On the Freyd Cover of a Topos

IEKE MOERDIJK*

A theory is said to have the disjunction-property (DP) if whenever a disjunction $\phi \lor \psi$ is provable in the theory, either $\phi$ or $\psi$ must be provable. As is well-known, many theories for intuitionistic arithmetic and analysis have the DP. The DP for intuitionistic type theory was first established by Friedman. More recently, a purely topos theoretic proof has been given by Freyd. An extensive discussion of both methods can be found in [4]. Although Freyd's construction is much more elegant, A. Ščedrov and P. Scott have shown that the two methods are essentially the same in [7].

A question that arises immediately is the following: If one adds new symbols and a particular set of axioms $T$ to the logical axioms and rules, does the resulting higher-order theory still have the DP? Some instances of this question in which $T$ consists of a single axiom have been considered in [5]. In this note, we will obtain a syntactic description of a class of theories that have the DP by investigating some of the logical properties of the Freyd cover, thus extending the results of [5].

The results will not cover many of the higher-order analogues of theories of intuitionistic arithmetic and analysis which are known to have the DP. One reason for this is that, from a more logical point of view, the Freyd cover lacks many nice properties. For an alternative type of cover that fills this gap, the reader is referred to [6].

In the first section of this paper, we will motivate the Freyd cover from a more logical perspective. There is probably nothing new in this, but it still is important to realize that what is really going on is a straightforward generalization of more traditional methods used in the model theory of first-order

*The contents of this paper and of [6] were first presented at the Brouwer conference, June 1981. I am indebted to Josje Lodder for helpful discussions, and to the referee for his careful comments.

Received December 7, 1981; revised October 11, 1982
intuitionistic logic. Thus, the above-mentioned result of Ščedrov and Scott should not come as a surprise. This perspective also opens the way to connections with, for example, (higher-order analogues of) the Aczel-slash, and the Kleene-slash (see [8]).

In the second section, we examine preservation-properties of the Freyd cover, and prove the main result.

1 Motivating the Freyd cover Everybody knows how to prove the disjunction property for intuitionistic propositional logic (or Heyting’s Arithmetic, etc.): If $\phi$ and $\psi$ are two nonprovable formulas, just take two Kripke models $K_1 \not\models \phi$ and $K_2 \not\models \psi$, and add a new bottom node (this operator on Kripke models is called the Smorynski operator).

Then the bottom node cannot force $\phi \lor \psi$, so $\phi \lor \psi$ is not provable either (for details, see [8]).

Looking at this topologically, what we did was take two sheaf-models over spaces $X_1$ and $X_2$, take their topological sum $X_1 + X_2$, and define a new space $X = (X_1 + X_2) \cup \{\ast\}$, where $\ast \notin X_1 + X_2$ is a closed point of $X$ whose only neighbourhood is the whole space $X$.

But this is precisely the situation for applying the theorem of Artin glueing [2], which says that you can get $\text{Sh}(X)$, the category of sheaves over $X$, by glueing along the global sections functor $\Gamma$,

$$\text{Sh}(X_1 + X_2) \cong \text{Sh}(X_1) \times \text{Sh}(X_2) \xrightarrow{\text{Sets}} \text{Sets} \cong \text{Sh}(\ast).$$

This is easily generalized for topoi, using the elementary form of Artin glueing ([3], Section 4.2): Given two topoi $\mathcal{E}_1$ and $\mathcal{E}_2$, let $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\mathcal{E}} \text{Sets}$ be the global sections-functor $(1,-)$, and glue along $\Gamma$, i.e., make the comma category $(\text{Sets} \downarrow \Gamma)$. This topos $(\text{Sets} \downarrow \Gamma)$ is the Freyd cover of $\mathcal{E}_1 \times \mathcal{E}_2$, and will be denoted by $\mathcal{E}_1 \ast \mathcal{E}_2$. Objects of this topos are triples $(X, E, \phi)$, where $X$ is a set, $E = (E_1, E_2)$ is an object of $\mathcal{E}_1 \times \mathcal{E}_2$, and $\phi$ is a function $X \rightarrow \Gamma_E$. Recall (see [9]) that we have a geometric morphism

$$\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\mathcal{E}} \mathcal{E}_1 \ast \mathcal{E}_2$$

with inverse image the forgetful functor $\mathcal{E}_1 \ast \mathcal{E}_2 \xrightarrow{\mathcal{E}} \mathcal{E}_1 \times \mathcal{E}_2$, $U(X, E, \phi) = E$, and with direct image the cofree coalgebra functor $\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\mathcal{E}} \mathcal{E}_1 \ast \mathcal{E}_2$, $GE = (\Gamma_E, E, \text{id}_{\Gamma_E})$. This geometric morphism is an open inclusion, so $U$ is logical, and $G$ preserves exponents.

We now want to reason as in the case of the Smorynski operator, roughly as follows: given two nonprovable formulas $\phi$ and $\psi$ of intuitionistic higher-order logic, find topoi $\mathcal{E}_1$ and $\mathcal{E}_2$ with interpretations $J_1$ in $\mathcal{E}_1$ and $J_2$ in $\mathcal{E}_2$
such that \( \mathcal{E}_1 \models \phi \) and \( \mathcal{E}_2 \models \psi \). Then the product \( \mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \) is an interpretation in \( \mathcal{E}_1 \times \mathcal{E}_2 \) such that \( \mathcal{E}_1 \times \mathcal{E}_2 \models \phi \) and \( \mathcal{E}_1 \times \mathcal{E}_2 \models \psi \). We now want to transport this interpretation \( \mathcal{J} \) along \( G \) and obtain an interpretation \( \mathcal{J} \) in \( \mathcal{E}_1 \times \mathcal{E}_2 \) with the property that \( U \circ \mathcal{J} = \mathcal{J} \). Since \( U \) is logical (and therefore preserves validity), \( \mathcal{E}_1 \times \mathcal{E}_2 \models \phi \) and \( \mathcal{E}_1 \times \mathcal{E}_2 \models \psi \). From a simple inspection of the subobject-classifier in the comma-topos \( \mathcal{E}_1 \times \mathcal{E}_2 \) (the terminal object in \( \mathcal{E}_1 \times \mathcal{E}_2 \) is indecomposable, see [5]) it then follows that \( \mathcal{E}_1 \times \mathcal{E}_2 \models \phi \) and \( \mathcal{E}_1 \times \mathcal{E}_2 \models \psi \).

Below, we will discuss the problem of:

1. how to make \( \mathcal{J} \) out of \( \mathcal{J} \)?

Often, one starts with a theory \( T \) and two nonprovable formulas \( T \models \phi \) and \( T \models \psi \), and finds \( \mathcal{E}_1, \mathcal{J}_1 \) and \( \mathcal{E}_2, \mathcal{J}_2 \) such that \( \mathcal{E}_1 \models T \) and \( \mathcal{E}_2 \models T \), \( \mathcal{E}_2 \models \phi \), \( \mathcal{E}_1 \models \psi \). To show that \( T \) has the DP, one then wants \( \mathcal{E}_1 \times \mathcal{E}_2 \) to be a model of \( T \) under the interpretation \( \mathcal{J} \), too. So we want to know:

2. for which theories \( T \) does it hold that whenever \( \mathcal{E}_1, \mathcal{J}_1 \) and \( \mathcal{E}_2, \mathcal{J}_2 \) are models of \( T \), so is \( \mathcal{E}_1 \times \mathcal{E}_2, \mathcal{J} \)?

(1) and (2) will be dealt with in the next section.

But before we turn to this, let us be more explicit about interpretations. We take a version of higher-order logic of the kind described in [1], which is sound and complete for interpretations in topoi. The language has two ingredients: sorts and constants. We have a set of ground sorts \( \{s_i | t \in I\} \), from which we can build up the set of sorts inductively: every groundsort is a sort, and if \( s_1, \ldots, s_n \) are sorts, \( [s_1, \ldots, s_n] \) is a sort (the sort of \( n \)-place relations taking arguments of sorts \( s_1, \ldots, s_n \), respectively), and \( [s_1, \ldots, s_n] \rightarrow t \) is a sort (the sort of functions taking \( n \) arguments of sorts \( s_1, \ldots, s_n \), respectively, to a value of sort \( t \)). We also have a set of constants \( \{c_j | j \in J\} \), together with an assignment \( c \mapsto \#(c) \) of a sort to each constant. An interpretation \( \mathcal{J} \) of the language in a topos \( \mathcal{E} \) assigns to each groundsort an object \( \mathcal{J}(s) \) of \( \mathcal{E} \); \( \mathcal{J} \) is then extended to all sorts by setting:

\[
\mathcal{J}([s_1, \ldots, s_n]) = \mathcal{J}(s_1) \times \ldots \times \mathcal{J}(s_n),
\]

\[
\mathcal{J}(s_1, \ldots, s_n \rightarrow t) = \mathcal{J}(t) \mathcal{J}(s_1) \times \ldots \times \mathcal{J}(s_n).
\]

Further, \( \mathcal{J} \) assigns an arrow \( \mathcal{J}(c) : 1 \rightarrow \mathcal{J}(\#c) \) to each constant \( c \). The interpretation of terms and formulas is then defined in the standard way (see, e.g., [1]).

Note that abstraction terms (terms of the form \( \langle x_1, \ldots, x_n | \phi \rangle \)) are eliminable in formulas. Therefore we will in the sequel assume that formulas do not contain abstraction terms.

Below, we will use the word term only in the following sense: variables and constants are terms, and if \( a_1, \ldots, a_n \) are terms and \( f \) is a functional term of the appropriate sort, \( f(a_1, \ldots, a_n) \) is a term. Thus, no quantifiers, connectives, or abstraction \( (\cdot | \cdot | \cdot) \) can occur in terms. Note that every formula of the higher-order language is equivalent to one which is built up from atomic formulas of the form \( R(a_1, \ldots, a_n) \) or \( a_1 = a_2 \), where \( a_1, \ldots, a_n \) are terms in this sense and \( R \) is a relational term in this sense, by the usual clauses for the
quantifiers and connectives. It is important to be explicit about this, as will appear in the sequel.

2 Preservation properties of the Freyd cover  We consider a slightly more general situation: let $\mathcal{E}$ and $\mathcal{F}$ be topoi, and let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ be a left-exact functor. We then have a geometric morphism $\mathcal{E} \to (\mathcal{F} \downarrow d)$ given by the forgetful functor $U: (\mathcal{F} \downarrow d) \to \mathcal{E}$ and the cofree coalgebra functor $G: \mathcal{E} \to (\mathcal{F} \downarrow d)$; $U$ is logical, $G$ preserves exponents, and $U \circ G = id_\mathcal{E}$. Suppose that we have an interpretation $\mathcal{J}$ of the logical language in $\mathcal{E}$. We want to construct an interpretation $\overline{\mathcal{J}}$ in $(\mathcal{F} \downarrow d)$ (cf. (1) above).

First note that $G\Omega_\mathcal{E}$ is a retract of $\Omega(\mathcal{F} \downarrow d)$: the classifying morphism $G\Omega_\mathcal{E} \xrightarrow{\delta} \Omega(\mathcal{F} \downarrow d)$ of $\text{Gtrue}$: $1 \simeq G1 \to G\Omega_\mathcal{E}$ is split mono, with splitting $\Omega(\mathcal{F} \downarrow d) \xrightarrow{\Omega(j \circ d)} G\Omega_\mathcal{E}$ (the transpose of $U\Omega(\mathcal{F} \downarrow d) \Rightarrow \Omega_\mathcal{E}$).

For a groundsort $s$ we define an object $\overline{\mathcal{J}}(s)$ of $(\mathcal{F} \downarrow d)$ by

$$\overline{\mathcal{J}}(s) = G\mathcal{J}(s)$$

$\overline{\mathcal{J}}$ is then uniquely (up to isomorphism) extended to all sorts. We then construct by induction on the sort $s$ morphisms $k_s$ and $e_s$

$$G\mathcal{J}(s) \xrightarrow{k_s} \overline{\mathcal{J}}(s) \xrightarrow{e_s} G\mathcal{J}(s)$$

with $e_s \circ k_s = 1_{G\mathcal{J}(s)}$, and $U(k_s) = U(e_s) = 1_{\mathcal{J}(s)}$. If $s$ is a groundsort, then $k_s = e_s = 1_{G\mathcal{J}(s)}$. If $s = [t_1, \ldots, t_n]$, and we have defined $k_{t_i}$ and $e_{t_i}(i = 1, \ldots, n)$, then $k_s$ and $e_s$ are defined as the compositions

$$\rho^{\overline{\mathcal{J}}(t_1) \times \ldots \times \overline{\mathcal{J}}(t_n)} \circ G\Omega_\mathcal{E}^{t_1 \times \ldots \times t_n}$$

and

$$\lambda^{\overline{\mathcal{J}}(t_1) \times \ldots \times \overline{\mathcal{J}}(t_n)} \circ \Omega_\mathcal{E}^{t_1 \times \ldots \times t_n}.$$ 

If $s = [t_1, \ldots, t_n \to r]$, and we have defined $k_{t_i}$, $e_{t_i}(i = 1, \ldots, n)$, $k_r$, $e_r$, then $k_s$ and $e_s$ are the following two compositions

$$\rho^{\mathcal{J}(r) \times \ldots \times \mathcal{J}(t_n) \circ G\mathcal{J}(t_1) \times \ldots \times G\mathcal{J}(t_n)}$$

and

$$\lambda^{\mathcal{J}(r) \times \ldots \times \mathcal{J}(t_n) \circ \mathcal{J}(t_1) \times \ldots \times \mathcal{J}(t_n)}.$$ 

$\overline{\mathcal{J}}$ is then defined for constants as follows: if $\#c = s$, then

$$\overline{\mathcal{J}}(c) = 1 \simeq G1 \xrightarrow{G\mathcal{J}(c)} G\mathcal{J}(s) \xrightarrow{k_s} \overline{\mathcal{J}}(s).$$

This completes the definition of $\overline{\mathcal{J}}$. Note that $U \circ \overline{\mathcal{J}} = \mathcal{J}$. Since $U$ is logical, we immediately have

2.1 Lemma  Let $\phi$ be an arbitrary formula, with free variables among $x_1, \ldots, x_n$. Then

$$U\left(\lfloor \phi \rfloor_\mathcal{J} \Rightarrow \prod_{i=1}^n \mathcal{J}(\#x_i)\right) = \left(\lfloor \phi \rfloor_\mathcal{J} \Rightarrow \prod_{i=1}^n \mathcal{J}(\#x_i)\right),$$

and similarly for terms.
For an atomic formula \( R(\tau_1, \ldots, \tau_n) \), where \( R \) is a relational constant, and \( \tau_1, \ldots, \tau_n \) are terms (recall the convention at the end of Section 1) with free variables among \( x_1, \ldots, x_k \), and \( \mathcal{J}(\# x_i) = A_i \), \( \| R(\tau_1, \ldots, \tau_n) \|_J \) defines a subobject of \( A_1 \times \cdots \times A_k \) in \( \mathcal{E} \), or a morphism \( A_1 \times \cdots \times A_k \to \Omega \), or \( 1 \to \Omega^{A_1 \times \cdots \times A_k} \). Now what is \( \| R(\tau_1, \ldots, \tau_n) \|_J \) in \( (\mathcal{F} \downarrow d) \)? We will show that the association

(1) \[ \| R(\tau_1, \ldots, \tau_n) \|_J \] corresponds to the following operation on subobjects

(2) \( \Phi: \mathcal{E}(A, \Omega) \to (\mathcal{F} \downarrow d)(\mathcal{A}, \Omega) \)

(here \( A = \mathcal{J}(s_1) \times \cdots \times \mathcal{J}(s_k) \), \( \mathcal{A} = \mathcal{J}(s_1) \times \cdots \times \mathcal{J}(s_k) \)), for suitable \( s_1, \ldots, s_k \): \( \Phi \) associates with \( 1 \to \Omega^A \) the composition

\[ 1 \simeq G1 \xrightarrow{\mathcal{F}^{-1}} (\Omega^A) \xrightarrow{k} \Omega^\mathcal{A} \]

where \( k \) is the splitmono for \([s_1, \ldots, s_n]\). (In the sequel, we will usually omit the indices on the morphisms \( k_s \) and \( e_s \).)

For the proof that (1) is the same as (2), first observe that for any term \( \sigma \) with free variables among \( x_1, \ldots, x_n \), \( \mathcal{J}(\# x_i) = A, \mathcal{J}(\# x_i) = \mathcal{A}, k_{A_i} = k_{#x_i} \), the following diagram commutes (the proof is an easy induction on \( \sigma \)):

Now suppose for ease of notation that \( R \) is a one-place relational constant, say with \( \mathcal{J}(R): 1 \to \Omega^B \), and write \( \mathcal{J}(\sigma): 1 \to B \) for the transpose of \( \| \sigma \|_J: A \to B \). Then the claim that

\[ \Phi(\| R(\sigma) \|_J) = \| R(\sigma) \|_J \]

follows easily, if we can show that the following compositions (i) and (ii) are identical:

(i) \[
\begin{align*}
& A \xrightarrow{1 \times G(\mathcal{J}(\sigma))} A \times G(B^A) \xrightarrow{1 \times k} A \times \overline{B^A} \xrightarrow{ev} \overline{B} \\
& \Omega^A \xrightarrow{ev} \Omega^B \xrightarrow{\mathcal{F}^{-1} d} \Omega^A
\end{align*}
\]

(ii) \[
\begin{align*}
& A \xrightarrow{1 \times G(\mathcal{J}(\sigma))} A \times G(\Omega^B) \xrightarrow{1 \times G(1_{\Omega^{G^B}})} A \times G(\Omega^A) \xrightarrow{1 \times k} A \times \overline{\Omega^A} \xrightarrow{ev} \Omega.
\end{align*}
\]

But from the definition of \( k \) it follows that (1) is identical to

\[
\begin{align*}
& A \xrightarrow{1 \times G(\mathcal{J}(\sigma))} A \times G(B^A) \xrightarrow{ex1} GA \times G(B^A) \xrightarrow{Gev} GB \xrightarrow{k} \overline{B} \\
& \Omega^A \xrightarrow{ev} GB \times G(\Omega^B) \xrightarrow{Gev} \Omega^B \xrightarrow{ev} \Omega
\end{align*}
\]

and since \( e \circ k = id \), this is identical to
Similarly, one shows that (2) is identical to
\[ A \xrightarrow{1 \times G\J(R)} A \times G(\Omega^B) \xrightarrow{\text{ev}_1} GA \times G(\Omega^B) \xrightarrow{G\1 \times \text{ev}_1} GB \times G(\Omega) \xrightarrow{G\Omega} \Omega. \]

And clearly, the latter two compositions are identical, since \( J(\sigma) \) is the transpose of \( [\sigma]_J \). As is easily seen, this proves the correspondence of (1) and (2) not only for \( R \) a single constant, but also more generally for \( R \) a term without variables (i.e., \( R \) built up from constants by functional application only).

Let us now turn to the properties of the operation \( \Phi \). First a notational convention: a subobject of \( A \) is either represented by a mono \( B \to A \), or its classifying morphism \( A \to \Omega \), or its transpose \( 1 \to \Omega \). In all these cases we will write \( \Phi(B) \), \( \Phi(f) \), \( \Phi(f') \) for the corresponding representation of the subobject given by the original definition of \( \Phi \).

2.2 Lemma \( \Phi \) preserves conjunction (and hence \( \Phi \) is orderpreserving).

By "\( \Phi \) preserves conjunction" we mean that if \( f, g: A \to \Omega \) in \( \mathcal{C} \), then \( \Phi(\wedge_{\mathcal{C}} \circ (f, g)) = \wedge_{\mathcal{C}} (\Phi(f), \Phi(g)) \); similarly for the other cases to be considered below.

Proof: We have to show that
\[ G(\Omega^A \times \Omega^A) \xrightarrow{G(\wedge_{\mathcal{C}})} G(\Omega^A) \xrightarrow{\text{id}} \Omega^A = G(\Omega^A \times \Omega^A) \xrightarrow{\text{id} \times \text{id}} \Omega^A \times \Omega^A \xrightarrow{\wedge_{\mathcal{C}}} \Omega^A. \]

Passing to the transposed maps, the left-hand side becomes
\[
\begin{align*}
&\quad\frac{A \times G(\Omega^A \times \Omega^A)}{A \times G(\Omega^A)} \xrightarrow{1 \times G(\wedge_{\mathcal{C}})} A \times G(\Omega^A) \xrightarrow{\text{ev}_1} GA \times G(\Omega^A) \xrightarrow{G\1 \times \text{ev}_1} GB \times G(\Omega^A) \xrightarrow{G\Omega} G\Omega \xrightarrow{\Omega} \Omega \\
&= \frac{A \times G(\Omega^A \times \Omega^A)}{A \times G(\Omega^A)} \xrightarrow{(\text{ev}_1, \text{ev}_1)} GA \times GA \times G\Omega^A \\
&\quad\times G\Omega^A \xrightarrow{(G\text{ev}_{\sigma_1, \pi_1}, G\text{ev}_{\sigma_2, \pi_2})} G\Omega \times G\Omega \xrightarrow{G\wedge_{\mathcal{C}}} G\Omega \xrightarrow{\Omega} \Omega.
\end{align*}
\]

Similarly, the right-hand side becomes
\[
\begin{align*}
&\quad\frac{A \times G(\Omega^A \times \Omega^A)}{A \times G(\Omega^A)} \xrightarrow{(\text{ev}_1, \text{ev}_1)} GA \times GA \times G\Omega^A \xrightarrow{G\text{ev} \times G\text{ev}} G\Omega \\
&\quad\times G\Omega \xrightarrow{\rho \times \rho} \Omega \times \Omega \xrightarrow{\wedge_{\mathcal{C}} \circ \rho} \Omega.
\end{align*}
\]

Therefore, it suffices to show that
\[
\begin{align*}
G\Omega \times \Omega \xrightarrow{\rho \times \rho} \Omega \times \Omega \\
\xrightarrow{G\wedge_{\mathcal{C}}} G\Omega \xrightarrow{\wedge_{\mathcal{C}} \circ \rho} \Omega
\end{align*}
\]
commutes. But this follows easily from the fact that \( \rho \) classifies \( G1 \xrightarrow{\text{true}} G\Omega \).

Note that from the fact that \( \Phi: \text{Sub}_{\mathcal{C}}(A) \to \text{Sub}_{\mathcal{C}}(A) \) is orderpreserving, it immediately follows that for \( U \) and \( V \in \text{Sub}_{\mathcal{C}}(A) \),
\[
\begin{align*}
\Phi(U) \vee \Phi(V) &\leq \Phi(U \vee V) \\
\Phi(U \Rightarrow V) &\leq \Phi(U) \Rightarrow \Phi(V).
\end{align*}
\]
2.3 Lemma. \( \Phi \) preserves \( \top_A \), the largest subobject of \( A \). Also, \( \Phi \) preserves \( \bot_A \), the smallest subobject, provided \( d \) preserves the initial object \( 0 \).

Proof: Following the same method as in the proof of Lemma 2.2, we see that it suffices to show that \( G1 \xrightarrow{\text{true}} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{\text{true}} \Omega \) (which is clear from the definition of \( \rho \)) and that \( G1 \xrightarrow{\text{false}} G\Omega \xrightarrow{\rho} \Omega = 1 \xrightarrow{\text{false}} \Omega \). This latter identity only holds if \( G \) preserves the initial, or, equivalently, if \( d \) does. For in that case \( \rho \circ G\text{false} \) classifies the subobject \( G0 \cong 0 \Rightarrow 1 \cong G1 \), since both squares of the diagram below are pullback:

\[
\begin{array}{ccc}
G1 & \xrightarrow{G\text{false}} & G\Omega \\
\downarrow & & \downarrow \rho \\
G0 & \xrightarrow{G\text{true}} & 1
\end{array}
\]

2.4 Remark: The properties of \( \Phi \) that have been stated above also follow easily from the following alternative description of \( \Phi \): If \( U \Rightarrow A \) is a subobject of \( A \), then \( \Phi(U) \cong e^{-1}(GU) \); that is, the following diagram is pullback:

\[
\begin{array}{ccc}
\overline{A} & \xrightarrow{e} & GA \\
\downarrow & & \downarrow \\
\Phi(U) & \xrightarrow{GU} & 1
\end{array}
\]

2.5 Lemma. \( \Phi \) preserves negation (provided \( d \) preserves \( 0 \)).

Proof: From the fact that \( \Phi(U \Rightarrow V) \leq \Phi(U) \Rightarrow \Phi(V) \), and \( \Phi(\bot_A) = \bot_A \), it follows that \( \Phi(\neg U) \leq \neg\Phi(U) \).

As for the converse, it again suffices (as in the proof of Lemma 2.2) to show that the subobject classified by \( G\Omega \xrightarrow{\neg} \Omega \xrightarrow{\neg} \Omega \) is contained in the subobject classified by \( G\Omega \xrightarrow{G\neg} G\Omega \xrightarrow{\rho} \Omega \). So make two pullbacks:

\[
\begin{array}{ccc}
G\Omega & \xrightarrow{\rho} & \Omega \\
\downarrow g & & \downarrow \neg \\
P & \xrightarrow{!} & 1
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G\Omega & \xrightarrow{G\neg} & G\Omega \\
\downarrow G\text{false} & & \downarrow \rho \\
G1 & \xrightarrow{1} & 1
\end{array}
\]

Now \( P \leq G1 \) in \( \text{Sub}(G\Omega) \), for \( \rho \circ G\text{false} \circ ! = \text{false} \circ ! = \rho \circ g \), so \( G\bot \circ ! = g \), since \( \rho \) is mono.

We now turn to the quantificational structure. Let's first consider universal quantification. Recall that \( \Omega^B \xrightarrow{\forall B} \Omega \) is the classifier of the exponentially transposed of \( B \Rightarrow 1 \leadsto \Omega \). Universal quantification \( \text{Sub}_{\phi}(A \times B) \Rightarrow \text{Sub}_{\phi}(A) \)
is then defined by composing an arrow $1 \rightarrow \Omega^{A \times B}$ with $(\forall_B)^A$: $\Omega^{A \times B} \cong (\Omega^B)^A \rightarrow \Omega^A$.

2.6 Lemma $\Phi$ preserves universal quantification; that is, for a subobject $U \rightarrow A \times B$ in $E$, $\Phi(\forall_B(U)) \cong \forall_B(\Phi(U))$.

Proof: It again suffices to show that

\begin{enumerate}[(i)]
    \item $G(\Omega^{A \times B}) \xrightarrow{G((\forall_B)^A)} G(\Omega^A) \times G(\Omega^B) \xrightarrow{\kappa} G(\Omega^{A \times B}) \xrightarrow{G(\forall_B)} G(\Omega^B)$.
\end{enumerate}

It is easy to see that this would follow from

\begin{enumerate}[(ii)]
    \item $G(\Omega^B) \xrightarrow{G(\forall_B)} G(\Omega^B) \xrightarrow{\kappa} \Omega^B \xrightarrow{G(\forall_B)} \Omega$.
\end{enumerate}

Since the left-hand side in (ii) classifies $G(\text{true}_B)$,

\[
\begin{array}{ccc}
G(\Omega^B) & \xrightarrow{\rho} & \Omega \\
| & & | \\
G(\text{true}_B) & \xrightarrow{\text{true}} & \Omega \\
\end{array}
\]

it suffices to show that the left-hand square of the diagram below is pullback

\[
\begin{array}{ccc}
G(\Omega^B) & \xrightarrow{k} & \Omega^B \\
| & & | \\
G(\text{true}_B) & \xrightarrow{\text{true}_B} & \Omega \\
\end{array}
\]

But since $k$ is mono, we only have to show that it commutes which is easy.

As for the existential quantifier, recall that $\Omega^B \xrightarrow{\Omega_B} \Omega$ is the classifier of the image of $\exists_B \xrightarrow{\exists_B} \Omega^B \times G(\Omega^B) \xrightarrow{\pi_1} \Omega^B$. (We will write $\exists_B$ for this image.)

2.7 Lemma For a subobject $U \in \text{Sub}_A(A \times B)$, $\exists_B(\Phi(U)) \leq \Phi(\exists_B(U))$.

Proof: As before, we have to show that the subobject of $G(\Omega^B)$ classified by $G(\Omega^B) \xrightarrow{k} \Omega^B \xrightarrow{\exists_B} \Omega$ is contained in that classified by $G(\Omega^B) \xrightarrow{G(\exists_B)} G(\Omega) \xrightarrow{\delta} \Omega$.

Now $\rho \circ G\exists_B$ classifies the image of $G\subseteq_B \xrightarrow{G\exists_B} G\Omega^B \xrightarrow{\pi} G\Omega$. Let $P$ be the subobject of $G(\Omega^B)$ classified by $\exists_B$. Pullbacks preserve epi-mono-factorizations, so $P$ is the image of the pullback of $\exists_B \rightarrow G\Omega \xrightarrow{\pi} G\Omega^B$ along $k$, or, the image of $\pi \circ q$ in the diagram below

\[
\begin{array}{ccc}
Q & \xrightarrow{q} & B \times G\Omega^B \\
| & \downarrow{pb.} & | \\
\exists_B & \xrightarrow{1 \times k} & B \times G\Omega^B \\
| & \downarrow{k} & | \\
& \xrightarrow{\pi} & G\Omega^B \\
\end{array}
\]
$q$ is the pullback of $1 \xrightarrow{\text{true}} \Omega$ along $ev \circ (1 \times k) = \rho \circ Gev \circ (e \times 1)$

We have to show that $P \subseteq G(\exists_B \exists_B)$, or, that $\pi \circ q$ factors through $G(\exists_B \exists_B$), or, that $G(\exists_B \exists_B \circ \pi \circ q) = G(\text{true})$. But $\pi \circ q = \pi \circ (e \times 1) \circ q = G(\pi \circ Gev \circ s)$, and, by definition, $\exists_B \circ \pi \circ e = \text{true}$, so $G(\exists_B \circ G(\pi \circ Gev \circ s) = G(\text{true})$.

2.8 Lemma Let $\Delta_A \Rightarrow A \times A$ be the diagonal. Then

(i) $\Phi(\Delta_A) \geq \Delta_A$
(ii) if $e_A$ is iso, $\Phi(\Delta_A) = \Delta_A$.

Proof: Immediate from Remark 2.4.

We now return to question (2) of Section 1. Let us call an atomic formula simple if it is $T$ or $\bot$, or it is either of the form $a_1 = a_2$, where $a_1$ and $a_2$ are terms (in the sense explained at the end of Section 1!), or of the form $R(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are terms, and $R$ is a relational term without (free) variables occurring in it. Furthermore, we call an occurrence of $=$ in a formula basic if it occurs in a subformula $a_1 = a_2$, where $a_1$ and $a_2$ are terms whose sorts are nonrelational, that is, have been built up from groundsorts without using the rule to make $[s_1, \ldots, s_n]$ from $s_1, \ldots, s_n$.

2.9 Theorem Let $T$ be a theory which has a set of axioms of the form $\forall \exists(\phi(\vec{x}) \Rightarrow \psi(\vec{x}))$, where the atomic parts of $\phi$ and $\psi$ are simple, and

- $\exists, \forall$, and nonbasic $=$ occur only positively in $\phi$, and only negatively in $\psi$.
- $\rightarrow$ occurs only negatively in $\phi$, and only positively in $\psi$.

Then

(i) if $(\mathcal{C}, \mathcal{J})$ is a model of $T$ and $\mathcal{C} \xrightarrow{d} \mathcal{F}$ is a left-exact functor which preserves the initial object, then $((\mathcal{F} \downarrow d), \mathcal{J})$ is a model of $T$.
(ii) $T$ has the disjunction-property.

Proof: (ii) follows from (i), and (i) follows easily from the properties of $\Phi$ that have been collected in the preceding lemmas.

We conclude with some remarks. First of all, it should be pointed out that the same techniques can be used to prove a result similar to Theorem 2.9 for theories having the existence property. Secondly, observe that the axioms of Higher-order Heyting's Arithmetic (HHA) are not preserved. In other words, if the language has a basic sort $N$ for the natural numbers, and the theory $T$ includes HHA, $\mathcal{J}(N)$ must be the natural number object of $\mathcal{C}$ for $(\mathcal{C}, \mathcal{J})$ to be a model of $T$, but $\mathcal{J}(N) = G \mathcal{J}(N)$ is, in general, not the natural number object of
There are several ways to improve on this, one of them being contained in [6], so we will not go into this here.

Finally, a word about occurrences of the identity, which also illustrates the conditions on atomic formulas. Suppose, for example, that we have a constant $\xi$ of a functional sort $[[s] \to [s]]$ that is interpreted in $(\varphi, \xi)$ by $\xi(f) : \Omega^A \to \Omega^A$, and that $\xi(f)$ equals the identity. Then $\varphi \models \forall U : \Omega^A \cdot f(U) = U$, and the identity-symbol occurring in $\forall U : \Omega^A \cdot f(U) = U$ is nonbasic, so its preservation is not covered by the theorem. This is how it should be, since $\xi(f)$ is $\Omega^A \xrightarrow{\xi} G(\Omega^A) \xrightarrow{\xi} \Omega^A$ in this case, which is not the identity-arrow. Rewriting $\forall U : \Omega^A \cdot f(U) = U$ as $\forall U : \Omega^A \forall x : A(f(U)(x) \leftrightarrow U(x))$ does not help, since now the atomic part $f(U)(x)$ is not simple.

REFERENCES


Instituut voor Logica and Grondslagenonderzoek
Universiteit van Amsterdam
Roetersstraat 15
1018 WB Amsterdam
The Netherlands