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**Some topological spaces which are universal for intuitionistic predicate logic**

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Communicated by Prof. A.S. Troelstra at the meeting of April 25, 1981**ABSTRACT**

It is shown that for every formula  $\varphi$  which cannot be proved in intuitionistic predicate logic, and for every metrizable space  $X$  without isolated points one can find a sheaf model over  $X$  in which  $\varphi$  is not valid.

**INTRODUCTION AND SUMMARY**

This paper deals with sheaf-models for intuitionistic predicate logic. These models extend the wellknown Beth- and Kripke-models, and arise most naturally if one interprets intuitionistic predicate logic with an existence-operator, as in [S], [FS]. In the first section we quickly review the basic definitions and properties of sheaf-models. We will just restrict ourselves to ordinary intuitionistic predicate logic (henceforth IPC) without the existence-operator, mainly because the result of section 2 carries over automatically to Scott's logic with the existence operator, by the correspondence with ordinary intuitionistic logic proved in [S].

A complete Heyting algebra  $\Omega$  (or a topological space  $X$ ) is called *universal* for IPC if every formula which cannot be derived in IPC has a counterexample in a sheaf over  $\Omega$  (or over  $X$ ). In [T], Tarski proved that every separable metric space without isolated points is universal for intuitionistic *propositional* logic, by (essentially) showing that every finite Heyting algebra arises as subalgebra of (the algebra of open subsets of) an open subspace of every separable metric space. It has been noted that the separability-condition is actually superfluous, see for example chapter III of [RS].

In section 2 we adapt Tarski's method in order to show that every metric space contains the algebra of open subsets of the binary Kripke-tree as a subalgebra (with respect to the *infinite* joins and meets necessary for interpreting *predicate* logic); or, dually, we show that for every metrizable space  $X$  without isolated points there exists an open and continuous mapping onto the sobrification of the binary Kripke tree. Thus we obtain the following

**THEOREM.** Every metrizable space without isolated points is universal for intuitionistic predicate logic.

## 1. SHEAF-MODELS FOR INTUITIONISTIC LOGIC

This section outlines the basic theory of sheaf-models for first-order languages. For more details the reader is referred to [FS].

**1.1 DEFINITIONS.** A complete Heyting algebra (cHa) is a complete lattice  $(\Omega, \wedge, \vee, \perp, \top)$  satisfying the  $\wedge\vee$ -distributive law, i.e. for any  $u \in \Omega$  and  $\Delta \subseteq \Omega$ ,  $u \wedge \vee \Delta = \vee \{u \wedge v \mid v \in \Delta\}$ .  $\Omega$  then has a top-element  $\top = \vee \Omega$ , a bottom-element  $\perp = \wedge \Omega$ , and an implication-operator  $\Rightarrow$  defined by  $u \Rightarrow v = \vee \{w \in \Omega \mid u \wedge w \leq v\}$ . The pseudocomplement  $\neg u$  of  $u$  is  $u \Rightarrow \perp$ .

The cHa's are made into a category by defining morphisms to be functions which preserve finite meets and arbitrary joins; i.e., a function  $f^*: \Omega \rightarrow \Omega'$  from a cHa  $\Omega$  to a cHa  $\Omega'$  is a *cHa-morphism* iff  $f^*(\top) = \top$ ,  $f^*(u \wedge v) = f^*(u) \wedge f^*(v)$ , and  $f^*(\vee_{i \in I} u_i) = \vee_{i \in I} f^*(u_i)$  for all  $u, v, u_i \in \Omega$ .

The main example of a cHa is the lattice  $\mathcal{O}(X)$  of open subsets of a topological space  $X$  (partially ordered by inclusion). If  $f: X \rightarrow Y$  is a continuous function from a space  $X$  to a space  $Y$  then  $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  is a cHa-morphism.

A cHa-morphism is called *open* if it preserves arbitrary meets, too. Note that for a continuous function  $f: X \rightarrow Y$ ,  $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  is open in this sense iff  $f$  is an open map.

Let us now turn to sheaves. If  $\Omega$  is a cHa, an  $\Omega$ -set is a pair  $\mathbf{A} = \langle A, \llbracket \cdot = \cdot \rrbracket_{\mathbf{A}} \rangle$  where  $\llbracket \cdot = \cdot \rrbracket_{\mathbf{A}}: A \times A \rightarrow \Omega$  is a binary function on the set  $A$  (the Heyting-valued identity relation on  $A$ ) satisfying

- (i)  $\llbracket a = b \rrbracket = \llbracket b = a \rrbracket$
  - (ii)  $\llbracket a = b \rrbracket \wedge \llbracket b = c \rrbracket \leq \llbracket a = c \rrbracket$
- for all  $a, b, c \in A$ .

An  $\Omega$ -sheaf is an  $\Omega$ -set which is 'closed under singletons' in the following sense: if  $\mathbf{A}$  is an  $\Omega$ -set, a *singleton* of  $\mathbf{A}$  is a function  $\sigma: A \rightarrow \Omega$  satisfying

- (i)  $\sigma(a) \wedge \llbracket a = b \rrbracket \leq \sigma(b)$
- (ii)  $\sigma(a) \wedge \sigma(b) \leq \llbracket a = b \rrbracket$ .

An  $\Omega$ -sheaf is an  $\Omega$ -set having the *sheaf-property*: for every singleton  $\sigma$  of  $\mathbf{A}$  there is a *unique*  $a \in A$  such that for all  $b \in A$ ,  $\sigma(b) = \llbracket a = b \rrbracket_{\mathbf{A}}$  (written ' $\sigma = \{a\}$ '). If  $\mathbf{A}$  is an  $\Omega$ -set we can define  $\Omega$ -valued *extent*  $E: A \rightarrow \Omega$  and *equivalence*  $\llbracket \cdot \equiv \cdot \rrbracket$  by  $Ea: = \llbracket a = a \rrbracket$ ,  $\llbracket a \equiv b \rrbracket: = (Ea \wedge Eb) \Rightarrow \llbracket a = b \rrbracket$ .

1.2 EXAMPLES. If  $X$  is a topological space, the continuous real-valued partial functions with open domain form an  $\mathcal{O}(X)$ -sheaf by setting  $\llbracket f = g \rrbracket = \text{Int}\{x \in \text{domain}(f) \cap \text{domain}(g) \mid f(x) = g(x)\}$ . Then  $Ef = \text{domain}(f)$ . Another example is provided by taking the upwards-closed subsets of a partially ordered set  $\mathbb{P} = (P, \leq)$  ( $U \subseteq P$  is upwards closed iff  $p \geq q \in U$  implies  $p \in U$ ). These subsets form a cHa (partially ordered by inclusion), and sheaves on this cHa correspond to *Kripke-models* on  $\mathbb{P}$ .

1.3. Let  $\mathbf{A}$  be an  $\Omega$ -set. The set  $\hat{A}$  of singletons of  $A$  can be made into an  $\Omega$ -set  $\hat{\mathbf{A}}$  by setting

$$\llbracket \sigma = \tau \rrbracket_{\hat{\mathbf{A}}} = \bigvee_{a \in A} (\sigma(a) \wedge \tau(a)).$$

$\hat{\mathbf{A}}$  is always an  $\Omega$ -sheaf, called the *sheafification* of  $\mathbf{A}$ . Further,  $\mathbf{A}$  is embedded in  $\hat{\mathbf{A}}$  by the function  $a \rightarrow \{a\}$ . ( $\{a\}$  is the singleton defined by  $\{a\}(b) = \llbracket a = b \rrbracket_{\mathbf{A}}$ .)

1.4. Let  $\mathbf{A}$  be an  $\Omega$ -set. A (strict)  $n$ -place operation on  $\mathbf{A}$  is a function  $F: A^n \rightarrow A$  satisfying both

$$EF(a_1, \dots, a_n) \leq Ea_1 \wedge \dots \wedge Ea_n,$$

and

$$\llbracket a_1 = b_1 \rrbracket \wedge \dots \wedge \llbracket a_n = b_n \rrbracket \leq \llbracket F(a_1, \dots, a_n) = F(b_1, \dots, b_n) \rrbracket, \text{ for all } a_i, b_i \in A.$$

A (strict)  $n$ -place relation on  $\mathbf{A}$  is a function  $R: A^n \rightarrow \Omega$  satisfying both

$$R(a_1, \dots, a_n) \leq Ea_1 \wedge \dots \wedge Ea_n$$

and

$$\llbracket a_1 = b_1 \rrbracket \wedge \dots \wedge \llbracket a_n = b_n \rrbracket \wedge R(a_1, \dots, a_n) \leq R(b_1, \dots, b_n), \text{ for all } a_i, b_i \in A.$$

If  $F$  is an  $n$ -place operation on  $\mathbf{A}$ ,  $F$  extends uniquely to an  $n$ -place operation  $\hat{F}$  on  $\hat{\mathbf{A}}$  ( $\hat{F}(\sigma_1, \dots, \sigma_n)$  is the singleton

$$b \mapsto \bigvee \{ \llbracket b = F(a_1, \dots, a_n) \rrbracket \wedge \sigma_1(a_1) \wedge \dots \wedge \sigma_n(a_n) \mid a_1, \dots, a_n \in A \}).$$

Similarly, an  $n$ -place relation  $R$  on  $\mathbf{A}$  extends uniquely to an  $n$ -place relation  $\hat{R}$  on  $\hat{\mathbf{A}}$  ( $\hat{R}(\sigma_1, \dots, \sigma_n) = \bigvee \{ R(a_1, \dots, a_n) \wedge \sigma_1(a_1) \wedge \dots \wedge \sigma_n(a_n) \mid a_1, \dots, a_n \in A \}$ ).

1.5. Let us now turn to logic. Fix a first-order language  $L$  containing relation-symbols  $(R_i)_{i \in I}$ , function-symbols  $(F_j)_{j \in J}$ , individual constants  $(c_k)_{k \in K}$ , infinitely many variables  $\{x_n \mid n \in \omega\}$ , and an identity-symbol  $=$ . Terms and formulas of  $L$  are built up as usual. An *interpretation*  $\mathcal{I}$  of  $L$  in an  $\Omega$ -sheaf  $\mathbf{A}$  assigns to each individual constant  $c$  an element  $\mathcal{I}(c)$  of  $A$ , to each  $n$ -place relation-symbol  $R$  an  $n$ -place relation  $\mathcal{I}(R)$  on  $\mathbf{A}$ , and to each  $n$ -place function-symbol  $F$  an  $n$ -place operation  $\mathcal{I}(F)$  on  $\mathbf{A}$ . To interpret terms and formulas (relative to  $\mathcal{I}$ ) we add a constant  $\mathbf{a}$  for each element  $a \in A$ . For this expanded language the interpretation  $\mathcal{I}(\tau)$  of a closed term  $\tau$  is inductively defined by  $\mathcal{I}(\mathbf{a}) = a$  for  $a \in A$ ,  $\mathcal{I}(c) = \mathcal{I}(c)$  for  $c$  a constant of  $L$ , and if

$\tau = F(\tau_1, \dots, \tau_n)$ ,  $\mathcal{A}(\tau) = \mathcal{A}(F)(\mathcal{A}(\tau_1), \dots, \mathcal{A}(\tau_n))$ . For a sentence  $\phi$  of the expanded language we define its value  $\llbracket \phi \rrbracket_{\mathcal{I}} \in \Omega$  (relative to  $\mathcal{I}$ ; the subscript  $\mathcal{I}$  is usually omitted) by

$$\begin{aligned} \llbracket R(\tau_1, \dots, \tau_n) \rrbracket &= \mathcal{A}(R)(\mathcal{A}(\tau_1), \dots, \mathcal{A}(\tau_n)) \\ \llbracket \tau = \sigma \rrbracket &= \llbracket \mathcal{A}(\tau) = \mathcal{A}(\sigma) \rrbracket_{\mathbf{A}} \\ \llbracket \phi \overset{\Delta}{\rightarrow} \psi \rrbracket &= \llbracket \phi \rrbracket \overset{\Delta}{\rightarrow} \llbracket \psi \rrbracket, \llbracket \neg \phi \rrbracket = \neg \llbracket \phi \rrbracket \\ \llbracket \forall x \phi(x) \rrbracket &= \wedge \{ \overset{\Delta}{E} a \Rightarrow \llbracket \phi(\mathbf{a}) \rrbracket \mid a \in A \} \\ \llbracket \exists x \phi(x) \rrbracket &= \vee \{ E a \wedge \llbracket \phi(\mathbf{a}) \rrbracket \mid a \in A \} \end{aligned}$$

An  $L$ -formula  $\phi(c_1, \dots, c_m, x_1, \dots, x_n)$  is called *valid in  $\mathbf{A}$*  w.r.t.  $\mathcal{I}$  ( $\mathbf{A} \models_{\mathcal{I}} \phi$ ) iff for every  $a_1, \dots, a_n \in A$ ,  $E c_1 \wedge \dots \wedge E c_m \wedge E a_1 \wedge \dots \wedge E a_n \leq \llbracket \phi(c_1, \dots, c_m, a_1, \dots, a_n) \rrbracket$ . (Here  $c_1, \dots, c_m$  and  $x_1, \dots, x_n$  are the constants and variables in  $\phi$ .) A formula  $\phi$  is called *valid in  $\Omega$*  iff for every  $\Omega$ -sheaf  $\mathbf{A}$  and every interpretation  $\mathcal{I}$  in  $\mathbf{A}$ ,  $\mathbf{A} \models_{\mathcal{I}} \phi$ . Finally, we define the first-order theory  $\text{Th}(\Omega)$  of a cHa  $\Omega$  to be the set  $\{\phi \mid \Omega \models \phi\}$  of sentences valid in  $\Omega$ . (If  $\Omega = \mathcal{O}(X)$  for a topological space, we write  $\text{Th}(X)$  for  $\text{Th}(\mathcal{O}(X))$ .) The *soundness-theorem* says that for every cHa  $\Omega$ ,  $\text{IPC} \subseteq \text{Th}(\Omega)$ .

1.6. *The image construction.* Let  $f^*: \Omega \rightarrow \Omega'$  be a  $\wedge$ -preserving function. If  $\mathbf{A}$  is an  $\Omega$ -set, the *image*  $f^*(\mathbf{A})$  of  $\mathbf{A}$  is the  $\Omega'$ -set  $\langle A, \llbracket \cdot \rrbracket_{f^*(\mathbf{A})} \rangle$  defined by setting  $\llbracket a = b \rrbracket_{f^*(\mathbf{A})} = f^*(\llbracket a = b \rrbracket_{\mathbf{A}})$ . If  $F$  is an  $n$ -place operation on  $\mathbf{A}$ , it also is an  $n$ -place operation  $f^*(F) = F$  on  $f^*(\mathbf{A})$ . And if  $R$  is an  $n$ -place relation on  $\mathbf{A}$  we get an  $n$ -place relation  $f^*(R)$  on  $f^*(\mathbf{A})$  by defining  $f^*(R)(a_1, \dots, a_n) = f^*(R(a_1, \dots, a_n))$ .

If  $\mathbf{A}$  is an  $\Omega$ -sheaf,  $f^*(\mathbf{A})$  need not be one, but we call its sheafification  $(f^*(\mathbf{A}))^\wedge$  also the image of  $\mathbf{A}$ , and the cap  $\wedge$  is often omitted. Similarly, working within sheaves we write  $f^*(F)$  for  $(f^*(F))^\wedge$  and  $f^*(R)$  for  $(f^*(R))^\wedge$ . Using this, we can define the image  $f^*(\mathcal{I})$  of an interpretation  $\mathcal{I}$  of  $L$  in an  $\Omega$ -sheaf  $\mathbf{A}$  to be the interpretation in the  $\Omega'$ -sheaf  $f^*(\mathbf{A})$  obtained by applying  $f^*$  to the interpretation  $\mathcal{I}$  of constants, relation- and function-symbols. If we want to evaluate formulas of  $L$  w.r.t.  $f^*(\mathcal{I})$  in  $f^*(\mathbf{A})$  we only need to look at the original  $\Omega$ -set  $f^*(\mathbf{A})$ , and not necessarily to its sheafification, by theorem 5.18 of [FS].

1.7. *Subalgebras.* Below, by a subalgebra of a cHa  $\Omega$  we will always mean a subalgebra with respect to *all* operations  $\wedge, \vee, \top, \perp, \Rightarrow$ . This differs from the concept of subalgebra defined in [FS], which we will not use here.

## 2. UNIVERSALITY OF SOME TOPOLOGICAL SPACES

In this section we will be concerned with complete Heyting algebras of the form  $\mathcal{O}(X)$ , for a topological space  $X$ . All (undefined) terminology used is standard, as e.g. in [E]. A space  $X$  is called *universal for IPC* if  $\text{Th}(X) = \text{IPC}$ , i.e. for any first-order formula  $\phi$  which cannot be derived in intuitionistic predicate logic there is a sheaf  $\mathbf{A}$  on  $\mathcal{O}(X)$  and an interpretation  $\mathcal{I}$  in  $\mathbf{A}$  such that  $\llbracket \phi \rrbracket_{\mathcal{I}} \neq \top$ .

Unfortunately, the following problem will not be solved here:

PROBLEM. Give a topological characterization of the class  $\mathcal{U}$  of spaces which are universal for *IPC*.

Below we will show that  $\mathcal{U}$  includes the class of metrizable spaces without isolated points. First, let us note some closure-properties of  $\mathcal{U}$ .

2.1. PROPOSITION.

- (i) If  $X$  contains an open subspace  $Y$  such that  $Y \in \mathcal{U}$ , then  $X \in \mathcal{U}$ .
- (ii) If  $Y \in \mathcal{U}$  and  $f: X \rightarrow Y$  is an open continuous surjection, then  $X \in \mathcal{U}$ .
- (iii) If  $Y \in \mathcal{U}$  and  $\mathcal{O}(Y)$  is isomorphic to a subalgebra of  $\mathcal{O}(X)$  then  $X \in \mathcal{U}$ .

PROOF. (i) Consider  $i^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ ,  $i^*(U) = U$ . It is easy to check that  $i^*$  preserves the cHa-operations  $\wedge$ ,  $\vee$ , and  $\perp$ . Hence given an interpretation  $\mathcal{I}$  in an  $\mathcal{O}(Y)$ -sheaf  $\mathbf{A}$ , we can show by induction that

$$i^*(\llbracket \phi \rrbracket_{\mathcal{I}}) = \llbracket \phi \rrbracket_{i^*(\mathcal{I})} \cap Y$$

( $i^*(\mathcal{I})$  is the image interpretation in the  $\mathcal{O}(X)$ -sheaf  $i^*(\mathbf{A})$ .) Thus  $\text{Th}(X) \subseteq \subseteq \text{Th}(Y)$ , which proves (i).

(ii) Again use images: the map  $f^{-1}: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$  preserves all cHa-operations (always  $\wedge$ ,  $\vee$ ,  $\top$ ,  $\perp$ ;  $\wedge$  and  $\Rightarrow$  because  $f$  is open), so by induction it follows that for an interpretation  $\mathcal{I}$  in an  $\mathcal{O}(Y)$ -sheaf  $\mathbf{A}$ ,

$$f^{-1}(\llbracket \phi \rrbracket_{\mathcal{I}}) = \llbracket \phi \rrbracket_{f^{-1}(\mathcal{I})}.$$

From the surjectivity of  $f$  it then follows that  $\text{Th}(X) \subseteq \subseteq \text{Th}(Y)$ .

(iii) (This is the dual of (ii).) Take the embedding  $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ . By definition, it preserves all cHa-operations, so we can use images as in (ii) to show  $\text{Th}(X) \subseteq \subseteq \text{Th}(Y)$ .  $\square$

COROLLARY.  $\mathcal{U}$  is closed under topological products and sums.

2.2. *The binary Kripke-tree.* The binary Kripke-tree is the set of finite sequences of zeros and ones, partially ordered by inclusion: we write

$$K = \{f: n \rightarrow 2 \mid n \in \omega\},$$

and for  $f, g \in K$ ,  $f \leq g$  iff  $g$  extends  $f$ . The partial order  $\mathbb{K} = \langle K, \leq \rangle$  can also be considered as a topological space with  $K$  as its set of points and the upwards closed sets (cf. 1.2.) as opens. This space is also denoted by  $\mathbb{K}$ .

The following result is well-known.

THEOREM.  $\mathbb{K}$  is universal for intuitionistic predicate logic.

We are now able to prove the main result of this paper. Below, we write  $B(A, \varepsilon) = \{y \mid d(y, A) < \varepsilon\}$  for a subset  $A$  of a metric space  $(X, d)$ ;  $B(\{x\}, \varepsilon)$  is written as  $B(x, \varepsilon)$ . First we need a definition.

2.3. DEFINITION. Let  $(X, d)$  be a metric space,  $\varepsilon$  a positive real number, and  $A$  a subset of  $X$ .

- (i)  $A$  is called  $\varepsilon$ -dense if  $\forall x \in A \exists a \in A d(x, a) < \varepsilon$ .
- (ii)  $A$  is called nowhere  $\varepsilon$ -dense if  $\forall a, a' \in A, a \neq a'$  implies  $d(a, a') \geq \varepsilon$ .

Note that for each  $\varepsilon > 0$ , a dense set is  $\varepsilon$ -dense, and a nowhere  $\varepsilon$ -dense set is nowhere dense. Also, that every nowhere  $\varepsilon$ -dense set is *closed*.

2.4. LEMMA. Let  $(X, d)$  be a metric space. Then for each  $\varepsilon > 0$  there exists an  $\varepsilon$ -dense, nowhere  $\varepsilon/2$ -dense subset  $A$  of  $X$ .

PROOF. Choose  $\varepsilon > 0$ . Since the class of nowhere  $\varepsilon/2$ -dense subsets of  $X$  is non-empty and closed under unions of chains, we can find a maximal nowhere  $\varepsilon/2$ -dense subset  $A$  of  $X$ , by Zorn's lemma. Clearly,  $A$  must be  $\varepsilon$ -dense.

2.5. LEMMA. Let  $(X, d)$  be a metric space without isolated points, let  $\varepsilon > 0$ , and let  $U$  be an open, non-empty subset of  $X$ . Then there are disjoint, open, non-empty subsets  $V_0$  and  $V_1$  of  $U$  having the following properties:

1.  $\forall x \in U \exists y \in U \setminus (V_0 \cup V_1) d(x, y) < \varepsilon$
2.  $\bar{U} \subseteq \bar{V}_0 \cup \bar{V}_1$ , and  $\bar{U} \subseteq V_0 \cup \bar{V}_1$
3.  $\bar{U} \setminus U \subseteq \overline{U \setminus (V_0 \cup V_1)}$

PROOF. In order to satisfy 1. we take an  $\varepsilon$ -dense, nowhere  $\varepsilon/2$ -dense set  $A \subseteq U$  (lemma 2.4.). And to satisfy 3., we first define a sequence  $(F_n)_{n \in \omega}$  of subsets of  $U$  such that  $U \setminus (\bigcup_{n \in \omega} F_n)$  is dense in  $U$ , and  $\bar{U} \setminus U \subseteq \overline{\bigcup_{n \in \omega} F_n}$ . If we then define  $V_0$  and  $V_1$  such that  $V_0 \cup V_1 \subseteq U \setminus (\bigcup_{n \in \omega} F_n \cup A)$ , properties 1. and 3. of the lemma will hold.

To define the sets  $F_n$  ( $n \in \omega$ ), first fix a well ordering of  $\bar{U} \setminus U$ , say  $\bar{U} \setminus U = \{p_\eta \mid \eta < \alpha\}$ . Now fix  $n \in \omega$  and define by transfinite induction a sequence  $(x_\xi^n)_{\xi < \alpha}$  as follows. Suppose the  $x_\xi^n$  have been defined for all  $\xi < \eta$ ; define  $x_\eta^n$  to be any point such that

$$x_\eta^n \in U \cap (B(p_\eta, 2^{-n}) \setminus \bigcup_{\xi < \eta} B(x_\xi^n, 2^{-n}))$$

if this set is non-empty. Otherwise, let  $x_\eta^n$  equal  $x_0^n$ . Now let  $F_n$  be the set of points  $x_\xi^n$  thus defined.

Note that the sequence  $(F_n)_n$  has the following properties, for any  $n$ :

- (i)  $\forall x \in F_n \exists p \in \bar{U} \setminus U d(x, p) < 2^{-n}$
- (ii)  $F_n$  is a closed discrete subset of  $U$ , in fact, if  $x, y \in F_n$ ,  $x \neq y$  then  $d(x, y) \geq 2^{-n}$
- (iii)  $\forall p \in \bar{U} \setminus U \exists y \in F_n d(p, y) < 2 \cdot 2^{-n}$

Let  $F = \overline{\bigcup_{n \in \omega} F_n}$ . Then by (iii),  $\bar{U} \setminus U \subseteq F$ . Also,  $U \setminus (F \cup A)$  is a dense open subset of  $U$ , and  $\bigcup_n F_n \cup A$  is a discrete subset of  $U$  which is closed in  $U$ . Both facts follow by observing that if  $p \in U$  and  $\delta < 0$  is such that  $B(p, \delta) \subseteq U$ , then if  $2^{-n} < \frac{1}{2}\delta$ ,  $B(p, \delta) \cap F_k = \emptyset$  for  $k \geq n$ , by (i) above. Hence, using (ii) and the fact that  $A$  is nowhere  $\varepsilon$ -dense, we get that for  $\delta$  sufficiently small,  $|B(p, \delta) \cap (F \cup A)| \leq n + 1$ .

Set  $W = U \setminus (F \cup A)$ . We will now define two disjoint open subsets  $W_0$  and  $W_1$  of  $W$  such that  $F \cup A \subseteq \bar{W}_0 \cap \bar{W}_1$ . First note that because  $\bigcup_n F_n \cup A$  is a discrete set which is closed in  $U$ , it follows easily from paracompactness of the subspace

$U$  that we can find a  $\delta_x > 0$  for every  $x \in \bigcup_{n \in \omega} F_n \cup A$ , such that if  $x, y \in \bigcup_n F_n \cup A$  and  $x \neq y$  then  $B(x, \delta_x) \cap B(y, \delta_y) = \emptyset$ . Since such a point  $x$  is not isolated, we can find a sequence  $(U(x, n))_{n \in \omega}$  of disjoint non-empty open subsets of  $B(x, \delta_x) \setminus \{x\}$  converging to  $x$  (i.e., for each neighbourhood  $W_x$  of  $x$  there exists an  $n$  such that  $\forall k \geq n \ U(x, k) \subseteq W_x$ ), for each  $x \in \bigcup_n F_n \cup A$ . Let

$$W_x^0 = \bigcup_{n \in \omega} U(x, 2n), \quad W_x^1 = \bigcup_{n \in \omega} U(x, 2n+1).$$

Finally, define

$$W_i = \bigcup \{W_x^i \mid x \in \bigcup_{n \in \omega} F_n \cup A\}, \text{ for } i=0, 1.$$

Then  $W_0 \neq \emptyset \neq W_1$ ,  $W_0 \cap W_1 = \emptyset$ , and  $W_0 \cup W_1 \subseteq W$ . It is also clear from the construction that

$$(iv) \quad \bigcup_n F_n \cup A \subseteq \bar{W}_0 \cap \bar{W}_1.$$

Now define

$$V_0 = \text{Int}(\bar{W}_0), \quad V_1 = W \setminus \bar{V}_0$$

then also  $W \setminus \bar{V}_1 = V_0$ , since  $V_0$  is regular open.

It is now easy to show that  $V_0$  and  $V_1$  satisfy property 2. of the lemma. For  $W \subseteq (\bar{V}_0 \cup V_1) \cap (V_0 \cup \bar{V}_1)$ , and by (iv),  $F \cup A = \overline{\bigcup_n F_n \cup A} \subseteq (\bar{V}_0 \cup V_1) \cap (V_0 \cup \bar{V}_1)$ . Hence  $\bar{U} = W \cup (F \cup A) \subseteq (\bar{V}_0 \cup V_1) \cap (V_0 \cup \bar{V}_1)$ . This completes the proof.

**2.6. THEOREM.** Let  $(X, d)$  be a metric space without isolated points. Then there exists a subalgebra  $\mathbb{H}$  of  $\mathcal{O}(X)$  which is isomorphic to  $\mathcal{O}(\mathbb{K})$ .

**PROOF.** Using lemma 2.5 we can define by induction on the length of sequences a collection of open, non-empty subsets  $\{B_s \mid s \in K\}$  such that for each  $s \in K$ , the following properties hold ( $*$  denotes concatenation):

1.  $B_{\langle \rangle} = X$  ( $\langle \rangle$  is the empty sequence)
2. if  $\text{length}(s) \geq n$  then  $\forall x \in B_s \ \exists y \in B_s \setminus (B_{s*0} \cup B_{s*1})$  such that  $d(x, y) < 2^{-n}$ , for every  $n \in \omega$ .
3.  $B_{s*0} \cup B_{s*1} \subseteq B_s$ ,  $B_{s*0} \cap B_{s*1} = \emptyset$ .
4.  $\bar{B}_s \subseteq (\bar{B}_{s*0} \cup B_{s*1}) \cap (B_{s*0} \cup \bar{B}_{s*1})$ .
5.  $\bar{B}_s \setminus B_s \subseteq \overline{B_s \setminus (B_{s*0} \cup B_{s*1})}$ .

$$\text{Let } \mathbb{H} = \left\{ \bigcup_{s \in N} B_s \mid N \subseteq K \right\}.$$

As we will now show, we can equip  $\mathbb{H}$  with the cHa-operations inherited from  $\mathcal{O}(X)$ , because  $\mathbb{H}$  is closed under these operations:

- (i)  $\mathbb{H}$  is a subalgebra of  $\mathcal{O}(X)$ .

Below we will write  $B_N$  for an element  $\bigcup_{s \in N} B_s$  of  $\mathbb{H}$ . We have to show that  $\mathbb{H}$  is closed under  $\wedge, \vee, \cap, \cup, \top, \perp, \Rightarrow$ ; the only non-trivial cases are  $\wedge$  and  $\Rightarrow$ , so let's consider these in turn.

First,  $\mathbb{H}$  is closed under  $\wedge$ : take a collection  $\{N^\alpha \mid \alpha \in \mathcal{A}\} \subseteq \mathcal{P}(K)$ . We have to show that

- (ii)  $\text{Int}\left(\bigcap_{\alpha \in \mathcal{A}} B_{N^\alpha}\right) = \bigcup \{B_s \mid \forall \alpha \in \mathcal{A} \ B_s \subseteq B_{N^\alpha}\}.$



The inclusion  $\supseteq$  is clear. For  $\subseteq$ , take  $x \in \text{Int}(\bigcap_{\alpha \in \mathcal{A}} B_{N\alpha})$ . Then for some  $n$ ,  $B(x, 2^{-n}) \subseteq B_{N\alpha}$  for each  $\alpha \in \mathcal{A}$ . But from property 2. above we find that if  $B(x, 2^{-n}) \subseteq B_{N\alpha}$ , then  $\exists s \in N^\alpha (\text{length}(s) \leq n \text{ and } x \in B_s)$ . So  $x$  has the property

$$\exists n \forall \alpha \in \mathcal{A} \exists s \in N^\alpha (x \in B_s \text{ and } \text{length}(s) \leq n).$$

From this, the inclusion from left to right is immediate.

Secondly,  $\mathbb{H}$  is closed under  $\Rightarrow$ : first note that for each  $s \in K$ ,

$$(iii) \quad X \setminus \bar{B}_s = \cup \{B_r \mid B_r \cap B_s = \emptyset\}.$$

This follows easily by induction on  $\text{length}(s)$ : if  $\text{length}(s) = 0$ , (iii) is trivially true; and if  $s = s' * i$  ( $i = 0$  or  $1$ ), we get by 4. above that  $X \setminus \bar{B}_s = X \setminus \bar{B}_{s'} \cup B_{s'*(1-i)}$ ; applying the induction hypothesis to  $s'$  then proves (iii).

To show that  $\mathbb{H}$  is closed under  $\Rightarrow$  it suffices to show that for each  $N$  and  $M \subseteq K$ ,

$$(iv) \quad \text{Int}((X \setminus B_N) \cup B_M) = \cup \{B_r \mid \forall s \in N \cdot B_r \cap B_s \subseteq B_M\}.$$

Since

$$\begin{aligned} \text{Int}((X \setminus B_N) \cup B_M) &= \text{Int} \bigcap_{s \in N} ((X \setminus B_s) \cup B_M) \\ &= \text{Int} \bigcap_{s \in N} \text{Int}((X \setminus B_s) \cup B_M) \end{aligned}$$

we get by (ii) above that it suffices to show that every  $s \in K$  and  $M \subseteq K$ ,

$$(v) \quad \text{Int}((X \setminus B_s) \cup \bigcup_{t \in M} B_t) = \cup \{B_r \mid B_r \cap B_s \subseteq \bigcup_{t \in M} B_t\}.$$

Without loss of generality we can assume that

$$(vi) \quad \forall t \in M t > s.$$

Further, the inclusion from right to left in (v) is clear. So, to prove  $\subseteq$ , choose  $x \in \text{Int}((X \setminus B_s) \cup \bigcup_{t \in M} B_t)$ . If  $x \in X \setminus \bar{B}_s$ , we get by (iii) above that  $x \in \cup \{B_r \mid B_r \cap B_s \subseteq \bigcup_{t \in M} B_t\}$ . If this is not the case, we have  $x \in \overline{\bigcup_{t \in M} B_t}$ ; then, if  $x \in B_s$  we can choose  $\delta > 0$  so small that  $B(x, \delta) \subseteq B_s$  and  $B(x, \delta) \subseteq (X \setminus B_s) \cup \bigcup_{t \in M} B_t$ , hence  $x \in \bigcup_{t \in M} B_t$ ; and if  $x \notin B_s$ , we get  $x \in \overline{\bigcup_{t \in M} B_t} \setminus B_s \subseteq \bar{B}_s \setminus B_s$ , by (vi). So any neighbourhood of  $x$  meets  $B_s \setminus (B_{s*0} \cup B_{s*1}) \subseteq B_s \setminus \bigcup_{t \in M} B_t$  (by 5. and (vi)), contradicting  $x \in \text{Int}((X \setminus B_s) \cup \bigcup_{t \in M} B_t)$ . This shows that  $\mathbb{H}$  is closed under  $\Rightarrow$ .

From the characterization of  $\wedge$  and  $\Rightarrow$  ((ii) and (iv) above) in  $\mathbb{H}$  it is almost evident that  $\mathbb{H}$  is isomorphic to  $\mathbb{K}$ . This completes the proof of 2.6.

Combining 2.1.(iii), 2.2. and 2.6. we get

2.7. COROLLARY. Every metrizable space without isolated points is universal for intuitionistic predicate logic.

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