(\mathbb{T}_n(a), b, c).$$

We shall frequently use the propositions-as-types principle of type theory. It states that a type A can be regarded as a proposition, in which case we say that A is *true* if there is some element $a \in A$. Conversely, each proposition is also a type (of its proof objects). We introduce some useful notation. An object A of **Sets** is written $(\overline{A}, =_A)$ where the type is $\overline{A}$ and $=_A$ is the equivalence relation on $\overline{A}$. If $P(x)$ is a property of A that is preserved under the equality $=_A$, we use

$$\{x \in A \mid\mid P(x)\}$$

to denote the set $((\Sigma x \in \overline{A})P(x), \sim)$ with the equivalence $\sim$ given by $(x, p) \sim (x', p')$ iff $x =_A x'$. For a map $f : B \longrightarrow A$ in **Sets**, let $f^{-1}(a)$ denote the set $\{x \in B \mid\mid f(x) =_A a\}$, the *fiber of f over a*. We define the *discrete category* $A^{\#}$ corresponding to the set $A = (\overline{A}, =_A)$ by letting the collection of objects be $\overline{A}$ and the set of morphisms from a to b be the type $a =_A b$ together with an equality which identifies all morphisms. The proof objects for reflexivity, symmetry and transitivity then becomes the identity arrow, the inverse operation and the composition, respectively. The discrete category is thus a groupoid. We extend f^{-1} to a functor $A^{\#} \longrightarrow$ **Sets** by letting, for $p : (a =_A a')$, $f^{-1}(p) : f^{-1}(a) \longrightarrow f^{-1}(a')$ be the unique map $(x, p') \mapsto (x, p'')$.

In [15] we proved the following result.

Theorem 12.1 *In the type theory* $\mathbf{ML}_{<\omega}\mathbf{W}$*, the category* **Sets** *is a pretopos with dependent products and W-types.* ∎

The purpose of the remainder of the section is to strengthen this theorem by showing that the category **Sets** is a stratified pseudotopos within the same type theory; see Theorem 12.7 below. For each $n < \omega$, let $\mathbf{Sets}_n$ be the full subcategory of **Sets** where the objects are sets $A = (\overline{A}, =_A)$ and where $\overline{A} = \mathbb{T}_n(a)$, for some $a \in \mathbb{U}_n$, and $x =_A y$ is of the form $\mathbb{T}_n(e(x, y))$ for some $e \in (\mathbb{T}_n(a \times a) \longrightarrow \mathbb{U}_n)$. Clearly

$$\mathbf{Sets}_0 \subseteq \mathbf{Sets}_1 \subseteq \mathbf{Sets}_2 \subseteq \cdots$$

and since the hierarchy $\mathbb{U}_n, \mathbb{T}_n$ is cumulative, every set belongs to some $\mathbf{Sets}_n$. Let $\mathcal{S}_n$ be the class of maps $f : B \longrightarrow A$ in **Sets** such that for every $a \in A$ there is some set S in $\mathbf{Sets}_n$ and an isomorphism $\varphi : f^{-1}(a) \longrightarrow S$. Below we first show that **Sets** has enough (internal) projectives (Lemma 12.3) and then that $\mathcal{S}_n$ is representable by a universal map $\pi_{(n)} : E_n \longrightarrow U_n$ (Lemma 12.4). Next, in Lemma 12.5 we show that $\mathcal{S}_n$ is closed under composition and is locally full. Finally, Lemma 12.6 verifies the fiber axioms (F1–5).

To construct the universal map for $\mathcal{S}_n$ we use the universe $\mathbb{U}_n$, $\mathbb{T}_n$ and the identity type. We first take a closer look at the latter construction.

Identity types and projective objects. For any type A and any elements $a, b \in A$ we can in type theory form the (intensional) identity type $\mathrm{Id}_A(a, b)$ (also written $\mathrm{Id}(A, a, b)$). Its intended interpretation is as the type of proofs (if any) that a and b are identical. The introduction rule is

$$(\mathrm{Id} - \text{intro.}) \quad \frac{a \in A}{\mathsf{r}(a) \in \mathrm{Id}_A(a, a)},$$

and the elimination rule is

$$(\mathrm{Id} - \text{elim.}) \quad \frac{c \in \mathrm{Id}_A(a, b) \qquad \begin{array}{c}(x \in A)\\ \vdots \\ d(x) \in C(x, x, \mathsf{r}(x))\end{array}}{\mathsf{J}(c, d) \in C(a, b, c)}$$

where $C(x, y, z)$ is a family of types with $x, y \in A$, $z \in \mathrm{Id}_A(x, y)$, which we call the *eliminating family.* The computation rule connecting the two is

$$\mathsf{J}(\mathsf{r}(a), d) = d(a).$$

M. Hofmann and T. Streicher [8] discovered that the identity type induces a groupoid structure on each type. (We refer to Streicher [20] for a thorough investigation of identity types.) We define for each type A an associated groupoid $A^{\equiv}$. Let A be the objects of $A^{\equiv}$ and for any $a, b \in A$, let $\mathrm{Id}_A(a, b)$ be the type of morphisms from a to b. Two such morphisms p and q are considered equal if

$$\mathrm{Id}(\mathrm{Id}_A(a, b), p, q)$$

is true. The identity arrow for $a \in A$, is $1_a = \mathsf{r}(a) \in \mathrm{Id}_A(a, a)$. For $p \in \mathrm{Id}_A(a, b)$ and $q \in \mathrm{Id}_A(b, c)$ we define the composition by

$$q \circ p = \mathrm{Ap}(\mathsf{J}(p, t), q) \in \mathrm{Id}_A(a, c)$$

where $t(x) = \lambda u.u \in (\mathrm{Id}_A(x,c) \longrightarrow \mathrm{Id}_A(x,c))$ and where $C(x,y,z) \equiv \mathrm{Id}_A(y,c) \longrightarrow \mathrm{Id}_A(x,c)$ is the eliminating family. The computation rule gives immediately $q \circ 1_b = q$. The other axioms for a category are verified using Id-elimination. The inverse p^{-1} of $p \in \mathrm{Id}_A(a,b)$ is given by $p^{-1} = \mathsf{J}(p, \mathsf{r})$ where the eliminating family is $C(x,y,z) \equiv \mathrm{Id}_A(y,x)$. We have directly $1_a^{-1} = 1_a$. That p^{-1} is the inverse of p is checked using Id-elimination. We summarize this as a lemma.

Lemma 12.2 (Hofmann, Streicher) *For each type A, the structure $A^{\equiv}$ is a groupoid.* ∎

The category **Presets** is the full subcategory of **Sets** determined by the objects of the form $(A, \mathrm{Id}_A(\cdot,\cdot))$. We often just write A for such an object. The full subcategory given by the objects of this form in $\mathbf{Sets}_n$ is denoted $\mathbf{Presets}_n$. To any family of types $B(x)$ $(x \in A)$ there is a functor associated

$$F = F_{A,B} : A^{\equiv} \longrightarrow \mathbf{Presets} \tag{44}$$

defined as follows. Let $F(a) =_{\mathrm{def}} B(a)$ and for $p \in \mathrm{Id}_A(a,b)$, put $F(p) =_{\mathrm{def}} \mathsf{J}(p,t)$ where $t(x) = \lambda u.u \in (B(x) \longrightarrow B(x))$ and the eliminating family $C(x,y,z)$ is $B(x) \longrightarrow B(y)$. It is immediate that $F(1_a) = 1_{B(a)}$. To prove functoriality one uses (Id-elim) for the family $C(a,b,q)$ given by

$$(\forall p \in \mathrm{Id}_A(b,c))\,(\forall x \in B(a))\,\mathrm{Id}_{B(c)}(F(p)(F(q)(x)), F(p \circ q)(x)).$$

We may view the family $B(x)$ $(x \in A)$ as fibers of a map in the following way. Form the type $S = (\Sigma x \in A)B(x)$. Let $\pi_1 : (S, \mathrm{Id}_S(\cdot,\cdot)) \longrightarrow (A, \mathrm{Id}(\cdot,\cdot))$ be the first projection. Then for each $u \in A$ there is an isomorphism

$$\varphi_u : \pi_1^{-1}(u) \longrightarrow (B(u), \mathrm{Id}_{B(u)}(\cdot,\cdot))$$

given by $\varphi_u((x,y),p) = F(p)(y)$, where $p \in \mathrm{Id}_A(x,u)$.

Let $\mathcal{P}_n$ be the class of maps $f : B \longrightarrow A$ in **Presets** such that for each $x \in A$ there exists S in $\mathbf{Presets}_n$ and an isomorphism $g : f^{-1}(x) \longrightarrow S$.

Lemma 12.3 *(i) For every $f : B \longrightarrow A$ in $\mathcal{S}_n$ there exists a quasi-pullback*

$$\begin{tikzcd} Q \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "f"] \\ P \arrow[r, two heads, "m"'] & A \end{tikzcd} \tag{45}$$

where h belongs to $\mathcal{P}_n$ and m is epi. Moreover if A is a terminal, then P can be chosen to be terminal.

(ii) Each $h : Q \longrightarrow P$ in $\mathcal{P}_n$ is internally projective in $\mathbf{Sets}/P$.

Proof. (i) Suppose that $f : B \longrightarrow A$ is in $\mathcal{S}_n$. By the usual choice principle for pure types we find $S_x \in \mathbf{Sets}_n$ and $g_x : \overline{S_x} \longrightarrow f^{-1}(x)$ so that g_x is an isomorphism for each $x \in \overline{A}$. Let $P = (\overline{A}, \mathrm{Id}_{\overline{A}}(\cdot,\cdot))$. Put $\overline{Q} = (\Sigma x \in \overline{A})\overline{S}_x$, $Q = (\overline{Q}, \mathrm{Id}_{\overline{Q}}(\cdot,\cdot))$ and $h = \pi_1$. By the paragraph preceeding this lemma we have $h^{-1}(x) \cong (\overline{S}_x, Id_{\overline{S}_x})$, so h is in $\mathcal{P}_n$. Define $k : Q \longrightarrow B$ by $k(x,y) = \pi_1(g_x(y))$, and $m : P \longrightarrow A$ by $m = \lambda x.x$. These functions are trivially well-defined since both P and Q have identity as equality. The map m is also trivially epi. We check that the square (45) is a quasi-pullback. Let (p,b) be an element of the canonical pullback $P \times_A B$. Then $p \in P$, $b \in B$ and $m(p) = p =_A f(b)$. Thus $b' = (b,q) \in f^{-1}(p)$ for some q. The canonical map from Q to $P \times_A B$ is (h,k) and we have $(h,k)(p, g_p^{-1}(b')) = (p, \pi_1(g_p(g_p^{-1}(b')))) = (p, \pi_1(b')) = (p,b)$. The map (h,k) is thus epi, since (p,b) was arbitrarily chosen. If A is a terminal, then it is isomorphic to the canonical one element set $(\mathbb{N}_1, \mathrm{Id}_{\mathbb{N}_1}(\cdot,\cdot))$, which can be taken to be P.

(ii) Let $h : Q \longrightarrow P$ be an arrow in $\mathcal{P}_n$. Suppose that $t : T \longrightarrow P$, $r : X \longrightarrow P$ and $s : Y \longrightarrow P$ are objects of $\mathbf{Sets}/P$ and that $f : T \times_P Q \longrightarrow X$ and $k : Y \longrightarrow X$ are arrows of the same slice, i.e. $rf = t\pi_1$ and $rk = s$. Moreover, suppose that k is epi in $\mathbf{Sets}/P$. Thus $(\forall x \in \overline{X})\,(\exists y \in \overline{Y})\,k(y) =_X x$. By the choice principle for types, let $m : \overline{X} \longrightarrow \overline{Y}$ be a function so that $k(m(x)) =_X x$ for all $x \in \overline{X}$. Let $T' = (\overline{T}, \mathrm{Id}_{\overline{T}}(\cdot,\cdot))$, and put $t' = t$ and $e = \lambda x.x$. Thus e is epi and an arrow in $\mathbf{Sets}/P$. Define $g : T' \times_P Q \longrightarrow Y$ by

$$g(u,p) = m(f(e(u),p)).$$

Since equality in $T' \times_P Q$ amounts to identity in each of the components, g becomes automatically well-defined. One easily checks, using the above equations, that g is an arrow in $\mathbf{Sets}/P$. Also $k(g(u,p)) =_X k(m(f(e(u),p))) =_X f(e(u),p)$, proving that h is internally projective. ∎

Construction of universal small maps. Now we construct the universal small map $\pi_{(n)} : E_n \longrightarrow U_n$ for $\mathcal{S}_n$. Let $(\mathbb{U},\mathbb{T}) = (\mathbb{U}_n, \mathbb{T}_n)$ be the nth universe. By (44) above we have a functor $F_n = F_{\mathbb{U},\mathbb{T}}$ from $\mathbb{U}^{=}$ to **Presets**. Let U be the set $(\overline{U}, =_U)$ where $\overline{U}$ is the type

$$(\Sigma a \in \mathbb{U})\,(\Sigma e \in \mathbb{T}(a \times a) \longrightarrow \mathbb{U})\,P(a,e)$$

and $P(a,e)$ is a proposition stating that $\mathbb{T}(e(\cdot,\cdot))$ is an equivalence relation on $\mathbb{T}(a)$. For an element $u = (a,e,q) \in U$ we write $(u)_1 = a$ and $S_u = (\overline{S_u}, =_{(u)})$, where $\overline{S_u} = \mathbb{T}(a)$ and $x =_{(u)} y$ is $\mathbb{T}(e(x,y))$. The equality $=_U$ of U is given by: $u =_U v$ iff for some $p \in \mathrm{Id}_{\mathbb{U}}((u)_1, (v)_1)$ we have

$$(\forall x, y \in S_u)\,[x =_{(u)} y \Longleftrightarrow F_n(p)(x) =_{(v)} F_n(p)(y)]. \tag{46}$$

A proof object for this equality is thus a pair (p,m). Since F_n is a functor from a groupoid, the property (46) means, indeed, that $F_n(p)$ is an isomorphism from S_u to S_v, with inverse $F_n(p^{-1})$. Using the functoriality of F_n it is then straightforward to check that $=_U$ is an equivalence relation, where the reflexivity property follows by letting $p = \mathsf{r}(a)$. Next define $E = (\overline{E}, =_E)$ by putting

$$\overline{E} = (\Sigma u \in U)\mathbb{T}((u)_1)$$

and where the equality $=_E$ is given by: $(u, x) =_E (v, y)$ iff $F_n(p)(x) =_{(v)} y$ for some $p \in \mathrm{Id}_U((u)_1, (v)_1)$ such that (46) holds. Again using functoriality this is seen to be an equivalence relation. The projection $\pi : E \longrightarrow U$ given by $\pi(u, x) = u$ is clearly a well-defined map in $\mathbf{Sets}_{n+1}$. Finally, let $\pi_{(n)} = \pi$, $E_n = E$ and $U_n = U$ and the construction is complete.

Lemma 12.4 *For each n, $\pi_{(n)} : E_n \longrightarrow U_n$ represents $\mathcal{S}_n$, and moreover there is a pullback diagram*

$$\begin{tikzcd}
E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\pi_{(n)}"'] & E_{n+1} \arrow[d, "\pi_{(n+1)}"] \\
U_n \arrow[r, "\tau_n"'] & U_{n+1}
\end{tikzcd}$$

and $U_n \longrightarrow 1 \in \mathcal{S}_{n+1}$.

Proof. Let $u = (a, e, q) \in U_n$ so that $S_u = (\mathbb{T}_n(a), \mathbb{T}(e(\cdot, \cdot)))$. By definition every set in $\mathbf{Sets}_n$ is of this form. For the first statement, it is sufficient to establish an isomorphism φ as below, by the characterization of representability (see Section 3). Define

$$\varphi : S_u \longrightarrow \pi_{(n)}^{-1}(u) = \{(v, y) \in E_n \mid\mid v =_{U_n} u\}$$

by $\varphi(x) = ((u, x), q_x)$ where $q_x = (p_x, m_x)$ is a proof of $u =_{(u)} u$ with $p_x = \mathsf{r}((u)_1)$. Then $\varphi(x) = \varphi(x')$ iff $x =_{(u)} x'$. To prove that φ is onto, let $(v, y) \in E_n$ satisfy $v =_{U_n} u$. Hence for some $p \in \mathrm{Id}_{U_n}((v)_1, (u)_1)$ the equivalence (46) holds. Let $x = F_n(p)(y)$. Then $(u, x) =_{E_n} (v, y)$, so φ is also onto, and consequently an isomorphism.

To obtain the pullback square we define

$$\tau_n(a, e, q) = (t_n(a), t_n \circ e, q) \text{ and } \varepsilon_n(u, x) = (\tau_n(u), x).$$

These functions are checked to be well-defined using Id-elimination. Trivially, $\pi_{(n+1)} \circ \varepsilon_n = \tau_n \circ \pi_{(n)}$, so the square commutes.

Suppose now that $f : C \longrightarrow U_n$ and $g : C \longrightarrow E_{n+1}$ satisfy $\tau_n \circ f =_{U_{n+1}} \pi_{(n+1)} \circ g$. Write $g(u) = ((g_1 u, g_2 u, g_3 u), g_4 u)$ and $f(u) = (f_1 u, f_2 u, f_3 u)$, so by the definition of $=_{U_{n+1}}$ there is some $p_u \in \mathrm{Id}_{\mathbb{U}_{n+1}}(g_1 u, t_n(f_1 u))$ with

$$\mathbb{T}_{n+1}((g_2 u)(x, y)) \Longleftrightarrow \mathbb{T}_n((f_2 u)(F_{n+1}(p_u)(x), F_{n+1}(p_u)(y)))$$

for all $x, y \in \mathbb{T}_{n+1}(g_1 u)$. Note that $k(u) = F_{n+1}(p_u)(g_4 u) \in F_{n+1}(t_n(f_1 u)) = \mathbb{T}_n(f_1 u)$. Define $h : C \longrightarrow E_n$ by $h(u) = (f(u), k(u))$. It is straightforward, but somewhat tedious, to check that h is the unique map with $\pi_{(n)} \circ h = f$ and $\varepsilon_n \circ h = g$.

Finally, it is clear that $U_n \longrightarrow 1 \in \mathcal{S}_{n+1}$, since the construction of U_n is carried out within the universe $\mathbb{U}_{n+1}$. ∎

Lemma 12.5 *For maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *in* **Sets** *with* $g \in \mathcal{S}_n$,

$$f \in \mathcal{S}_n \Longleftrightarrow gf \in \mathcal{S}_n.$$

Proof. For each $z \in \overline{Z}$ there is B_z in $\mathbf{Sets}_n$ and an isomorphism $\psi_z : B_z \longrightarrow g^{-1}(z)$. There are canonical injections $\gamma_y : f^{-1}(y) \longrightarrow X$, $\xi_z : \underline{(gf)^{-1}(z)} \longrightarrow X$ and $\eta_z : g^{-1}(z) \longrightarrow Y$.

($\Rightarrow$) Suppose $f \in \mathcal{S}_n$. Thus for each $y \in \overline{Y}$ there is $A_y \in \mathbf{Sets}_n$ and an isomorphism $\varphi_y : A_y \longrightarrow f^{-1}(y)$. To show $gf \in \mathcal{S}_n$ it suffices to prove that $(gf)^{-1}(z)$ is isomorphic to some $C \in \mathbf{Sets}_n$. Let $D = B_z$, $\delta = \eta_z \circ \psi_z$ and define a functor $E : D^{\#} \longrightarrow \mathbf{Sets}_n$ by $E(u) = A_{\delta(u)}$ and for $p : (u =_D u')$ let $E(p) = \varphi^{-1}_{\delta(u')} \circ f^{-1}(t_{u,u'}(p)) \circ \varphi_{\delta(u)}$ where

$$t_{u,u'} : (u =_D u' \longrightarrow \delta(u) =_Y \delta(u')).$$

Note that $\underline{E(p)}$ does not depend on p, since $f^{-1} : Y^{\#} \longrightarrow \mathbf{Sets}$ is a functor. Let now $\overline{C} = (\Sigma u \in \overline{D})\overline{E}(u)$ and define the equality by $(u, v) =_C (u', v')$ iff $E(p)(v) =_{E(u')} v'$ for some $p : (u =_D u')$. By the functor property $=_C$ is indeed an equivalence relation. Thus $C \in \mathbf{Sets}_n$. Define a map $\theta : C \longrightarrow (gf)^{-1}(z)$ by $\theta(u, v) = (x, q)$ where $x = \gamma_{\delta(u)}(\varphi_{\delta(u)}(v))$ and q is a proof object for $g(f(x)) =_Z z$. It is readily checked that θ is an isomorphism.

($\Leftarrow$) Suppose $gf \in \mathcal{S}_n$. Thus for each $z \in Z$ there is $C_z \in \mathbf{Sets}_n$ and an isomorphism $\theta_z : C_z \longrightarrow (gf)^{-1}(z)$. Let $y \in Y$ be arbitrary and let $D = C_{g(y)}$. Let $(y, p) \in g^{-1}(g(y))$ and put

$$A = \{u \in D \mid\mid \psi^{-1}_{g(y)}(h_y(\theta_{g(y)}(u))) =_{B_{g(y)}} \psi^{-1}_{g(y)}(y, p)\},$$

where h_y is the canonical injection $(gf)^{-1}(g(y)) \longrightarrow g^{-1}(g(y))$. By the form of A is clear that it belongs to $\mathbf{Sets}_n$. It is straightforward to check that $\alpha : A \longrightarrow g^{-1}(y)$, given by $\alpha(u, p) = (\xi_{g(y)}(\theta_{g(y)}(u)), p')$, for some p' depending on (u, p), defines an isomorphism. ∎

Lemma 12.6 *The class* $\mathcal{S}_n$ *satisfies the axioms (F1-5) for slicing.*

Proof. This is straightforward by examining the constructions fiberwise, and by using the fact that $\mathbf{Sets}_n$ is closed under the pretopos operations, Π- and W-constructions. ∎

Now Lemma 12.3 – 12.6 yield the desired theorem.

Theorem 12.7 *In the type theory* $\mathbf{ML}_{<\omega}\mathbf{W}$*, the category* **Sets** *is a stratified pseudotopos, with enough internal projectives.* ∎

Remark 12.8 In [6] a σ-complete boolean algebra $\mathbb{B}$ was constructed within **Sets** such that $\mathcal{E} = \mathrm{Sh}_{\mathbf{Sets}}(\mathbb{B})$ validates the "numerical omniscience scheme"

$$\text{(NOS)} \qquad (\forall n \in \mathbb{N})\,(\varphi(n) \vee \neg\varphi(n)) \longrightarrow (\exists n \in \mathbb{N})\,\varphi(n) \vee (\forall n \in \mathbb{N})\,\neg\varphi(n),$$

for any formula φ of the internal logic. Moreover, countable and dependent choice (AC_0, DC) hold in $\mathcal{E}$. Since $\mathcal{E}$ is the class of internal sheaves in **Sets**, it is by Theorems 11.2 and 12.7 a stratified pseudotopos, and we have thus a constructive model for CZF^+ + NOS + AC_0 + DC, a set theory, PZF, that should be suitable for classical predicative mathematics. The model $\mathcal{E}$ is absolute for Π^0_2-formulae [6], so that effective content can be extracted from certain proofs in PZF.

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