Notre Dame Journal of Formal Logic
Volume 24, Number 4, October 1983

On the Freyd Cover of a Topos

IEKE MOERDIJK*

A theory is said to have the disjunction-property (*DP*) if whenever a disjunction $\phi \vee \psi$ is provable in the theory, either ϕ or ψ must be provable. As is well-known, many theories for intuitionistic arithmetic and analysis have the *DP*. The *DP* for intuitionistic type theory was first established by Friedman. More recently, a purely topos theoretic proof has been given by Freyd. An extensive discussion of both methods can be found in [4]. Although Freyd's construction is much more elegant, A. Ščedrov and P. Scott have shown that the two methods are essentially the same in [7].

A question that arises immediately is the following: If one adds new symbols and a particular set of axioms T to the logical axioms and rules, does the resulting higher-order theory still have the *DP*? Some instances of this question in which T consists of a single axiom have been considered in [5]. In this note, we will obtain a syntactic description of a class of theories that have the *DP* by investigating some of the logical properties of the Freyd cover, thus extending the results of [5].

The results will *not* cover many of the higher-order analogues of theories of intuitionistic arithmetic and analysis which are known to have the *DP*. One reason for this is that, from a more logical point of view, the Freyd cover lacks many nice properties. For an alternative type of cover that fills this gap, the reader is referred to [6].

In the first section of this paper, we will motivate the Freyd cover from a more logical perspective. There is probably nothing new in this, but it still is important to realize that what is really going on is a straightforward generalization of more traditional methods used in the model theory of first-order

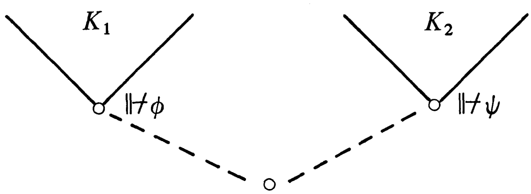
*The contents of this paper and of [6] were first presented at the Brouwer conference, June 1981. I am indebted to Josje Lodder for helpful discussions, and to the referee for his careful comments.

Received December 7, 1981; revised October 11, 1982

intuitionistic logic. Thus, the above-mentioned result of Ščedrov and Scott should not come as a surprise. This perspective also opens the way to connections with, for example, (higher-order analogues of) the Aczel-slash, and the Kleene-slash (see [8]).

In the second section, we examine preservation-properties of the Freyd cover, and prove the main result.

1 Motivating the Freyd cover Everybody knows how to prove the disjunction property for intuitionistic propositional logic (or Heyting's Arithmetic, etc.): If ϕ and ψ are two nonprovable formulas, just take two Kripke models $K_1 \nVdash \phi$ and $K_2 \nVdash \psi$, and add a new bottom node (this operator on Kripke models is called the Smorynski operator).



Then the bottom node cannot force $\phi \vee \psi$, so $\phi \vee \psi$ is not provable either (for details, see [8]).

Looking at this topologically, what we did was take two sheaf-models over spaces X_1 and X_2, take their topological sum $X_1 + X_2$, and define a new space $X = (X_1 + X_2) \cup \{*\}$, where $* \notin X_1 + X_2$ is a closed point of X whose only neighbourhood is the whole space X.

But this is precisely the situation for applying the theorem of Artin glueing [2], which says that you can get $Sh(X)$, the category of sheaves over X, by glueing along the global sections functor Γ,

$$Sh(X_1 + X_2) \cong Sh(X_1) \times Sh(X_2) \xrightarrow{\Gamma} Sets \cong Sh(*).$$

This is easily generalized for topoi, using the elementary form of Artin glueing ([3], Section 4.2): Given two topoi $\mathcal{E}_1$ and $\mathcal{E}_2$, let $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\Gamma} Sets$ be the global sections-functor $(1, -)$, and glue along Γ, i.e., make the comma category $(Sets \downarrow \Gamma)$. This topos $(Sets \downarrow \Gamma)$ is the Freyd cover of $\mathcal{E}_1 \times \mathcal{E}_2$, and will be denoted by $\mathcal{E}_1 * \mathcal{E}_2$. Objects of this topos are triples (X, E, ϕ), where X is a set, $E = (E_1, E_2)$ is an object of $\mathcal{E}_1 \times \mathcal{E}_2$, and ϕ is a function $X \to \Gamma E$. Recall (see [9]) that we have a geometric morphism

$$\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_1 * \mathcal{E}_2$$

with inverse image the forgetful functor $\mathcal{E}_1 * \mathcal{E}_2 \xrightarrow{U} \mathcal{E}_1 \times \mathcal{E}_2$, $U(X, E, \phi) = E$, and with direct image the cofree coalgebra functor $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{G} \mathcal{E}_1 * \mathcal{E}_2$, $GE = (\Gamma E, E, id_{\Gamma E})$. This geometric morphism is an open inclusion, so U is logical, and G preserves exponents.

We now want to reason as in the case of the Smorynski operator, roughly as follows: given two nonprovable formulas ϕ and ψ of intuitionistic higher-order logic, find topoi $\mathcal{E}_1$ and $\mathcal{E}_2$ with interpretations $\mathcal{I}_1$ in $\mathcal{E}_1$ and $\mathcal{I}_2$ in $\mathcal{E}_2$

such that $\mathscr{E} \underset{\mathscr{I}_1}{\not\models} \phi$ and $\mathscr{E}_2 \underset{\mathscr{I}_2}{\not\models} \psi$. Then the product $\mathscr{I} = \mathscr{I}_1 \times \mathscr{I}_2$ is an interpretation in $\mathscr{E}_1 \times \mathscr{E}_2$ such that $\mathscr{E}_1 \times \mathscr{E}_2 \underset{\mathscr{I}}{\not\models} \phi$ and $\mathscr{E}_1 \times \mathscr{E}_2 \underset{\mathscr{I}}{\not\models} \psi$. We now want to transport this interpretation $\mathscr{I}$ along G and obtain an interpretation $\overline{\mathscr{I}}$ in $\mathscr{E}_1 * \mathscr{E}_2$ with the property that $U \circ \overline{\mathscr{I}} = \mathscr{I}$. Since U is logical (and therefore preserves validity), $\mathscr{E}_1 * \mathscr{E}_2 \underset{\overline{\mathscr{I}}}{\not\models} \phi$ and $\mathscr{E}_1 * \mathscr{E}_2 \underset{\overline{\mathscr{I}}}{\not\models} \psi$. From a simple inspection of the subobject-classifier in the comma-topos $\mathscr{E}_1 * \mathscr{E}_2$ (the terminal object in $\mathscr{E}_1 * \mathscr{E}_2$ is indecomposable, see [5]) it then follows that $\mathscr{E}_1 * \mathscr{E}_2 \underset{\overline{\mathscr{I}}}{\not\models} \phi \vee \psi$. Below, we will discuss the problem of

(1) how to make $\overline{\mathscr{I}}$ out of $\mathscr{I}$?

Often, one starts with a theory T and two nonprovable formulas $T \not\vdash \phi$ and $T \not\vdash \psi$, and finds $\mathscr{E}_1$, $\mathscr{I}_1$ and $\mathscr{E}_2, \mathscr{I}_2$ such that $\mathscr{E}_1 \underset{\mathscr{I}_1}{\models} T$ and $\mathscr{E}_2 \underset{\mathscr{I}_2}{\models} T$, $\mathscr{E}_2 \underset{\mathscr{I}_1}{\not\models} \phi$, $\mathscr{E}_1 \underset{\mathscr{I}_2}{\not\models} \psi$. To show that T has the *DP*, one then wants $\mathscr{E}_1 * \mathscr{E}_2$ to be a model of T under the interpretation $\overline{\mathscr{I}}$, too. So we want to know

(2) for which theories T does it hold that whenever $(\mathscr{E}_1, \mathscr{I}_1)$ and $(\mathscr{E}_2, \mathscr{I}_2)$ are models of T, so is $(\mathscr{E}_1 * \mathscr{E}_2, \overline{\mathscr{I}})$?

(1) and (2) will be dealt with in the next section.

But before we turn to this, let us be more explicit about *interpretations.* We take a version of higher-order logic of the kind described in [1], which is sound and complete for interpretations in topoi. The language has two ingredients: sorts and constants. We have a set of ground sorts $\{s_i | i \in I\}$, from which we can build up the set of sorts inductively: every groundsort is a sort, and if $s_1, \ldots, s_n$, t are sorts, $[s_1, \ldots, s_n]$ is a sort (the sort of n-place relations taking arguments of sorts $s_1, \ldots, s_n$, respectively), and $[s_1, \ldots, s_n \to t]$ is a sort (the sort of functions taking n arguments of sorts $s_1, \ldots, s_n$, respectively, to a value of sort t). We also have a set of constants $\{c_j | j \in J\}$, together with an assignment $c \mapsto \#(c)$ of a sort to each constant. An interpretation $\mathscr{I}$ of the language in a topos $\mathscr{E}$ assigns to each groundsort an object $\mathscr{I}(s)$ of $\mathscr{E}$; $\mathscr{I}$ is then extended to all sorts by setting

$$\mathscr{I}([s_1, \ldots, s_n]) = \Omega^{\mathscr{I}(s_1) \times \ldots \times \mathscr{I}(s_n)};$$
$$\mathscr{I}(s_1, \ldots, s_n \to t) = \mathscr{I}(t)^{\mathscr{I}(s_1) \times \ldots \times \mathscr{I}(s_n)}.$$

Further, $\mathscr{I}$ assigns an arrow $\mathscr{I}(c)$: $1 \to \mathscr{I}(\#c)$ to each constant c. The interpretation of terms and formulas is then defined in the standard way (see, e.g., [1]).

Note that abstraction terms (terms of the form $\{\langle x_1, \ldots, x_n \rangle | \phi\}$) are eliminable in formulas. Therefore we will in the sequel assume that *formulas do not contain abstraction terms.*

Below, we will use the word *term* only in the following sense: variables and constants are terms, and if $\sigma_1, \ldots, \sigma_n$ are terms and f is a functional term of the appropriate sort, $f(\sigma_1, \ldots, \sigma_n)$ is a term. Thus, no quantifiers, connectives, or abstraction $(\{\cdot | \cdot\})$ can occur in terms. Note that every formula of the higher-order language is equivalent to one which is built up from atomic formulas of the form $R(\sigma_1, \ldots, \sigma_n)$ or $\sigma_1 = \sigma_2$, where $\sigma_1, \ldots, \sigma_n$ are terms in this sense and R is a relational term in this sense, by the usual clauses for the quantifiers and connectives. It is important to be explicit about this, as will appear in the sequel.

*2 **Preservation properties of the Freyd cover*** We consider a slightly more general situation: let $\mathscr{E}$ and $\mathscr{F}$ be topoi, and let $\mathscr{E} \xrightarrow{d} \mathscr{F}$ be a left-exact functor. We then have a geometric morphism $\mathscr{E} \to (\mathscr{F} \downarrow d)$ given by the forgetful functor U: $(\mathscr{F} \downarrow d) \to \mathscr{E}$ and the cofree coalgebra functor G: $\mathscr{E} \to (\mathscr{F} \downarrow d)$; U is logical, G preserves exponents, and $U \circ G = id_{\mathscr{E}}$. Suppose that we have an interpretation $\mathscr{I}$ of the logical language in $\mathscr{E}$. We want to construct an interpretation $\overline{\mathscr{I}}$ in $(\mathscr{F} \downarrow d)$ (cf. (1) above).

First note that $G\Omega_{\mathscr{E}}$ is a retract of $\Omega_{(\mathscr{F}\downarrow d)}$: the classifying morphism $G\Omega_{\mathscr{E}} \xrightarrow{\rho} \Omega_{(\mathscr{F}\downarrow d)}$ of $Gtrue$: $1 \simeq G1 \to G\Omega_{\mathscr{E}}$ is splitmono, with splitting $\Omega_{(\mathscr{F}\downarrow d)} \xrightarrow{\lambda} G\Omega_{\mathscr{E}}$ (the transpose of $U\Omega_{\mathscr{F}\downarrow d} \xrightarrow{\cong} \Omega_{\mathscr{E}}$).

For a groundsort s we define an object $\overline{\mathscr{I}}(s)$ of $(\mathscr{F} \downarrow d)$ by

$$\overline{\mathscr{I}}(s) = G\mathscr{I}(s)$$

$\overline{\mathscr{I}}$ is then uniquely (up to isomorphism) extended to all sorts. We then construct by induction on the sort s morphisms k_s and e_s

$$G\mathscr{I}(s) \overset{k_s}{\rightarrowtail} \overline{\mathscr{I}}(s) \overset{e_s}{\twoheadrightarrow} G\mathscr{I}(s)$$

with $e_s \circ k_s = 1_{G\mathscr{I}(s)}$, and $U(k_s) = U(e_s) = 1_{\mathscr{I}(s)}$. If s is a groundsort, then $k_s = e_s = 1_{G\mathscr{I}(s)}$. If $s = [t_1, \ldots, t_n]$, and we have defined k_{t_i} and $e_{t_i} (i = 1, \ldots, n)$, then k_s and e_s are defined as the compositions

$$\rho^{\overline{\mathscr{I}}(t_1)\times\ldots\times\overline{\mathscr{I}}(t_n)} \circ G\Omega_{\mathscr{E}}^{e_{t_1}\times\ldots\times e_{t_n}}$$

and

$$\lambda^{G\mathscr{I}(t_1)\times\ldots\times G\mathscr{I}(t_n)} \circ \Omega_{(\mathscr{F}\downarrow d)}^{k_{t_1}\times\ldots\times k_{t_n}}.$$

If $s = [t_1, \ldots, t_n \to r]$, and we have defined k_{t_i}, $e_{t_i} (i = 1, \ldots, n)$, k_r, e_r, then k_s and e_s are the following two compositions

$$\mathscr{I}(r)^{e_{t_1}\times\ldots\times e_{t_n}} \circ k_r^{G\mathscr{I}(t_1)\times\ldots\times G\mathscr{I}(t_n)}$$

and

$$G\mathscr{I}(t)^{k_{t_1}\times\ldots\times k_{t_n}} \circ e_r^{\overline{\mathscr{I}}(t_1)\times\ldots\times\overline{\mathscr{I}}(t_n)}.$$

$\overline{\mathscr{I}}$ is then defined for constants as follows: if $\#c = s$, then

$$\overline{\mathscr{I}}(c) = 1 \simeq G1 \xrightarrow{G\mathscr{I}(c)} G\mathscr{I}(s) \xrightarrow{k_s} \overline{\mathscr{I}}(s).$$

This completes the definition of $\overline{\mathscr{I}}$. Note that $U \circ \overline{\mathscr{I}} = \mathscr{I}$. Since U is logical, we immediately have

2.1 Lemma *Let ϕ be an arbitrary formula, with free variables among $x_1, \ldots, x_n$. Then*

$$U\left([\![\phi]\!]_{\overline{\mathscr{I}}} \rightarrowtail \prod_{i=1}^{n} \mathscr{I}(\#x_i)\right) = \left([\![\phi]\!]_{\mathscr{I}} \rightarrowtail \prod_{i=1}^{n} \mathscr{I}(\#x_i)\right),$$

and similarly for terms.

For an atomic formula $R(\tau_1, \ldots, \tau_n)$, where R is a relational constant, and $\tau_1, \ldots, \tau_n$ are terms (recall the convention at the end of Section 1) with free variables among $x_1, \ldots, x_k$, and $\mathcal{I}(\#x_i) = A_i$, $[\![R(\tau_1, \ldots, \tau_n)]\!]_{\mathcal{I}}$ defines a subobject of $A_1 \times \ldots \times A_k$ in $\mathcal{E}$, or a morphism $A_1 \times \ldots \times A_k \to \Omega$, or $1 \to \Omega^{A_1 \times \ldots \times A_k}$. Now what is $[\![R(\tau_1, \ldots, \tau_n)]\!]_{\overline{\mathcal{I}}}$ in $(\mathcal{F} \downarrow d)$? We will show that the association

(1) $[\![R(\tau_1, \ldots, \tau_n)]\!]_{\mathcal{I}} \mapsto [\![R(\tau_1, \ldots, \tau_n)]\!]_{\overline{\mathcal{I}}}$

corresponds to the following operation on subobjects

(2) $\Phi\colon \mathcal{E}(A, \Omega) \to (\mathcal{F} \downarrow d)(\overline{A}, \Omega)$

(here $A = \mathcal{I}(s_1) \times \ldots \times \mathcal{I}(s_k)$, $\overline{A} = \overline{\mathcal{I}}(s_1) \times \ldots \times \overline{\mathcal{I}}(s_k)$, for suitable $s_1, \ldots, s_k$): Φ associates with $1 \xrightarrow{f} \Omega^A$ the composition

$$1 \simeq G1 \xrightarrow{Gf} (\Omega^A) \xrightarrow{k} \Omega^{\overline{A}}$$

where k is the splitmono for $[s_1, \ldots, s_n]$. (In the sequel, we will usually omit the indices on the morphisms k_s and e_s.)

For the proof that (1) is the same as (2), first observe that for any term σ with free variables among $x_1, \ldots, x_n$, ($\mathcal{I}(\#x_i) = A$, $\overline{\mathcal{I}}(\#x_i) = \overline{A}$, $k_{A_i} = k_{\#x_i}$) the following diagram commutes (the proof is an easy induction on σ):

$$\begin{tikzcd}
\overline{A}_1 \times \ldots \times \overline{A} \arrow[r, "{[\![\sigma]\!]_{\overline{\mathcal{I}}}}"] & \overline{B} \\
GA_1 \times \ldots \times GA_n \arrow[u, "k_{A_1} \times \ldots \times k_{A_n}"] \arrow[r, "{[\![\sigma]\!]_{\mathcal{I}}}"] & GB \arrow[u, "k_B"']
\end{tikzcd}$$

Now suppose for ease of notation that R is a one-place relational constant, say with $\mathcal{I}(R)\colon 1 \to \Omega^B_{\mathcal{E}}$, and write $\mathcal{I}(\sigma)\colon 1 \to B^A$ for the transpose of $[\![\sigma]\!]_{\mathcal{I}}\colon A \to B$. Then the claim that

$$\Phi([\![R(\sigma)]\!]_{\mathcal{I}}) = [\![R(\sigma)]\!]_{\overline{\mathcal{I}}}$$

follows easily, if we can show that the following compositions (i) and (ii) are identical:

(i) $\overline{A} \xrightarrow{1 \times G\mathcal{I}(\sigma)} \overline{A} \times G(B^A) \xrightarrow{1 \times k} \overline{A} \times \overline{B}^{\overline{A}} \xrightarrow{ev} \overline{B} \xrightarrow{1 \times G\mathcal{I}(R)} \overline{B} \times G(\Omega^B_{\mathcal{E}}) \xrightarrow{1 \times k} B \times \Omega^{\overline{B}}_{\mathcal{F} \downarrow d} \xrightarrow{ev} \Omega_{\mathcal{F} \downarrow d}$

(ii) $\overline{A} \xrightarrow{1 \times G\mathcal{I}(R)} \overline{A} \times G(\Omega^B) \xrightarrow{1 \times G(\Omega^{[\![\sigma]\!]_{\mathcal{I}}})} \overline{A} \times G(\Omega^A) \xrightarrow{1 \times k} \overline{A} \times \Omega^{\overline{A}} \xrightarrow{ev} \Omega.$

But from the definition of k it follows that (1) is identical to

$$\overline{A} \xrightarrow{1 \times G\mathcal{I}(\sigma)} \overline{A} \times G(B^A) \xrightarrow{e \times 1} GA \times G(B^A) \xrightarrow{Gev} GB \xrightarrow{k} \overline{B} \xrightarrow{1 \times G\mathcal{I}(R)} \overline{B} \times G(\Omega^B)$$
$$\xrightarrow{e \times 1} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega$$

and since $e \circ k = id$, this is identical to

$$\bar{A} \xrightarrow{e} GA \xrightarrow{1\times G\mathcal{J}(\sigma)} GA \times G(B^A) \xrightarrow{Gev} GB \xrightarrow{1\times G\mathcal{J}(R)} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega.$$

Similarly, one shows that (2) is identical to

$$\bar{A} \xrightarrow{1\times G\mathcal{J}(R)} \bar{A} \times G(\Omega^B) \xrightarrow{e\times 1} GA \times G(\Omega^B) \xrightarrow{G[\![\sigma]\!]_{\mathcal{J}}\times 1} GB \times G(\Omega^B) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega.$$

And clearly, the latter two compositions are identical, since $\mathcal{J}(\sigma)$ is the transpose of $[\![\sigma]\!]_{\mathcal{J}}$. As is easily seen, this proves the correspondence of (1) and (2) not only for R a single constant, but also more generally for R a term without variables (i.e., R built up from constants by functional application only).

Let us now turn to the properties of the operation Φ. First a notational convention: a subobject of A is either represented by a mono $B \rightarrowtail A$, or its classifying morphism $A \xrightarrow{f} \Omega$, or its transpose $1 \xrightarrow{\hat{f}} \Omega^A$. In all these cases we will write $\Phi(B)$, $\Phi(f)$, $\Phi(\hat{f})$ for the corresponding representation of the subobject given by the original definition of Φ.

2.2 Lemma *Φ preserves conjunction (and hence Φ is orderpreserving).*

By "Φ preserves conjunction" we mean that if $f, g\colon A \to \Omega$ in $\mathcal{E}$, then $\Phi(\wedge_{\mathcal{E}} \circ (f, g)) = \wedge_{(\mathcal{F}\downarrow d)} \circ (\Phi(f), \Phi(g))$; similarly for the other cases to be considered below.

Proof: We have to show that

$$G(\Omega^A \times \Omega^A) \xrightarrow{G(\wedge^A)} G(\Omega^A) \xrightarrow{k} \Omega^{\bar{A}} = G(\Omega^A \times \Omega^A) \xrightarrow{k\times k} \Omega^{\bar{A}} \times \Omega^{\bar{A}} \xrightarrow{\wedge^{\bar{A}}} \Omega^{\bar{A}}.$$

Passing to the transposed maps, the left-hand side becomes

$$\begin{aligned} &\bar{A} \times G(\Omega^A \times \Omega^A) \xrightarrow{1\times G(\wedge^A)} \bar{A} \times G(\Omega^A) \xrightarrow{e\times 1} GA \times G(\Omega^A) \xrightarrow{Gev} G\Omega \xrightarrow{\rho} \Omega \\ &= \bar{A} \times G(\Omega^A \times \Omega^A) \xrightarrow{(e,e)\times 1} GA \times GA \times G\Omega^A \\ &\quad \times G\Omega^A \xrightarrow{(Gev\circ(\pi_1,\pi_3),\, Gev\circ(\pi_2,\pi_4))} G\Omega \times G\Omega \xrightarrow{G\wedge} G\Omega \xrightarrow{\rho} \Omega. \end{aligned}$$

Similarly, the right-hand side becomes

$$\begin{aligned} &\bar{A} \times G(\Omega^A \times \Omega^A) \xrightarrow{(e,e)\times 1} GA \times GA \times G\Omega^A \times G\Omega^A \xrightarrow{Gev\times Gev} G\Omega \\ &\quad \times G\Omega \xrightarrow{\rho\times\rho} \Omega \times \Omega \xrightarrow{\wedge} \Omega. \end{aligned}$$

Therefore, it suffices to show that

$$\begin{tikzcd} G\Omega \times \Omega \arrow[r, "\rho\times\rho"] \arrow[d, "G\wedge_{\mathcal{E}}"] & \Omega \times \Omega \arrow[d, "\wedge_{\mathcal{F}\downarrow d}"] \\ G\Omega \arrow[r, "\rho"] & \Omega \end{tikzcd}$$

commutes. But this follows easily from the fact that ρ classifies $G1 \xrightarrow{Gtrue} G\Omega$.

Note that from the fact that $\Phi\colon Sub_{\mathcal{E}}(A) \to Sub_{\mathcal{F}\downarrow d}(\bar{A})$ is orderpreserving, it immediately follows that for U and $V \in Sub_{\mathcal{E}}(A)$,

$$\begin{aligned} \Phi(U) \vee \Phi(V) &\leqslant \Phi(U \vee V) \\ \Phi(U \Rightarrow V) &\leqslant \Phi(U) \Rightarrow \Phi(V). \end{aligned}$$

2.3 Lemma *Φ preserves $\top_A$, the largest subobject of A. Also, Φ preserves $\bot_A$, the smallest subobject, provided d preserves the initial object* 0.

Proof: Following the same method as in the proof of Lemma 2.2, we see that it suffices to show that $G1 \xrightarrow{Gtrue} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{true} \Omega$ (which is clear from the definition of ρ) and that $G1 \xrightarrow{Gfalse} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{false} \Omega$. This latter identity only holds if G preserves the initial, or, equivalently, if d does. For in that case $\rho \circ Gfalse$ classifies the subobject $G0 \cong 0 \rightarrowtail 1 \cong G1$, since both squares of the diagram below are pullback

$$\begin{tikzcd}
G1 \arrow[r, "G\,false"] & G\Omega \arrow[r, "\rho"] & \Omega \\
G0 \arrow[u] \arrow[r] & G1 \arrow[u, "G\,true"'] \arrow[r] & 1 \arrow[u, "true"']
\end{tikzcd}$$

2.4 Remark: The properties of Φ that have been stated above also follow easily from the following alternative description of Φ: If $U \rightarrowtail A$ is a subobject of A, then $\Phi(U) \cong e^{-1}(GU)$; that is, the following diagram is pullback

$$\begin{tikzcd}
\bar{A} \arrow[r, two heads, "e"] & GA \\
\Phi(U) \arrow[u, tail] \arrow[r] & GU \arrow[u, tail]
\end{tikzcd}$$

2.5 Lemma *Φ preserves negation (provided d preserves* 0*).*

Proof: From the fact that $\Phi(U \Rightarrow V) \leqslant \Phi(U) \Rightarrow \Phi(V)$, and $\Phi(\bot_A) = \bot_{\bar{A}}$, it follows that $\Phi(\neg U) \leqslant \neg\Phi(U)$.

As for the converse, it again suffices (as in the proof of Lemma 2.2) to show that the subobject classified by $G\Omega \xrightarrow{\rho} \Omega \xrightarrow{\neg} \Omega$ is contained in the subobject classified by $G\Omega \xrightarrow{G\neg} G\Omega \xrightarrow{\rho} \Omega$. So make two pullbacks:

$$\begin{tikzcd}
G\Omega \arrow[r, tail, "\rho"] & \Omega \arrow[r, "\neg"] & \Omega \\
P \arrow[u, tail, "g"] \arrow[r, tail, "!"] & 1 \arrow[u, "false"'] \arrow[r] & 1 \arrow[u, "true"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G\Omega \arrow[r, "G\neg"] & G\Omega \arrow[r, "\rho"] & \Omega \\
G1 \arrow[u, "G\,false"'] \arrow[r] & 1 \arrow[u, "G\,true"'] \arrow[r] & 1 \arrow[u, "true"']
\end{tikzcd}$$

Now $P \leqslant G1$ in $Sub(G\Omega)$, for $\rho \circ Gfalse \circ ! = false \circ ! = \rho \circ g$, so $G\bot \circ ! = g$, since ρ is mono.

We now turn to the quantificational structure. Let's first consider universal quantification. Recall that $\Omega^B \xrightarrow{\forall_B} \Omega$ is the classifier of the exponentially transposed of $B \to 1 \xrightarrow{true} \Omega$. Universal quantification $Sub_{\mathcal{E}}(A \times B) \to Sub_{\mathcal{E}}(A)$

is then defined by composing an arrow $1 \to \Omega^{A \times B}$ with $(\forall_B)^A$: $\Omega^{A\times B} \cong (\Omega^B)^A \to \Omega^A$.

2.6 Lemma *Φ preserves universal quantification; that is, for a subobject $U \rightarrowtail A \times B$ in E, $\Phi(\forall_B(U)) \cong \forall_{\bar{B}}(\Phi(U))$.*

Proof: It again suffices to show that

$$\text{(i)}\quad G(\Omega^{A\times B}) \xrightarrow{G((\forall_B)^A)} G(\Omega^A) \xrightarrow{k} \Omega^{\bar{A}} = G(\Omega^{A\times B}) \xrightarrow{k} \Omega^{\bar{A}\times\bar{B}} \xrightarrow{(\forall_{\bar{B}})^{\bar{A}}} \Omega^{\bar{A}}.$$

It is easy to see that this would follow from

$$\text{(ii)}\quad G(\Omega^B) \xrightarrow{G(\forall_B)} G\Omega \xrightarrow{\rho} \Omega = G(\Omega^B) \xrightarrow{k} \Omega^{\bar{B}} \xrightarrow{\forall_{\bar{B}}} \Omega.$$

Since the left-hand side in (ii) classifies $G(\ulcorner true_B \urcorner)$,

$$\begin{tikzcd} G(\Omega^B) \arrow[r, "G(\forall_B)"] & G\Omega \arrow[r, "\rho"] & \Omega \\ G1 \arrow[u, "G(\ulcorner true_B \urcorner)"] \arrow[r] & G1 \arrow[u, "G\ true"'] \arrow[r] & 1 \arrow[u, "true"'] \end{tikzcd}$$

it suffices to show that the left-hand square of the diagram below is pullback

$$\begin{tikzcd} G(\Omega^B) \arrow[r, "k"] & \Omega^{\bar{B}} \arrow[r, "\forall_{\bar{B}}"] & \Omega \\ G1 \arrow[u, "G(\ulcorner true_B \urcorner)"] \arrow[r] & 1 \arrow[u, "\ulcorner true_{\bar{B}} \urcorner"'] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

But since k is mono, we only have to show that it commutes which is easy.

As for the existential quantifier, recall that $\Omega^B \xrightarrow{\exists_B} \Omega$ is the classifier of the image of $\in_B \overset{e_B}{\rightarrowtail} \Omega^B \times B \xrightarrow{\pi_1} \Omega^B$. (We will write $\exists_B$ for this image.)

2.7 Lemma *For a subobject $U \in Sub_{\mathcal{E}}(A \times B)$, $\exists_{\bar{B}}\Phi(U) \leqslant \Phi(\exists_B(U))$.*

Proof: As before, we have to show that the subobject of $G(\Omega^B)$ classified by $G(\Omega^B) \xrightarrow{k} \Omega^{\bar{B}} \xrightarrow{\exists_{\bar{B}}} \Omega$ is contained in that classified by $G(\Omega^B) \xrightarrow{G(\exists_B)} G\Omega \xrightarrow{\rho} \Omega$.

Now $\rho \circ G\exists_B$ classifies the image of $G\in_B \rightarrowtail GB \times G\Omega^B \xrightarrow{\pi} G\Omega^B$. Let P be the subobject of $G(\Omega^B)$ classified by $\exists_{\bar{B}}$. Pullbacks preserve epi-mono-factorizations, so P is the image of the pullback of $\in_{\bar{B}} \to \Omega^{\bar{B}} \times \bar{B} \xrightarrow{\pi} \Omega^{\bar{B}}$ along k, or, the image of $\pi \circ q$ in the diagram below

$$\begin{tikzcd} Q \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "pb."] & \bar{B} \times G\Omega^B \arrow[r, "\pi"] \arrow[d, "1 \times k"] & G\Omega^B \arrow[d, "k"] \\ \in_{\bar{B}} \arrow[r] & \bar{B} \times \Omega^{\bar{B}} \arrow[r, "\pi"] & \Omega^{\bar{B}} \end{tikzcd}$$

q is the pullback of $1 \xrightarrow{true} \Omega$ along $ev \circ (1 \times k) = \rho \circ Gev \circ (e \times 1)$

$$\begin{tikzcd}
B \times G\Omega^B \arrow[r, twoheadrightarrow, "e \times 1"] & GB \times G\Omega^B \arrow[r, "Gev"] & G\Omega \arrow[r, "\rho"] & \Omega \\
Q \arrow[u, rightarrowtail, "q"'] \arrow[r, twoheadrightarrow] & G{\in_B} \arrow[u, "Ge_B"'] \arrow[r] & G1 \arrow[u, "Gtrue"'] \arrow[r] & 1 \arrow[u, "true"']
\end{tikzcd}$$

We have to show that $P \leqslant G(\exists_B)$, or, that $\pi \circ q$ factors through $G\exists_B$, or, that $G\exists_B \circ \pi \circ q = Gtrue$. But $\pi \circ q = \pi \circ (e \times 1) \circ q = G\pi \circ Ge_B \circ s$, and, by definition, $\exists_B \circ \pi \circ e_B = true$, so $G\exists_B \circ G\pi \circ Ge_B \circ s = Gtrue$.

2.8 Lemma *Let $\Delta_A \rightarrowtail A \times A$ be the diagonal. Then*

(i) $\Phi(\Delta_A) \geqslant \Delta_{\bar{A}}$
(ii) *if e_A is iso,* $\Phi(\Delta_A) = \Delta_{\bar{A}}$.

Proof: Immediate from Remark 2.4.

We now return to question (2) of Section 1. Let us call an atomic formula *simple* if it is $\top$ or $\bot$, or it is either of the form $\sigma_1 = \sigma_2$, where σ_1 and σ_2 are terms (in the sense explained at the end of Section 1 !), or of the form $R(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n$ are terms, and R is a relational term without (free) variables occurring in it. Furthermore, we call an occurrence of $=$ in a formula *basic* if it occurs in a subformula $\sigma_1 = \sigma_2$, where σ_1 and σ_2 are terms whose sorts are nonrelational, that is, have been built up from groundsorts without using the rule to make $[s_1, \ldots, s_n]$ from $s_1, \ldots, s_n$.

2.9 Theorem *Let T be a theory which has a set of axioms of the form $\forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$, where the atomic parts of ϕ and ψ are simple, and*

- *$\exists$, $\forall$, and nonbasic $=$ occur only positively in ϕ, and only negatively in ψ*
- *$\rightarrow$ occurs only negatively in ϕ, and only positively in ψ.*

Then

(i) *if $(\mathscr{E}, \mathscr{J})$ is a model of T and $\mathscr{E} \xrightarrow{d} \mathscr{F}$ is a left-exact functor which preserves the initial object, then $((\mathscr{F} \downarrow d), \bar{\mathscr{J}})$ is a model of T,*
(ii) *T has the disjunction-property.*

Proof: (ii) follows from (i), and (i) follows easily from the properties of Φ that have been collected in the preceding lemmas.

We conclude with some remarks. First of all, it should be pointed out that the same techniques can be used to prove a result similar to Theorem 2.9 for theories having the existence property. Secondly, observe that the axioms of Higher-order Heyting's Arithmetic (*HHA*) are not preserved. In other words, if the language has a basic sort N for the natural numbers, and the theory T includes *HHA* $\mathscr{J}(N)$ must be the natural number object of $\mathscr{E}$ for $(\mathscr{E}, \mathscr{J})$ to be a model of T, but $\bar{\mathscr{J}}(N) = G\mathscr{J}(N)$ is, in general, not the natural number object of

$(\mathscr{F} \downarrow d)$. There are several ways to improve on this, one of them being contained in [6], so we will not go into this here.

Finally, a word about occurrences of the identity, which also illustrates the conditions on atomic formulas. Suppose, for example, that we have a constant f of a functional sort $[[s] \to [s]]$ that is interpreted in $(\mathscr{E}, \mathscr{I})$ by $\mathscr{I}(f)\colon \Omega^A \to \Omega^A$, and that $\mathscr{I}(f)$ equals the identity. Then $\mathscr{E} \models_{\mathscr{I}} \forall U\colon \Omega^A \cdot f(U) = U$, and the identity-symbol occurring in $\forall U\colon \Omega^A \cdot f(U) = U$ is nonbasic, so its preservation is not covered by the theorem. This is how it should be, since $\overline{\mathscr{I}}(f)$ is $\Omega^{\overline{A}} \xrightarrow{e} G(\Omega^A) \xrightarrow{k} \Omega^{\overline{A}}$ in this case, which is not the identity-arrow. Rewriting $\forall U\colon \Omega^A \cdot f(U) = U$ as $\forall U\colon \Omega^A\ \forall x\colon A(f(U)(x) \longleftrightarrow U(x))$ does not help, since now the atomic part $f(U)(x)$ is not simple.

REFERENCES

[1] Boileau, A. and A. Joyal, "La logique des topos," *The Journal of Symbolic Logic*, vol. 46 (1981), pp. 6-16.

[2] Grothendieck, A., *Théorie des Topos et Cohomologie Étale des Schémas*, Springer, New York, 1972. (See esp. IV, Section 9.5.)

[3] Johnstone, P., *Topos Theory*, Academic Press, New York, 1977.

[4] Lambek, J. and P. J. Scott, "Intuitionistic type theory and the free topos," *Journal of Pure and Applied Algebra*, vol. 19 (1980), pp. 215-257.

[5] Lambek, J. and P. J. Scott, "Independence of premisses and the free topos," pp. 109-227 in *Constructive Mathematics*, ed., F. Richman, Springer-Verlag, New York, 1980.

[6] Moerdijk, I., "Glueing topoi and higher order disjunction and existence," to appear in *Proceedings of the Brouwer Centenary Conference*, eds., D. van Dalen and A. Troelstra, North-Holland, Amsterdam, 1982.

[7] Ščedrov, A. and P. J. Scott, "Comparing the proofs of the disjunction and the existence property," to appear in *Proceedings of the Brouwer Centenary Conference*, eds., D. van Dalen and A. Troelstra, North-Holland, Amsterdam, 1982.

[8] Smorynski, C., "Applications of Kripke models," pp. 324-391 in *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, ed., A. Troelstra, Springer, New York, 1973.

[9] Wraith, G., "Artin glueing," *Journal of Pure and Applied Algebra*, vol. 4 (1974), pp. 345-348.

Instituut voor Logika and Grondslagenonderzoek
Universiteit van Amsterdam
Roetersstraat 15
1018 WB Amsterdam
The Netherlands