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AN ELEMENTARY PROOF OF THE DESCENT THEOREM FOR
GROTHENDIECK TOPOSES

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The key theorem of Joyal & Tierney [1] is the descent theorem for geometric
morphisms of Grothendieck toposes (over a fixed base topos \( \mathcal{S} \)). This theorem says
that open surjections are effective descent morphisms – a fact which has remarkable
consequences (see loc. cit.). Joyal and Tierney prove the descent theorem by first
developing descent theory for ‘modules’ (suplattices) over locales, parallel to des-
cent theory for commutative rings. In this way they provide an algebraic explanation
for the theorem. The purpose of this note is to give a direct proof of the descent
theorem.

1. Formulation of the descent theorem (see Joyal & Tierney [1])

Let \( \mathcal{E} \xrightarrow{f} \mathcal{D} \) be a geometric morphism of Grothendieck toposes over \( \mathcal{S} \), and
consider the diagram

\[
\begin{array}{c}
\mathcal{E} \times_\mathcal{S} \mathcal{E} \times_\mathcal{S} \mathcal{E} \\
\downarrow p_{12} \\
\mathcal{E} \times_\mathcal{S} \mathcal{E} \\
\downarrow p_{23} \\
\mathcal{E} \\
\downarrow p_{13} \\
\mathcal{S} \\
\delta
\end{array}
\]

Descent-data on an object \( X \in \mathcal{E} \) consists of a morphism \( \theta : p_1^*(X) \to p_2^*(X) \) such that
\( \delta^*(\theta) = \text{id} \) and \( p_{13}^*(\theta) = p_{23}^*(\theta) \circ p_{12}^*(\theta) \) (the cocycle condition). \( \text{Des}(f) \) denotes the
category of pairs \( (X, \theta) \), \( \theta \) descent-data on \( X \in \mathcal{E} \), where morphisms \( (X, \theta) \to (X', \theta') \) are
morphisms \( X \xleftarrow{f} X' \) in \( \mathcal{E} \) which commute with descent-data in the obvious way.
Any object \( f^*(D) \), \( D \in \mathcal{D} \), can be equipped with descent-data in a canonical way,
and this gives a commutative diagram
where $U$ is the forgetful functor. $f$ is called an effective descent morphism if $\mathcal{D} \to \text{Des}(f)$ is an equivalence of categories. The descent theorem states that every open surjection is an effective descent morphism.

Note that by working inside $\mathcal{D}$, it suffices to prove this theorem for the special case that $\varepsilon \to \mathcal{D}$ is the canonical geometric morphism $\varepsilon \to \mathcal{D}$; accordingly, we will only consider this case.

2. Some preliminary remarks

Let $\varepsilon = \text{Sh}(C)$, $\mathcal{C}$ a site in $\mathcal{D}$. Then a site for $\varepsilon \times \varepsilon = \varepsilon \times_{\mathcal{D}} \varepsilon$ is given by the product-category $\mathcal{C} \times \mathcal{C}$ with the coarsest topology making the projections

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{P_1} & \mathcal{C} \\
\downarrow \quad \downarrow \\
\mathcal{C} & \xrightarrow{P_2} & \mathcal{C}
\end{array}
$$

continuous, i.e. the topology is generated by covers of the form

$$
\{(C_i, D) \xrightarrow{(f_i, \text{id})} (C, D)\}_i \quad \text{and} \quad \{(C, D_j) \xrightarrow{(\text{id}, g_j)} (C, D)\}_j,
$$

where $\{C_i \xrightarrow{f_i} C\}_i$ and $\{D_j \xrightarrow{g_j} D\}_j$ are covers in $\mathcal{C}$. The inverse image $p_1^*$ of the geometric morphism $\varepsilon \times \varepsilon \xrightarrow{P_1} \varepsilon$ comes from composing with $P_1$, followed by sheafification. Similarly for $p_2^*$. The inverse image $\delta^*$ of the diagonal $\varepsilon \xrightarrow{\delta} \varepsilon \times \varepsilon$ comes from composing with $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ followed by sheafification: given $Y \in \text{Sh}(\mathcal{C} \times \mathcal{C}) = \varepsilon \times \varepsilon$, $\delta^*(Y)$ is the sheaf associated to the presheaf $C \to Y(C, C)$. So for $Y = p_1^*(X)$, $\delta^*p_1^*(X) \cong X$, and we have a canonical natural transformation $\eta_C : p_1^*(X)(C, C) \to X(C)$, which is the unit of the associated sheaf adjunction. Similarly for $p_2^*$.

3. The case of connected locally connected geometric morphisms

As a warming up exercise, let us point out that the descent theorem is trivial when $\varepsilon \to \mathcal{D}$ is connected, locally connected (this is not needed for the proof of the general case). Indeed, let $\mathcal{C}$ be a molecular site for $\varepsilon$ (with a terminal, since $\gamma$ is connected). Constant presheaves on $\mathcal{C}$ are sheaves, and $p_1^*, p_2^*$ are just given by composition with $P_1$ and $P_2$ respectively (no sheafification needed). Now suppose $X$ is a sheaf on $\mathcal{C}$, with descent-data $X \circ P_1 \xrightarrow{\theta} X \circ P_2$. This means that we are given
functions $\theta_{CD} : X(C) \to X(D)$ for every pair of objects $C$ and $D$ of $C$. Naturality of $\theta$ means that for any $C' \xrightarrow{f} C$, $D' \xrightarrow{g} D$, $X(g) \circ \theta_{CD} = \theta_{CD'} \circ X(f)$. $\delta^*(\theta) = \text{id}$ means that for any $C$, $\theta_{CC} : X(C) \to X(C)$ is the identity. And the cocycle condition means that for any triple $C, D, E$ of objects of $C$, $\theta_{DE} \circ \theta_{CD} = \theta_{CE}$. So in particular, taking $C = E$, $\theta_{CD}$ is inverse to $\theta_{DC}$, i.e. $\theta$ is an isomorphism. From this it easily follows that $X$ is isomorphic to the constant sheaf $\gamma^*(X(1))$: define

$$
\begin{array}{ccc}
X & \xleftarrow{\psi} & \gamma^*(X(1)) \\
\varphi& \end{array}
$$

by the components $\varphi_C = \theta_{1C}; \psi_C = \theta_{C1}$. $\varphi$ and $\psi$ are inverse to each other, and are natural in $C$ by naturality of $\theta$. It remains to show that any morphism $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ which is compatible with the canonical descent-data comes from a map $T \to T'$. But this is clear from the fact that $\gamma^*$ is full and faithful.

4. A proof of the descent theorem

This is essentially the same as 3, but we have to keep track of sheafification all the time. Let $\mathcal{E} \xrightarrow{Y} \mathcal{C}$ be an open surjection, and let $\mathcal{C}$ be an open site for $\mathcal{E}$; i.e. $\mathcal{C}$ has a terminal object 1, and every cover in $\mathcal{C}$ is inhabited. We have to show that

(a) every object $X \in \mathcal{E}$ equipped with descent-data is isomorphic to a constant sheaf;

(b) every morphism $\gamma^*(T) \xrightarrow{\tau} \gamma^*(T')$ which commutes with the canonical descent-data is of the form $\tau = \gamma^*(f)$.

To prove (a), choose $X \in \mathcal{E}$ with descent-data $\theta$. Write $\mathcal{E} \times \mathcal{E} \xrightarrow{p_1} \mathcal{E}$ and $\mathcal{E} \times \mathcal{E} \times \mathcal{E} \xrightarrow{p_1} \mathcal{E}$ for the projections. Identifying $p_1^*(X)(C, D)$ with $p_1^*(X)(D, C)$ in the canonical way, we may regard $\theta$ as a system of functions (in $\mathcal{C}$)

$$
\theta_{CD} : p_1^*(X)(C, D) \to p_1^*(X)(D, C)
$$

which are natural in $C, D$: for $C' \to C$ and $D' \to D$,

$$
\begin{array}{ccc}
p_1^*(X)(C, D) & \xrightarrow{\theta_{CD}} & p_1^*(X)(D, C) \\
\downarrow & & \downarrow \\
p_1^*(X)(C', D') & \xrightarrow{\theta_{CD'}} & p_1^*(X)(D', C')
\end{array}
$$

commutes. This implies that $\theta_{CD}$ is determined by its restriction $\theta_{CD} \circ i_1$,

$$
\begin{array}{ccc}
X(C) & \xrightarrow{i_1} & p_1^*(X)(C, D) \\
\downarrow & & \downarrow \\
p_1^*(X)(C, D) & \xrightarrow{\theta_{CD}} & p_1^*(X)(D, C)
\end{array}
$$

for which we also write $\theta_{CD}$. The condition $\delta^*(\theta) = \text{id}$ means that
commutes for every $C$, while the cocycle condition means that
\[
p^\ast(X)(C, E, D) \xrightarrow{\theta_{CD}(E)} p^\ast(X)(D, C, E)
\]
where $\theta_{CD(E)}$ is the obvious map induced by $\theta_{CD}$, etc.

We will use the following lemma, to be proved below.

**Lemma.** For $X \in \mathcal{C} = \text{Sh}(C)$, and objects $C, D, E$ of $C$, the canonical square
\[
p^\ast(X)(C, D) \xrightarrow{\theta_{CD}^c} p^\ast(X)(C, D, E)
\]
is a pullback in $\mathcal{C}$.

Let $S = \{x \in X(1) \mid \theta_{11}(i_1(x)) = i_1(x)\}$, where $i_1 : X(1) \to p^\ast(X)(1, 1)$ as above. We claim that $X \cong \gamma^\ast(S)$ via
\[
X \xleftarrow{\varphi} \gamma^\ast(S),
\]
where $\varphi$ is the transpose of $S \to X(1)$, and $\psi$ is the map defined by the components
\[
X(C) \xrightarrow{\psi_C} \gamma^\ast(S)(C)
\]
where $j_C$ is the obvious embedding, natural in $C$. The nontrivial thing is to show that $\psi_C$ is well-defined, i.e. that $\theta_{C1} \circ i_1$ factors through $j_C$. (Naturality of $\psi_C$ is then obvious.) So take $x \in X(C)$, and write $y = \theta_{C1}(i_1(x)) \in p^\ast(X)(1, C)$. We have to show that $y$ "locally does not depend on the $C$-coordinate", $y$ is given as a compatible family $\{y_a\}_a$, $y_a \in X(D_a)$, for a cover $\{(D_a, C_a) \xrightarrow{(D_a, f_a)} (1, C)\}_a$ in $C \times C$. 

Fix $\alpha$, and let $x_\alpha = x_1 f_\alpha \in X(C_\alpha)$. Then $\theta_{C_\alpha D_\alpha}(x_\alpha) = y_\alpha$, and by the cocycle condition, we have for any object $E$ of $C$ that $\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha)$ in $p_1^\star(X)(E, D_\alpha, C_\alpha)$. So by the lemma,

$$\theta_{C_\alpha E}(x_\alpha) = \theta_{D_\alpha E}(y_\alpha) \in X(E).$$

Choosing $E = C_\alpha$, we find that $\theta_{C_\alpha C_\alpha}(x_\alpha) \in X(C_\alpha)$, and hence since $\eta_{C_\alpha}$ is the identity on $X(C_\alpha)$, $p_1^\star(X)(C_\alpha, C_\alpha)$, that $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$. Now let $E$ run over all the objects $D_\beta, \beta \in \mathcal{A}$. Clearly by naturality of $\theta$, if

$$
\begin{array}{ccc}
D_\beta \\
| \\
\downarrow h \\
F \\
| \\
\downarrow k \\
D_\gamma
\end{array}
$$

then $\theta_{C_\alpha D_\beta}(x_\alpha) h = \theta_{C_\alpha E}(x_\alpha) = \theta_{C_\alpha D_\alpha}(x_\alpha) k$, so since $\{D_\beta \to 1\}_{\beta \in \mathcal{A}}$ is a cover in $C$ (by openness), there is a unique $z_\alpha \in X(1)$ with $z_\alpha h D_\beta = \theta_{C_\alpha D_\beta}(x_\alpha)$. So by naturality of $\theta$ again,

$$z_\alpha = \theta_{C_\alpha 1}(x_\alpha) \in X(1),$$

while moreover since $\theta_{C_\alpha C_\alpha}(x_\alpha) = x_\alpha$,

$$z_\alpha 1 C_\alpha = x_\alpha \in X(C_\alpha).$$

We claim that $\{z_\alpha\}_\alpha$ determines an element $z \in \psi_\star(S)(C)$. (Note that clearly if this is so, $\eta_C(z) = \theta_{C_1 1}(x)$. ) Indeed, the $z_\alpha$ are compatible in the sense that if

$$
\begin{array}{ccc}
C_\alpha \\
| \\
\downarrow h \\
E \\
| \\
\downarrow k \\
C_\alpha'
\end{array}
$$

commutes, then $z_\alpha = z_{\alpha'} \in X(1)$ -- this is obvious from naturality of $\theta$. Moreover, each $z_\alpha \in S$. For if $E$ is any object of $C$, we have $\theta_{1 E}(z_\alpha) = \theta_{C_\alpha E}(x_\alpha)$ in $p_1^\star(X)(1, C_\alpha, E)$ by the cocycle condition, so by the lemma, $\theta_{1 E}(z_\alpha) \in X(E)$. Since $\eta_1 X(1)$ is the identity, we find for $E = 1$ that $\theta_{1 1}(z_\alpha) = z_\alpha$. This proves that $\psi_C$ is well-defined.

It is now clear that $\phi$ and $\psi$ are inverse to each other: One way round, it suffices to show that $\psi_1 \phi_1(s) = s$ for $s \in S$. But $\phi_1(s) = s \in X(1)$, and $\theta_{1 1}(s) = i_1(s)$ by definition of $S$, so this is clear. The other way round, take $x \in X(C)$. Then
\[ \psi_C(x) \in \gamma^*(S)(C) \] is the element \( z \) as above with \( z \upharpoonright \mathcal{A}_a = z_a \in S \subset X(1) \). So by definition, \( \varphi_C(z) \in X(C) \) is given by \( \varphi_C(z) \upharpoonright \mathcal{A}_a = z_a \mathcal{A}_a \). But \( z_a \mathcal{A}_a = x_a \) as we have seen. So \( \varphi_C(z) = x \), i.e. \( \varphi_C \psi_C = \text{id} \). This proves (a).

To prove (b), suppose \( \gamma^*(T) \xrightarrow{\tau} \gamma^*(T') \) is compatible with the canonical descent-data \( \theta \) and \( \theta' \) on \( \gamma^*(T), \gamma^*(T') \). It is trivial to check that \( T = \{ t \in \gamma^*(T)(1) \mid \theta_1(t) = t \} \), and similarly for \( T' \). So if \( t \in T \cap \gamma^*(T)(1) \), then \( \theta'_1(t) = t \), \( \tau_1(t) = \tau_1(t) \), so \( \tau_1(t) \in T' \). Therefore \( \tau \) comes from a map \( T \to T' \), proving (b).

It remains to prove the lemma. To this end, suppose \( x \in p_1^*(X)(C, D) \) and \( y \in p_1^*(X)(C, E) \) are equal in \( p_1^*(X)(C, D, E) \). Write \( x = \{ x_a \}_{\mathcal{A}_a}, \ x_a \in X(C_a) \) a compatible family for a cover \( \mathcal{A} = \{(C_a, D_a) \to (C, D)\}_{\alpha \in \mathcal{A}} \) in \( C \times C \), and \( y = \{ y_\beta \}_{\mathcal{A} \beta}, \ y_\beta \in X(C_\beta) \), a compatible family for a cover \( \mathcal{B} = \{(C_\beta, E_\beta) \to (C, E)\}_{\beta \in \mathcal{B}} \) in \( C \times C \). Equality of \( x \) and \( y \) in \( p_1^*(X)(C, D, E) \) means that there is a common refinement \( \mathcal{A} = \{(C_i, D_i, E_i) \to (C, D, E)\}_{i \in I} \) of \( \{(C_a, D_a, E) \to (C, D, E)\}_{\alpha} \) and \( \{(C_\beta, D_\beta, E_\beta) \to (C, D, E)\}_{\beta} \) in \( C \times C \times C \) on which \( x \) and \( y \) agree. Replacing \( \mathcal{A} \) by \( \{(C_i, D_i) \to (C, D)\} \) and \( \mathcal{B} \) by \( \{(C_i, E_i) \to (C, E)\} \), we get the following notationally more manageable situation: we are given \( x_i \in X(C_i), \ y_i \in X(C_i) \), such that whenever we have a commutative diagram

\[
\begin{array}{ccc}
(C_i, D_i) & (A, B) & (C, D) \\
\downarrow & \downarrow & \downarrow \\
(C_j, D_j) & &
\end{array}
\]

then \( x_j \upharpoonright A = x_j \upharpoonright A \), and a similar condition for compatibility of \( \{ y_i \} \) with \( D \) replaced by \( E \). Moreover, since \( x \) and \( y \) agree on the cover \( \mathcal{A} \), \( x_i = y_i \) for every \( i \). We now have to show that \( x = \{ x_i \} \) comes from an element of \( X(C) \), i.e. that \( \{ x_i \} \) is compatible for the cover \( \{ C_i \to C \} \) in \( C \). So suppose

\[
\begin{array}{ccc}
A & C \\
\downarrow & \downarrow \\
C_{i_1} & & C_{i_2}
\end{array}
\]

commutes. Take a cover \( \{(P_\alpha, Q_\alpha, R_\alpha) \to (A, D_{i_1}, E_{i_2})\}_{\alpha} \) refining \( \mathcal{A} \); i.e. for each \( \alpha \) there is a \( j_\alpha \in I \) such that
commutes. By openness, \([P_{\alpha}, Q_{\alpha}] \rightarrow (A, D_{i_{i}})\) is a cover in \(C \times C\), while moreover,

\[
\begin{align*}
x_i \cdot 1_{P_{\alpha}} &= x_{j_{i}} \cdot 1_{P_{\alpha}} \\
&= y_{j_{i}} \cdot 1_{P_{\alpha}} \\
&= y_{i_{i}} \cdot 1_{P_{\alpha}} \\
&= x_{i_{i}} \cdot 1_{P_{\alpha}}
\end{align*}
\]

(by compatibility of \(\{x_i\}\) over \((C, D)\))

(by \(x = y\) over \((C, D, E)\))

(by compatibility of \(\{y_i\}\) over \((C, E)\))

(by \(x = y\) over \((C, D, E)\)).

The family \(\{P_{\alpha} \rightarrow A\}_a\) covers \(A\), so \(x_i \cdot 1_{A} = x_{i_{i}} \cdot 1_{A}\). This completes the proof of the lemma.

Reference