Some remarks on extending bar induction

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Communicated by Prof. A.S. Troelstra at the meeting of May 30, 1983

ABSTRACT

Extensions of bar induction considered in the literature (e.g. [T]) concern induction over trees, for example, subtrees of $\mathbb{N}^{<\mathbb{N}}$. We look here at more general "substructures" of $\mathbb{N}^{<\mathbb{N}}$ (which lead to more general families of paths than closed ones), for which the proper notion of induction is given by the concept of *formal space* of an infinitary geometric propositional theory ([FG]). In fact, much of what we say here is simply a specialisation of that general theory. On the other hand, the "completeness" theorem of [FG], which presents countably axiomatised formal spaces as images of subspaces of (formal) Baire space, enhances the interest of these special cases.

1. FORMAL SUBSPACES OF BAIRE SPACE

The theory of formal Baire space $\mathbb{B}$ is axiomatised in the natural way (see [FG]) by sequents of the form $u \Rightarrow \vee\Gamma$, where $u \in \mathbb{N}^{<\mathbb{N}}$ and $\Gamma \subseteq \mathbb{N}^{<\mathbb{N}}$. Subspaces of $\mathbb{B}$ are given by *theories* T, by adding further such axioms. It is convenient to note that axioms of the form $\langle\ \rangle \Rightarrow \vee\Gamma'$ suffice, since $u \Rightarrow \vee\Gamma$ is equivalent to $\langle\ \rangle \Rightarrow \vee(\Gamma \cup \{v \mid u, v \text{ incomparable}\})$.

In more topological terms, then, such a subspace is specified by stating that each of a family $\{V_i\}_{i\in I}$ of open subsets of Baire space should *cover* the whole.

A *point* of the subspace will then be a sequence α $(\in \mathbb{N}^{\mathbb{N}})$ belonging to $\bigcap_{i\in I} V_i$; more generally, α is a *point of* u iff α is a point with $u \leq \alpha$. Then (point-) *completeness* of the formal theory T says that the subspace has *enough points* (to distinguish opens), i.e. $u \vdash_T \vee\Gamma$ iff every point of u is a point of some $v \in \Gamma$.

Now ordinary bar induction *BI* is equivalent to completeness of the basic theory ([FG]), and for general T we get the following "extended" bar induction:

Suppose T axiomatised by $\{\langle\ \rangle \Rightarrow \bigvee\Gamma_i\}_{i\in I}$ where each Γ_i is closed downwards ("monotone"); let $K \subseteq \mathbb{N}^{<\mathbb{N}}$ also be monotone and let

(i) K bar points of T: $\forall\alpha\in Pt(T)\exists n\cdot\bar{\alpha}(n)\in K$

(ii) K be inductive for T: for each $i\in I$, $v\in\mathbb{N}^{<\mathbb{N}}$, if $\{u\in\Gamma_i|u\supseteq v\}\subseteq K$, then $v\in K$.

Then *bar induction for T*, $BI(T)$, says that for all such K, $\langle\ \rangle\in K$.

2. EXAMPLES

Note that our notion of bar induction depends on the given theory T, and not just on its set of points. There seems to be no natural form of "bar induction" applicable to arbitrary subsets of $\mathbb{N}^{\mathbb{N}}$; on the other hand, for certain such subsets there are canonical theories presenting them (e.g. (a) following).

(a) *Trees.* Any inhabited closed subset X of $\mathbb{N}^{\mathbb{N}}$ has a subtree A of $\mathbb{N}^{<\mathbb{N}}$ (in the sense of Tree′ in [T, 1.7]) associated with it, namely $A=\{u|\exists\alpha\in X\cdot u\subseteq\alpha\}$. The corresponding theory T is axiomatised by $\{\Rightarrow\bigvee\Gamma_n\}_{n\in\mathbb{N}}$ where $\Gamma_n=\{u\in A|lth(u)=n\}$. Then it is easy to see that $BI(T)$ corresponds to $EBI_0''(A)$ as in [T, 1.7], i.e. the natural form of bar induction for the tree A.

(b) *Closed sublocales.* The "localic" notion of "closed" subspace of formal Baire space does not in general coincide with the "pointwise" one in (a) (though it does if the various sets of sequences are decidable). A *closed sublocale* is given rather as the "complement" of an open set; in terms of theories, this means axioms of the form $u_i\Rightarrow\bigvee\phi$ (alternatively, $\Rightarrow\bigvee\{v|u_i, v$ incomparable$\}$), whose points are the complement of $\bigcup_{i\in I}V(u_i)$ where $V(u_i)=$ $=\{\alpha|u_i\subseteq\alpha\}$. Then a subset K is *inductive* (and monotone) for such a theory iff it is inductive in the usual sense (for *BI*) and $\forall i\in I\cdot u_i\in K$.

(c) *Arithmetic* G_δ. In the general theory of formal spaces an important rôle is played by *absolute* theories, i.e. those whose interpretations in sheaf models are "constant" (i.e. essentially the same as externally). Among these are those which are *arithmetically* defined ("without choice parameters" in the language of [T]), that is, the axioms have the form $\Rightarrow\bigvee\{u|A(u,n)\}$, for $n\in\mathbb{N}$, A arithmetic. The points are then $\{\alpha|\forall n\exists mA(\bar{\alpha}(m),n)\}$, hence form an *arithmetic* G_δ set (countable intersection of opens); we will denote by $EBI_0(G_\delta^{ar})$ extended bar induction for arithmetic G_δ. Our main purpose will be to point out the consistency of this with intuitionistic principles, and the corresponding derived rule for intuitionistic higher-order logic.

(d) *Classical theory.* If the index set (of the axioms) I is countable, a straightforward generalisation of the usual proof of *BI* gives $BI(T)$ classically. If I is allowed to be uncountable, however, T may have no points while remaining consistent (whence $BI(T)$ fails). For example, let T consist of axioms $\Rightarrow\bigvee\Gamma_\alpha$, for $\alpha\in\mathbb{N}^{\mathbb{N}}$, where $\Gamma_\alpha=\{u|u\not\subseteq\alpha\}$; the (opens of the) formal space of T then just consist of the *coperfect* opens of Baire space (FH, 4.1).

(a) For *arithmetic* G_δ (2 (c)) we get *EBI* holding in sheaves over many spaces, though there are spaces over which even *BI* fails ([FH, 3.8]). For a constructive treatment we can generalise [FH, 3.5] and show that, assuming $EBI_0(G_\delta^{ar})$ externally, the same holds internally in sheaves *over* any arithmetic G_δ subspace of $\mathbb{N}^{\mathbb{N}}$. For if A, B are arithmetic G_δ, the product $(A \times B)$ is also (homeomorphic to) one. Hence in $Sh(A)$ the (formal) space B_A has enough points, while the absoluteness of the theory for B ensures that B_A is the interpretation of the formal space of this theory *in* $Sh(A)$. The same holds when A is any formal space for which $EBI_o(G_\delta^{ar})$ implies that A has enough points (e.g. reals etc.).

(b) On the other hand, $EBI_0(G_\delta)$ fails generally in sheaf models. A counterexample even for trees of the form $A^{<\mathbb{N}}$ in sheaves over the reals is given in [FH, 3.10] and translated into a counterexample in the theory of lawless sequences in [T, 1.8]. This does not work even for *arbitrary* subtrees of $\mathbb{N}^{<\mathbb{N}}$ in sheaves over *Baire* space, however; in fact, one can show $KS+AC-NF+BI \Rightarrow EBI_0''$ (with parameters; it does not help though with the consistency of CS and EBI_0 with parameters), by combining results of [T] (KS makes all sets enumerable, and we use $AC-NF+KS \Rightarrow APC$, and 2.2(i), 2.5, 2.6 from [T]). For completeness, though, we give an ad hoc proof by some simple coding:

By KS and $AC-NF$ choose sequences α_u such that $u \in A$ iff $\exists x \cdot \alpha_u(x) \neq 0$, A a given subtree of $\mathbb{N}^{<\mathbb{N}}$. Let

$$u \in B \text{ iff } \forall i < lth(u) \cdot \alpha_{\bar{u}_1(i+1)}(j_2(u(i))) \neq 0$$

(and $\langle\ \rangle \in B$ as well), where j_1, j_2 are "unpairing functions" and

$$u_i = \langle j_i(u(0)), \ldots, j_i(u(lth\ u-1))\rangle.$$

Thus, for $u \in B$, $u_1 \in A$ and u_2 carries along the information guaranteeing this.

Then B is a decidable subtree of $\mathbb{N}^{<\mathbb{N}}$, so ordinary *BI* suffices to prove bar induction for B. Now let P be a monotone inductive bar of A, and set $u \in Q$ iff $u_1 \in P$. It is easy to check that Q is a monotone inductive bar of B, so that *BI* gives $\langle\ \rangle \in Q$, whence $\langle\ \rangle \in P$, as required.

(c) In contrast to the preceding result, EBI_0 for arbitrary *closed sublocales* (2 (b)) (hence also $EBI_0(G_\delta)$) contradicts WC^{cl}, weak continuity without parameters, which also holds in sheaves over Baire space. This holds even for sublocales of the finitary tree $3^{<\mathbb{N}}$.

We define first, for a given sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$, sequences $u_\alpha^n \in 3^{<\mathbb{N}}$, as follows:

(i) If $\forall k < n \cdot \alpha(k) = 1$ then $u_\alpha^n = \langle\ \rangle$

(ii) If k is least $< n$ with $\alpha(k) \neq 1$,

$$\begin{cases} \text{if } \alpha(k) = 0, & u_\alpha^n = 0^n \\ \text{if } \alpha(k) > 1, & u_\alpha^n = 2^n. \end{cases}$$

If $\exists k \cdot \alpha(k) \neq 1$ and, at the least such k, $\alpha(k)=0$, we write $\alpha < \mathbf{1}$; if $\alpha(k)>1$, we write $\alpha > \mathbf{1}$.

Each basic neighbourhood $V(u_\alpha^n)$ is clopen; we want then to consider the closed sublocale corresponding to (the complement of) the open set $\bigcup_n \neg V(u_\alpha^n)$. This is simply axiomatised by $T_\alpha = \{\Rightarrow u_\alpha^n\}_{n \in \mathbb{N}}$.

Clearly, if $\alpha < \mathbf{1}$, T_α has just the one point $\mathbf{0}$; and if $\alpha > \mathbf{1}$, just the one point 2. Now, if $\forall \alpha \cdot BI(T_\alpha)$ holds, then $\forall \alpha.\ T_\alpha$ has a point, provided that T_α can be shown to be *proper* [FG, 2.12–13], that is, whenever $\vdash_{T_\alpha} \vee \Gamma$, Γ is inhabited. But clearly not even the *first* value of any point of T_α can be found continuously in α on any neighbourhood of 1; since the form of the assertion has no further parameters, WC^{cl} must then fail.

All that remains is the properness of T_α, which is established by proving by an induction on proofs that, if $0^m \vdash_{T_\alpha} \vee \Gamma$ and $2^n \vdash_{T_\alpha} \vee \Delta$, then $\exists k[0^k \in \Gamma \vee 2^k \in \Delta]$.

4. EXTENDED BAR INDUCTION RULE

The techniques of [FJ] give a "completeness rule" for arbitrary formal spaces with absolute, definable axiomatisations, relative to intuitionistic higher-order logic. Their method refines the semantic proofs described in [H] for ordinary *BIR*, by avoiding a preliminary "recursive choice rule" ([H, Lemma 1]). Specialised to subspaces of formal Baire space we obtain an EBI_0-rule for arithmetic G_δ sets, strengthening [H, Appendix] which deals with certain trees only. The same method yields this rule also for the basic system extended by the axiom of partial choice for arithmetic subsets of $\mathbb{N}$.

ACKNOWLEDGEMENTS

This note was written in April 1982, when the second author was visiting the University of Münster. We would like to thank professor Diller for making this visit possible. The first author was supported by the Alexander von Humboldt-Foundation.

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