Toposes and Groupoids

IEKE MOERDIJK*
Department of Mathematics
University of Chicago

The aim of this paper is to explain to what extent the category of Grothendieck toposes can be described in terms of groupoids (in the category of locales). In the first section, I describe how every groupoid G gives rise to a topos BG, and in section 2, I discuss some of the functorial properties of this constrution $G \mapsto BG$. After having introduced two completeness properties of groupoids, we will see that toposes can be obtained by localizing groupoids (§4) or by considering geometric morphisms as obtained by tensoring with something analogous to a bimodule (§5). In section 6 I briefly discuss the fundamental group of a topos.

This paper provides a summary of my earlier papers [**M2**], [**M3**], [**M4**]. Since the proofs given there are often long and technical, and involve extensive use of change-of-base methods, I believe it is worthwhile to present these results all together in a more directly accessible way, and save the reader from being distracted by perhaps less digestible technicalities.

1. Equivariant sheaves.

We will be concerned with groupoids in the category of spaces (i.e., locales), briefly called *continuous groupoids*. If G is such a continuous groupoid, we write G_0 (resp.G_1) for the space of objects (morphisms) of G, d_0 for the domain, d_1 for the codomain, m for composition and s for the map associating the identity-morphism to a given object. So G is given as a diagram of locales

$$G_1 \underset{G_0}{\times} G_1 \; \overset{\xrightarrow{\pi_0}}{\underset{\xrightarrow[\pi_1]{}}{\xrightarrow{m}}} \; G_1 \; \overset{\xrightarrow{d_0}}{\underset{\xrightarrow[d_1]{}}{\xleftarrow{s}}} \; G_0$$

having the usual properties. In the particular case where $G_0 = 1$, we have a group-object in locales, or a continuous group. A *homomorphism* of groupoids $G \xrightarrow{f} H$ consists of two continuous maps $G_0 \xrightarrow{f_0} H_0$ and $G_1 \xrightarrow{f_1} H_1$ satisfying the usual identities. A groupoid G is called *open* if d_0 and $d_1 : G_1 \to G_0$

---

*Research supported by a Huygensfellowship of the Z.W.O.

are both open maps (this implies that m is open as well). *In this paper, all groupoids are assumed to be open.*

If G is a continuous groupoid, a G-space is a space E over G_0 equipped with a (contravariant) action of G on the right; i.e., there are maps $E \xrightarrow{p} G_0$ and $E \underset{G_0}{\times} G_1 \xrightarrow{\cdot} E$ satisfying the usual conditions. (Rather than expressing these by commutative diagrams, we just write the equations in set-theoretic language; so the domain of the action map $E \underset{G_0}{\times} G_1 \xrightarrow{\cdot} E$ is the set of pairs (e, g) with $d_1 g = p(e)$, and the equations are $p(e \cdot g) = d_0 g$, $e \cdot s(p(e)) = e$, $(e \cdot g) \cdot h = e \cdot (g \circ h)$ where $g \circ h = m(g, h)$.) A *morphism* of G-spaces $E \to E'$ is a map of spaces (locales) over G_0 which preserves the action. This defines a category (G-spaces).

A G-space E is *open* if its projection $E \xrightarrow{p} G_0$ is an open map; this implies that the action $E \underset{G_0}{\times} G_1 \xrightarrow{\cdot} E$ is open (recall that G is assumed to be an open groupoid).

A G-space is called *étale* if $E \xrightarrow{p} G_0$ is a local homeomorphism. This means that E is a *sheaf* on G_0, such that the morphisms of G act on the fibers of E: if $x \xrightarrow{g} y$ is a point of G_1, the action defines a map

$$g^* : E_y \to E_x, \quad g^*(e) = e \cdot g$$

1.1 PROPOSITION. *The category of étale G-spaces or equivariant G-sheaves, is a Grothendieck topos.*

This topos is denoted by BG, and called the *classifying topos* of G.

We remark that the definition of the topos BG also makes sense if G is just a *continuous category* (a category object in locales = spaces), rather than a groupoid.

Let me give some examples.

(1) The easiest case is where G is an abstract group ($G_0 = 1$ and G_1 is a discrete space). The topos BG is then simply the category of G-sets; i.e., objects are sets X equipped with an action $X \times G \xrightarrow{\cdot} X$, and morphisms are functions which preserve the action. Simple as they are, these toposes arises naturally in many contexts. BG is a $K(G, 1)$-topos, and it classifies G-torsors, so (writing [- , -] for isomorphism classes of geometric morphisms)

$$[\mathcal{E}, BG] \cong H^1(\mathcal{E}, G)$$

(see [**JW**]). Moreover for any abstract group, G is an *atomic* topos [**BD**].
(2) If G is a topological group, BG is the category of *continuous G-sets*, i.e., sets X equipped with an action $X \times G \stackrel{\cdot}{\rightarrow} X$ which is continuous if X is given the discrete topology. Continuity of the action is equivalent to the requirement that all *stabilizer subgroups*

$$S_x = \{g \in G \mid x \cdot g = x\}$$

are open. For a topological group G, BG is still an atomic topos, since one can simply write a given continuous G-set X as the sum of its orbits: for $x \in X$, the orbit $\mathcal{O}(x) = \{x \cdot g \mid g \in G\}$ is the smallest subobject of X containing x. If $U \subset G$ is an open subgroup, the set of right cosets G/U is an object of BG, and clearly

$$\mathcal{O}(x) \cong G/S_x$$

as objects of BG. Consequently, the full subcategory of objects of the form G/U (U an open subgroup of G) equipped with the atomic Grothendieck topology (all maps are covers) is a site for BG.

Toposes of continuous G-sets form the natural setting for so-called permutation-models in set-theory ([**Fo**], [**Fr**]). On the other hand, it is hard to say what a topos BG classifies, for a general topological group G. This is related to the fact that many different topological groups G determine the same topos BG; see [**M3**], and section 6 below.

(3) Let G be a profinite topological group, with a fundamental system of open normal subgroups $\{U_i\}$. So $G_i = G/U_i$ is a finite group and $G = \varprojlim G_i$ is a filtered inverse limit of finite groups and surjections. An object X of BG, i.e., a continuous G-set, can be written as a union

$$X = \bigcup X_i$$

where X_i is a G_i-set, and the actions of G_i on X_i for the different i satisfy an obvious compability requirement: simply let $X_i = \{x \in X \mid S_x \supseteq U_i\}$.

The cohomology of BG is precisely the Galois cohomology of G:

$$H^*(BG, A) \cong H^*_{\mathrm{Gal}}(G, A)$$

for any (discrete) G-module A (see [**S**]), and BG is still a $K(G,1)$-topos, i.e., the profinite fundamental group of BG is G, and the other homotopy groups vanish (see e.g. [**AM**]).

(4) More generally, if $\underline{G} = \{G_i\}_i$ is a filtered inverse system of discrete groups and *surjective* homomorphisms (Grothendieck calls this a *strict* progroup), one may consider sets X which can be written as a union $X = \bigcup X_i$ of compatible G_i-sets X_i, just like in the preceding example (see [**SGA4**, p. 319]). Such sets X again form a topos $B\underline{G}$, which is still a $K(\pi, 1)$-topos if one interprets the fundamental group as a progroup [**AM**]. $B\underline{G}$ is an example of a topos of equivariant sheaves: although the inverse limit $\varprojlim G_i$ may be trivial as a topological group, it is not so when one computes the inverse limit

$$G = \varprojlim G_i$$

in the category of locales. One thus obtains a *prodiscrete* continuous group G, and BG is the same as $B\underline{G}$; in fact

$$BG = B(\varprojlim G_i) \simeq \varprojlim BG_i = B\underline{G}.$$

(This example is discussed in detail in [**M4**]; see also section 6 below.)

(5) A well-known particular case of (2) above is the Schanuel topos $B(\mathrm{Aut}(\mathbb{N}))$, where $\mathrm{Aut}(\mathbb{N})$ is the group of permutations of the natural numbers with the usual (product) topology. If you compute a site for the Schanuel topos consisting of coset-objects $\mathrm{Aut}(\mathbb{N})/U$ for U a (basic) open subgroup, you will find that the Schanuel topos is precisely sheaves on (the opposite category of) the category of finite sets and monomorphisms, equipped with the atomic topology.

One may also put all finite sets and inclusions together, and look at the topological monoid $\mathrm{Mono}(\mathbb{N})$ of monomorphisms $\mathbb{N} \to \mathbb{N}$. Let $B(\mathrm{Mono}(\mathbb{N}))$ be the topos of sets equipped with a continuous action of $\mathrm{Mono}(\mathbb{N})$. It is an instructive exercise to verify that the inclusion $\mathrm{Aut}(\mathbb{N}) \hookrightarrow \mathrm{Mono}(\mathbb{N})$ induces an equivalence

$$B(\mathrm{Aut}(\mathbb{N})) \simeq B(\mathrm{Mono}(\mathbb{N})).$$

This is an instance of a general phenomenon, discussed in §3 below.

$B(\mathrm{Aut}(\mathbb{N}))$ classifies the notion of an infinite decidable set. From the point of view of homotopy and cohomology however, nothing like the situation in (3) and (4) holds, since $B(\mathrm{Aut}(\mathbb{N}))$ is contractible (see [**JW**]).

(6) Naturally, if X is a space,

$$\mathrm{Sh}(X) = BX,$$

where on the right, X stands for the trivial groupoid whose only morphisms are identities. More generally, if G is a topological group acting as a group of transformations on a topological space X, by $X \times G \xrightarrow{\cdot} X$ say, one can construct a topological groupoid

$$X_G = (X \times G \underset{\pi_1}{\overset{\cdot}{\rightrightarrows}} X)$$

i.e., X is the space of objects of X_G, $X \times G$ the space of morphisms, and $d_0(x,g) = x \cdot g$, $d_1(x,g) = x$, etc. Then BX_G is precisely the topos of sheaves E on X equipped with an action by the group G which lifts the original action of G on X, i.e.,

$$\begin{array}{ccc} E \times G & \longrightarrow & E \\ \downarrow & & \downarrow \\ X \times G & \longrightarrow & X \end{array}$$

commutes.

(7) Let M be a foliated manifold, and let $\mathrm{Hol}(M)$ be the holonomy groupoid of M: its space of objects is M, and a morphism from x to y in this groupoid is a homotopy class of paths $I \xrightarrow{\alpha} L$, where $L \subset M$ is the leaf of x and y (if x and y are on different leaves, there are no morphisms from x to y). This set has the structure of a manifold, so $\mathrm{Hol}(M)$ is a continuous (differentiable) groupoid. $B\mathrm{Hol}(M)$ is the category of sheaves on M which are locally constant on each leaf.

(8) An étendue is a topos $\mathcal{E}$ such that for some cover $B \twoheadrightarrow 1$ in $\mathcal{E}$, $\mathcal{E}/B$ is equivalent to $\mathrm{Sh}(G_0)$ for some space G_0. The diagram

$$B \times B \times B \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} B \times B \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} B \tag{1}$$

is a (trivial) groupoid in $\mathcal{E}$, and gives rise to a continuous groupoid

$$G = (G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0) \tag{2}$$

where $\mathrm{Sh}(G_1) = \mathrm{Sh}(G_0) \underset{\mathcal{E}}{\times} \mathrm{Sh}(G_0) = \mathcal{E}/(B \times B)$; i.e., the functor $X \mapsto \mathrm{Sh}(X)$ from spaces (locales) to toposes sends this groupoid (2) to the groupoid

$$\mathcal{E}/(B \times B \times B) \substack{\rightrightarrows \\ \longrightarrow} \mathcal{E}/(B \times B) \rightrightarrows \mathcal{E}/B$$

obtained from (1) by slicing. Thus, an object of BG is an object $E \xrightarrow{p} B$ of $\mathcal{E}/B$, equipped with an isomorphism $\theta : E \times B \to B \times E$ over $B \times B$, which can be described in set-theoretic notation (using the internal logic of $\mathcal{E}$!) as follows: for points b_1, b_2 of B there is an isomorphism

$$\theta_{b_1,b_2} : p^{-1}(b_1) \xrightarrow{\sim} p^{-1}(b_2)$$

such that

$$\theta_{b,b} = id, \quad \theta_{b_2,b_3} \circ \theta_{b_1,b_2} = \theta_{b_1,b_3}. \tag{3}$$

But then clearly $E \xrightarrow{p} B$ is just a projection: let $E/\sim$ be the quotient of E obtained by identifying e with $\theta_{p(a),b}(e)$ for any $e \in E$ and $b \in B$. Then, writing $[e]$ for the equivalence class of e in $E/\sim$, the map

$$E \xrightarrow{\varphi} (E/\sim) \times B, \qquad \varphi(e) = ([e], p(e))$$

is an isomorphism, with (well-defined, by (3)) inverse map ψ defined by $\psi([e], b) = \theta_{p(e),b}(e)$. Thus, we obtain an equivalence

$$BG \simeq \mathcal{E}.$$

(Notice that G is a groupoid whose domain and codomain maps are local homeomorphisms.)

The situation described in (8) is discussed in SGA4, exposé IV. The equivalence $BG \simeq \mathcal{E}$ is a very simple case of a result due to A. Joyal and M. Tierney, which asserts that any topos is of the form BG:

2.1 Representation Theorem. *([**JT**]) For any Grothendieck topos $\mathcal{E}$ there exists an open continuous groupoid G such that $\mathcal{E}$ is equivalent to BG.*

2. Basic Properties.

In this section I will describe some of the elementary functorial properties. Detailed proofs of the results in this section can be found in [**M2**,I,§4-6].

First of all, the construction of the topos BG can be performed over any base topos. More precisely, if $\mathcal{E}$ is a topos and G is a continuous groupoid in $\mathcal{E}$, one may consider étale G-spaces inside $\mathcal{E}$; these form a topos $B(\mathcal{E}, G)$ over $\mathcal{E}$, and we write

$$B(\mathcal{E}, G) \xrightarrow{\gamma} \mathcal{E}$$

for the canonical geometric morphism. A most important fact is that one can use change-of-base methods in the context of toposes of the form $B(\mathcal{E}, G)$, since this construction is stable:

2.1 STABILITY THEOREM. *Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be a geometric morphism, and let G be a continuous groupoid in $\mathcal{E}$. Then $f^{\#}(G)$ is a continuous groupoid in $\mathcal{F}$, and there is a canonical equivalence*

$$B(\mathcal{F}, f^{\#}G) \simeq \mathcal{F} \underset{\mathcal{E}}{\times} B(\mathcal{E}, G).$$

In this theorem, $f^{\#}$ denotes the pullback functor from spaces in $\mathcal{E}$ to spaces in $\mathcal{F}$.

In what follows, we fix an arbitrary base topos $\mathcal{S}$, and just write BG for $B(\mathcal{S}, G)$. Some basic properties of the topos BG follow from properties of G; e.g.

2.2 PROPOSITION. *Let G be a continuous groupoid in the base topos $\mathcal{S}$.*

(1) *If G_0 is an open space, then $BG \xrightarrow{\gamma} \mathcal{S}$ is open.*
(2) *If G_0 is locally connected and $d_0, d_1 : G_1 \rightrightarrows G_0$ are both open, then $BG \xrightarrow{\gamma} \mathcal{S}$ is locally connected.*
(3) *If G_0 is an open space and $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ is an open map, then $BG \xrightarrow{\gamma} \mathcal{S}$ is atomic.*

(For open maps, see [**J2**], [**JT**]; for locally connected maps see [**BP**], [**M1**, Appendix]; for atomic maps see [**BD**].) To prove 2.2, one may use properties of the projection map

$$\mathrm{Sh}(G_0) \xrightarrow{\pi_G} BG$$

whose inverse π_B^* is the forgetful functor. Here we have

2.3 PROPOSITION. (1) *If $d_0, d_1 : G_1 \rightrightarrows G_0$ are both open, so is $Sh(G_0) \xrightarrow{\pi_G} BG$.*
(2) *If d_0, d_1 are both locally connected, so is $Sh(G_0) \xrightarrow{\pi_G} BG$.*
(3) *If $d_0, d_1 : G_1 \rightrightarrows G_0$ are both local homeomorphisms, then $Sh(G_0) \xrightarrow{\pi_G} BG$ is atomic.*

One can apply 2.2 to get sharper forms of the representation Theorem 1.2 (2.4(4) appears in SGA4,*loc. cit*).

2.4 COROLLARY. *Let $\mathcal{E}$ be a topos over $\mathcal{S}$. Then*

(1) *$\mathcal{E} \to \mathcal{S}$ is open if and only if $\mathcal{E} \simeq BG$ for some continuous groupoid with $G_1 \rightrightarrows G_0 \to 1$ all open maps.*

(2) *$\mathcal{E} \to \mathcal{S}$ is (connected) locally connected if and only if $\mathcal{E} \simeq BG$ for a continuous groupoid with $G_1 \rightrightarrows G_0 \to 1$ all (connected) locally connected maps.*

(3) *$\mathcal{E} \to \mathcal{S}$ is (connected) atomic if and only if $\mathcal{E} \simeq BG$ for a continuous groupoid G with $G_0 \to 1$ open (and surjective) and $G_1 \xrightarrow{(d_0,d_1)} G_0 \times G_0$ open (and surjective).*

(4) *$\mathcal{E}$ is an étendue if and only if $\mathcal{E} \simeq BG$ for a groupoid G with both $G_1 \rightrightarrows G_0$ local homeomorphisms.*

The construction is functorial in G: if $G \xrightarrow{\varphi} H$ is a continuous homomorphism, the pullback of an étale H-space $E \xrightarrow{p} H_0$ along $G_0 \xrightarrow{\varphi_0} H_0$ has an obvious induced action by G, and this gives the inverse image of a geometric morphism

$$B(\varphi) : BG \to BH.$$

$G \xrightarrow{\varphi} H$ is called *open* if both $G_0 \xrightarrow{\varphi_0} H_0$ and $G_1 \xrightarrow{\varphi_1} H_1$ are open maps. Moreover, imitating the usual categorical notions, φ is called *essentially surjective* if $G_0 \underset{H_0}{\times} H_1 \xrightarrow{d_1\pi_2} H_0$ is an open surjection, *full* if $G_1 \to H_1 \underset{(H_0 \times H_0)}{\times} (G_0 \times G_0)$ is an open surjection, and *fully faithful* if

$$\begin{array}{ccc} G_1 & \longrightarrow & H_1 \\ \big\downarrow & & \big\downarrow \\ G_0 \times G_0 & \longrightarrow & H_0 \times H_0 \end{array}$$

is a pullback; φ is an *essential equivalence* if φ is open, fully faithful and essentially surjective.

The basic properties are

2.5 THEOREM. *Let $B\varphi : BG \to BH$ be the geometric morphism induced by a continuous homomorphism $G \xrightarrow{\varphi} H$.*

(1) *If φ_0 is open, $B\varphi$ is open.*

(2) *If φ is essentially surjective then $B\varphi$ is surjective.*

(3) *If φ is essentially surjective and full, then $B\varphi$ is connected.*

(4) *If φ is open and full, then $B\varphi$ is atomic.*

(5) *If φ is an essential equivalence, $B\varphi$ is an equivalence of toposes.*

And of a somewhat different nature:

(6) *If φ_0 and $G_1 \rightrightarrows G_0$ are locally connected and $H_1 \rightrightarrows H_0$ are open, then $B\varphi$ is locally connected.*

(7) *If φ_0 and $G_1 \rightrightarrows G_0$ are local homeomorphisms and $H_1 \rightrightarrows H_0$ are open, then $B\varphi$ is atomic.*

3. Completions of Continuous Groupoids.

Let G be a continuous groupoid. I will describe a continuous groupoid $\widehat{G}$ and a continuous category $\gamma(G)$, together with maps

$$G \xrightarrow{\theta} \widehat{G} \subset \gamma(G),$$

such that these all define the same topos:

$$BG \xrightarrow{\sim} B\widehat{G} \xrightarrow{\sim} B\gamma(G).$$

3.1 Construction of $\gamma(G)$ (cf [**M2**,II,§3]). Consider the lax pullback

$$(1) \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "d_1"] \arrow[d, "d_0"'] & \mathrm{Sh}(G_0) \arrow[d, "\pi_G"] \arrow[dl, phantom, "\swarrow"] \\ \mathrm{Sh}(G_0) \arrow[r, "\pi_G"'] & BG \end{tikzcd}$$

$\mathcal{L}$ is actually a localic topos, i.e., there is a unique locale $\gamma(G)_1$ such that $\mathcal{L} \simeq \mathrm{Sh}(\gamma(G)_1)$. $\gamma(G)$ is the category with $\gamma(G)_0 = G_0$ as space of objects, $\gamma(G)_1$ as space of morphisms, and d_0, d_1 from diagram (1) as domain and codomain.

By the universal property of (1), points of $\gamma(G)_1$ are triples (x, y, α), where x and y are points of G_0, and $\alpha : \mathrm{ev}_y \to \mathrm{ev}_x$ is a natural transformation ($\mathrm{ev}_x : \mathrm{Sh}(G_0) \to$ Sets takes the fiber at x). $d_0(x, y, \alpha) = x$, $d_1(x, y, \alpha) = y$. It is clear how to define composition in $\gamma(G)$ on *points*. By change of base (Yoneda lemma) this actually defines the structure of a continuous category on $\gamma(G)$.

The universal property of (1) gives a homomorphism of continuous categories

$$G \xrightarrow{\theta} \gamma(G)$$

which is the identity on objects; on morphisms, θ sends a point g of G_1 to the triple $(d_0 g, d_1 g, g^*)$.

3.2 PROPOSITION. $G \xrightarrow{\theta} \gamma(G)$ *induces an equivalence of toposes*

$$BG \xrightarrow{\sim} B\gamma(G).$$

3.3 Construction of $\widehat{G}$. $\widehat{G}$ is the subcategory of $\gamma(G)$ with the same objects, but only the isomorphisms of $\gamma(G)$ as arrows: so $\widehat{G}$ is in fact a continuous groupoid, and $G \xrightarrow{\theta} \gamma(G)$ factors through $\widehat{G} \subset \gamma(G)$. $\widehat{G}_1$ can be directly described by the pullback

$$\begin{tikzcd} \mathrm{Sh}(\widehat{G}_1) \arrow[r] \arrow[d] & \mathrm{Sh}(G_0) \arrow[d] \arrow[dl, phantom, "\Swarrow\sim"] \\ \mathrm{Sh}(G_0) \arrow[r] & BG \end{tikzcd} \tag{2}$$

We call G *étale complete* if $G \xrightarrow{\theta} \widehat{G}$ is an isomorphism.

3.4 PROPOSITION. $G \xrightarrow{\theta} \widehat{G}$ *induces an equivalence of toposes*

$$BG \xrightarrow{\sim} B\widehat{G}$$

This is a consequence of the descent theorem for open geometric morphisms ([**JT**]).

3.5 EXAMPLE: (cf [**M3**]) Let G be a topological group with a countable base at the identity element e. Then

$$\gamma(G) = \varprojlim_{U} G/U$$

where U ranges over open subgroups (ordered by inclusion), and G/U is the set of right cosets Ux (the quotient topology on G/U is the discrete one). $\gamma(G)$ is a topological monoid, with multiplication defined by the formula

$$(\bar{x} \cdot \bar{y})_U = Ux_U \cdot y_{x_U^{-1}Ux_U}$$

where we write a point of $\gamma(G)$ as a sequence $\bar{x} = \{U \cdot x_U\}_U$. For instance, for the group $\mathrm{Aut}(S)$ of isomorphisms of some infinite set S, we have

$$\gamma(\mathrm{Aut}(S)) = \{1-1 \text{ maps } S \to S\}$$

Notice also that if $G = \varprojlim G_i$ is profinite, then $\gamma(G) \cong G$; this holds in fact for arbitrary localic prodiscrete groups (§1, example (4); and §6 below).

4. Toposes as a localization of groupoids.

Let [Top] denote the category of Grothendieck toposes (over some fixed base topos) and isomorphisms classes of geometric morphisms. I will describe how [Top] can be considered as a localization of a category of continuous groupoids. (Detailed proofs are given in [**M2**, I,§7].)

First of all, the category of continuous groupoids is a 2-category in a natural way: 1-cells $G \to H$ are continuous homomorphisms, and if $G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} M$ are two such, a 2-cell $\alpha : \varphi \Rightarrow \psi$ is the localic analogue of a natural transformation, i.e., a continuous map $G_0 \xrightarrow{\alpha} H_1$ such that $d_0\alpha = \varphi$, $d_1\alpha = \psi$, and

$$\begin{tikzcd} G_1 \arrow[r, "(\alpha d_1, \varphi_1)"] \arrow[d, "(\psi, \alpha d_0)"'] & H_1 \underset{H_0}{\times} H_1 \arrow[d, "m"] \\ H_1 \underset{H_0}{\times} H_1 \arrow[r, "m"] & H_1 \end{tikzcd}$$

commutes. Let [Groupoids] denote the category of continuous groupoids and isomorphism classes of continuous homomorphisms. Let $ECG \subseteq$ [Groupoids] be the full subcategory given by the étale complete groupoids.

4.2 PROPOSITION. *The class $\underline{E}$ of isomorphism classes of essential equivalences (cf §2) admits a calculus of right fractions (in the sense of [***GZ***]) in the category [Groupoids], as well as in the subcategory ECG of étale complete groupoids.*

4.2 LOCALIZATION THEOREM. *The functor B from continuous groupoids to toposes induces an equivalence*

$$ECG[\underline{E}^{-1}] \xrightarrow{\sim} [Top].$$

4.3 REMARK: In the proof of 4.2, one uses the following construction. If $BG \xrightarrow{f} BH$ is any geometric morphism, define a space K_0 by the pullback

$$\begin{tikzcd}
\mathrm{Sh}(K_0) \arrow[rr, "\psi_0"] \arrow[d, "\varphi_0"'] & & \mathrm{Sh}(H_0) \arrow[d, "\pi_H"] \\
\mathrm{Sh}(G_0) \arrow[r, "\pi_G"] & BG \arrow[r, "f"] & BH
\end{tikzcd}$$

and make K_0 into a groupoid by defining K_1 as the pullback

$$\begin{tikzcd}
K_1 \arrow[r, "\varphi_1"] \arrow[d] & G_1 \arrow[d] \\
K_0 \times K_0 \arrow[r] & G_0 \times G_0
\end{tikzcd}$$

so that we obtain an essential equivalence $K \xrightarrow{\varphi} G$. If H is étale complete, then f gives a homomorphism $K \xrightarrow{\psi} H$ such that

$$\begin{tikzcd}
BK \arrow[d, "B\varphi"'] \arrow[dr, "B\psi"] & \\
BG \arrow[r, "f"] & BH
\end{tikzcd}$$

commutes (up to natural isomorphism).

We remark here that it can be shown that K_0 can be equipped with an action of G on the right and one of H on the left, and that the induced functor

$$BH \to BG, \quad E \mapsto E \underset{H}{\otimes} K_0$$

is naturally isomorphic to the inverse image f^* of the given geometric morphism f. Thus, every geometric morphism comes from tensoring by a "bispace", a space with two actions as above. However, this particular construction does not take care of 2-cells, i.e., natural transformations between geometric morphisms. In the next section, a more careful construction will be given that does take these 2-cells into account.

5. Geometric Morphisms as Tensor Products.

In this section I will describe the category $\underline{\text{Top}}(BH, BG)$ of geometric morphisms and natural transformations between their inverse image functors, completely in terms of the completions γH and γG, and spaces equipped with an action by each of these completions. The reader can find detailed proofs in [**M2**,II,§4–6].

5.1 Bispaces. Let G and H be continuous groupoids, with completions γG and γH (cf. 3.1). A *γG- γH-bispace* (or briefly *bispace*) is a space R which is at the same time a left γG-space and a right γH-space, such that the two actions commute with each other. So there are projections $p_G : R \to G_0$ and $p_H : R \to H_0$, and action maps $\gamma(G)_1 \underset{G_0}{\times} R \xrightarrow{*} R$ (pullback along $\gamma(G)_1 \xrightarrow{d_0} G_0$), $R \underset{H_0}{\times} \gamma(H)_1 \xrightarrow{\cdot} R$ (pullback along $\gamma(H)_1 \xrightarrow{d_1} H_0$) satisfying the usual identities for an action (covariant for $\cdot$, contravariant for $*$), as well as three compatibility conditions expressed by the following commutative diagrams:

(1)
$$\gamma(G)_1 \underset{G_0}{\times} R \underset{\pi_2}{\overset{*}{\rightrightarrows}} R \xrightarrow{p_H} H_0$$

(2)
$$R \underset{H_0}{\times} \gamma(H)_1 \underset{\pi_1}{\overset{\cdot}{\rightrightarrows}} R \xrightarrow{p_G} G_0$$

(3)
$$\begin{array}{ccc} \gamma(G)_1 \underset{G_0}{\times} R \underset{H_0}{\times} \gamma(H)_1 & \xrightarrow{1\times\cdot} & \gamma(G)_1 \underset{G_0}{\times} R \\ {\scriptstyle *\times 1}\downarrow & & \downarrow{\scriptstyle *} \\ R \underset{H_0}{\times} \gamma(H)_1 & \xrightarrow{\cdot} & R \end{array}$$

A *homomorphism* of bispaces $R \to R'$ is a continuous map of spaces which is compatible with both actions. This defines a category

$$(\gamma G\text{- } \gamma H\text{-bispaces}).$$

5.2 Tensor products. If R is a bispace as above, and E is a (right) $\gamma(G)$-space, the tensor product $E \underset{\gamma(G)}{\otimes} R$ is defined by the usual coequalizer diagram

$$(4) \qquad E \underset{G_0}{\times} \gamma(G)_1 \underset{G_0}{\times} R \underset{\cdot\times R}{\overset{E\times *}{\rightrightarrows}} E \underset{G_0}{\times} R \twoheadrightarrow E \underset{\gamma G}{\otimes} R$$

This coequalizer can be pretty unmanageable, mainly because it need not be stable. However, if the actions of $\gamma(G)$ on E and R are both given by open maps, then (4) is stable.

5.3 Open bispaces, flat bispaces. A bispace R as in 5.1 is *open* if $p_H : R \to H_0$ is open, both actions $\gamma(G)_1 \underset{G_0}{\times} R \xrightarrow{*} R$ and $R \underset{H_0}{\times} \gamma(H)_1 \xrightarrow{\bullet} R$ are open, and the diagonal action $\gamma(G)_1 \underset{G_0}{\times} \gamma(G)_1 \underset{G_0}{\times} R \xrightarrow{\mu} R \underset{H_0}{\times} R$ ("$\mu(\xi, \xi', r) = (\xi * r, \xi' * r)$") is open.

LEMMA. *If R is an open bispace and E is an étale G-space (which can be considered as an étale $\gamma(G)$-space, cf 3.2) then $E \underset{\gamma(G)}{\otimes} R$ is an étale H- (or $\gamma(H)$-) space.*

So an open bispace R induces a functor

$$g(R)^* = - \underset{\gamma G}{\otimes} R : BG \to BH$$

R is called *flat* (on the left) if $g(R)^* = - \underset{\gamma G}{\otimes} R$ is left-exact. We denote the full subcategory of (γG-γH-bispaces) consisting of the flat ones by

$$\underline{\mathrm{Flat}}(\gamma G, \gamma H).$$

So we obtain a functor

$$\underline{\mathrm{Flat}}(\gamma G, \gamma H) \xrightarrow{g} \underline{\mathrm{Top}}(BH, BG).$$

5.4 THEOREM. *The functor g has a fully faithful right adjoint*

$$\underline{R} : \underline{Top}(BH, BG) \to \underline{Flat}(\gamma G, \gamma H).$$

The construction of $\underline{R}$ is easy enough to describe: given a geometric morphism $BH \xrightarrow{f} BG$, $\underline{R}(f)$ is constructed as the lax pullback

(5)
$$\begin{tikzcd} \mathrm{Sh}(\underline{R}(f)) \arrow[rr] \arrow[d] & & \mathrm{Sh}(G_0) \arrow[d, "\pi_G"] \\ \mathrm{Sh}(H_0) \arrow[r, "\pi_H"] & BM \arrow[r, "f"] & BG \end{tikzcd}$$

(this lax pullback is a localic topos, so determines a unique space $\underline{R}(f)$). By the universal property of (5), $\underline{R}(f)$ can be equipped with the structure of a γG-γH-bispace. It follows from 5.4 as stated that tensoring with $\underline{R}(f)$ defines a functor which is naturally isomorphic to f^* (the counit of $g \dashv \underline{R}$ is an isomorphism):

5.5 COROLLARY. *For every geometric morphism $BH \xrightarrow{f} BG$ there is a natural isomorphism of functors $BG \to BH$:*

$$f^* \simeq - \underset{\gamma G}{\otimes} \underline{R}(f).$$

Let us call a flat bispace R complete if $\eta_R : R \to \underline{R}g(R)$ is an isomorphism. One can then form a large bicategory whose objects are continuous groupoids G, whose 1-cells $G \to H$ are complete flat γG-γH-bispaces, and whose 2-cells are homomorphisms of such. The tensor product = composition of 1-cells is given by first taking the tensor product of flat bispaces and then completing; i.e., for $G \xrightarrow{R} H \xrightarrow{S} K$, we define

$$R \hat{\otimes} S =_{def} \underline{R}g(R \underset{\gamma(H)}{\otimes} S) \simeq \underline{R}(g(S) \circ g(R))$$

5.6 COROLLARY. *This bicategory of continuous groupoids and complete flat bispaces is equivalent to the dual of the bicategory of toposes and geometric morphisms.*

6. Pointed atomic toposes, Galois toposes, and the fundamental group.

In this section, I come back to toposes of the form BG for a continuous group G. Such a topos is connected, atomic, and has a canonical point $\mathcal{S} \xrightarrow{p_G} BG$ whose inverse image p_G^* is the forgetful functor ($\mathcal{S}$ is an arbitrary base topos). It is proved in [**JT**] that the converse also holds:

6.1 Theorem. *(see [**JT**]) For any atomic connected $\mathcal{S}$-topos $\mathcal{E} \to \mathcal{S}$ with a point $\mathcal{S} \xrightarrow{p} \mathcal{E}$, there exists a continuous group G in $\mathcal{S}$ such that $\mathcal{E} \simeq BG$ as $\mathcal{S}$-toposes, and p corresponds to the canonical point p_G of BG under the equivalence.*

If $(\mathcal{E}, p)$ and $(\mathcal{F}, q)$ are pointed $\mathcal{S}$-toposes, I write $\underline{\mathrm{Top}}.((\mathcal{F}, q), (\mathcal{E}, p))$ for the category of pointed maps; i.e. the objects are pairs $(f, \alpha), \mathcal{F} \xrightarrow{f} \mathcal{E}$ over $\mathcal{S}$ and $\alpha : fq \xrightarrow{\sim} p$ (over $\mathcal{S}$), and the maps $\beta : (f, \alpha) \to (f', \alpha')$ are natural transformations $f^* \xrightarrow{\beta} f'^*$ compatible with the α's, i.e., for each $E \in \mathcal{E}$,

$$\begin{tikzcd} b^* f^*(E) \arrow[r, "\alpha_E", "\sim"'] \arrow[d, "q^*(\beta_E)"'] & p^*(E) \arrow[dl, "\alpha'_E", "\sim"'] \\ q^* f'^*(E) & \end{tikzcd}$$

commutes. Note that if q is faithful, there can be at most one such β, and β is an isomorphism. So for continuous groups G and H, the 2-cells define an equivalence relation on $\underline{\mathrm{Top}}.((BH, p_H), (BG, p_G))$, with quotient denoted by $[(BH, p_H), (BG, p_G)]_.$.

I will also consider the category $\underline{\mathrm{Top}}_+((\mathcal{F}, q), (\mathcal{E}, p)$ whose objects are maps of pointed toposes $(f, \alpha$ as above, and whose morphisms $(f, \alpha) \to (f', \alpha')$ are natural transformations $f^* \to f'^*$ (no compatibility with α and α' required).

From 5.4 one can deduce the following result (see [**M3**]).

6.2 Theorem. *For continuous groups G and H, the canonical functor*

$$(1) \qquad Hom(\gamma H, \gamma G) \to [(BH, p_H), (BG, p_G)].$$

is an isomorphism of sets, and

$$(2) \qquad \underline{Hom}(\gamma H, \gamma G) \to \underline{Top}_+((BH, p_H), (BG, p_G))$$

is an equivalence of categories.

In (1), $\mathrm{Hom}(\gamma H, \gamma G)$ is the set of continuous homomorphisms; in (2), $\underline{\mathrm{Hom}}(\gamma H, \gamma G)$ is the same set made into a category, where for homomorphisms φ and $\psi : \gamma H \to \gamma G$, a map $\varphi \to \psi$ is a point $\overline{g}$ of $\gamma(G)$ such that $\varphi(x) \cdot \overline{g} = \overline{g} \cdot \psi(x)$.

I now briefly discuss how Grothendieck's theory of the fundamental group (see [**SGA1**]), interpreted as a progroup as in [**AM**], fits into this context. For details I refer the reader to [**M4**]; some of what follows has independently been considered by J. Kennison, see [**K**].

A prodiscrete group is a continuous group G which is the inverse limit (in locales!) of a filtered inverse system of discrete groups (and surjective homomorphisms, as one may without loss of generality assume). Since clearly for a prodiscrete group G, $G \cong \widehat{G} \cong \gamma(G)$, we have

6.3 COROLLARY. *The embedding of prodiscrete groups into pointed toposes is fully faithful; in fact*

$$\underline{Hom}(H, G) \simeq \underline{Top}_+((BH, p_H), (BG, p_G))$$

whenever G and H are continuous groups with G prodiscrete.

The image of the embedding in 6.3 can be characterized as follows. Recall that an atom A in a topos $\mathcal{E}$ is *normal* if it is an $\mathrm{Aut}_{\mathcal{E}}(A)$-torsor in $\mathcal{E}$. A *Galois topos* is a pointed connected atomic topos which is generated by its normal atoms. From [**M3**,§3], I quote:

6.4 THEOREM. *A pointed topos $(\mathcal{E}, p)$ is a Galois topos if and only if there exists a prodiscrete continuous group G such that $\mathcal{E} \simeq BG$ (p corresponding to p_G as in 6.1).*

Let $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ be a connected locally connected topos over $\mathcal{S}$. An object E of $\mathcal{E}$ is *locally constant* if there are an S in $\mathcal{S}$ and a $V \twoheadrightarrow 1$ in $\mathcal{E}$ such that $E \times V \cong \gamma^*(S) \times V$ over V. The connected locally constant objects form a normal atomic site, and if one closes off under sums, one obtains a topos $\pi_1(\mathcal{E})$, and the inclusion $\pi_1(\mathcal{E}) \subset \mathcal{E}$ is the inverse image of a surjective geometric morphism

$$\varphi : \mathcal{E} \twoheadrightarrow \pi_1(\mathcal{E}).$$

If $\mathcal{E}$ has a point p, $\pi_1(\mathcal{E})$ has a point $\varphi(p)$, and therefore by 6.4 there exists a prodiscrete group $\pi_1(\mathcal{E}, p)$ such that

$$(\pi_1(\mathcal{E}), \varphi(p)) \simeq (B\pi_1(\mathcal{E}, p), \overline{p}),$$

where $\overline{p} = p_{\pi_1(\mathcal{E}, p)}$ is the canonical point of $B(\pi_1(\mathcal{E}, p))$.

6.5 PROPOSITION. *Let $(\mathcal{E},p)$ be a connected locally connected pointed topos. $(\mathcal{E},p) \xrightarrow{\varphi} (B\pi_1(\mathcal{E},p),\overline{p})$ is the universal map into a Galois topos; i.e., for every Galois topos $\mathcal{G}$, composition with φ defines an equivalence*

$$\underline{Top}\Big(B(\pi_1(\mathcal{E},p)),\mathcal{G}\Big) \xrightarrow{\sim} \underline{Top}(\mathcal{E},\mathcal{G})$$

(which restricts to equivalences

$$\underline{Top}_+\Big(B(\pi_1(\mathcal{E},p)),\mathcal{G}\Big) \simeq \underline{Top}_+(\mathcal{E},\mathcal{G})$$

and

$$\underline{Top}_{\bullet}(B\pi_1(\mathcal{E},p),\mathcal{G}) \simeq \underline{Top}_{\bullet}(\mathcal{E},\mathcal{G})).$$

In the parenthetical remark, we have suppressed the points from the notation. Putting 6.3, 6.4, and 6.5 together, we obtain

6.6 COROLLARY. *Let $(\mathcal{E},p)$ be a connected locally connected topos. For any prodiscrete group G there is an equivalence (natural in G)*

$$\underline{Top}_+\Big((\mathcal{E},p),(BG,p_G)\Big) \simeq \underline{Hom}(\pi_1(\mathcal{E},p),G)$$

Since for a discrete group G, any two points of BG are isomorphic, we conclude

6.7 COROLLARY. *Let $(\mathcal{E},p)$ be a connected locally connected pointed topos. (a) For any discrete group G,*

$$\underline{Top}(\mathcal{E},BG) \simeq \underline{Hom}(\pi_1(\mathcal{E},p),G)$$

(b) For any abelian group A,

$$H^1(\mathcal{E},A) \cong Hom(\pi_1(\mathcal{E},p),A)$$

Note that (b) follows from (a) and the fact that BA classifies A-torsors, i.e., $H^1(\mathcal{E},A) \cong \mathrm{Top}(\mathcal{E},BA)$ (cf. eg [**JW**], [**J1**]).

REFERENCES

[SGA4] M. Artin, A. Grothendieck, J. L. Verdier, "Théorie des Topos et Cohomologie des Schémas," SLN 269, 1972.
[AM] M. Artin, B. Mazur, "Etale Homotopy," SLN 100, 1969.
[BD] M. Barr, R. Diaconescu, *Atomic toposes*, JPAA **17** (1980), 1–24.
[BP] M. Barr,R. Paré, *Molecular toposes*, JPAA **17** (1980), 127–152.
[Fo] M. Fourman, *Sheaf models for set theory*, JPAA **19** (1980), 91–101.
[Fr] P. Freyd, *The axiom of choice*, JPAA **19** (1980), 103-125.
[GZ] P. Gabriel, M. Zisman, "Calculus of Fractions and Homotopy Theory," Springer-Verlag, 1967.
[SGA1] A. Grothendieck, "Revêtements Étales et Groupe Fondamental," SLN 224, 1971.
[J1]] P. T. Johnstone, "Topos Theory," Academic Press, 1977.
[J2] ——————, *Open maps of toposes*, Manuscripta Math **31** (1980), 217-247.
[JT] A. Joyal, M. Tierney, *An extension of the Galois theory of Grothendieck*, Memoirs AMS **309** (1984).
[JW] A. Joyal, G. Wraith, $K(\pi, n)$*-toposes*, in Contemporary Mathematics **30** (1984).
[K] J. Kennison, *The fundamental group of a molecular topos*, JPAA **46** (1987), 187–215.
[M1] I. Moerdijk, *Continuous fibrations and inverse limits of toposes*, Comp. Math. **58** (1986), 45–72.
[M2] ——————, *The classifying topos of a continuous groupoid, I and II*, (part I is to appear in Transactions AMS).
[M3] ——————, *Morita equivalence of continuous groups*, (to appear in Math. Proc. Cambridge Phil. Soc.).
[M4] ——————, *Prodiscrete groups and Galois toposes,*, submitted.
[S] J. P. Serre, "Cohomologie Galoisienne," SLN 5, 1964.

University of Chicago, Chicago, IL 60637

This paper is in final form and will not be published elsewhere.