SHEAF MODELS FOR CHOICE SEQUENCES*

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Contents

Introduction ...................... 63
1. Monoid models .................... 67
  1.1. Sheaves over monoids .............. 67
  1.2. Forcing ......................... 69
2. Modelling CS and its relativizations ............... 70
  2.1. The theory CS ................... 71
  2.2. The model for CS ............... 72
  2.3. Relativizations of CS ........... 82
3. The connection with the elimination translation ........ 85
  3.1. Constructive metatheory ........... 85
  3.2. Forcing and the elimination translation .......... 88
4. Spatial models ................... 91
5. Lawlessness ...................... 95
  5.1. Open data as analytic data ........ 96
  5.2. A sheaf model for LS .......... 98
  5.3. Projection models are Beth models ........ 103
References ......................... 106

Introduction

Sheaf models and toposes have by now become an important means for studying intuitionistic systems. They provide a unifying generalization of earlier semantic notions, such as Kripke models, topological (Beth) models, and realizability interpretations. Moreover, higher order languages with arbitrary function- and power-types can be interpreted naturally in these models.

In this paper we investigate sheaf models for intuitionistic theories of choice sequences. We will be mainly concerned here with sheaf models for the theories LS and CS in the language of elementary analysis with variables for numbers and sequences. Both systems are theories for (parts of) intuitionistic Baire space. The part of CS not involving lawlike function variables coincides with the system FIM of [12], which was intended as a codification of intuitionistic mathematical practice.

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The axioms of CS and LS are based on an analysis of how certain kinds of choice sequences are presented: thus, the conceptional viewpoint behind these systems is the 'analytic' one (as opposed to the 'holistic' viewpoint).

From the holistic viewpoint, the universe of choice sequences is grasped as a whole, and quantification over this domain is intuitively clear. From the analytic viewpoint, one sees choice sequences as individual objects, each given by a possibly non-predetermined construction process. Subdomains of choice sequences can be distinguished, according to the sort of information about a sequence that may become available at the various stages of its construction process. (For a discussion of holistic vs. analytic see [19].)

Extreme examples of subdomains of intuitionistic Baire space are the lawlike and the lawless sequences. Lawlike sequences are given by a set of rules which tell us how to construct a value for each given argument. These rules are the 'available data' on the sequence, they do not change during the construction process. The construction process of a lawless sequence, on the other hand, is comparable to the casting of an infinite-sided die, with the stipulation that an initial segment of the sequence may be deliberately fixed in advance. The available data on a lawless sequence consist at each stage of its construction of an initial segment of the sequence only.

LS is the formal theory of lawless sequences. The advantage of lawless sequences is that the relative simplicity of the available data makes it possible to justify rigorously (though informally) the validity of the traditional intuitionistic continuity axioms for this subdomain. The drawback of lawless sequences lies in the fact that the subdomain is not closed under any non-trivial continuous operation. LS is therefore not suited as a formal basis for intuitionistic analysis. The formal system CS is adequate for this purpose, it combines strong continuity axioms with closure under continuous operations. In general, on the analytic approach "one starts with (a conceptual analysis of) the idea of an individual choice sequence of a certain type (say 7) and attempts to derive from the way such a choice sequence is supposed to be given to us (i.e. from the type of data available at any given moment of its generation) the principles which should hold for the choice sequences of type 7" ([20, p. 5]).

The CS-axioms arise from the presupposition that there exists a notion of individual choice sequence for which the available data consist of lawlike continuous operations. The problem is to justify this presupposition, that is, to find a subdomain of intuitionistic Baire space for which the available data of its individual elements are the continuous operations (or any other subdomain of Baire space of which the CS-axioms can be seen to hold, cf. [10], [7]).

A common and important feature of LS and CS is, that their axioms give a full explanation of quantification over a subdomain of choice sequences in terms of quantification over lawlike objects. This is formally reflected in the elimination theorems for both systems.

Lawless sequences (of zero's and one's) first appear (as absolutely free se-
Sheaf models for choice sequences

quences) in [13]. In [14] lawless sequences of natural numbers are treated, with a sketch of the elimination theorem. An extensive treatment of LS can be found in [18].

The elimination translation provides a model for LS: it is a syntactic interpretation of LS in its lawlike part, which is a subsystem of classical analysis. [17] gives an 'internal' model for LS: it is shown that there exists a universe of sequences $\mathcal{U}_\alpha$, constructed from a single lawless $\alpha$, of which we can prove in LS that it is a model of LS. In [1] an LS-model is presented based on forcing techniques and Beth models. In the appendix to [1] it is shown that the 'internal model' construction of [17] is in fact equivalent to a Beth model construction.

CS was introduced and discussed extensively in [15]. A concise treatment can also be found in [18]. The elimination translation for CS (in [15]) gives a syntactic interpretation of this theory. [7] and [10] give models for relativized variants of CS. More specifically, universes $\mathcal{U}_\alpha$, constructed from a single lawless $\alpha$, are presented for which one can prove in LS that they are models for variants of CS. Such projection models correspond to Beth models in the ordinary sense. The motivation behind the 'reductionist program' of constructing such internal models for complex notions of choice sequence inside LS is discussed in [10].

The emphasis in this paper lies with the system CS. In fact, our original aims were

(a) to see whether it was possible to obtain monoid models for the system CS (and possibly also LS),

(b) to deny or confirm the first impression that there might be a connection between monoid forcing and the elimination translation, and

(c) to try and simplify the construction of models for variants of CS as presented in [7].

We briefly outline the contents of the paper: in Section 1 we give the basic concepts relevant for the interpretation of intuitionistic theories in sheaves over a site $(M, \mathcal{F})$, $M$ a monoid, and $\mathcal{F}$ a Grothendieck topology on $M$. In particular, we define 'Grothendieck topology $\mathcal{F}$ on a monoid $M$', 'sheaf over $(M, \mathcal{F})$', and we give the inductive clauses of Beth–Kripke–Joyal forcing over $(M, \mathcal{F})$. The material in this section is standard, and proofs are not given in detail. Readers familiar with such interpretations can skip Section 1. It is intended for those less at home in toposes. We assume all readers to be familiar with interpretations in sheaves over complete Heyting algebras (or over topological spaces). Such models occur in Sections 4 and 5. A good introduction to such models is [4].

One of the main results of this paper is that sheaves over monoids give us a new and very simple model for the theory CS. This will be proved in Section 2, where we also show how to obtain similar models for variants of CS (Section 2.3).

There are essentially two ways to explain the naturalness and simplicity of these models. On the one hand it can be shown that forcing over the monoid of Section 2.2 coincides (at lower types) with the elimination translation of [15] (cf. 3.2), while the elimination translation is in fact the canonical interpretation prescribed
by the axioms (cf. 3.1). On the other hand, the closure properties of the universe of choice sequences described by the CS-axioms (whatever that universe may be), can be captured in a geometric theory. The generic model in the classifying topos (in the sense of [16]) for this theory again coincides with the monoid model of Section 2.2. This correspondence will be worked out in [9]. (The relation between monoid models, elimination translations and classifying toposes described here for CS, also holds for the relativizations of CS discussed in 2.3.)

It should be remarked here that the techniques exploited in Section 2 can also be applied to theories which are analogous to CS or one of its relativized versions, but with Baire space replaced by the space of Dedekind reals. One then obtains models in which the Dedekind reals appear as the sheaf of continuous functions \( \mathbb{R} \to \mathbb{R} \). As in Section 2.2, a model satisfying the axiom of real-analytic data may then be constructed; and as in Section 2.3, one can construct a model in which there is a dense subset \( D \) of \( \mathbb{R} \) satisfying real open data

\[ \forall d \in D (Ad \to \exists d_1, d_2 (d_1 < d < d_2 \land \forall e \in D (d_1 < e < d_2 \to Ae))), \]

(this axiom has been considered in [8]).

We will not work out these models separately. One reason for this is that – as far as things go through – the proofs are completely analogous to those given in Section 2.2. Another, more fundamental, reason is that some of the results obtained in 2.2, like ‘all functions from \( \mathbb{R} \) to \( \mathbb{R} \) are continuous’ do not go through. One should not consider monoids of real functions, but sites of open subspaces of finite products \( \mathbb{R}^n \), in order to be able to obtain a full parallel with Sections 2 and 3 of this paper.

Returning to the subject matter of Section 3, we stress that the elimination theorem (and hence also, monoid models) give us a formal interpretation of choice sequences (“quantification over choice sequences as a figure of speech” [18]). In this respect, the models presented in Section 2 are completely different from the ones in [7], which grew out of a conceptual analysis of a primitive notion of choice sequence. As explained in [19], from the conceptual point of view sheaf models for choice sequences over (subspaces of) Baire space are of a particular interest. Therefore we will prove in the fourth section that with each of the monoid models one may associate a spatial model which is first-order equivalent to it. For countable monoids, the space will be a subspace of Baire space.

In Section 5 models for lawless sequences are discussed. We will first explain that although approximations of LS can be modelled in sheaves over monoids, LS itself cannot. We will then give an LS-model in sheaves over a topological space instead. This model is inspired by the model discussed in the appendix of [1]. The proofs we give, however, are semantical (in contrast to Troelstra’s original proofs), and our treatment works for a language with arbitrary function and powertypes (these are not contained in the original LS-language). We conclude this final section with a discussion of the elimination translation for LS, and its connection to projection models, and to our own model for LS.
Sheaf models for choice sequences

How to read this paper. We repeat that readers who are familiar with forcing over sites can skip Section 1. As will be apparent from this introduction, we have made some efforts to explain the connections with the existing intuitionistic literature on choice sequences. This will be done in the expository Sections 3 and 5.3. Readers who are mainly interested in seeing classical models for intuitionistic theories of choice sequences are advised to read Sections 2, 4, 5.1 and 5.2 only.

1. Monoid models

In this section we present some basic definitions and facts of sheaf semantics, for the particular case of sheaves over a monoid. The material is standard, and proofs are omitted or only briefly outlined.

1.1. Sheaves over monoids

A monoid $M$ is a category with just one object, or equivalently, a triple $M = (M, \cdot, 1)$, where $M$ is a set with an associative binary operation $\cdot$ which has a two-sided unit $1$. If $X$ is a set, an action of a monoid $M$ on $X$ is an operation $\cdot : M \times X \to X$ such that for any $x \in X$ and $f, g \in M$,

(i) $x \cdot 1 = x$,
(ii) $(x \cdot f) \cdot g = x \cdot (f \circ g)$.

Such pairs $(X, \cdot)$ are called $M$-sets; the element $x \cdot f$ of $X$ is called the restriction of $x$ to (or along) $f$. A morphism of $M$-sets $(X, \cdot) \to (Y, \cdot)$ is a function $\alpha : X \to Y$ which preserves the action; i.e., $\alpha(x \cdot f) = \alpha(x) \cdot f$ for any $x \in X$, $f \in M$. A sub-$M$-set of $(X, \cdot)$ is a subset $Y \subseteq X$ which is closed under $\cdot$; equivalently, a subset $Y \subseteq X$ with action $\cdot_Y$ such that the inclusion $Y \to X$ is a morphism of $M$-sets.

We give some examples of $M$-sets that will be used later. The set $\mathbb{N}$ of natural numbers can be made into an $M$-set by giving it the trivial action: $n \cdot f = n$ for $n \in \mathbb{N}$, $f \in M$. All elements of $\mathbb{N}$ are 'constant' for this action. Another $M$-set, which usually has hardly any constant elements, is the set of sieves (or cribles, or right-ideals) on $M$: a sieve on $M$ is a subset $S \subseteq M$ such that if $f \in S$ and $g \in M$ then also $f \circ g \in S$. The set of sieves is made into an $M$-set by setting $S \cdot f = \{g \in M | f \circ g \in S\}$.

Finally, note that $M$ itself may be regarded as an $M$-set, with action $f \cdot g = f \circ g$.

A (Grothendieck-) topology on $M$ is a family $\mathcal{G}$ of sieves on $M$ with the following properties:

(i) $M \in \mathcal{G}$,
(ii) if $S \in \mathcal{G}$ and $f \in M$, then $S \cdot f \in \mathcal{G}$,
(iii) if $R \subseteq M$, and there exists an $S \in \mathcal{G}$ such that $\forall f \in S \ (R \cdot f \in \mathcal{G})$ then $R \in \mathcal{G}$. 
The elements of $\mathcal{G}$ are called (\(\mathcal{G}\)-)covers, or (\(\mathcal{G}\)-)covering sieves. (When $S$ is a subset of $M$ (but not necessarily a sieve), we will often say that $S$ is a cover while we actually mean that the sieve $\{ s \circ f \mid s \in S, f \in M \}$ generated by $S$ is a cover.) It can be shown from (ii) and (iii) that if $S$ and $S'$ are covering sieves, so is $S \cap S'$.

An $\mathcal{M}$-set $(X, \uparrow)$ is called (\(\mathcal{G}\)-)separated if for each $S \in \mathcal{G}$, $\forall f \in S \ (x \uparrow f = y \uparrow f)$ implies $x = y$, for all $x, y \in X$. We now define sheaves: a collection $(x_f \mid f \in S)$ of an $\mathcal{M}$-set $(X, \uparrow)$ indexed by a sieve $S \in \mathcal{G}$ is called compatible if for each $g \in M$, $x_f \uparrow g = x_{reg}$. Now an $\mathcal{M}$-set $(X, \uparrow)$ is a (\(\mathcal{G}\)-)sheaf if for each compatible collection $(x_f \mid f \in S)$ there exists a unique $x$ (called the join of $(x_f \mid f \in S)$) with $x \uparrow f = x_f$ for each $f \in S$. By the uniqueness of joins, sheaves are separated.

Conversely, with a separated $\mathcal{M}$-set $(X, \uparrow)$ we can associate a sheaf $L(X, \uparrow)$ (the sheafification of $(X, \uparrow)$) as follows: the elements of $L(X, \uparrow)$ are equivalence-classes of compatible families $(x_f \mid f \in S)$ indexed by a cover $S$, where we identify two such families $(x_f \mid f \in S)$ and $(y_g \mid g \in T)$ if there exists a cover $R \subset S \cap T$ such that $x_f = y_f$ for each $f \in R$. The action of $\mathcal{M}$ on $L(X, \uparrow)$ is defined by

$$(x_f \mid f \in S) \uparrow g = (x_{g \circ h} \mid h \in S \uparrow g).$$

\(\uparrow\) is well-defined on equivalence-classes, and $L(X, \uparrow)$ is a sheaf. $L$ is functorial, in the sense that a morphism $(X, \uparrow) \rightarrow (Y, \uparrow)$ can be uniquely extended to a morphism $L\alpha : L(X, \uparrow) \rightarrow L(Y, \uparrow)$. (In fact, all this can be done also for $\mathcal{M}$-sets which are not necessarily separated. For details, see [16].)

For a monoid $\mathcal{M}$ with a topology $\mathcal{G}$ on it, the collection of sheaves and morphisms between them form a category $\text{Sh}(\mathcal{M}, \mathcal{G})$. This category is a topos, which means that it is possible to interpret higher-order intuitionistic logic in this category. Before we turn to this interpretation, let us indicate how to construct products, exponents, and powersets in $\text{Sh}(\mathcal{M}, \mathcal{G})$.

The product of two $\mathcal{M}$-sets $X = (X, \uparrow)$ and $Y = (Y, \uparrow)$ is simply the Cartesian product $XXY$ with pointwise action, $(x, y) \uparrow f = (x \uparrow f, y \uparrow f)$. It is easy to see that $XXY$ is a sheaf if $X$ and $Y$ are.

The exponent (function-space) $Y^X$ (or sometimes $(X \to Y)$) is defined to be the set of morphisms

$$\alpha : \mathcal{M} \times X \rightarrow Y$$

(where $\mathcal{M}$ is regarded as an $\mathcal{M}$-set), with action by $(\alpha \uparrow f)(g, x) = \alpha(f \circ g, x)$.

This makes the evaluation $(-) : Y^X \times X \rightarrow Y$, $\alpha(x) = \alpha(1, x)$ into a morphism of $\mathcal{M}$-sets. One can check that $Y^X$ is a sheaf whenever $Y$ is. There is a natural 1-1 correspondence between morphisms $Z \to Y^X$ and morphisms $Z \times X \to Y$ induced by the evaluation.

The $\mathcal{M}$-set of truthvalues ("the subobject classifier") $\Omega$ is the $\mathcal{M}$-set of $\mathcal{G}$-closed sieves on $\mathcal{M}$: A sieve $R$ on $\mathcal{M}$ is $\mathcal{G}$-closed if for any $f \in M$, $\exists S \in \mathcal{G} \forall s \in S \ (f \circ s \in R)$ implies $f \in R$. $\Omega$ is a sub-$\mathcal{M}$-set of the $\mathcal{M}$-set of sieves on $\mathcal{M}$, and $\Omega$ is a sheaf. There is a natural 1-1 correspondence between morphisms $\mathcal{M} \rightarrow \Omega$ and sub-sieves (sub-$\mathcal{M}$-sets which are sheaves) $U \subseteq X$: given $\alpha$, the corresponding $U$ is defined by $x \in U \iff 1 \in \alpha(x)$. Conversely, given $U \subseteq X$, $\alpha$ is defined by $\alpha(x) = \{ f \in M \mid x \uparrow f \in U \}$. Powerobjects $\mathcal{P}(X)$ are now constructed as exponents $\Omega^X$. 


1.2. Forcing

A language for higher-order logic consists of two parts, the set of sorts and the set of constants. The set of sorts can be built up inductively: the basic sort is the sort \( N \) of natural numbers; and if \( s_1, \ldots, s_n \) and \( t \) are sorts, then so are \( \mathcal{P}(s_1 \times \cdots \times s_n) \) (the sort of \( n \)-place relations taking arguments of sorts \( s_1, \ldots, s_n \) respectively), and \( I^n(s_1 \times \cdots \times s_n) \) (the sort of \( n \)-place functions taking arguments of sorts \( s_1, \ldots, s_n \) to a value of sort \( t \)). The other part is a set of constants \( \{ c_i \mid i \in I \} \), together with an assignment of a sort \( \#(c) \) to each constant \( c \). We also take the language to contain infinitely many variables of each sort.

A (standard-)interpretation \( \mathcal{I} \) of such a language in a topos of sheaves on a monoid \( \text{Sh}(\mathcal{M}, \mathcal{I}) \) assigns to each sort \( s \) a sheaf \( \mathcal{I}(s) \), according to the following rules:

(i) \( \mathcal{I}(N) \) is the sheafification of the constant \( \mathcal{M} \)-set \( N \) (we will usually write \( N \) for this sheaf).

(ii) \( \mathcal{I}(\mathcal{P}(s_1 \times \cdots \times s_n)) = \mathcal{I}(\mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n)) \), and \( \mathcal{I}(I^n(s_1 \times \cdots \times s_n)) = \mathcal{I}(I^n(s_1) \times \cdots \times \mathcal{I}(s_n)) \).

Further, \( \mathcal{I} \) assigns an element \( \mathcal{I}(c) \) of \( \mathcal{I}(\#c) \) to each constant \( c \), which is a fixed point of the action on \( \mathcal{I}(\#c) \) (this is the same as a morphism from the one-point \( \mathcal{M} \) set \( 1 \) to \( \mathcal{I}(\#c) \)). By the correspondences given at the end of 1.1, one may also think of the interpretation as assigning a subsheaf of \( \mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \) to a constant of sort \( \mathcal{P}(s_1 \times \cdots \times s_n) \), and a morphism \( \mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \rightarrow \mathcal{I}(t) \) to a constant of sort \( I^n(s_1 \times \cdots \times s_n) \). The empty product is 1, so the interpretation \( \mathcal{I}(\mathcal{P}() \) is the \( \mathcal{M} \)-set of truthvalues \( \Omega \).

Terms of the language are built up as usual. Terms of sort \( \mathcal{P}(() \) are called formulas. If \( \tau(x_1, \ldots, x_n) \) is a term of sort \( t \) with free variables among \( x_i \) of sort \( s_i \) (\( i = 1, \ldots, n \)), its interpretation (relative to \( x_1, \ldots, x_n \) will be a morphism \( \mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \rightarrow \mathcal{I}(t) \), for which we write \( \llbracket \tau \rrbracket_{s_1, \ldots, x_n} \) (or, just \( \llbracket \tau \rrbracket \)). It is defined inductively. First consider terms built up from variables and non-logical constants: we let \( \llbracket x \rrbracket \) be the projection \( \mathcal{I}(s_1) \times \cdots \times \mathcal{I}(s_n) \rightarrow \mathcal{I}(s_i) \); and if \( [\sigma] \) and \( [\tau_i] \) have been defined for \( i = 1, \ldots, n \), and \( \sigma \) and \( \tau_1, \ldots, \tau_n \) are of the appropriate sorts, then we let \( [\sigma(\tau_1, \ldots, \tau_n)] = [\sigma]([\tau_1], \ldots, [\tau_n]) \). For formulas we also have the possibility of making new formulas by use of logical constants. If \( A(x_1, \ldots, x_n) \) is a formula with \( x_i \) free, and \( \mathcal{I}(\#x) - Y_i, \llbracket A \rrbracket \) will be a morphism \( Y_1 \times \cdots \times Y_n \rightarrow \Omega \). Alternatively, \( \llbracket A \rrbracket \) is interpreted as a subsheaf of \( Y_1 \times \cdots \times Y_n \), and the correspondence is given by

\[
y = (y_1, \ldots, y_n) \in \llbracket A \rrbracket \iff 1 \in \llbracket A \rrbracket(y_1, \ldots, y_n).
\]

We will write \( \vdash A(y_1, \ldots, y_n) \) for \( 1 \in \llbracket A \rrbracket(y_1, \ldots, y_n) \). The definition of the interpretation can then be completed as follows: \( \vdash \sigma_1(y_1, \ldots, y_n) = \sigma_2(y_1, \ldots, y_n) \) iff \( [\sigma_1](y_1, \ldots, y_n) = [\sigma_2](y_1, \ldots, y_n) \),

\( \vdash R(\tau_1(y_1, \ldots, y_n), \ldots, \tau_k(y_1, \ldots, y_n)) \) iff \( (y_1, \ldots, y_n) \in \llbracket R(\tau_1, \ldots, \tau_k) \rrbracket \),

\( \vdash A \land B(y) \) iff \( \vdash A(y) \) and \( \vdash B(y) \),

\( \vdash A \lor B(y) \) iff there exists an \( S \in \mathcal{I} \) such that for each \( f \in S \) either \( \vdash A(y \upharpoonright f) \) or \( \vdash B(y \upharpoonright f) \),
\[ \vdash \neg A(y) \iff \text{for each } f \in M, \not\vdash A(y \uparrow f), \]
\[ \vdash A \rightarrow B(y) \iff \text{for each } f \in M \text{ such that } \vdash A(y \uparrow f), \text{ also } \vdash B(y \uparrow f), \]
\[ \vdash \forall x \ A(x)(y) \iff \text{for each } a \in \mathcal{I}(\#x) \text{ and each } f \in M \vdash A(a, y \uparrow f), \]
\[ \vdash \exists x \ A(x)(y) \iff \text{there exists an } S \in \mathcal{I} \text{ such that for each } f \in S \text{ we can find an } a_f \in \mathcal{I}(\#x) \text{ with } \vdash A(a_f, y \uparrow f). \]

Finally, we list some properties of the interpretation; the easy proofs are left to the reader.

**1.2.1. Lemma**

(i) \( \vdash A(y_1, \ldots, y_n) \) implies \( \vdash A(y_1 \uparrow f, \ldots, y_n \uparrow f) \), for each \( f \in M \).

(ii) If \( S \in \mathcal{I} \), and for each \( f \in S \), \( \vdash A(y_1 \uparrow f, \ldots, y_n \uparrow f) \), then also \( \vdash A(y_1, \ldots, y_n) \).

(iii) For closed \( A \), either \( \vdash A \) or \( \vdash \neg A \).

\( \vdash A(y_1, \ldots, y_n) \) is defined as \( (y_1, \ldots, y_n) \in \llbracket A \rrbracket \subseteq Y_1 \times \cdots \times Y_n \), so (i) says that \( \llbracket A \rrbracket \) is a sub-\( M \)-set, (ii) says that it is in fact a subsheaf, while (iii) says that the one-point \( M \)-set \( \llbracket 1 \rrbracket \) has only two subsheaves.

If \( X \) is a sheaf, a subset \( Y \) of \( X \) is said to generate \( X \) if every element of \( X \) is locally the restriction of an element of \( Y \); that is, for each \( x \in X \) we can find a cover \( S \in \mathcal{I} \) such that

\[ \forall f \in S \exists g \in M \exists y \in Y \ x \uparrow f = y \uparrow g. \]

Note that if the generating set \( Y \) is closed under restrictions, we may as a consequence of the preceding lemma restrict ourselves to \( Y \) when verifying whether a formula of the form \( \forall x : X A \) or \( \exists x : X A \) is forced. More precisely,

\[ \vdash \forall x : X A(x)(p) \iff \text{for all } y \in Y \text{ and all } f \in M, \vdash A(x)(y, p \uparrow f), \]
\[ \vdash \exists x : X A(x)(p) \iff \text{there is a cover } S \in \mathcal{I} \text{ such that for each } f \in S \text{ we can find a } y_f \in Y \text{ with } \vdash A(x)(y_f, p \uparrow f). \]

**1.2.2. Lemma.** For any standard-interpretation,

(i) \( \vdash \forall x : s \exists! y : t A(x, y) \iff \exists! f : t^1 \forall x : s A(x, f(x)), \)
\[ \vdash \exists! y : s \exists x \forall x : s (A(x) \leftrightarrow y(x)). \]

(ii) Adding constants \( 0 \) and \( S \) with their obvious interpretations, we obtain a model of higher-order Heyting’s Arithmetic (HAH) with full induction:

\[ \vdash \forall X : s \mathcal{P} N (X(0) \land \forall n : N(X(n) \rightarrow X(Sn)) \rightarrow X = N). \]

**2. Modelling CS and its relativizations**

In this section we will describe monoid models for the system CS (Section 2.2), and for the relativizations of CS which are considered in [7] (Section 2.3). We
shall reason classically about the models. Later on (in Section 3) we consider refinements using an intuitionistic metatheory. But first, we introduce some notation and state the CS-axioms.

2.1. *The theory CS*

CS was introduced and extensively discussed in [15]. The motivation behind its introduction was to give an adequate formal system for the foundation of intuitionistic analysis from the analytic viewpoint. The domain of choice sequences described by CS will be called $B_C$. We will write $B_L$ for the domain of lawlike sequences. Before stating the axioms, we introduce some notation.

We use $k, n, m, \ldots$ as variables for natural numbers, $e, \eta, \xi, \ldots$ as variables for elements of $B_C$, $u, v, w, \ldots$ as variables for finite sequences of natural numbers, and $a, b, c, \ldots$ as variables for lawlike mappings from $N$ to $N$, or from $N^{\leq N}$ to $N$. $x, y, z, \ldots$ are variables ranging over the whole of Baire space $B$, $\leq$ is used for the natural ordering between finite sequences. $*$ denotes concatenation. If $x \in B$ and $u \in N^{\leq N}$, then "$x \in u$" stands for "$x$ has initial segment $u$", and we often write $u$ for the basic open $\{x \mid x \in u\}$ of Baire space. If $x \in B$ and $n \in N$, then $\bar{x}(n)$ denotes the initial segment $(x(0), \ldots, x(n-1))$ of $x$ of length $n$.

Besides $B_C$ and $B_L$ there is a third set playing an important role in the theory CS, namely the set $K$ of lawlike inductive neighbourhood-functions (mappings from $N^{\leq N}$ to $N$). An element $a$ of $K$ has the following properties:

$$\forall x \in B \exists n \in N \ a(\bar{x}(n)) > 0,$$
$$\forall u, v \in N^{\leq N} \ (u \leq v \land a(u) > 0 \rightarrow a(u) = a(v)).$$

Such a function $a$ codes a continuous $g : N^N \rightarrow N^N$ by

$$g(x)(n) = m \text{ iff } \exists k a((n) * \bar{x}(k)) = m + 1.$$  

We put

$$K = \{a \mid a \in K\}.$$  

Thus $K \subseteq \text{cts}(N^N, N^N)$, the set of continuous functions from Baire space to Baire space. In fact, classically, $K$ is the set of all continuous functions; intuitionistically, 'continuous' is in this context usually defined as 'being an element of $K$'.

The system CS consists of the following axioms and schemata.

1. (closure and pairing)

$$\forall f \in K \forall c : \exists \eta f(\eta) = f(c),$$
$$\forall e, \eta f, g \in K : \exists \xi (e = f(\xi) \land \eta = g(\xi)).$$

2. (analytic data)

$$\forall e \ (A(e) \rightarrow \exists f \in K \ (\exists \eta (e = f(\eta)) \land \forall \eta A(f(\eta))).$$

3. (continuity for lawlike objects) For $p$ ranging over $N, B_L$, or $K$:

$$\forall e \exists p A(e, p) \rightarrow \exists a \in K \forall u (au \neq 0 \rightarrow \exists p \forall e \in u A(e, p)).$$
4. \((\forall \varepsilon \exists \eta \text{ continuity})\)

\[ \forall \varepsilon \exists \eta \ A(\varepsilon, \eta) \rightarrow \exists f \in \mathcal{K} \ \forall \varepsilon \ A(\varepsilon, f(\varepsilon)). \]

And finally a schema of lawlike countable choice

5. \((\text{AC–NF})\)

\[ \forall n \exists a \in B L A(n, a) \rightarrow \exists \text{ lawlike } N \xrightarrow{\text{f}} B L \forall n A(n, F n). \]

In the schemata, there are no free variables except possibly lawlike ones.

Observe that

(a) combination of CS\(3\) and AC–NF yields a principle of continuous choice analogous to CS\(4\);

(b) if \(i : N \times N \rightarrow N\) is bijective, and \(h\) is the induced homeomorphism from \(N^N \times N^N\) to \(N^N\), \(h(x, y)(n) = j(x(n), y(n))\), then \(\pi_1 \circ h^{-1}, \pi_2 \circ h^{-1} \in \mathcal{K}\), and for all \(f, g \in \mathcal{K}\), \(h \circ (f, g) \in \mathcal{K}\); hence \(B_C \times B_C \cong B_C\) via \(h\), by CS\(1\).

2.2. The model for CS

Consider the monoid \(\text{cts}(B, B)\) of endomorphisms of Baire space, equipped with the open cover topology \(\mathcal{g}\): for a sieve \(S\) we set \(S \in \mathcal{g}\) iff there is an open cover \(\{U_i : i \in I\}\) of \(B\) together with homeomorphisms \(B \rightarrow U_i\) such that each of the composites \(u_i : B \Rightarrow U_i \leftarrow B\) is in \(S\).

In connection with Section 3, we note the following. Let \(\mathcal{K}\) be the set of external neighbourhood functions, i.e. the set of functions \(f : N^N \rightarrow N\) which satisfy \(\forall x \in B \exists n \in N \ f(x(n)) > 0\) and \(\forall u, v (u \leq v \land f(u) > 0 \rightarrow f(v) = f(u))\). Then each cover \(S \in \mathcal{g}\) has a ‘characteristic function’ in \(\mathcal{K}\), i.e. with each \(S \in \mathcal{g}\) there is an \(f_S \in \mathcal{K}\) such that for all \(u \in N^N\), there is a homeomorphism \(B \Rightarrow \{x \in B \mid x \in u\}\) in \(S\) whenever \(f_S(u) \neq 0\). Conversely, with each \(f \in \mathcal{K}\) we may associate a cover \(S_f \in \mathcal{g}\), namely

\[ S_f = \{g \mid \exists u (f(u) \neq 0 \land \text{im}(g) \subseteq u)\}. \]

Our model will be the standard interpretation in \(\text{Sh}(\text{cts}(B, B), \mathcal{g})\). We start by identifying the sheaf of natural numbers \(N\) and internal Baire space \(N^N\) in this model.

2.2.1. Lemma. (a) \(N\) is isomorphic to the sheaf \(\text{cts}(B, N)\) of continuous functions \(B \rightarrow N\), with the monoid action given by composition, \(a \downarrow f = a \circ f\).

(b) \(N^N\) is isomorphic to \(\text{cts}(B, B)\), the monoid itself, with composition as the monoid action, \(f \downarrow g = f \circ g\).

Proof. (a) According to the definition of the standard interpretation given in Section 1.2, elements of \(N\) are equivalence classes of collections \((n_f \mid f \in S), n_f \in \mathbb{N}, S \in \mathcal{g}\), which are compatible (i.e., \(n_f = n_{fg}\) for all \(f \in S\) and \(g \in \text{cts}(B, B)\)). If \(S\) is a cover, \(S\) contains continuous functions \(u_i : B \Rightarrow U_i \leftarrow B\) for some open cover \(\{U_i\}\)
of \( B \), and with a compatible \((n_f \mid f \in S)\) we may associate a function \( a : B \to \mathbb{N} \) by
\[
a(x) = n_a \quad \text{iff} \quad x \in U_a.
\]
Then \( a \) is well-defined (by compatibility), and continuous. Conversely, each continuous function \( a : B \to \mathbb{N} \) determines an open cover \( \{ a^{-1}(n) \mid n \in \mathbb{N} \} \) of \( B \), and hence a cover \( S_a = \{ f \mid \text{im}(f) \subseteq \text{some } a^{-1}(n) \} \in \mathcal{G} \), together with a compatible collection \((n_f \mid f \in S_a)\), where \( n_f = m \) iff \( \text{im}(f) \subseteq a^{-1}(m) \). These two constructions are each others inverses (up to equivalence) and they both preserve the monoid-action.

(b) The exponent \( \mathbb{N}^N \) is the set of morphisms \( \tau : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}) \to \text{cts}(B, \mathbb{N}) \) with monoid-action given by \( (\tau \upharpoonright f)(g, a) = \tau (f \circ g, a) \) (see 1.1).

With such a \( \tau \) we associate the continuous function \( f_\tau : B \to B \) defined by
\[
f_\tau(x)(n) = \tau(1, \bar{n})(x),
\]
where \( \bar{n} : B \to \mathbb{N} \) is the constant function with value \( n \). Conversely, with \( f \in \text{cts}(B, B) \) we associate the morphism \( \tau_f \) defined by
\[
\tau_f(g, a)(x) = f(g(x))(a(x)).
\]
As in part (a), these two constructions are inverse to each other, and they preserve the monoid-action. \( \square \)

2.2.2. **Remark.** If \( f \in N^N, a \in N \), then functional application in the model is given by \( f(a) = \lambda x. f(x)(a(x)) \). Thus \( \models f(a) = b \) iff for all \( x \in B, f(x)(a(x)) = b(x) \).

\( N^{< N}, \) the sheaf of internal finite sequences of natural numbers, can be identified as \( \text{cts}(B, \mathbb{N}^{< N}) \), in a way analogous to 2.2.1(a). If \( f \in N^N \) and \( a \in N \), then \( \bar{f}(a) \), the initial segment of \( f \) with length \( a \), is \( \lambda x. \bar{f}(x)(a(x)) \in \text{cts}(B, \mathbb{N}^{< N}) \), and if \( u \in N^{< N} \), then \( \models f \in u \) (i.e. \( f \) has initial segment \( u \)) iff for all \( x, f(x) \in u(x) \). \( \square \)

Next we turn to the interpretation of lawlike objects. Intuitively one may think of the application of the monoid-action to an element of a sheaf as a step in a construction process. For example, one may regard an element \( f \in N^N \) as 'a choice sequence at some stage of its construction'. The information we have at that stage is, that the sequence lies in \( \text{im}(f) \). After restricting \( f \) to \( g \), we have the information that the sequence lies in \( \text{im}(f \circ g) \).

Lawlike elements are elements whose construction is completed. Therefore we put

2.2.3. **Definition.** Let \( X \) be any sheaf. \( X_L \) is the smallest subsheaf of \( X \) which contains the set \( \{ x \in X \mid x \text{ is invariant under the monoid-action} \} \), i.e. \( x \in X_L \) iff there is a cover \( S \) such that
\[
\forall f \in S \forall g \in \text{cts}(B, B) \ x \upharpoonright f = x \upharpoonright f \circ g.
\]
We call the elements of \( X_L \) the lawlike elements of \( X \).
Observe that \((N)_L = N, (N^{<\omega})_L = N^{<\omega}\); natural numbers and finite sequences of natural numbers are all lawlike. An element \(f\) of \(N^N \cong \text{cts}(B, B)\) is invariant under restrictions iff \(f\) is a constant function. Thus \(B_L\) is interpreted as the sheaf of locally constant functions from \(B\) to \(B\).

If \(x\) is an element of a sheaf \(X\), and \(f : B \to B\) is constant, then \(x \upharpoonright f\) is lawlike. In other words, each element has a lawlike restriction, so

\[ \vdash \forall x \in X \quad \neg \neg (x \text{ is lawlike}). \]

An immediate consequence of this observation is the following Specialization Property:

\((SP)\) \[ \vdash \exists x \in X \text{ A}(x) \to \exists x \in X_L \text{ A}(x) \]

for formulae \(A\) containing only lawlike parameters besides \(x\) (cf. 1.2.1). For \(X = N^N\), this property was formulated in [15].

We now consider internal neighbourhood functions. The exponent \(N^{(N^{<\omega})}\) is the set of morphisms \(f : \text{cts}(B, B) \times \text{cts}(B, N^{<\omega}) \to \text{cts}(B, N)\) with restrictions defined as \((f \upharpoonright g)(h, b) = f(g \circ h, b)\). We put

\[ K_0 \text{ is the sheaf } \{ f \in N^{(N^{<\omega})} \mid \vdash \forall g \in N^N \exists a \in N \, f(g(a)) > 0 \quad \land \forall u, v \in N^{<\omega} \, (u \leq v \land f(u) > 0 \to f(v) = f(u)) \}, \]

and we interpret \(K\) as the sheaf \((K_0)_L\) of lawlike elements of \(K_0\). Below we will show that the model satisfies Bar Induction, of which induction over \(K\) (and over \(K_0\)) is a well-known corollary.

Observe that an element of \(K\) which is invariant under restrictions is in fact a morphism \(\text{cts}(B, B) \to \text{cts}(B, N)\).

One easily proves the following. If \(f \in \mathcal{K}\) (the external set of neighbourhood functions) then \(\tilde{f} : \text{cts}(B, N^{<\omega}) \to \text{cts}(B, N)\) defined by \(\tilde{f}(b)(x) = f(b(x))\) is an element of \(K\), and conversely, if \(f \in K\) is invariant under restrictions, then \(f = \tilde{g}\) for some \(g \in \mathcal{K}\). Hence \(K\) is the sheaf of morphisms \(\text{cts}(B, B) \times \text{cts}(B, N^{<\omega}) \to \text{cts}(B, N)\) which are locally of the form \(\tilde{f}\) for some \(f \in \mathcal{K}\).

Let us look at internal functions on Baire space. The exponent \(N^N \to N^N\) is the set of morphisms \(F : \text{cts}(B, B) \times \text{cts}(B, B) \to \text{cts}(B, B)\), with restrictions defined by \((F \upharpoonright f)(g, h) = (F \circ f \circ g)(h)\). An \(F \in N^N \to N^N\) preserves the monoid-action: \(F(f \circ h, g \circ h) = F(f, g) \circ h\). Let \(h : B \to B \times B\) be a homeomorphism, and write \(\alpha = F(\pi_1 h, \pi_2 h)\). Then \(F(f, g) = \alpha \circ h^{-1} \circ (f, g)\) for any \(f, g \in \text{cts}(B, B)\), since \(f = \pi_1 \circ h \circ h^{-1} \circ (f, g)\) and \(g = \pi_2 \circ h \circ h^{-1} \circ (f, g)\). So \(F\) is completely determined by \(F(\pi_1 h, \pi_2 h)\).

An \(F \in N^N \to N^N\) which is invariant under restrictions is in fact a morphism from \(\text{cts}(B, B)\) to \(\text{cts}(B, B)\). Such an \(F\) is of the form \(F(f) = \alpha \circ f\), where \(\alpha = F(1)\). So lawlike elements of \(N^N \to N^N\) are locally of the form \(f \mapsto \alpha \circ f\) for some \(\alpha \in \text{cts}(B, B)\).

Each \(\alpha \in \text{cts}(B, B)\) has (externally) a neighbourhood function \(f_\alpha \in \mathcal{K}\), i.e. a func-
tion such that
\[ \forall x \ (\alpha(x)(n) = m \iff \exists k \ f_a((n) * x(k)) = m + 1). \]
With \( f_a \in K \) we may associate an internal neighbourhood function \( f_a \in K \) as above. One easily verifies that for \( F : g \mapsto \alpha \circ g \) in \( (N^N \to N^N)_L \),
\[ \vdash \ " f_a \text{ is a neighbourhood function for } F " \].
Hence for all \( F \in (N^N \to N^N)_L \),
\[ \vdash \ " F \text{ is continuous} " \].
We will prove below that \( \vdash \forall F : N^N \to N^N \ (F \text{ is continuous}) \).
\( K \) is the sheaf of all lawlike mappings from \( N^N \) to \( N^N \) which have a neighbourhood function in \( K \). It will be clear from the foregoing that \( K = (N^N \to N^N)_L \).

The last step in the definition of the CS-model is the interpretation of the universe of choice sequences \( B_C \). We interpret \( B_C \) as \( N^N \), internal Baire space.
Under this interpretation, the axiom of closure (the first part of CS1) is obviously true. The verification of the axiom of pairing (the other half of CS1) is straightforward. We state this explicitly in the following lemma.

2.2.4. Lemma. The standard interpretation in \( Sh(cts(B, B), f) \), with the interpretation of \( B_L, B_C, \) and \( K \) as described above, gives a model of CS1, i.e.
\[ \vdash \forall f \in K \ \forall \epsilon \in B_C \ (f(\epsilon) \in B_C), \]
\[ \vdash \forall \epsilon, \eta \in B_C \ \exists f, g \in K \ \exists \xi \in B_C \ (\epsilon = f(\xi) \land \eta = g(\xi)) \].

The following observation will help to simplify the proofs in the sequel.

2.2.5. Lemma. Let \( X \) be a sheaf, and let \( A(p_1, \ldots, p_n, x) \) be a formula (possibly containing parameters \( p_1, \ldots, p_n \)). Then if \( \vdash \exists x \in X \ A(p_1, \ldots, p_n, x) \), there exists a \( q \in X \) such that \( \vdash A(p_1, \ldots, p_n, q) \).
Proof. If \( \vdash \exists x \in X \ A(x) \), then there is an \( f \in K \) such that for all \( u \in N^N \),
\[ f(u) \neq 0 \Rightarrow \exists x_u \in X \vdash (A \upharpoonright \hat{u})(x_u), \]
where \( \hat{u} : B \to \{ y \mid y \in u \} \leftarrow B \), and \( A \upharpoonright \hat{u} \) stands for the formula \( A \) with all parameters restricted to \( \hat{u} \). Let \( \{ u_i \} \) be the set of minimal finite sequences such that \( f(u_i) \neq 0 \), and let \( S \) be the cover \( \{ g \in cts(B, B) \mid \exists i \ (im(g) \subseteq u_i) \} \). For each \( g \in S \) there is a (unique) \( i \) and a (unique) \( h \in cts(B, B) \), such that \( g = \hat{u}_i \circ h \). Let \( x_g = x_{u_i} \upharpoonright h \). Then the collection \( (x_g \mid g \in S) \) is compatible, so there is a unique \( q \in X \) with \( \forall g \in S : x_g = q \upharpoonright g \). For this \( q \) we have \( \vdash (A \upharpoonright g)(q \upharpoonright g) \) for each \( g \in S \); hence also \( \vdash A(q) \) (cf. 1.2.1). 

Each of the next three Theorems 2.2.6, 7, 9 consists of two parts, one stating the validity of a lawlike schema in the model, the other the validity of a related
axiom. We will briefly consider the connection between these two parts below, cf. Remark 2.2.13. From now on, in this section "\textsc{ff}" always refers to forcing in the interpretation in \( \text{Sh}(\text{cts}(B, B), \mathcal{G}) \) described above.

### 2.2.6. Theorem

(i) The schema of lawlike countable choice AC\(\text{-}N\) holds (\(X\) any sort; besides \(x, A\) contains lawlike parameters only):

\[
\text{\textsc{ff}} \forall n \in N \exists x \in X A(n, x) \rightarrow \exists F \in (X^N)_L \ \forall n A(n, F_n).
\]

(ii) The axiom of countable choice AC\(\text{-}N^*\) holds (\(X\) any sort):

\[
\text{\textsc{ff}} \forall P \in \mathcal{P}(N \times X) (\forall n \exists x P(n, x) \rightarrow \exists F \in X^N \ \forall n P(n, F_n)).
\]

**Proof.** (i) Suppose \(\text{\textsc{ff}} \forall n \in N \exists x \in X A(n, x)\). By 2.2.5 and the specialization property, for each \(n \in N\) we can find an \(x_n \in X_L\) with \(\text{\textsc{ff}} A(n, x_n)\), and by SP we may even assume that \(x_n\) is invariant under restrictions. Let \(F: \text{cts}(B, B) \times \text{cts}(B, N) \rightarrow X\) be the unique morphism determined by

\[
F(1, n) = x_n, \quad \text{for each } n \in N.
\]

Then \(F\) is lawlike, i.e. \(\text{\textsc{ff}} F \in (X^N)_L\), and \(\text{\textsc{ff}} \forall n A(n, F_n)\).

(ii) Choose \(P: \text{cts}(B, B) \times \text{cts}(B, N) \times X \rightarrow \Omega\) with \(\text{\textsc{ff}} \forall n \exists x P(n, x)\). For each \(n\) we may find (by Lemma 2.2.5) an \(x_n \in X\) such that \(\text{\textsc{ff}} P(n, x_n)\), i.e. \(P(1, \bar{n}, x_n) = T\). As in (i), let \(F: \text{cts}(B, B) \times \text{cts}(B, N) \rightarrow X\) be the morphism determined by \(F(1, n) = x_n\). Then \(\text{\textsc{ff}} \forall n P(n, F_n)\), for if \(n \in N\) and \(f \in \text{cts}(B, B)\), then

\[
P(f, \bar{n}, (F \uparrow f)(\bar{n})) = P(f, \bar{n}, x_n \uparrow f) = P(1, \bar{n}, x_n) \uparrow f = T \uparrow f \uparrow T. \quad \Box
\]

### 2.2.7. Theorem

(i) The schema of lawlike continuity for natural numbers C\(\text{-}N\) holds: \(A \) has all non-lawlike parameters shown

\[
\text{\textsc{ff}} \forall \varepsilon \in N^N \exists n \in N A(\varepsilon, n) \rightarrow \exists F \in K \ \forall u \in N^{<N} \ (Fu > 0 \rightarrow \exists n \ \forall \varepsilon \in u A(\varepsilon, n)).
\]

(ii) The axiom of continuity for natural numbers C\(\text{-}N^*\) holds:

\[
\text{\textsc{ff}} \forall P \in \mathcal{P}(N^N \times N) (\forall \varepsilon \exists n P(\varepsilon, n) \rightarrow \exists F \in K_0 \ \forall u \in N^{<N} \ (Fu > 0 \rightarrow \exists n \ \forall \varepsilon \in u P(\varepsilon, n))).
\]

**Proof.** (i) Suppose \(\text{\textsc{ff}} \forall \varepsilon \in N^N \exists n \in N A(\varepsilon, n)\). Then in particular, choosing \(\varepsilon\) the identity mapping, we find a continuous \(a: B \rightarrow \mathbb{N}\) such that \(\text{\textsc{ff}} A(1, a)\). Externally, \(a\) has a neighbourhood function \(g \in K\) determined by

\[
g(u) = m + 1 \quad \text{iff} \quad \forall x \in u \ a(x) = m.
\]

Internalizing this neighbourhood function gives us the \(F \in K\) with the required properties. More precisely, let

\[
F: \text{cts}(B, N^{<N}) \rightarrow \text{cts}(B, N), \quad F(\bar{u}) = g \circ \bar{u}.
\]

Choose any \(u \in N^{<N}\) such that \(\text{\textsc{ff}} F(\bar{u}) > 0\). Then \(a\) is constant on \(\{x \in B \mid x \in u\}\), say with value \(n\), and it follows easily from \(\text{\textsc{ff}} A(1, a)\) that \(\text{\textsc{ff}} \forall \varepsilon \in u A(1, \varepsilon)\).
(ii) Choose $P : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}) \times \text{cts}(B, B) \to \Omega$ such that $\vdash \forall \varepsilon \exists n P(n, \varepsilon)$. Fix any homeomorphism $h : B \to B \times B$, and find a continuous $a : B \to \mathbb{N}$ such that $P(a_1 h, a, a_2 h) = T$. Let $\{u_i : B \to U_i \hookrightarrow B\}$ be a disjoint cover such that $a \circ u_i$ is constant, say with value $n_i$. We now have to find an $F : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \text{cts}(B, \mathbb{N})$ such that

1. $F \in K_0$,
2. $\vdash \forall v (Fv \neq 0 \to \exists n \forall \varepsilon \in v P(n, \varepsilon))$.

Define, for $f \in \text{cts}(B, B)$ and $v \in \mathbb{N}^{<\mathbb{N}}$,

$$F(f, \bar{v}) = \begin{cases} 1, & \text{if for some } i, f(x)(\text{lth}(v_i)) \times v \subseteq h(U_i) \\ 0, & \text{otherwise.} \end{cases}$$

Note that $F(f, \bar{v})$ is continuous, and that $F(fg, \bar{v}) = F(f, \bar{v}) \circ g$, so $F$ determines a well-defined morphism $\text{cts}(B, B) \times \text{cts}(B, \mathbb{N}^{<\mathbb{N}}) \to \text{cts}(B, \mathbb{N})$.

We show that now (1) and (2) hold:

For (1), the only thing that is perhaps not immediately clear is that $\vdash \forall \varepsilon \exists u \in u \land Fv \neq 0$. To show this, choose $f$ and $g$ in $\text{cts}(B, B)$. Then $\forall x \in B \exists i \langle f(x), g(x) \rangle \in h(u_i)$, so

$$\forall x \exists i \exists u_x \exists f(x) \exists v_x \exists g(x) u_x \times v_x \subseteq h(U_i),$$

and we may assume $\text{lth}(u_i) = \text{lth}(v_x)$. Now choose for every $x$ a neighbourhood $w_x$ of $x$ such that $\forall y \in w_x f(y) \in u_x$; then

$$\forall x \exists i \forall y \in w_x \langle f(x), \text{lth}(v_x) \rangle \times v_x \subseteq h(U_i).$$

Thus, we have found a cover $\{w_i\}_i$ and finite sequences $v_i$ such that for each $x$, $g(w_i(x)) \in v_i$, and $\overline{f(w_i(x))(\text{lth}(v_i)) \times v_i} \subseteq h(U_i)$, in other words

$$\vdash \exists v \in \mathbb{N}^{<\mathbb{N}} (g \in v \land (F \upharpoonright f)(v) \neq 0).$$

Hence (1) holds.

For (2), choose $v$ and $f$ such that $\vdash F(f, \bar{v}) \neq 0$, i.e.,

$$\forall x \exists i \overline{f(x)(\text{lth}(v)) \times v} \subseteq h(U_i).$$

Now fix a cover $\{w_i : B \to W_i \hookrightarrow B\}_i$ such that for each $j$ there is one particular $U_i$ such that $\forall x \in W_j \overline{f(x)(\text{lth}(v)) \times v} \subseteq h(U_i)$. It suffices to show that for each $w_i$,

$$\vdash \exists n \forall \varepsilon \in \bar{v} \upharpoonright w_i \langle f \upharpoonright w_i \circ g(x), k(x) \rangle \in h(U_i).$$

To this end, let $n = n_i$, and choose $g$ and $k$ in $\text{cts}(B, B)$ such that $\vdash k \in \bar{v} \upharpoonright w_i \upharpoonright g$, i.e. $\forall x \in B k(x) \in v$. Then $\forall x \in B \langle f \circ w_i \circ g(x), k(x) \rangle \in h(U_i)$, so we can find a continuous $\psi : B \to B$ such that $\langle f w_i g, k \rangle = H \circ u_i \circ \psi$.

But then

$$P(f w_i g, n_i, k) = P(\pi_1 h u_i \psi, a u_i \psi, \pi_2 h u_i \psi)$$

$$= P(\pi_1 h, a, \pi_2 h) \upharpoonright u_i \psi$$

$$= T \upharpoonright u_i \psi = T,$$

so $\vdash (P \upharpoonright f w_i g)(n_i, k)$, which completes the proof of (2). □
Note that in the proofs of C–N and C–N* we did not use special properties of N, except that natural numbers are lawlike. Therefore,

**2.2.8. Corollary.** The model satisfies the schema of lawlike continuity for lawlike objects, and the axiom of continuity for lawlike objects. In particular, the schema CS3 holds in the model.

**2.2.9. Theorem.** (i) The scheme of lawlike continuity for sequences C–C holds in the model (where $A(\epsilon, \eta)$ is a formula with all non-lawlike parameters shown): \[
\mbox{\vdash \forall \epsilon \in N^N \exists \eta \in N^N A(\epsilon, \eta) \rightarrow \exists F \in K \forall \epsilon \in N^N A(\epsilon, F\epsilon).}
\]

(ii) The axiom of continuity for sequences C–C* holds:
\[
\mbox{\vdash \forall \epsilon \in N^N \exists \eta \in N^N A(\epsilon, \eta) \rightarrow \exists F : N^N \rightarrow N^N \forall \epsilon P(\epsilon, F\epsilon)).}
\]

**Proof.** (i) Suppose \(\mbox{\vdash \forall \epsilon \exists \eta A(\epsilon, \eta)}\). If we choose \(\epsilon = 1\), we find by Lemma 2.2.5 an \(f \in cts(B, B)\) such that \(\mbox{\vdash A(1, f)}\). Hence also \(\mbox{\vdash A(h, f \circ h)}\) for all \(h \in cts(B, B)\) (by Lemma 1.2.1, since all other parameters in \(A\) are lawlike). Thus letting \(F\) be the morphism “compose with \(f\)” : \(cts(B, B) \times cts(B, B) \rightarrow cts(B, B)\), \(F(g, h) = f \circ h\), proves (i) (cf. the discussion of the internal set \(K\) at the beginning of this subsection).

(ii) Choose a morphism \(P : cts(B, B) \times cts(B, B) \rightarrow \Omega\); suppose that \(\mbox{\vdash \forall \epsilon \exists \eta P(\epsilon, \eta)}\). In particular, we find that \(\mbox{\vdash \exists \eta (P \uparrow \pi_1 h)(\pi_2 h, \eta)}\), for a homeomorphism \(h : B \rightarrow B \times B\). By 2.2.5, there exists an \(f \in cts(B, B)\) such that \(\mbox{\vdash (P \uparrow \pi_1 h)(\pi_2 h, f)}\), i.e. \(P(\pi_1 h, \pi_2 h, f) = T\). Define a morphism
\[
F : cts(B, B) \times cts(B, B) \rightarrow cts(B, B)
\]
by
\[
F(g_1, g_2) = f \circ h^{-1} \circ (g_1, g_2).
\]
then for all \(g_1, g_2 \in cts(B, B)\),
\[
P(g_1, g_2, (F \uparrow g_1)(1, g_2)) = P(\pi_1 h^{-1}(g_1, g_2), \pi_2 h^{-1}(g_1, g_2), fh^{-1}(g_1, g_2)) = (P(\pi_1 h, f, \pi_2 h) \uparrow h^{-1})(g_1, g_2) = T,
\]
by choice of \(f\). So
\[
\mbox{\vdash \forall \epsilon P(\epsilon, F\epsilon)}.
\]
Finally, \(F\) is continuous, since all internal functions \(N^N \rightarrow N^N\) are continuous, by Theorem 2.2.15 below.

In the next theorem, we do not state the schema separately, since it follows from the axiom.
2.2.10. Theorem (Full Bar Induction BI*).

\[ \vdash \forall P \in \mathcal{P}(N^{<N})(\forall e \exists u \in P (e \in u) \land \forall u (\forall n u \ast \langle n \rangle \in P \leftrightarrow u \in P) \Rightarrow \langle \rangle \in P). \]

In the proof of this theorem, we will externally use the principle of 'double Bar Induction', which says that if \( U \) is a subset of \( N^{<N} \times N^{<N} \) barring each pair of sequences of natural numbers, and monotone and inductive in both arguments (separately), then \( (\langle \rangle, \langle \rangle) \in U \). This principle follows (constructively) from ordinary Bar Induction.

Proof of 2.2.10. Choose \( P : \text{cts}(B, B) \times \text{cts}(B, N^{<N}) \to \Omega \), such that

\[ \vdash \forall e \exists u \in P (e \in u) \land \forall u (\forall n u \ast \langle n \rangle \in P \leftrightarrow u \in P). \]

Now let

\[ \hat{P} = \{(u, v) \in N^{<N} \times N^{<N} \mid P(u, v) = T} \]

(here \( T \) is the top-element of \( \Omega \), \( \hat{u} \) is the function \( x \mapsto u \ast x \in \text{cts}(B, B) \), and \( \hat{v} \) is the constant function \( B \to N^{<N} \) with value \( v \)).

Clearly, \( \hat{P} \) is monotone and inductive in each of its arguments. For each \( x \in B \), let \( e_x \) be the constant function \( B \to B \) with value \( x \). Then \( \vdash \exists n \bar{e}_x(n) \in P \), hence for some continuous \( a : B \to N \), \( \vdash \bar{e}_x(a) \in P \), i.e. \( P(1, \lambda y \bar{e}_x(y)) = T \). If \( y \) is any element of \( B \), choose an initial segment \( u \) of \( y \) such that \( a \) is constant on \( \{ z \in B \mid z \in u \} \), say with value \( n \). Then \( P(\hat{u}, \hat{x}(n)) = T \), i.e. \( \langle u, \hat{x}(n) \rangle \in \hat{P} \). This shows that \( \hat{P} \) bars pairs of sequences, so by double Bar Induction, \( (\langle \rangle, \langle \rangle) \in \hat{P} \), i.e. \( \vdash (\langle \rangle) \in P \). □

2.2.11. Theorem (i) (Analytic Data). Let \( A(e) \) be a formula with all parameters lawlike, except for \( e \). Then

\[ \vdash \forall e \in N^N (A(e) \to \exists F \in K (\exists \eta F = F(\eta) \land \forall \zeta A(F(\zeta)))). \]

(ii) (Generalized Analytic Data). Let \( X_1, \ldots, X_n \) be arbitrary sorts, and let \( A(x_1, \ldots, x_n) \) be a formula with all parameters lawlike, except for the variables \( x_i \) of sort \( X_i \) (\( i = 1, \ldots, n \)). Then

\[ \vdash \forall x_1 \cdots \forall x_n (A(x_1, \ldots, x_n) \to \exists F \in (\mathcal{X}_1 \times \cdots \times \mathcal{X}_n)^{N^N}, \exists \eta (x_1, \ldots, x_n) = F \eta \land \forall \eta A(F(\eta))). \]

Proof. Since \( K = (N^N \rightarrow N^N)_{1 \times} \), (i) is a special case of (ii). To prove (ii), we may assume that \( n = 1 \), by taking the product \( X = X_1 \times \cdots \times X_n \) of sheaves. So suppose \( \vdash A(x) \), with \( x \in X \). Let \( F : \text{cts}(B, B) \times \text{cts}(B, B) \to X \) be the morphism defined by \( F(f, g) = x \uparrow g \). Then \( F \) is lawlike, \( \vdash F(1) = x \), and \( \vdash \forall \eta A(F(\eta)). \) □

Reviewing the properties of the model that have now been proved, we see that the CS-axioms are satisfied: CS1 was proved in 2.2.4, CS2 is 2.2.11(i), CS3 is 2.2.8, CS4 is 2.2.9(i), and AC–NF is a special case of 2.2.6(i). Thus,
2.2.12. Corollary. The standard interpretation in \( \text{Sh}(\text{cts}(B, B), \emptyset) \) (with \( B_L, B_C \) and \( K \) interpreted as described in the beginning of this section) gives a model of the theory CS.

2.2.13. Remark. We promised above to say a word about the relation between the lawlike schemata and the axioms. We have given separate proofs of the starred axioms AC-N*, C-N*, C-C* and BI* here in order to make our discussion of the properties of the model self-contained. Readers familiar with [15] will be aware of the fact that variants of the schemata AC-N, C-N, C-C, and BI with an additional parameter of type \( N^N \) follow logically from the schemata without a parameter via analytic data. We shall indicate briefly how the starred axioms follow from the lawlike schemata via generalized analytic data. For example, consider the relation between C-N and C-N*: to prove C-N*

\[
\varphi(V \in \text{lawlike} F: N^N \rightarrow \mathcal{P}(N^N \times N) \forall \eta \exists n F(\eta)(\varepsilon, \eta)
\]

it suffices, by generalized analytic data, to show that

\[
\varphi(\forall \exists n A(\eta, \varepsilon, n))
\]

for formulas \( A \) with no other non-lawlike parameters than \( \eta \) and \( \varepsilon \). In a similar way, AC-N* (C-C*, BI*) is reduced to AC-N (C-C, BI) with an additional parameter. The derivation of schemata with an additional parameter is treated in [15], Section 5.7. The proofs there use analytic data in the form

\[
\varphi(\forall \eta (A \eta \rightarrow B \eta) \leftrightarrow \forall f \in K (A \eta f(\eta) \rightarrow B \eta f(\eta)))
\]

A consequence of Theorem 2.2.7(ii) is that the model satisfies strong continuity principles for functions between metric spaces.

2.2.14. Theorem. Let \( (X, \rho) \) be an internal metric space, which is separable, i.e.

\[
\varphi(\exists d \in X^N \{d_n \mid n \in \mathbb{N}\} \text{ is dense in } X).
\]

Then

\[
\varphi(\text{"all functions } N^N \rightarrow X \text{ are continuous"}).
\]

Proof. Given a morphism \( F \in X^{N^N} \), consider the predicates

\[
P_k = \{(n, \varepsilon) \mid \rho(d_n, F(\varepsilon)) < 2^{-k}\} \in \mathcal{P}(N \times N^N),
\]

and apply C-N*.
2.2.15. Theorem. \( \vdash " \) if \( X \) is a complete separable metric space, and \( Y \) is a separable metric space, then all functions \( X \to Y \) are continuous".

Proof. In case \( X \) is Baire space, this is immediate from 2.2.14. The general case follows from the fact that AC–N* (logically) implies that every complete separable metric space \( X \) is a quotient of Baire space (i.e., \( \vdash \) "there exists a \( g : \mathbb{N}^N \to X \) such that for all \( V \subseteq X \), \( V \) is open in \( X \) iff \( g^{-1}(V) \) is open in \( \mathbb{N}^N \)"), see e.g. [20]. \( \square \)

Thus, for example, all functions from Baire space to itself, and all real functions are continuous in the model. It is perhaps illustrative to see what real numbers look like in the model: note that by countable choice (2.2.6) all reals are Cauchy. Since \( \mathbb{N} \) appears as \( \text{cts}(B, \mathbb{N}) \), the sheaf \( Q \) of internal rationals is the sheaf of locally constant functions \( B \to Q \), for which we write \( \text{loco}(B, Q) \). Sequences of rationals are morphisms \( \alpha : \text{cts}(B, B) \times \text{cts}(B, \mathbb{N}) \to \text{loco}(B, Q) \), and we can show that such \( \alpha \) are determined by their values \( \alpha(1, n) \), so \( Q^N \equiv \text{loco}(B, Q)^N \equiv \text{loco}(B \times \mathbb{N}, Q) \). The sheaf of Cauchy sequences \( C \subseteq Q^N \) is the subsheaf given by

\[
\alpha \in C \quad \text{iff} \quad \vdash \forall k \exists n \forall n' > n \: |\alpha n - \alpha n'| < 1/k,
\]

while elements of \( C \) are identified according to

\[
\vdash \alpha \sim \beta \quad \text{iff} \quad \vdash \forall k \exists n \forall n' > n \: |\alpha n' - \beta n'| < 1/k,
\]

We write \( \mathcal{R} \) for the sheaf of internal reals, which is the (internal) quotient \( C/\sim \).

2.2.16. Proposition. \( \mathcal{R} \) is isomorphic to the sheaf of continuous real valued functions on Baire space.

Proof. If \( \alpha \in C \), then by AC–N, \( \vdash \exists f \in \mathbb{N}^N \forall k \forall n' > f(k) \: |\alpha f(k) - \alpha n'| < 1/k \), hence (by 2.2.5) there exists a continuous \( f : B \to \mathbb{N}^N \) such that

\[
\vdash \forall k \forall n' > f(k) \: |\alpha f(k) - \alpha n'| < 1/k,
\]

or equivalently,

\[
(1) \quad \forall x \in B \forall k \in \mathbb{N} \forall n' > f(x)(k) \: |\alpha(x, f(x)(k)) - \alpha(x, n')| < 1/k.
\]

But (1) expresses that for each \( x \in B \), \( \{\alpha(x, n)\} \) is an (external) Cauchy-sequence, hence we have a function

\[
F_{\alpha} : B \to \mathbb{R}, \quad x \mapsto \lim_n \alpha(x, n)
\]

and it is straightforward to check that \( F_{\alpha} \) is continuous, and that \( \alpha \sim \beta \) implies that \( F_{\alpha} = F_{\beta} \).

Conversely, for each continuous \( g : B \to \mathbb{R} \) we can construct an internal Cauchy-sequence \( \sigma_g \in \text{loco}(B \times \mathbb{N}, Q) \) such that

\[
(2) \quad \sigma_{F_{\alpha}} \sim \alpha, \quad \text{and} \quad F_{\sigma_{g}}f = g,
\]
as follows. Given \( g \), fix for each \( n \) a cover \( \mathcal{U}^n = \{ U^n_k \} \) of \( B \) consisting of disjoint clopen subsets, with the property that

\[
\forall x, g \in U^n_k |g(x) - g(y)| < 1/n, \quad \text{and} \quad \mathcal{U}^{n+1} \text{ refines } \mathcal{U}^n.
\]

For each \( n \) and \( k \) we choose a rational \( q(n, k) \) such that

\[
\forall x \in U^n_k |g(x) - q(n, k)| < 2/n.
\]

Now let

\[
\sigma_g(x, n) = q(n, k(x)),
\]

where \( k(x) \) is the unique \( k \) with \( x \in U^n_k \). Then \( f : B \to \mathbb{N} \) defined by \( f(x)(n) = 4n \) is a modulus of convergence for \( \sigma_g \), i.e.

\[
\forall x \in B \forall k \forall n' > f(x)(k) |\sigma_g(x, f(x)(k)) - \sigma_g(x, n')| < 4/k,
\]

so we have that \( \sigma_g \) is a Cauchy-sequence'. Clearly, \( \lim_{n \to \infty} \sigma_g(x, n) = g(x) \), so the latter half of (2) holds. It is also straightforward to check that \( \sigma_g \sim \alpha \). To conclude the proof of the fact that \( \sigma \) and \( F \) are isomorphisms, it suffices to observe that they preserve the monoid-actions (the action on \( \text{loco}(B \times \mathbb{N}, \mathcal{Q}) \) is given by \( \alpha \upharpoonright f = \alpha \circ (f \times 1) \)), which is obvious.  \( \square \)

This concludes our discussion of the model. We will return to it from a different point of view in Section 3.

2.3. Relativizations of CS

In \([7]\), relativizations of CS are studied, which are obtained by the following procedure: when \( M \subseteq K \) is a monoid of neighbourhood functions, (with a corresponding submonoid \( M = \{ f \mid f \in M \} \) of \( \text{cts}(B, B) \)), we can restrict the quantifiers over (lawlike) elements of \( K \) in the CS-axioms to \( M \). This leads to the following axioms:

\begin{align*}
\text{CS}(M) & 1. \ a \ (\text{closure of } B_C) \quad \forall f \in M \forall \epsilon \exists \eta \ (f(\epsilon) = \eta), \\
& \quad \text{b \ (pairing)} \quad \forall \epsilon, \eta \exists f, g \in M \exists \zeta \ (\epsilon = f(\zeta) \land \eta = g(\zeta)). \\
\text{CS}(M) & 2. \ (\text{analytic data}) \quad \forall \epsilon (A(\epsilon) \rightarrow \exists f \in M (\exists \eta \epsilon = f(\eta) \land \forall \zeta A(f(\zeta))). \\
\text{CS}(M) & 3. \ = \text{CS3} \\
\text{CS}(M) & 4. \ (\forall \epsilon \exists \eta \text{ continuity}) \quad \forall \epsilon \exists \eta A(\epsilon, \eta) \rightarrow \forall \epsilon \exists f \in M A(\epsilon, f(\epsilon))
\end{align*}

and lawlike AC–NF as before.

For countable sets \( M \) these relativizations come up naturally if one tries to model CS in sheaves over Baire space (see Section 4 below).

\( \text{CS}(M)4 \) may seem rather unusual. Note first that it is non-trivial: elements of \( M \) are lawlike, and if \( \eta \) is a non-lawlike element of \( B_C \), then there is no lawlike \( f \) such that \( \forall \epsilon f(\epsilon) = \eta \). Secondly, an \( f \in M \) has a lawlike neighbourhood function, so we can apply CS3 to \( \forall \epsilon \exists f \in M A(\epsilon, f(\epsilon)) \). This yields an open cover \( \{ U_i \mid i \in I \} \) of \( N^N \) by disjoint basic open sets, such that for all \( i \) there is an \( f_i \in M \) satisfying
∀ε ∈ u, A(ε, f₁(ε)). Finally, through AC-NF we can piece the fᵢ’s together and find an f ∈ K such that ∀ε ∈ A(ε, f(ε)). That is to say, CS(M)+ CS₃+ AC-NF+ CS₄. A consequence of this is that CS(K) coincides with CS.

Note that the converse implication of CS(M)+ follows from CS(M)₁a. If Bₐ is closed under application of elements of M only and M is a proper subset of K, then the converse of CS₄ may fail.

In the sequel, j is some fixed bijection N×N → N with inverses j₁ and j₂:N → N. This induces a homeomorphism h:Nⁿ → Nⁿ×Nⁿ, h(x)(m) = (j₁x(m), j₂x(m)), with inverse h⁻¹ such that h⁻¹(x, y)(m) = j(x(m), y(m)).

We call M pairing-closed iff π₁ ∘ h, π₂ ∘ h ∈ M, and for all f and g ∈ M, h⁻¹ ∘ (f, g) ∈ M. If M is pairing-closed, then one can prove in CS(M) that Bₓ × Bₓ = Bₓ via h.

We shall briefly indicate how models for CS(M) can be obtained by the methods of Section 2.2.

Let M be a submonoid of cts(B, B), such that:

1. For all finite sequences u, the function u:x → u | x is in M. (u | x denotes the sequence obtained from x by replacing the initial segment lth(u) of x by u.)

Let ℱ be the collection of sieves S ≤ M satisfying

2. For all x ∈ B there is a u ∈ S such that x ∈ im(u).

Then ℱ is a Grothendieck topology on M, and we interpret CS(M) in sheaves over (M, ℱ).

Before we do so, however, a word on the condition (1) and the definition (2) seems in order. The open cover topology ℱ on cts(B, B) is characterized by the fact that with each set S ∈ ℱ there is a collection {Ui}, of opens of B which cover B, and such that for each i, S contains an embedding B → Ui ⊆ B. To preserve this characteristic property, we must restrict our attention to monoids which contain sufficiently many open embeddings. For reasons of simplicity we consider only monoids which satisfy (1), and we define the topology (2) accordingly.

In the model over (M, ℱ), N, Nⁿ, Bₓ, K and K appear just as before (cf. Section 2.2, pp. 72–75). Let nbf(M) be the set of neighbourhood functions for elements of M. Then nbf(M) ≤ K. M is interpreted as the set of locally constant maps B → nfb(M), and M as the sheaf of morphisms cts(B, B) × cts(B, B) → cts(B, B) generated by the set of morphisms which are of the form F(f, g) = h ∘ g, for some fixed h ∈ M (cf. the discussion of the interpretation of K and K in Section 2.2). Note that M has the same properties internally as M has externally; in particular, M is closed under pairing iff M is, and M contains j₁ and j₂ iff M does. Finally, Bₓ is interpreted as the smallest subsheaf of Nⁿ (= cts(B, B)) which contains 1; in other words, Bₓ is interpreted as the set of functions in cts(B, B) which are locally in Mₓ.

As in Section 2.2 we consider the specialization principle, continuity of lawlike functions Bₓ → Nⁿ, countable choice, ∀ε ∈ n-continuity (C-N and C-N⁺), continuity of arbitrary functions Bₓ → Nⁿ, ∀ε ∈ n-continuity (CS(M)+) and bar induction (BI and BI⁺) in Sh(M, ℱ) under the interpretation described above.
If \( \mathcal{M} \) contains all constant functions \( B \rightarrow B \), i.e. if \( B_L \subseteq B_C \), then the specialization property holds in \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) by the same argument as in 2.2.

All lawlike mappings \( F : B_C \rightarrow N^N \) are locally of the form \( F(f) = \alpha \circ f \), where \( \alpha \in \text{cts}(B, B) \) is \( F(1) \). So the elements of \( (B_C \rightarrow N^N)_L \) are continuous and have a neighbourhood function in \( K \). Hence they can be extended naturally to continuous functions \( N^N \rightarrow N^N \) in \( K \).

\( \text{Sh}(\mathcal{M}, \mathcal{G}) \) is a model for the axiom of countable choice \( AC-N^* \) (proof as in 2.2.6). If \( \mathcal{M} \) contains all constant functions \( B \rightarrow B \) then the schema \( AC-N \) holds in \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) (cf. 2.2.6). In any case, \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) is a model for the schema \( AC-N \) of lawlike countable choice to lawlike objects:

\[
\vdash \forall n \exists x \in X_L A(n, x) \rightarrow \exists \text{lawlike } N^L \forall n A(n, F(n)),
\]

where \( X \) is an arbitrary sheaf.

The schema \( C-N \) is valid in \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) (cf. 2.2.7). If \( \mathcal{M} \) is pairing-closed (i.e. if \( B_C \times B_C = B_C \)) then the corresponding axiom \( C-N^* \) holds as well, by the same argument as in 2.2.7.

If \( \mathcal{M} \) is pairing-closed then all functions \( F : B_C \rightarrow N^N \) are continuous. This follows immediately from \( C-N^* \).

In \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) the schema \( CS(M)^4 \) holds:

\[
\vdash \forall \varepsilon \exists \eta A(\varepsilon, \eta) \rightarrow \forall \varepsilon \exists F \in \mathcal{M} A(\varepsilon, F(\varepsilon)).
\]

The proof deviates slightly from the one for \( C-C \) in 2.2.9. Assume \( \vdash \forall \varepsilon \exists \eta A(\varepsilon, \eta) \), then in particular \( \vdash A(1, f) \) for some \( f \in B_C \). This \( F \) is locally in \( \mathcal{M} \), hence there is a cover \( \{ \bar{u}_i \} \) such that each \( f \circ \bar{u}_i \in \mathcal{M} \), and of course \( \vdash A(\bar{u}_i, f \circ \bar{u}_i) \). Define \( F_i \in \mathcal{M} \) by \( F_i(g, h) = f \circ \bar{u}_i \), so \( \vdash A(\bar{u}_i, F_i(\bar{u}_i)) \) for all \( i \), and therefore \( \vdash \exists F \in \mathcal{M} A(1, F(1)) \). Hence also \( \vdash \forall \varepsilon \exists F \in \mathcal{M} A(\varepsilon, F(\varepsilon)) \).

If \( \mathcal{M} \) is pairing-closed, then the axiom \( C-C^* \) holds in the form

\[
\vdash \forall \varepsilon \exists P \in \mathcal{P}(B_C \times B_C) \ (\forall \varepsilon \exists \eta P(\varepsilon, \eta) \rightarrow \exists F : B_C \rightarrow B_C \forall \varepsilon P(\varepsilon, F(\varepsilon))).
\]

To see this, let \( P \in \mathcal{P}(B_C \times B_C) \), i.e. \( P : M \times B_C \times B_C \rightarrow \Omega \), and assume \( \vdash \forall \varepsilon \exists \eta P(\varepsilon, \eta) \). Then there is an \( f \in B_C \) such that \( P(\pi_1 \circ h, \pi_2 \circ h, f) = T \), where \( h : B \Rightarrow B \times B \) is induced by \( j \). Define the morphism \( F : M \times B_C \rightarrow B_C \) by \( F(g_1, g_2) = f \circ h^{-1} \circ (g_1, g_2) \). One easily verifies that \( \vdash \forall \varepsilon P(\varepsilon, F(\varepsilon)) \). Moreover, by the previous remark \( F \) is continuous.

\( \text{Sh}(\mathcal{M}, \mathcal{G}) \) is a model for relativized analytic data and for generalized analytic data. Note that it suffices to prove \( CS(M)^3 \) for the global elements of \( B_C \), i.e. the elements of \( \mathcal{M} \). Thus, assume that \( \vdash A(f) \) for some \( f \in \mathcal{M} \), and define \( F : M \times B_C \rightarrow B_C \) by \( F(g, h) = f \circ h \). Then trivially \( \vdash A(F(1)) \), hence also \( \vdash \forall \varepsilon A(F(\varepsilon)) \). Generalized analytic data is proved as in 2.2.11.

Finally, we consider \( BI \) and \( BI^* \). \( \text{Sh}(\mathcal{M}, \mathcal{G}) \) is a model for the schema \( BI \), independent of the properties of \( \mathcal{M} \) (the proof is left to the reader). If \( \mathcal{M} \) contains all constant functions \( B \rightarrow B \), then \( BI^* \) holds by the argument of 2.2.10. There is an alternative way of proving \( BI^* \) however, which leads to the following result: if
\( \mathcal{M} \) is pairing-closed, then BI* holds in \( \text{Sh}(\mathcal{M}, \mathcal{S}) \). To see this, choose \( P \in \mathcal{P}(N^{< N}) \) such that

\[ \vdash \forall \epsilon \exists u (\epsilon \in u \land Pu) \land \forall u (\forall n P(u \ast \langle n \rangle) \iff Pu). \]

Then in particular \( \vdash \exists u (\pi_1 \circ h \in u \land P(\pi_2 \circ h, u) \) (where \( h : B \to B \times B \) is the homeomorphism induced by \( j \)), so we can find an \( a \in \text{cts}(B, N^{< N}) \) such that \( \forall x \in B \pi_1 \circ h(x) \in a(x) \) and \( P(\pi_2 \circ h, a) = T \). Let \( \{ \tilde{u}_i \} \) be a cover such that \( a \circ \tilde{u}_i = \tilde{u}_i \), constant for each \( i \). For a finite sequence \( \omega \), let us write \( \omega_1, \omega_2 \) for the finite sequences such that \( \omega = \omega_1 \omega_2 \). We may without loss assume that \( a \circ \tilde{u}_i = \tilde{u}_i \), the constant function with value \( u_i \). Obviously \( \pi_2 \circ h \circ \tilde{u}_i = \tilde{u}_i \), so we have that \( P(\tilde{u}_i, \tilde{u}_i) = T \) for all \( i \). Now consider the predicate \( Qu = [P(\tilde{u}_2, \tilde{u}_1) = T] \). \( Q \) now satisfies \( \forall u (\forall n P(u \ast \langle n \rangle) \iff Qu) \) (note that \( h : u \ast \langle n \rangle \to (u_1 \ast \langle j_1, n \rangle) \times (u_2 \ast \langle j_2, n \rangle) \)). Hence we may apply BI to \( Q \) and find \( P(\langle \rangle, \langle \rangle) = T \), i.e. \( \vdash P(\langle \rangle) \).

As an immediate corollary to the observations just made, we obtain

**2.3.1. Theorem.** \( \text{Sh}(\mathcal{M}, \mathcal{S}) \) is a model for CS(\( \mathcal{M} \)). □

### 3. The connection with the elimination translation

We now want to investigate the connection between the interpretation of CS provided by the elimination translation of [15] and the monoid models. The interpretation of CS through the elimination translation is an interpretation in a constructive metatheory. Therefore we will first (Section 3.1) outline a constructive treatment of the monoid models presented earlier, before actually comparing the two interpretations (Section 3.2).

#### 3.1. Constructive metatheory

We restrict ourselves to the interpretation of what we shall call the *minimal language*. This is a four-sorted language of predicate logic, with sorts \( N \) (natural numbers), \( B_L \) (lawlike sequences), \( K \) (lawlike inductive neighbourhood functions) and \( B_C \) (choice sequences). It does not have a sort \( N^N \). It is implicit in the rules of term-formation that both \( B_L \) and \( B_C \) are subsorts of \( N^N \).

Note that there is a conceptual difference between the treatment of \( B_C \) and \( B_L \) as subsets of \( N^N \) and their treatment as separate sorts. Being of sort \( B_C \) or \( B_L \) is an intensional property of an object: it is given to us as an object of that sort. Being an element of the subset \( B_C \) or \( B_L \) is an extensional property of an object: from the way it is given to us we can prove that it satisfies the extensional \( \epsilon \)-relation w.r.t. that subset.

The minimal language contains constants which make it possible to represent each primitive recursive \( f : N^p \to N \) by a term \( t[n_1, \ldots, n_p] \) in \( p \) numerical parameters. In particular there is a bijective \( j : N^2 \to N \) in the language, with
inverses $j_1, j_2 : N \to N$. Through $j, j_1, j_2$ elements of $N$ can be viewed as codes for elements of $N^p$ and $N^{<\infty}$. $K$ is treated as a subsort of $B_1$, i.e. the domain $N^{<\infty}$ of the inductive neighbourhood functions is coded in $N$.

We shall treat the minimal language rather loosely below. E.g. we use quantifiers $\forall f \in K, \exists f \in K$ and write equations $f(\varepsilon)(n) = m$ which are not in the minimal language. Note however that quantification over $K$ can be replaced by quantification over $K$ and that atomic formulae $f(s)(n) = m$ can be translated into their ‘definition’ $\exists k ((\varepsilon(n) * \varepsilon(k)) = m + 1)$ where $a \in K$ is the neighbourhood function of $f \in K$.

The treatment of CS in [15] is much more precise. The formal language used there to formulate the axioms in is an extension of the minimal language. The main difference is that it has constants $\text{app}_p : K \times B_C^p \to B_C$, where $\text{app}_p(a, \varepsilon_1, \ldots, \varepsilon_p)$ is written as $a \mid (\varepsilon_1, \ldots, \varepsilon_p)$. Among the CS-axioms in [15] is one specifying that $\text{app}_p(a \mid (\varepsilon_1, \ldots, \varepsilon_p)) = f\nu_p(\varepsilon_1, \ldots, \varepsilon_p)$ where $f \in K$ has neighbourhood function $a \in K$ and $\nu_p$ is a homeomorphism $(N^p)^p \to N^N$. Note that this axiom makes our CSI-axioms of closure and pairing redundant. In fact, closure and pairing are almost implicit in the presence of the constants $\text{app}_p$. The minimal language is entirely neutral in this respect. It can be used therefore to formulate all kinds of theories of choice sequences.

For our metatheory we use the theory IDB, or rather a definitional extension of this system. Strictly speaking, IDB is a two-sorted system, with variables $k, l, m, n, \ldots$ of sort $N$, and variables $x, y, z, \ldots$ of sort $B$. The language has the same constants as the language of CS for the definition of primitive recursive functions from $N^p$ to $N$. In particular we have $j : N^2 \to N$ with inverses $j_1$ and $j_2 : N \to N$ in the language as above, so $N^p$ and $N^{<\infty}$ can be treated codewise. We shall consider $N^p$ and $N^{<\infty}$ as separate sorts here. Another constant of the language of IDB is the constant $K$, for the set of neighbourhood functions. Formally these are treated as maps from $N$ to $N$, but we refrain from this coding and continue to look upon external neighbourhood functions as maps from $N^{<\infty}$ to $N$. Working within IDB, continuous functions from $B$ to $B$ are the functions coded by elements of $K$. We add a constant $\kappa$ to the language for these continuous functions. When working within IDB, we will often write $\text{cts}(B, B)$ for $\kappa$. ($\kappa$ is defined from $K$ as $K$ is from $K$, see the beginning of 2.2.)

The axioms of IDB are the usual arithmetical axioms, the ‘defining’ axioms for its constants (in particular, the axiom of induction over $\kappa$), and the choice axiom AC–NF. Bar induction is not an axiom of IDB, nor does it have any of the typical intuitionistic continuity axioms for Baire space. Thus, IDB is just a subsystem of classical analysis.

We must adapt the interpretation of the language of CS in sheaves over $\kappa = \text{cts}(B, B)$ with the open cover topology to allow its treatment in IDB.

First we look at the definition of the open cover topology. As noted in Section 2, each open cover has a characteristic function in $K$. In the constructive metatheory, we use this observation as the definition of open cover: a sieve
S \subseteq \text{cts}(B, B) \text{ is a cover iff there is an } a \in K \text{ such that for all } u \in \mathbb{N}^{<\mathbb{N}}, \lambda x \cdot u \mid x \text{ is in } S \text{ whenever } a(u) \neq 0. \text{ (Recall that } \lambda x \cdot u \mid x \text{ is the function "replace the initial segment of length } lth(u) \text{ by } u'. \text{ In [15] it is shown that this function has a neighbourhood function in } K, \text{ i.e. } \lambda x \cdot u \mid x \in \text{cts}(B, B).)\]

The formal covers thus defined form a Grothendieck topology:

(i) \(\lambda n \cdot m + 1 \in K\), so \(\text{cts}(B, B)\) is a cover.

(ii) If \(S\) is a cover with characteristic function \(a \in K\), and \(f \in \text{cts}(B, B)\) has a neighbourhood function \(b \in K\), then \(S \uparrow f\) has characteristic function \(a ; b \in K\). (For \(a ; b\) see [15]; \(a ; b\) is defined in such a way that if \(a ; b(v) \neq 0\), then there is a \(u \in \mathbb{N}^{<\mathbb{N}}\) such that \(a(u) \neq 0\) and \(\text{im}(f \circ \lambda x \cdot v \mid x) \subseteq u).\)

(iii) If \(R \subseteq \text{cts}(B, B)\) is a sieve, \(S\) a cover with characteristic function \(a \in K\), and \(R \uparrow f\) is a cover with characteristic function \(b_f \in K\) for each \(f \in S\), then \(R\) is a cover, with characteristic function \(a/b\), where \(b : \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}\) is such that \(\lambda v \cdot b((u) * v) = b_{\lambda x \cdot u}((x) u)\). (For \(a/b\) see [15].)

Next we look at the sheaves that are needed to interpret the CS-language. As in Section 2, we interpret \(B_c\) as \(\text{cts}(B, B)\). All other sorts and predicate constants are to be interpreted as sheaves of lawlike objects, i.e. objects which are locally invariant under restrictions. Such sheaves are completely determined by their global elements, the elements which are totally invariant under restrictions. Quantification over sheaves of lawlike objects reduces to quantification over the global elements of such sheaves, because

\[ \vdash \forall x \in X_L \; A(x) \iff \forall x \in \bar{X}_L \; \vdash A(x), \]

and

\[ \vdash \exists x \in X_L \; A(x) \iff \exists S \in J \forall f \in S \exists x \in \bar{X}_L \; \vdash (A \uparrow f)(x), \]

where \(\bar{X}_L\) is the set of global elements of \(X_L\). Consequently, we can interpret \(N, B_l\) and \(K\) by the external sets \(\mathbb{N}, B,\) and \(K\) (modulo coding of finite sequences), respectively.

Finally, we reformulate the forcing clauses. Prime formulas of CS are of the form \(t = s\) (\(t\) and \(s\) numerical terms). Equations \(t = s\) are basically of the form \(en - m\) or of the form \(an - m, e\) of sort \(B_{c_l}\), \(a\) of sort \(B_{l_c}\). (Constants are interpreted by 'themselves', more complex equations can be replaced by equivalent formulas in which only these simple equations occur.) \(B_c\) is interpreted as \(\text{cts}(B, B)\), \(B_{l_c}\) as \(B\), so we can put

\[ \vdash fn = m \iff \forall x \in B f(x)(n) = m, \]
\[ \vdash xn = m \iff xn = m. \]

We then proceed by induction:

\[ \vdash A \land B \iff \vdash A \text{ and } \vdash B, \]
\[ \vdash A \lor B \iff \exists a \in K \forall u (au \neq 0 \Rightarrow (\vdash A \uparrow (\lambda x \cdot u \mid x)) \]
\[ \text{or } \vdash B \uparrow (\lambda x \cdot u \mid x)), \]
\[ \vdash A \rightarrow B \iff \forall f \in \text{cts}(B, B) (\vdash A \uparrow f \Rightarrow \vdash B \uparrow f), \]
The language restrictions make it rather tedious to verify that the proofs we gave for the validity of CS in sheaves over \( \text{Cts}(B, B) \) with the open cover topology in Section 2 can be given in IDB with respect to the adapted forcing definition above. It may be instructive to look at bar induction. Note first of all that the language does not permit the formulation of this principle as an axiom. Instead one can look at the schema with an additional parameter of sort \( B_\infty \),

\[ \forall \epsilon \exists n A(\bar{e}(n), \eta) \land \forall u(A(u, \eta) \leftrightarrow \forall n A(u \cdot \langle n \rangle, \eta)) \rightarrow A(\langle \rangle, \eta). \]

To prove in IDB that this schema is forced one uses the same argument as in Section 2.3, except that external bar induction is replaced by induction over unsecured sequences (which is a corollary of induction over \( K \),

\[ \forall a \in K (\forall u (au \neq 0 \rightarrow B(u)) \land \forall u (B(u) \leftrightarrow \forall n B(u \cdot \langle n \rangle)) \rightarrow B(\langle \rangle)). \]

Another problem here is that one has to show in IDB that the forcing interpretation is sound, in order to have a full constructive proof that forcing over \( \text{Cts}(B, B) \) yields a CS model. Both the validity of the axioms and the soundness follow from the observations in the next subsection.

We close this subsection with the following remark. Let \( A \) be a lawlike sentence in the minimal language, i.e. all quantifiers in \( A \) are of sort \( N \) or sort \( B_\infty \). Let \( A^* \) be the IDB-formula obtained by replacing quantifiers over \( K \) by quantifiers over \( K \).

\[ \exists.1.1. \text{Lemma.} \quad \text{IDB} \vdash A^* \iff \text{IDB} \vdash "\text{II-A}". \]

In other words, the theory of the lawlike part of CS under the forcing interpretation is just IDB. Since in the definition of CS in \([15]\) IDB is the lawlike part of CS, the treatment of forcing in an intuitionistic metatheory yields an interpretation which is in this respect more faithful than the classical treatment.

3.2. Forcing and the elimination translation

Convention. In this section we assume that all choice parameters in a formula are shown in notation.

In \([15]\) a translation \( \tau \) is defined which maps sentences of the language of CS to lawlike sentences. This translation is called the elimination translation. The elimination theorem shows that \( \tau \) provides a sound interpretation of CS in IDB. We give a short account of this interpretation here.
Sheaf models for choice sequences

The characteristic property of the CS-axioms is that they give an explanation of choice quantifiers in terms of quantifiers over lawlike objects. This characteristic property is exploited in the elimination translation.

Consider a formula $\exists \varepsilon A(\varepsilon)$. By the specialization property, it is equivalent to $\exists a \in B_L A(a)$. Thus existential quantification over $B_C$ (in the absence of choice parameters) is explained as existential quantification over $B_L$. (In Section 2 we have shown that the specialization property is true under the forcing interpretation; in the form here, namely $\exists \varepsilon A(\varepsilon) \leftrightarrow \exists a \in B_L A(a)$, it follows logically from analytic data.)

Next we look at a formula $\forall \varepsilon \exists p A(\varepsilon, p)$ ($p$ ranging over $N$, $B_L$, or $K$). By CS3 and CS1a it is equivalent to $\exists a \in K \forall u \ (au \neq 0 \rightarrow \exists p \forall \varepsilon A(u \mid \varepsilon, p))$, so universal quantification over $B_C$ in the context of a lawlike existential quantifier is explained in terms of a lawlike quantifier over $K$ and a universal choice quantifier over a formula of lower complexity. A similar observation holds for $\forall \varepsilon \ (A(\varepsilon) \vee B(\varepsilon))$. By logic it follows that a universal choice quantifier in the context of a lawlike universal quantifier or a conjunction can be pushed inside, i.e.

$$\forall \varepsilon \forall p A(\varepsilon, p) \leftrightarrow \forall p \forall \varepsilon A(\varepsilon, p), \quad \text{and} \quad \forall \varepsilon \ (A(\varepsilon) \wedge B(\varepsilon)) \leftrightarrow (\forall \varepsilon A(\varepsilon) \wedge \forall \varepsilon B(\varepsilon));$$

so universal choice quantification in this context reduces to universal choice quantification over a formula of lower complexity.

Analytic data may equivalently be formulated as

$$\forall \varepsilon \ (A(\varepsilon) \rightarrow B(\varepsilon)) \leftrightarrow \forall f \in K (\forall \varepsilon A(f(\varepsilon)) \rightarrow \forall \varepsilon B(f(\varepsilon))),$$

so universal choice quantification in the context of an implication is explained in terms of lawlike universal quantification over $K$ and universal choice quantification over formulas of lower complexity.

By CS1b we have $\forall \varepsilon \forall \eta A(\varepsilon, \eta) \leftrightarrow \forall f, g \in K \forall \varepsilon A(f(\varepsilon), g(\varepsilon))$, i.e. a pair of universal choice quantifiers can be reduced to a single universal choice quantifier and a pair of lawlike quantifiers over $K$.

Consider a formula $\forall \varepsilon \exists \eta A(\varepsilon, \eta)$. By CS4 and CS1a this is equivalent to $\exists f \in K \forall \varepsilon A(\varepsilon, f(\varepsilon))$, i.e. universal choice quantification in the context of an existential choice quantifier is explained in terms of a lawlike quantifier over $K$ and a universal choice quantifier over a formula of lower complexity.

Finally a formula $\forall \varepsilon \ (f(\varepsilon)(n) = m)$, where $f \in K$, is easily seen to be equivalent to $\forall b \in B_L \ (f(b)(n) = m)$, so universal choice quantification over an atomic formula is explained as lawlike quantification.

One may summarize this by saying that the explanation of choice quantifiers consists of a procedure to push universal choice quantifiers over the other logical signs and to replace them eventually by universal lawlike quantifiers in front of equations $t = s$, and to replace existential choice quantifiers not in the scope of a universal one by existential lawlike ones straightaway.
This procedure is the elimination translation. \( \tau \) namely is defined inductively as follows:

\[\tau(\exists \varepsilon A(\varepsilon)) = \exists \varepsilon = B_L \tau(A(\varepsilon)),\]
\[\tau(\forall \varepsilon (f(\varepsilon)(n) = m)) = \forall \varepsilon \in B_L f(a)(n) = m,\]
\[\tau(\forall \varepsilon (A(\varepsilon) \land B(\varepsilon))) = \tau(\forall \varepsilon A(\varepsilon)) \land \tau(\forall \varepsilon A(\varepsilon)),\]
\[\tau(\forall \varepsilon \forall \varepsilon A(\varepsilon, p)) = \forall \varepsilon \forall \varepsilon A(\varepsilon, p)),\]
\[\tau(\forall \varepsilon A(\varepsilon) \rightarrow B(\varepsilon)) = \forall \varepsilon \in K (\tau(\forall \varepsilon A(f(\varepsilon))) \rightarrow \tau(\forall \varepsilon B(f(\varepsilon)))),\]
\[\tau(\forall \varepsilon (A(\varepsilon) \lor B(\varepsilon))) = \exists \varepsilon \in K \forall \varepsilon (au \neq 0 \rightarrow \tau(\forall \varepsilon A(\varepsilon)) \lor \tau(\forall \varepsilon B(\varepsilon))),\]
\[\tau(\forall \varepsilon \exists \varepsilon A(\varepsilon, p)) = \exists \varepsilon \in K \forall \varepsilon (au \neq 0 \rightarrow \exists \varepsilon \tau(\forall \varepsilon A(\varepsilon, p))),\]
\[\tau(\forall \varepsilon \forall \eta A(\varepsilon, \eta)) = \forall \varepsilon, \eta \in K \tau(\forall \varepsilon A(f(\varepsilon), g(\varepsilon))),\]
\[\tau(\forall \varepsilon \exists \eta A(\varepsilon, \eta)) = \exists \varepsilon \in K \tau(\forall \varepsilon A(\varepsilon, f(\varepsilon))).\]

(In [15], the clauses for \( \lor \) and \( \exists p \) contain an implicit application of AC–NF. Our presentation is slightly different from but equivalent to the one given in [15].)

The elimination theorem states that the interpretation of CS in IDB via \( \tau \) is sound, i.e.

(a) \( \text{CS} \vdash A \Rightarrow \text{IDB} \vdash \tau(A) \), for all sentences \( A \) in the language of CS,

and that is faithful, in the sense that

(b) \( \text{CS} \vdash A \iff \tau(A) \), for all sentences \( A \) in the language of CS.

The obvious question to ask now is whether the forcing- and the elimination-interpretation are in any sense related to one another. The answer is given by the following theorem.

3.2.1. Theorem. Let \( A \) be a sentence in the minimal language, and let \( \tau(A)^* \) be obtained from the elimination translation \( \tau(A) \) as indicated at the end of the preceding Section 3.1. Then \( \tau(A)^* \) and \( \vdash A \) are provably equivalent in IDB. In fact one can show that \( \tau(\forall \varepsilon A(f_1(\varepsilon), \ldots, f_n(\varepsilon)))^* \) is literally the same as \( \vdash A(f_1, \ldots, f_n) \).

Proof. The second claim is proved by a straightforward induction on the logical complexity of \( A(\varepsilon_1, \ldots, \varepsilon_n) \). From this, the equivalence of \( \tau(A)^* \) and \( \vdash A \) for arbitrary sentences \( A \) follows easily, using the soundness of \( \tau \). \( \square \)

This theorem shows that elimination and monoid forcing are essentially the same interpretation.

As a corollary to the elimination theorem and Theorem 3.2.1 we now find that the monoid-forcing interpretation of CS (in the original CS-language) is classifying for CS, in the sense that
3.2.2. Corollary. \( \text{IDB} \vdash \text{"}\vdash \text{A} \text{""} \iff \text{CS} \vdash \text{A} \). □

The monoid forcing interpretation is also classifying in the sense of [16]; this will be extensively discussed in [9].

We have thus shown that the elimination theorem is in fact a special case of the standard method of interpreting intuitionistic theories in sheaves over a category equipped with a Grothendieck topology. This result also shows that the elimination procedure is not just a syntactical trick.

(It should perhaps be remarked here that it is not claimed in [15] that the underlying idea is syntactical; the explanation of the elimination translation given above even suggests the contrary. The syntactic flavour of [15] rather seems inherent to the attention paid to the metatheory.)

A similar connection between monoid models and elimination translations can be formulated for relativizations of CS. We trust that, with the monoid models of Section 2.3 in mind, the interested reader can work out the details of an elimination translation “which expresses monoid forcing” for relativizations of CS.

4. Spatial models

We have now seen how CS and its relativizations can be interpreted in sheaves over (a submonoid of) \( \text{cts}(B, B) \) with the open cover topology. In the preceding section it has been shown that this interpretation corresponds to the elimination translation for CS, i.e. the interpretation is in a sense the one ‘prescribed’ by the axioms, and the monoid models are in a strong sense the classifying models for CS and its relativizations. But still, the monoid models do not help to solve the problem of finding an informally described class of construction processes (a subdomain of the universe of choice sequences) for which the validity of CS-axioms can be rigorously justified. As has already been said in the introduction, the monoid models are formally motivated, not conceptually.

It therefore remains of interest to find models for CS (or relativizations) which are spatial, and then preferably over spaces ‘resembling’ Baire space. The interest of such spaces lies in their relation to internal ‘projection’ models: a model over Baire space (treated in an intuitionistic metatheory) is equivalent to a projection model of the form \( \mathcal{U}_\alpha = \{ f(\alpha) \mid f \in S \} \), where \( S \) is a subset of \( \text{cts}(B, B) \) (cf. Section 5.3 below). Such a \( \mathcal{U}_\alpha \) is a subdomain of intuitionistic Baire space, i.e. it is a ‘conceptual model’. (For more discussion see [10] and especially [19].)

In fact, the Diaconescu cover [2] yields a general procedure for obtaining a cHa which is first-order equivalent to any given site (cf. [11]), but it seems to be difficult to describe the cHAs thus obtained in terms of familiar spaces. We will therefore not apply the Diaconescu cover here, but instead we give a more direct construction, which yields for each of the monoids \( \mathcal{M} \) discussed in Section 2.3 a
topological space $X_M$, which is first-order equivalent to $M$ with the open cover topology. For countable $M$, $X_M$ is homeomorphic to a subspace of Baire space. In general, $X_M$ is a subspace of $M^N$ with the product topology, where $M$ is regarded as a discrete space.

Let $M$ be a submonoid of $\text{cts}(B, B)$, of the form described in 2.3. If $F = (F_n)_n$ is a sequence of elements of $M$, we define $F_m$ by induction: $F_0 = 1$ (the identity-mapping) and $F_{m+1} = F_m \circ F_{n+m}$. (Thus, if $m > 0$, $F_m = F_n \circ \cdots \circ F_{n+m-1}$.) We will call a sequence admissible if for any composition $F_m$ of $m$ successive elements of $F$ the first $m$ numbers of the sequence $F_m(x)$, $x \in B$, do not depend on $x$; i.e., $F$ is admissible iff for all $m$ and $n$, $\lambda x F_{m+n+1}(x)(m) : B \to \mathbb{N}$ is constant.

For a sequence $F$, being admissible means that we can define points $\lim_n(F)$ of Baire space, for each $n \in \mathbb{N}$, by setting

$$\lim_n(F)(m) = F_m(x)(m), \text{ for some (all) } x \in B.$$  

Let $X_M$ be the space whose points are the admissible elements of $M^N$, with the product topology, regarding $M$ as a discrete space; thus basic opens are the sets

$$\tilde{F}_n = \{G \mid G \text{ is admissible, and } F_i = G_i \text{ for } i = 0, \ldots, n-1\}.$$  

Note that this topology makes the functions $\lim_n : X_M \to B$ continuous.

The language that we will consider is the minimal language, with an additional constant $M$ (for a subset of $K$). Thus, we have a sort of natural numbers $N$, a sort of lawlike sequences $B_L$ (a subsort of $N^N$), a sort of lawlike neighbourhood functions $K$ (a subsort of $B_L$), and a sort $B_C$ of choice-sequences.

In sheaves over $X_M$, Baire space $N^N$ is interpreted as the sheaf of continuous $B$-valued functions. We will interpret $B_C$ as the sheaf generated by (global) elements of the form

$$f \circ \lim_n : X_M \to B,$$

where $n \in \mathbb{N}$, and $f$ is (locally) an element of $M$ (i.e. $\exists f \in B_C$ in the monoid model over $M$, as in Section 2.3). The lawlike types are interpreted in sheaves over $X_M$ as the sheaves of locally constant functions with the appropriate range.

We will show by formula-induction that forcing over the monoid $M$ and forcing over the space $X_M$ are equivalent (Theorem 4.3 below). But first we need a lemma to be able to compare covers in $M$ and covers of $X_M$.

4.1. Lemma. Let $F \in X_M$, $n \in \mathbb{N}$. Then for each $\alpha \in B$ there exists a sequence $G(\alpha)$ such that

(i) $G(\alpha)$ is admissible, and $G(\alpha) \in \tilde{F}_n$.  
(ii) If $m > k \geq n$, range($G(\alpha)_k \circ \cdots \circ G(\alpha)_{m-1}$) $= V_{\alpha(r)} := \{x \in B \mid \alpha(r) \text{ is an initial segment of } x\}$, for some $r$ strictly increasing in $m$.  
(iii) If $k \geq n$, then $\lim_k G(\alpha) = \alpha$.

Proof. If $\alpha \in B$, then for each $k < n$ there exists a function $g_{k}$ such that for each
m, the first \( m + 1 \) values of \( F_k \circ \cdots \circ F_{n-1} \) are constant on \( V_{\alpha(n)}(m) \), by continuity of \( F_k \circ \cdots \circ F_{n-1} \) at \( \alpha \).

Let \( g^\alpha(m) = \max_{k \leq n} g^\alpha(k) \). Then

\[
(a) \quad \forall k < n \forall x, x' \in V_{\alpha(g^\alpha(n))} \forall i \leq m F_k \circ \cdots \circ F_{n-1}(x)(i) = F_k \circ \cdots \circ F_{n-1}(x')(i)
\]

and without loss we may assume

\[
(b) \quad g^\alpha \text{ is strictly increasing.}
\]

Now let

\[
G(\alpha)_k(x) = F_k(x) \quad \text{if } k < n,
\]

\[
G(\alpha)_k(x) = \alpha(g^\alpha(k)) \mid x \quad \text{if } k \geq n.
\]

(Recall that if \( x \in B, U \subseteq \mathbb{N}^\omega \), then \( u \mid x \) denotes the sequence obtained from \( x \) by replacing the initial segment of \( x \) of length \( l(u) \) by \( u \).) Then (i)-(iii) hold: the only thing that is perhaps not immediately clear is that \( G(\alpha) \) is admissible. Consider a composition \( G(\alpha)_k \circ \cdots \circ G(\alpha)_k + m \) of \( m + 1 \) successive elements of \( G(\alpha) \): if \( k + m < n \), then there is nothing to prove since \( F \) is admissible. If \( k \geq n \), then \( G(\alpha)_k \circ \cdots \circ G(\alpha)_k + m(x) = \alpha(g^\alpha(k + m)) \mid x \), and it is immediate from (b) that the first \( m + 1 \) values of this output do not depend on \( x \). And if \( k < n \leq k + m \), then

\[
G(\alpha)_k \circ \cdots \circ G(\alpha)_k + m(x) = F_k \circ \cdots \circ F_{n-1} \circ G(\alpha)_n \circ \cdots \circ G(\alpha)_k + m(x) = G(\alpha)_k \circ \cdots \circ F_{n-1}(\alpha(g^\alpha(k + m)) \mid x),
\]

and by (a), the first \( m + 1 \) values of this output do not depend on \( x \). \( \square \)

We now list some properties that we need in the inductive steps of Theorem 4.3 below:

**4.2. Lemma.** (a) If \( U \subseteq \{ G_m \mid G \in F_m, n \geq m \} \) and \( \forall G \in F_n \exists m \geq n G \in U \), then the set \( \{ G_m^{n-m} \circ f \mid F_m \in U, f \in M \} \) is a cover of \( M \).

(b) If a sieve \( S \) covers in \( M \), then \( S \) bars each \( F_n \), i.e. \( \forall G \in F_n \exists m \geq n G_m^{n-m} \in S \).

(c) If \( g \in M \), then \( \{ f \mid \exists m \geq n \exists G \in F_n G_m^{n-m} = g \circ f \} \) is a cover of \( M \).

**Proof.** (a) is immediate from Lemma 4.1; (b) follows trivially from the definition of admissibility; (c) is a combination of (a) and part (ii) of the definition of a Grothendieck topology. \( \square \)

**4.3. Theorem.** Let \( A(\varepsilon_1, \ldots, \varepsilon_p) \) be a formula in the restricted language for \( CS(M) \) described above, (where \( \varepsilon_1, \ldots, \varepsilon_p \) are the non-lawlike variables occurring in this formula,) and suppose \( m_1, \ldots, m_p \leq n \). Then

\[
F_n \vDash A(f_1 : m_1, \ldots, f_p : m_p) \iff \vDash A(f_1 \circ F_{m_1}^{n-m_1}, \ldots, f_p \circ F_{m_p}^{n-m_p})
\]

where we write \( f_i : m_i \) for \( f_i \circ \text{lim}_{m_i} \). (\( \vDash \) on the left is forcing in sheaves over the space \( X_M \), \( \vDash \) on the right is forcing over the monoid \( M \) with the open cover topology).
Proof. By induction on $A$:

1. $A(\varepsilon) = \varepsilon_{m_1} = m_2$. Then if $k \leq n$,
   \[ \bar{F}_n \vdash \varepsilon_{m_1} = m_2[f : k] \iff \forall G \in \bar{F}_n f \circ \lim_k (G(m_1)) = m_2. \]
   But $f \circ \lim_k = f \circ F^{-k} \circ \lim_m$, so (using 4.1) this is equivalent to
   \[ \forall x \in G \iff \exists m \in m_1 \lim_k (G)(m_1) = m. \]
   i.e. \( \vdash \varepsilon_{m_1} = m_2[f \circ F^{-k}]. \)

2. $A$ is $B \land C$. This step is trivial.

3. $A$ is $B \lor C$. Then we have
   \[ \bar{F}_n \vdash B \lor C \iff [f : m] \]
   iff \( \forall G \in \bar{F}_n \exists m \geq n (\bar{G}m \vdash B[f_i : m_i] \lor \bar{G}m \vdash C[f_i : m_i]) \]
   iff \( \forall g \in \bar{F}_n \exists m \geq n (\vdash B[f_i \circ G_{n_i}^{m_i-n}] \lor \vdash C[f_i \circ G_{n_i}^{m_i-n}]) \).
   But $G_{n_i}^{m_i-n} = F_{n_i}^{m_i-n} \circ G_{n_i}^{m_i-n}$, so by 4.2(a) and (b) this is equivalent to
   \[ \exists \text{ cover } S \in \mathbb{M} \forall f \in S \vdash B[f_i \circ F_{n_i}^{m_i-n} \circ f] \lor \vdash C[f_i \circ F_{n_i}^{m_i-n} \circ f] \]
   iff \( \vdash B \lor C[f_i \circ F_{n_i}^{m_i-n}] \).

4. $A$ is $B \rightarrow C$. In this case,
   \[ \bar{F}_n \vdash B \rightarrow C[f_i : m_i] \]
   iff \( \forall G \in \bar{F}_n \forall m \geq n (\bar{G}m \vdash B[f_i : m_i] \Rightarrow \bar{G}m \vdash C[f_i : m_i]) \]
   iff \( \forall G \in \bar{F}_n \forall m \geq n (\vdash B[f_i \circ F_{n_i}^{m_i-n} \circ G_{n_i}^{m_i-n}] \Rightarrow \vdash C[f_i \circ F_{n_i}^{m_i-n} \circ G_{n_i}^{m_i-n}]) \)
   and by 4.2(c) this is equivalent to
   \[ \forall g \in \mathbb{M} (\vdash B[f_i \circ F_{n_i}^{m_i-n} \circ g] \Rightarrow \vdash C[f_i \circ F_{n_i}^{m_i-n} \circ g]), \]
   i.e. \( \vdash B \rightarrow C[f_i : m_i] \).

5, 6. The case of universal quantification over lawlike types is obvious. The case of existential lawlike quantifications is analogous to case (3) above.

7. $A$ is $\forall \eta B(\varepsilon_1, \ldots, \varepsilon_p, \eta)$. Now if $\bar{F}_n \vdash A[f_i : m_i]$, i.e.
   \[ \forall f \in B_C \forall m \bar{F}_n \vdash B[f_i : m_i, f : m], \]
   then also
   \[ \forall f \in B_C \forall G \in \bar{F}_n \forall m \geq n \bar{G}m \vdash B[f_i : m_i, f : m] \]
   (\( B_C \) is the sheaf of functions $f$ with $\vdash f \in B_C$ in the monoid-model); so by the induction hypothesis,
   \[ \forall f \in B_C \forall G \in \bar{F}_n \forall m \geq n \vdash B[f_i \circ F_{n_i}^{m_i-n} \circ G_{n_i}^{m_i-n}, f]. \]
   But then, if $f \in B_C$ and $g \in \mathbb{M}$ are arbitrary, we derive that for the cover $S$ defined in 4.2(c),
   \begin{align*}
   \text{for each } f' \in S, \quad &\vdash B[f_i \circ F_{n_i}^{m_i-n} \circ g \circ f', f \circ f'], \\
   \text{then also} \quad &\forall f \in B_C \forall G \in \bar{F}_n \forall m \geq n \bar{G}m \vdash B[f_i \circ F_{n_i}^{m_i-n} \circ G_{n_i}^{m_i-n}, f].
   \end{align*}
hence also \( \vdash B[fi \circ F_{m_i}^n \circ g, f] \). This shows that \( \vdash \forall \eta B(e_i, \eta) [fi \circ F_{m_i}^n] \). Conversely, if \( \vdash \forall \eta B(e_i, \eta) [fi \circ F_{m_i}^n] \), then if \( f \) and \( m \) are arbitrary, it follows immediately from the induction hypothesis that for each \( G \in \bar{F}n \) and each \( k \geq n, \bar{G}k \vdash B[fi : m_i, f : m] \). Hence also \( \bar{F}n \vdash B[fi : m_i, f : m] \). Thus \( \bar{F}n \vdash \forall \eta B(e_i, \eta) [fi \circ F_{m_i}^n] \).

(8) Finally, take \( A \) is \( 3nB(e_1, \ldots, e_m, \eta) \). Suppose that \( \bar{F}n \vdash \exists \eta B(e_i, \eta) [fi : m_i] \), i.e.

\[
\forall G \in \bar{F}n \exists m \geq n \exists k, f \bar{G}m \vdash B[fi : m_i, f : k].
\]

We may assume \( k \leq m \), so by induction hypothesis this is equivalent to

\[
\forall G \in \bar{F}n \exists m \geq n \exists k, f \bar{G}m \vdash B[fi \circ F_{m_i}^n \circ F_m^p, f \circ F_k^q \circ G_{m_i}^q].
\]

Using Lemma 4.2(a), we then obtain \( \vdash \exists \eta B(e_i, \eta) [fi \circ F_{m_i}^n] \). Conversely, if \( \vdash \exists \eta B(e_i, \eta) [fi \circ F_{m_i}^n] \), Lemma 4.2(b) gives us for each \( G \in \bar{F}n \) an \( m \geq n \) and a function \( f_G \) such that \( \vdash B[fi \circ F_{m_i}^n \circ G_{m_i}^p, f_G] \), or, using the induction hypothesis, \( \bar{G}m \vdash B[fi : m_i, f_G : m] \). Thus \( \bar{F}n \vdash \exists \eta B(e_i, \eta) [fi : m_i] \). This completes the proof. \( \square \)

5. Lawlessness

A universe of sequences which satisfies the CS-axioms has strong closure properties: it is closed under the application of all lawlike continuous operations. For sequences satisfying the CS(M)-axioms, these closure properties are somewhat weaker. On the far other end we find the universe of lawless sequences, which has no closure properties at all (application of a lawlike continuous operation other than the identity to a lawless sequence never yields a lawless sequence again!) An important axiom here is the axiom of open data, which roughly says that the extension of a property of lawless sequences is always an open subset of the space of lawless sequences (as a subspace of intuitionistic Baire space. For a precise formulation, see 5.2 below).

In this part of the paper, we first (Section 5.1) return to the models of Section 2.3, focusing attention on those which satisfy a version of open data. We also describe how to obtain models for the theory of lawless sequences LS by an internal model construction ('projection models', iterated forcing). The theory of lawless sequences is formulated here in a language without arbitrary function and power types (the minimal language), and the internal model-construction is essentially the construction of [17].

Unfortunately, the proof of the correctness of this construction in [17] involves a long formula induction, and is rather complex. Moreover, it is not easy to see whether this proof can be extended to a higher order language. Therefore we will in Section 5.2 present a sheaf model over (a space homeomorphic to) Baire space for the higher order theory of lawless sequences. The proofs given in 5.2 are purely semantical, and the model seems to be more perspicuous than the model for LS presented in [1].
Our construction of a model for the higher order theory of lawless sequences was actually inspired by Troelstra's appendix to [1], and it seems worth the effort of explaining this in more detail. This will be done in Section 5.3.

5.1. Open data as analytic data

As a first example of a CS(M)-model, consider the monoid $M_h$ of local homeomorphisms of Baire space into itself. As has been said in 2.3, in sheaves over this monoid Baire space internally appears as $\text{cts}(B, B)$, and the sheaf $B_{\text{ct}}$ of 'M-choice sequences' is interpreted as the subsheaf of $\text{cts}(B, B)$ generated by 1, i.e. the sheafification of $M_h$, which is just $M_h$ in this case. In this particular model, analytic data 'is' open data (without choice parameters):

5.1.1. Proposition. In the model just described,

$$\vdash \forall \varepsilon \in M_h \forall \varepsilon \in (\exists \eta \in \varepsilon \land \forall \eta \in \exists \eta \in \varepsilon)$$

where $A$ has all non-lawlike parameters shown.

Proof. We have to show that

$$\vdash \forall \psi \in M_h \forall \varepsilon \in (\exists \eta \in \psi \eta \in \exists \eta \in \psi).$$

It suffices to choose $\psi$ with $\psi(f, g) = \hat{\psi} \circ g$ for some fixed local homeomorphism $\hat{\psi}$ (since such morphisms $\psi : M_h \times M_h \rightarrow M_h$ generate the sheaf $M_h$ of internal lawlike operations $\mathbb{N}^N \rightarrow \mathbb{N}^N$, see 2.3). So suppose $f$ and $g$ are local homeomorphisms such that $\psi f = (\psi \circ h) \eta$, i.e. (Lemma 2.2.3) there exists a local homeomorphism $h$ such that $f = \hat{\psi} \circ h$. Find a cover $\{W_i : B \rightarrow W_i \subset B\}$ such that $f \upharpoonright W_i$ is a homeomorphism, and let for each $x \in W_i$, $n_i \in \mathbb{N}$ be such that

$$u_i := f(x)(n_i) \subseteq f(U_i),$$

and let $v_i^x$ be an initial segment of $x$ such that $f(v_i^x) \subseteq f(x)(n_i)$. The $\{v_i^x\}_{i,x}$ form a cover, and $\vdash f \upharpoonright v_i^x \in u_i^x$ for each $i$ and each $x \in W_i$. Also $\vdash \forall \eta \in u_i^x \exists \eta \in \psi(\eta)$, for if $k$ and $l$ are local homeomorphisms such that $k \upharpoonright l \subseteq u_i^x$, then $\forall y \in B, k(y) \in u_i^x = f(x)(n_i)$, so range$(k) \subseteq U_i$, and therefore $k = \hat{\psi} h^{-1} k$, i.e. $\vdash \exists \eta \in k = \psi(\eta)$. \qed

Sheaves over the monoid of local homeomorphisms were considered by Fourman in his talk at the Brouwer conference. He defined the subsheaf $L$ of internal Baire space ($= \text{cts}(B, B)$), his sheaf of 'lawless sequences', to be the sheaf of local projections. More precisely, let $j : B \times B \rightarrow B$ be a fixed homeomorphism, and define $L$ to be the subsheaf of $\text{cts}(B, B)$ generated by $j_1 = \pi_1 j$. Observe that this sheaf $L$ becomes definable in the particular theory $\text{CS}(M_h)$ under discussion, namely as $\{1 \varepsilon | \varepsilon \in B_{\text{ct}}\}$. Thus, this model may be regarded as a 'projection model', projected from a CS(M)-model of the type described in 2.3. In this model, the sequences in $L$ satisfy various conditions which are similar in character to the axioms of LS (as formulated in 5.2 below), but there are some striking differences.
For example, a crucial role is played by the notion of independence: two lawless sequences \( \alpha \) and \( \beta \) are said to be independent iff \( (\alpha, \beta) : = i \circ (\alpha, \beta) \in L \) (or in \( \text{CS}(M_0) \)-terms, \( j_1e \) and \( j_1\eta \) are independent iff there exists a \( \xi \in B_{M_0} \) such that \( j_1e = j_1j_1\xi, j_1\eta = j_1j_2\xi \)). This notion of independence is necessary, for example, to formulate the multiple-parameter version of the open data axiom which is valid in this model. Thus, there is an essential difference between this axiom of open data, and the more traditional axiom, where instead of independence one has just inequality. It does not seem to be possible to modify this model so as to obtain a monoid model for 'ordinary' open data.

The problem is that sheaves over monoids have to satisfy some non-trivial closure-conditions (provided the sheaf and the monoid are non-trivial). For example in Fourman's model, the sheaf \( L \) is closed under projecting \( (\alpha, \beta) \in L \rightarrow \alpha \in L \land \beta \in L \). Such closure conditions are incompatible with the ordinary multiple-parameter version of open data. This strongly suggests that it is impossible to obtain monoid models for the theory LS.

Let us take a different approach for obtaining an LS-model, by starting with a monoid model for \( \text{LS}^1 \). (\( \text{LS}^1 \) is the theory with axioms (schemas) just like those of LS, but with the schemas \( \text{LS}3, \text{LS}4 \) restricted to formulas containing at most one parameter over choice sequences (lawless sequences), see [1], [18].) In fact, the sheaf \( L \) above is a domain satisfying the \( \text{LS}^1 \)-axioms. A simpler \( \text{LS}^1 \)-model can be obtained as follows. Let \( M_0 \) be the monoid of continuous functions of the form \( u, u(x) = u \mid x \), for finite sequences \( u \). The open cover topology is just the 'bar topology' \( \{\{u_i \mid i \in T\} \) covers iff \( \{u_i \mid i \in I\} \) is a bar in \( \mathbb{N}^{=N} \). In Section 2.3 it was shown that sheaves over this monoid yield a model for this instance \( \text{CS}(M_0) \) of relativized CS, and it is clear that analytic data comes down to open data without (non-lawlike) parameters in this case, and the sheaf \( B_{M_0} \) of ‘\( M_0 \)-choice sequences’ (the subsheaf of \( \text{cts}(B, B) \) generated by the identity) gives a model for the theory \( \text{LS}^1 \).

It is not an LS-model, of course, again because the monoid action on the sheaf gives us too many closure properties. For example, in sheaves over \( M_0 \), \( \vdash \exists \xi, n \in B_{M_0} \), \( (\xi \not= \eta \land \exists n \forall k > n \, \bar{k} = \bar{n}k) \) (take to different sequences \( u \) and \( v \) of equal length), which clearly contradicts (the two parameter case of) open data.

At this point we may invoke a method of Troelstra's for constructing an LS-model from an \( \text{LS}^1 \)-model: In [17], Troelstra shows that if \( L^1 \) is a subspace of Baire space satisfying the \( \text{LS}^1 \)-axioms, then for each \( \alpha \in L^1 \),

\[
q_\alpha = \{u \ast \pi_u(\alpha) \mid u \in N^{\ast N}\}
\]

can intuitionistically be shown to be a model of LS (here \( \pi_u(\alpha)(n) = \alpha(u \ast (n)) \)).

Thus, within the monoid model \( \text{Sh}(M_0) \) under discussion, we have many LS-models, but they are not definable externally. An easy way out here is to construct internally the direct product \( \mathcal{U} = \prod_{\alpha \in B_c} q_\alpha \). Then \( \text{Sh}(M_0) \vdash "\mathcal{U} \vdash \text{LS}^1" \) by Troelstra's result, and it is possible to reduce this two-step forcing to a single step. One then obtains a sheaf-model over a site \( \mathcal{S} \) which is neither a monoid, nor a
topological space. We will not describe the construction of $S$ in detail: the reader who is familiar with models over sites will be able to work it out for himself.

It should be stressed that the proof of Troelstra’s result uses induction on formulas, and holds only for the first-order language in which LS is usually formulated. We have not been able to find a direct proof of the validity of the open data axiom in sheaves over the site $S$ without this restriction on the language.

5.2. A sheaf model for LS

We start by formulating the LS-axioms. They are formulated in a higher order language (with arbitrary function- and powersorts, as in Section 1), with in addition, sorts $B_L$ for lawlike sequences, $K$ for lawlike neighbourhoodfunctions, and $L$ for lawless sequences; these are all subsorts of $N^N$. We use $\alpha, \beta, \gamma, \ldots$ as variables ranging over $L$. The axioms are

LS1 (decidable equality)
\[ \forall \alpha, \beta \in L \ (\alpha = \beta \lor \alpha \neq \beta). \]

LS2 (density)
\[ \forall u \in N^{<N} \exists \alpha \alpha \in u. \]

LS3 (higher order open data) For each $n \in \mathbb{N}$,
\[ \forall \alpha_1 \cdots \forall \alpha_n \ (\neq (\alpha_1, \ldots, \alpha_n)) \land A(\alpha_1, \ldots, \alpha_n) \rightarrow \exists u_1 \exists \alpha_1 \cdots \exists u_n \exists \alpha_n \]
\[ \forall \beta_1 \in u_1 \cdots \forall \beta_n \in u_n \ (\neq (\beta_1, \ldots, \beta_n)) \rightarrow A(\beta_1, \ldots, \beta_n)) \]

where $\neq (\alpha_1, \ldots, \alpha_n)$ abbreviates $\bigwedge_{1 \leq i < j \leq n} \alpha_i \neq \alpha_j$.

LS4 (higher order continuity) For each $n \in \mathbb{N}$,
\[ \forall \alpha_1 \cdots \forall \alpha_n \ (\neq (\alpha_1, \ldots, \alpha_n)) \rightarrow \exists a \ A(\alpha_1, \ldots, \alpha_n, a) \]
\[ \rightarrow \exists e \in K_n \forall u_1 \cdots \forall u_n \ e(u_1, \ldots, u_n) \neq 0 \]
\[ \rightarrow \exists a \forall \alpha_1 \in u_1 \cdots \forall \alpha_n \in u_n \ (\neq (\alpha_1, \ldots, \alpha_n)) \rightarrow A(\alpha_1, \ldots, \alpha_n, a)) \]

(where $K_n$ is the set of $n$-place lawlike neighbourhoodfunctions, defined in the obvious way).

In LS3 and LS4, the formula $A$ contains no other non-lawlike parameters than the ones shown.

Our sheaf model will in fact be an interpretation in ‘sheaves with a group action’, as described in e.g. the appendix of [5]. Let $(u_n : n \in \mathbb{N})$ be an enumeration of $N^{<N}$ in which each sequence occurs infinitely many times. Let $T$ be the space $\prod_{n \in \mathbb{N}} V_{u_n}$ equipped with the product topology. In this section, we will write $V_u$ instead of just $u$ for the basic open subset $\{ x \mid x \in u \}$ of $B$, for $u$ a finite sequence. If $u_1, \ldots, u_n$ are finite sequences, then we write $\langle V_{u_1}, \ldots, V_{u_n} \rangle$ for the basic open subset $\cap_{i=1}^n \pi_i^{-1}(V_{u_i})$ of $T$. $T$ is obviously homeomorphic to $B$, but for present purposes $T$ is notationally more convenient than $B$ is.
We now define a group $G$ of auto(homeo)morphisms of $T$ as follows. Consider the following two types of automorphisms of $T$:

1. For each $n, m \in \mathbb{N}$ and $u \in \mathbb{N}^{< \mathbb{N}}$ such that $v_n \preceq u$ and $v_m \preceq u$, the automorphism $h = h[n, m, u]$ of $T$ which interchanges the $n$th and $m$th coordinate of a point $x \in T$, provided both coordinates begin with $u$; i.e.

$$h(x)_k = \begin{cases} 
  x_m, & \text{if } k = n \text{ and } x_n \in u, x_m \in u, \\
  x_n, & \text{if } k = m \text{ and } x_n \in u, x_m \in u, \\
  x_k, & \text{otherwise.}
\end{cases}$$

2. For each $n \in \mathbb{N}, f, g : \mathbb{N} \rightarrow \mathbb{N}$ with

$$n < g0 < f0 < g1 < f1 < g2 < f2 < \cdots$$

such that

- $\{v_{g(m)}\}_m$ and $\{v_{f(m)}\}_m$ are constant sequences in $\mathbb{N}^{< \mathbb{N}}$
- $v_{g(0)}$ and $v_{f(0)}$ are incompatible extensions of $v_n$, the automorphism $h = h[n, f, g]$ of $T$ defined as follows (cf. the picture below):

(a) If $x \in T$ is such that $x_n \in v_{f(0)}$ and $x_{g(0)} \in v_n$, then $h(x) = y$, where $y_n = x_{g(0)}$, $y_{f(0)} = x_n$, $y_{f(m+1)} = x_{f(m)}$ for each $m \in \mathbb{N}$, $y_{g(m)} = x_{g(m+1)}$ for each $m \in \mathbb{N}$, and $y_i = x_i$ for all $i \not\in \{n\} \cup \{g(m) | m \in \mathbb{N}\} \cup \{f(m) | m \in \mathbb{N}\}$.

(b) If $y \in T$ is such that $y_n \in v_{g(0)}$ and $y_{f(0)} \in v_n$, then $h(y) = x$, where $y$ and $x$ are related as in (a).

(c) If $z \in T$ is a point to which neither (a) nor (b) applies, then $h(z) = z$.

\[
\begin{array}{cccccccc}
  & n & g(0) & f(0) & g(1) & f(1) & g(2) & f(2) & \cdots \\
\hline
x & & & & & & & & \\
\hline
\rightarrow & & & & & & & & \\
\rightarrow & & & & & & & & \\
y & n & g(0) & f(0) & g(1) & f(1) & g(2) & f(2) & \cdots
\end{array}
\]

$G$ is the subgroup of the group of automorphisms of $T$ generated by all homeomorphisms of the form (1) or (2).

Recall (cf. [5], appendix) that if $G$ is a group of automorphisms of $T$, a ‘sheaf with $G$-action’ on $T$ is a sheaf $A$ on $T$ with an action of $G$ on the sections of $A$, written $a \mapsto a^g$, such that

$$a^1 = a, \quad a^{gh} = (a^g)^h, \quad [a^g = b^g] = g^{-1}([a = b])$$

(and hence, $E(a^g) = g^{-1}(E(a))$, and $(a \uparrow U)^g = a^g \uparrow g^{-1}(U)$).

In the ‘standard interpretation’ in such sheaves with a group action, the sheaf of natural numbers $\mathbb{N}$ appears as the sheaf of continuous partial functions $U \rightarrow \mathbb{N}$, $U \in \mathcal{C}(T)$, with right composition

$$\left( U \xrightarrow{a} \mathbb{N} \right) \mapsto \left( h^{-1}(U) \xrightarrow{a^g h} \mathbb{N} \right)$$
as action. Similarly, internal Baire space \( N^N \) appears as the sheaf of continuous partial functions \( U \to B \) with right composition as action.

If \( A \) is a sheaf with \( G \)-action, a global element of \( A \) is a global section \( a \) of \( A \) which is invariant under the action of \( G \) \((a^g = a \text{ for } g \in G)\). We define the sheaf \( A_L \) of lawlike elements of \( A \) to be the subsheaf of \( A \) generated by the global elements of \( A \). (In fact, this is what we also did in Section 2.)

Our model will be the standard interpretation in sheaves over the space \( T \) with \( G \)-action, where the space \( T \) and the group \( G \) are as defined above. Further, we specify the interpretation of the additional constants: \( B, \) and \( K \) are interpreted as the sheaf of locally constant partial functions \( U \to B \) and \( U \to K \) respectively (where \( K \subset B \) is the set of external neighbourhood functions), with right composition as action. The sheaf of lawless sequences \( L \) is the sheaf generated by the projections \( \pi_n : T \to B \) \((n \in \mathbb{N})\), again with right composition as action. Note that each of the homeomorphisms in \( G \) locally either is the identity, or interchanges coordinates. Hence the sheaf of partial functions \( U \to B \) \((U \in \mathcal{O}(T))\) which are locally some \( \pi_n \) is indeed closed under the action of \( G \).

The rest of this section will consist of the proof of the following theorem.

5.2.1. Theorem. The interpretation just described yields a model for the higher order theory \( LS \).

Verification of \( LS_1 \) and \( LS_2 \) is trivial. For the axioms of open data and continuity, however, we have to do some work. First note that if \( A(\alpha_1, \ldots, \alpha_n, p_1, \ldots, p_k) \) is a formula with \( \alpha_1, \ldots, \alpha_n \) as lawless parameters of sort \( L \), and we interpret all other parameters \( p_1, \ldots, p_k \) by global sections \( \tilde{p}_1, \ldots, \tilde{p}_k \) of the appropriate sheaves, then

\[
[A(\alpha_1, \ldots, \alpha_n, \tilde{p}_1, \ldots, \tilde{p}_k)]
\]

is a global section of the powersheaf \( \mathcal{P}(L^n) \); that is a function \( P : L^n \to \mathcal{O}(T) \) which is strict and extensional

\[
(P(\alpha_1, \ldots, \alpha_n)) \subseteq E(\alpha_1, \ldots, \alpha_n),
\]

\[
P(\alpha_1 \upharpoonright U, \ldots, \alpha_n \upharpoonright U) = P(\alpha_1, \ldots, \alpha_n) \cap U,
\]

and moreover preserves the action, i.e. \( P(\alpha_1, \ldots, \alpha_n) = g^{-1}P(\alpha_1, \ldots, \alpha_n) \).

By the interpretation of lawlike elements described above, such functions generate the extensions of the formulas \( A \) occurring in the \( LS \)-axioms, and therefore we may restrict our attention to strict extensional functions \( P \) which preserve the action, as we do in the following two lemmas.

5.2.2. Lemma. Let \( P : I^p \to \mathcal{O}(T) \) be a global section of \( \mathcal{P}(I^p) \), and let \( x \) and \( y \) be two points of \( T \) such that \( x_{n_i} = y_{n_i} \) for each \( i = 1, \ldots, p \). Then \( x \in P(\pi_{n_1}, \ldots, \pi_{n_p}) \) iff \( y \in P(\pi_{n_1}, \ldots, \pi_{n_p}) \).

Proof. Suppose \( x \in P(\pi_{n_1}, \ldots, \pi_{n_p}) \), and choose sequences \( u_1, \ldots, u_k \) such that
Sheaf models for choice sequences

Let \( x \in \langle V_{u_1}, \ldots, V_{u_p} \rangle \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p}) \). We may assume that each \( u_i \geq v_i \), and that \( k \geq n_p \). We now define an \( h \in G \) and a point \( z \in \langle V_{u_1}, \ldots, V_{u_p} \rangle \) such that \( h(z) = y \) and \( \pi_{n_i} \circ h = \pi_{n_i} \) for \( i = 1, \ldots, p \). This suffices to prove the lemma since then

\[
y = h(z) \in h \left( P(\pi_{n_1}, \ldots, \pi_{n_p}) \right) = P(\pi_{n_1} \circ h^{-1}, \ldots, \pi_{n_p} \circ h^{-1}) = P(\pi_{n_1}, \ldots, \pi_{n_p}).
\]

Let \( \{a_1, \ldots, a_l\} = \{1, \ldots, k\} \setminus \{n_1, \ldots, n_p\} \). Choose for each \( i \leq l \), two incompatible extension \( w_i \) and \( w_i' \) of \( u_{a_i} \), and let \( \{f^i(m)\}_m \) and \( \{g^i(m)\}_m \) be sequences of natural numbers such that

\[
v_{f(m)} = w_i \quad \text{and} \quad v_{g(m)} = w_i' \quad \text{for each} \quad i \leq l,
\]

and

\[
k < g^i(0) < f^i(0) < g^i(1) < f^i(1) < \cdots \quad \text{for each} \quad i \leq l.
\]

and such that the ranges of the \( g^i \)'s, and those of the \( f^i \)'s, are mutually disjoint. Now set

\[
h = h[a_1, f^1, g^1] \circ \cdots \circ h[a_1, f^1, g^1]
\]

and let \( z \) be the point defined by

\[
z_{a_i} = y_{f(0)}, \quad z_{g(0)} = y_{u_i}, \quad z_{f(m)} = y_{f(m+1)} \quad \text{for each} \quad m \in \mathbb{N},
\]

\[
z_{g(m+1)} = y_{g(m)} \quad \text{for each} \quad m \in \mathbb{N},
\]

\[
z_n = y_n \quad \text{for all other} \quad n.
\]

Then \( z \in \langle V_{u_1}, \ldots, V_{u_p} \rangle \), and \( h(z) = y \). \( \square \)

### 5.2.3. Lemma

Let \( P : L^p \to \mathcal{O}(T) \) be a global section of \( \mathcal{P}(L^p) \), and let \( U = \langle V_{u_1}, \ldots, V_{u_p} \rangle \) be a basic open subset of \( T \) with \( U \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p}) \). If \( f : \mathbb{N} \to \mathbb{N} \) is a function with \( f([m_1, \ldots, m_p]) \cap \{n_1, \ldots, n_p\} = \emptyset \), and \( W = \langle V_{w_1}, \ldots, V_{w_p} \rangle \) is a basic open of \( T \) such that \( w_{f(n_i)} \geq u_{n_i} \) \( (i = 1, \ldots, p) \), then \( W \subseteq P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}) \).

**Proof.** Let \( U \) and \( f \) be as described in the lemma. By 5.2.2, we find

\[
(1) \quad \pi_{n_1}^{-1}(V_{u_1}) \cap \cdots \cap \pi_{n_p}^{-1}(V_{u_p}) \subseteq P(\pi_{n_1}, \ldots, \pi_{n_p}).
\]

It suffices to show that

\[
(2) \quad \pi_{f(n_1)}^{-1}(V_{u_1}) \cap \cdots \cap \pi_{f(n_p)}^{-1}(V_{u_p}) \subseteq P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}),
\]

but by 5.2.2 again, (2) already follows from (3),
where \( s_i \) is the shortest sequence with

\[
(4) \quad s_i \supseteq u_n_i \quad \text{and} \quad s_i \supseteq v_{f(n)}.
\]

(We may of course assume that \( u_n_i \) and \( v_{f(n)} \) are compatible, since otherwise (2) is trivially true.)

To prove (3), choose \( y \in T \) such that

\[
y_{f(n)} \in u_n_i, \quad y_n \in u_n_i, \quad \text{and} \quad y_n \in v_{f(n_i)} \quad (i = 1, \ldots, p).
\]

We will define a point \( x \in \bigcap_{i=1}^{p} \pi_{n_i}^{-1}(V_{u_{n_i}}) \), and an automorphism \( h \in G \) such that

\[
h(x) = y, \quad \text{and} \quad \pi_{f(n)} \circ h = \pi_{n_i} \text{ on a neighbourhood of } x, \ i = 1, \ldots, p.
\]

This suffices to prove the lemma, since \( x \in P(\pi_{n_1}, \ldots, \pi_{n_p}) \) implies that also

\[
x \in P(\pi_{f(n_1)} \circ h, \ldots, \pi_{f(n_p)} \circ h) = h^{-1}P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}),
\]

hence \( y = h(x) \in P(\pi_{f(n_1)}, \ldots, \pi_{f(n_p)}) \).

Let

\[
h = h[n_p, f(n_p), s_p] \circ \cdots \circ h[n_1, f(n_1), s_1].
\]

\( h \in G \), since \( s_i \supseteq u_n_i \) (and without loss \( u_n_i \supseteq u_{n_i} \)), and \( s_i \supseteq v_{f(n)} \). Let \( x \) be defined by

\[
x_n_i = y_{f(n_i)}, \quad x_{f(n_i)} = y_n, \quad \text{for } i = 1, \ldots, p
\]

\[
x_m = y_m, \quad \text{for other coordinates } m.
\]

Then \( x_n_i \in s_i \) and \( x_{f(n_i)} \in s_i \), so \( h[n_p, f(n_p), s_p] \) interchanges the \( n_i \)th and \( f(n_i) \)th coordinates on a neighbourhood of \( x \). It is clear that \( h(x) = y \), and that \( x \in \bigcap_{i=1}^{p} \pi_{n_i}^{-1}(V_{u_{n_i}}) \).

**5.2.4. Lemma.** As Lemma 5.2.3, but without the requirement that \( f(\{n_1, \ldots, n_p\}) \cap \{n_1, \ldots, n_p\} = \emptyset \).

**Proof.** This follows from Lemma 5.2.3 by factoring \( f \) as a composition of injections that do satisfy the hypothesis of 5.2.3.

**Proof of 5.2.1.** It has already been observed that LS1 and LS2 are trivial, and the validity of open data (LS3) follows immediately from the preceding lemma. So we only have to check LS4. Now if \( \forall \alpha_1 \cdots \forall \alpha_p \exists a A(\alpha_1, \ldots, \alpha_p, a) \) is a formula with all (lawlike) parameters interpreted by global elements, then \( \forall \alpha_1 \cdots \forall \alpha_p (\neq (\alpha_1, \ldots, \alpha_p) \rightarrow \exists a A(\alpha_1, \ldots, \alpha_p, a)) \) is a global truth value, i.e. an open subset \( U \) of \( T \) such that \( g^{-1}(U) = U \) for all automorphisms \( g \in G \). But (using a composition of automorphisms of type (2)) it is easily seen that the only such \( U \) are \( \emptyset \) and \( T \).

We may thus assume that \( \forall \alpha_1 \cdots \forall \alpha_p (\neq (\alpha_1, \ldots, \alpha_p) \rightarrow \exists a A(\alpha_1, \ldots, \alpha_p, a)) \)
Sheaf models for choice sequences

103

= T. In particular, if we let \( n_1, \ldots, n_p \) be distinct natural numbers such that 
\[ v_{n_i} = {} \langle \cdot \rangle \text{ for } i = 1, \ldots, p, \]
we find that
\[ T = \{ a \mid \forall a A(\pi_{n_1}, \ldots, \pi_{n_p}, a) \}. \]

Let \( e \) be a \( p \)-place (external) neighbourhood function such that 
\[ e(w_1, \ldots, w_p) \neq 0 \]
implies that for some \( a \in B_L \),
\[ (V_{w_1}, \ldots, V_{w_p}) \subseteq \{ a \mid \forall a \in B_L \forall \beta_1 \in w_1 \cdots \forall \beta_p \in w_p \]
\[ (\neq (\beta_1, \ldots, \beta_p) \rightarrow A(\beta_1, \ldots, \beta_p, a)) = T. \]

To see this, choose \( w_1, \ldots, w_p \in \mathbb{N}^{\mathbb{N}^n} \) with 
\[ e(w_1, \ldots, w_p) \neq 0. \]
Choose \( a \in B_L \) such that 
\[ (V_{w_1}, V_{w_p}) \subseteq \{ a \mid \forall a \in B_L \forall \beta_1 \in w_1 \cdots \forall \beta_p \in w_p \]
\[ (\neq (\beta_1, \ldots, \beta_p) \rightarrow A(\beta_1, \ldots, \beta_p, a)) = T. \]

Hence (*) holds. □

5.3. Projection models are Beth models

In the foregoing we have used the word \textquote{projection-model} to refer to universes of the form \( \mathcal{U}^M_\alpha = \{ f(\alpha) \mid f \in M \} \), where \( M \) is a subset of \( K \) and \( \alpha \) is a lawless sequence or a sequence in a domain which satisfies the LS\(^1\) axioms. In this section we give our own exposition of the fact that validity in such a projection model is equivalent to constructive validity in a topological model over (formal) Baire space (cf. [18], and the appendix to [1]). By doing so, we hope to clarify the remarks made in the introduction to Section 4, as well as to explain the relation between the model presented in Section 5.2 and the appendix to [1].

As in Section 3, we restrict ourselves to the four sorted minimal language. As formal language for the treatment of interpretations of this minimal language in projection models we take the same language, but with the sort \( B_C \) replaced by \( L \) (for lawless sequences). We use \( \alpha, \beta, \gamma, \ldots \) as variables of sort \( L \). Moreover, we add a constant \( K \) for the sort of continuous functions \( N^N \to N^N \) with neighbourhood functions in \( K \). As constructive metatheory for the treatment of Beth-models we use the system IDB (cf. Section 3).

Let \( A(e_1, \ldots, e_n) \) be a formula in the minimal language, and let \( \mathcal{U}^M_\alpha \) be a projection model. \( \mathcal{U}^M_\alpha \models A(f_1(\alpha), \ldots, f_n(\alpha)) \) expresses that \( A \) holds if we interpret (a) the parameters \( e_1, \ldots, e_n \) by \( f_1(\alpha), \ldots, f_n(\alpha) \) respectively (\( f_i \in M \)); (b) the sort \( B_C \) by \( \mathcal{U}_\alpha \), i.e. quantifiers over \( B_C \) are interpreted as quantifiers over \( \mathcal{U}_\alpha \); and (c) the sorts \( B_L \) and \( N \) by themselves. (So the satisfaction sign \( \models \) is treated in the
traditional Tarskian sense here, be it within the theory LS¹, or within an LS¹-model. We will write $A^\alpha(f_1, \ldots, f_n)$ for the LS-formula in the single parameter $\alpha$ of sort $L$ which denotes $\mathcal{U}_M^\alpha \models A(f_1(\alpha), \ldots, f_n(\alpha))$; thus $A^\alpha(f_1, \ldots, f_n)$ is obtained from $A(\varepsilon_1, \ldots, \varepsilon_n)$ by substituting $f_i(\alpha)$ for $\varepsilon_i$, $i = 1, \ldots, n$, replacing bound variables $\varepsilon$ by suitably chosen $f(\alpha)$, and replacing the quantifiers $\forall \varepsilon$, $\exists \varepsilon$ by the corresponding $\forall f \in M$, $\exists f \in M$. We say that a sentence $A$ holds in $\mathcal{U}_M$ iff $LS¹ \vdash \forall \alpha A^\alpha$.

The LS¹-axioms provide a full explanation of universal lawless quantification over formulas $B(\alpha)$, in which $\alpha$ is the only choice parameter, and in which no quantifiers over $L$ occur. This explanation proceeds along the same lines as the explanation of quantification over choice sequences in CS (cf. Section 3.2), but since we restrict ourselves to the explanation of universal quantifiers and avoid nested quantification, there is no need for the explanation of $\exists \alpha$, $\forall \alpha \exists \beta$, $\forall \alpha \forall \beta$. The main difference with the CS-explanation lies in the treatment of formulas of the form $\forall \alpha (A(\alpha) \rightarrow B(\alpha))$. By open data in a single parameter, $\forall \alpha (A(\alpha) \rightarrow B(\alpha))$ is equivalent to $\forall u (\forall \alpha \subset u \Lambda(\alpha) \rightarrow \forall \alpha \in u B(\alpha))$, i.e., universal lawless quantification is explained in terms of universal quantification over finite sequences and universal lawless quantification over formulas of lower complexity.

The explanation leads to the following elimination translation for sentences $\forall \alpha \in u B(\alpha)$, $B(\alpha)$ not containing lawless quantifiers:

$$
\tau(\forall \alpha \in u f(\alpha)(n) = m) = \forall \alpha \subset u f(\alpha)(n) = m,
$$

$$
\tau(\forall \alpha \in u (A(\alpha) \wedge B(\alpha))) = \tau(\forall \alpha \in u A(\alpha)) \wedge \tau(\forall \alpha \in u B(\alpha)),
$$

$$
\tau(\forall \alpha \in u (A(\alpha) \vee B(\alpha))) = \exists a \in K \forall v (av \neq 0
\rightarrow (\tau(\forall \alpha \in u \ast v A(\alpha))
\vee \tau(\forall \alpha \in u \ast v B(\alpha))),
$$

$$
\tau(\forall \alpha \in u (A(\alpha) \rightarrow B(\alpha))) = \forall v (\tau(\forall \alpha \in u \ast v A(\alpha))
\rightarrow \tau(\forall \alpha \in u \ast v B(\alpha))),
$$

$$
\tau(\forall \alpha \in u \forall p A(\alpha, p)) = \forall p \tau(\forall \alpha \in u A(\alpha, p)), p \text{ of a lawlike sort},
$$

$$
\tau(\forall \alpha \in u \exists p A(\alpha, p)) = \exists a \in K \forall v (av \neq 0
\rightarrow \exists p \tau(\forall \alpha \in u \ast v A(\alpha, p))).
$$

The translation $\tau$ defined here is just a fragment of the full elimination translation for LS. $\tau$ has the following property (cf. [18]):

1. If $B(\alpha)$ is free of lawless quantifiers and lawless parameters other than $\alpha$, then $LS¹ \vdash (\forall \alpha B(\alpha) \leftrightarrow \tau(\forall \alpha B(\alpha)))$.

   $\tau(\forall \alpha B(\alpha))$ is a formula of lawlike IDB (i.e. IDB with $B_L$ for $B$, $K$ for $K$, $K$ for $cts(B, B)$, and the LS¹-axioms are conservative over IDB (in fact LS is conservative over IDB), so we also have

2. If $B(\alpha)$ is a formula as in (1) above, then

   $LS¹ \vdash \forall \alpha B(\alpha) \text{ iff } IDB \vdash \tau(\forall \alpha B(\alpha))$. 

Now let $A(e_1, \ldots, e_n)$ be a formula in the CS-language. Then $A^o(f_1, \ldots, f_n)$ is a formula in a single lawless parameter, without lawless quantifiers. Hence we can apply the previous elimination theorem to $\forall \alpha A^o(f_1, \ldots, f_n)$. Let us (suggestively) write $u \models A(f_1, \ldots, f_n)$ for $\tau(\forall \alpha \in u A^o(f_1, \ldots, f_n))$. The relation $u \models A(f_1, \ldots, f_n)$ then satisfies the following equivalences (provable in IDB):

\begin{align*}
\text{If } f(n) = m & \iff \forall a \in B_L(a \in u \to f(a)(n) = m), \\
\text{if } a(n) = m & \iff an = m (a \text{ of sort } B_1), \\
\text{if } A & \iff u \models A \wedge u \models B, \\
\text{if } \exists a \in K \forall v (av \neq 0 \to (u \ast v \models A \vee u \ast v \models B)), \\
\text{if } A & \to B \iff \forall v (u \ast v \models A \to u \ast v \models B), \\
\text{if } \forall p A(p) & \iff \forall p \ u \models A(p), \\
\text{if } \exists p A(p) & \iff \exists a \in K \forall v (av \neq 0 \to \exists p \ u \ast v \models A(p)), \\
\text{if } \forall e A(e) & \iff \forall f \in M \ u \models A(f), \\
\text{if } \exists e A(e) & \iff \exists a \in K \forall v (av \neq 0 \to \exists f \in M \ u \ast v \models A(f)).
\end{align*}

Inspection of these clauses shows that they are exactly the clauses defining 'formal' Beth-forcing for the minimal language, formulated in the language of lawlike IDB, where $N, B_L, \text{ and } K$ are interpreted by themselves and $B_C$ is interpreted as (the subsheaf of internal Baire space generated by) $M$. The word 'formal' in this context refers to the fact that the clauses for $\vee$ and $\exists$ are formulated in terms of existential quantification over $K$. The clauses are as for forcing in sheaves over Baire space, but we do not mention points. We just talk about finite sequences, and bars defined via $K$. In the absence of external bar induction this is a sensible adaption: instead of BI we can now use induction over unsecured sequences. The distinction between lawlike IDB and IDB itself is just a matter of notation. Hence the elimination theorem for $LS^1$ (properties (1) and (2) above) yields the following theorem.

5.3.1. Theorem. Let $A(e_1, \ldots, e_n)$ be a formula in the CS-language. Then $\forall \alpha^M \models A(f_1(\alpha), \ldots, f_n(\alpha))$, i.e. $LS^1 \models \forall \alpha A^o(f_1, \ldots, f_n)$, iff it is provable in IDB that $A(f_1, \ldots, f_n)$ holds in sheaves over formal Baire space, where $N$ is interpreted as $\text{cts}(B, N)$, $B_L$ as the sheaf generated by the constant functions $B \to B$. $K$ by the sheaf generated by the constant functions $B \to K$, and $B_C$ by the subsheaf of $\text{cts}(B, B)$ generated by $M$. $\Box$

A simple application of this result is the following. Let $M$ be the set $\{id\}$. Let $\forall \alpha A(\alpha)$ be an LS-sentence without other lawless quantifiers. Then $A(\alpha)$ and $A^o(id)$ are equivalent in $LS^1$. So $LS^1 \models \forall \alpha A(\alpha)$ iff $A(\alpha)$ holds in sheaves over formal Baire space, where $\alpha$ is interpreted as the generic element id. In this sense lawless sequences are generic.

Another application is the one mentioned in the appendix to [1]: In [17] it is shown that for $M = \{f_n : \alpha \mapsto n \ast (\alpha)_n \mid n \in \mathbb{N}\}$, $\forall \alpha^M$ is an LS-model, provably in
LS$^1$. Hence the sheaf generated by $\mathcal{M}$ is an LS-model over Baire space, provably in IDB. One easily verifies that there is a homeomorphism $h : B \to T'$, where $T'$ is the product of all basic opens of Baire space (without repetitions), and that $h$ can be chosen in such a way that $f_n \circ h^{-1} = \tau_n$. This is obviously the origin of the LS-model in 5.2 above.

**Acknowledgements**

Mike Fourman's talk at the Brouwer Symposium (June 1981) inspired our investigations into the subject. We would like to thank Robin Grayson for his helpful correspondence. Fourman, Grayson, and the authors independently and almost simultaneously (October 1981) observed that sheaves over the monoid of continuous functions from Baire space to itself with the open cover topology give a model for CS. Grayson also observed (independently from us) the connection with the elimination translation (cf. [3], [6]).

**Additional notes** (added September 1983)

(1) We would like to thank the referee, who spotted a large number of misprints in the original typescript.

(2) In Section 5.1 we discussed the model Sh($\mathcal{M}_h$) for a theory of lawless sequences, which was the topic of M. Fourman's talk at the Brouwer conference. After this paper had been submitted, we received a copy of Fourman's article (Notions of choice sequence, in: A.S. Troelstra and D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium, North-Holland, Amsterdam, 1982), which is rather more general in content and, among other things, puts some of the results of this paper in a wider setting.

**References**


