Two-dimensional causal dynamical triangulations provide a definition of the path integral for projectable two-dimensional Hořava-Lifshitz quantum gravity. We solve the theory coupled to gauge fields.  

\[ Z_{\text{P}} = \langle U_P^\dagger | &\rangle = \int dU_P Z_{\text{P}}[U_P] = Z_{\text{P}}^{I_{\text{P}} + I_{\text{P}}^*}, \]  

where we to each link \( \tilde{\ell} \) associate a \( U_\ell \in G \), and \( U_P \) is the product of the \( U_\ell \)'s around the plaquette. One has a large choice for \( Z_{\text{P}}[U_P] \), but for the purpose of extracting the Hamiltonian it is convenient to use the so-called heat kernel action,

\[ Z_P[U_P] = \langle U_P | e^{-\frac{g^2}{4} A_\mu \Delta G} | U \rangle = \sum_R d_R \chi(R) e^{-\frac{g^2}{4} A_\mu C_{\mu}(R)}, \]

(2)

where \( A_\mu = a_\mu A_\mu \) denotes the area of the plaquette with spatial lattice link length \( a_\mu \) and timelike link length \( a_\mu \) (we will usually think of \( a_\mu = a_\mu \)). \( \Delta G \) is the identity element in \( G \) and \( \Delta G \) the Laplace-Beltrami operator on \( G \). The convenient property of the heat kernel action in 2D is that it is additive, i.e. if we integrate over a link in (1) the action is unchanged: write \( U = U_{\ell_1} U_{\ell_2} U_{\ell_3} U_{\ell_4} \), then

\[ \int dU_{\ell_1} Z_{\text{P}}[U_{\ell_1}] Z_{\text{P}}[U_{\ell_2}] Z_{\text{P}}[U_{\ell_3}] Z_{\text{P}}[U_{\ell_4}] = Z_{\text{P}}^{I_{\text{P}} + I_{\text{P}}^*}, \]

(3)

where \( U_{\ell_1} U_{\ell_2} = U_{\ell_2} U_{\ell_3} U_{\ell_4} U_{\ell_5} \); see Fig. 1.

Let us now consider a lattice with \( \tau \) links in the time direction and \( \ell \) links in the spatial direction. We have two boundaries, with gauge field configurations \( \{ U_\ell \} \) and \( \{ U_{\ell'} \} \), which we choose to keep fixed [Dirichlet-like boundary conditions]. We can then write

\[ Z(g, \{ U_\ell \}, \{ U_{\ell'} \}) = \langle \{ U_\ell \} | \mathbf{T} | \{ U_{\ell'} \} \rangle, \]

where \( \mathbf{T} \) is the transfer matrix, giving us the transition amplitude between link configurations at neighboring time slices. However in 2D we can restrict \( \mathbf{T} \) to be an operator only acting on the holonomies since we can use (3) to integrate out the temporal links \( U_{\ell_0} \) which connect two time slices. We obtain

\[ \langle U'| \mathbf{T} | U \rangle = \langle U'| e^{-a_{\mu}(a_\mu \Delta G)} \mathbf{P} | U \rangle = \langle U'| \mathbf{P} e^{-a_{\mu}(a_\mu \Delta G)} \mathbf{P} | U \rangle, \]

(5)

where the projection operator \( \mathbf{P} \) is defined by

\[ \mathbf{P} | U \rangle = \int dG | GUG^{-1} \rangle, \]

(6)
and it appears in (5) as the result of integration over the last temporal link connecting the two time slices.

Denote the length of the lattice \( L = a \ell \). From (4) and (5) it follows that

\[
\hat{H} = \frac{1}{2} g^2 L \Delta_G
\]

(7)

if we restrict to the gauge invariant subspace (i.e. the subspace of class functions) projected out by \( \hat{\Pi} \).

III. COUPLING TO GEOMETRY

The covariant version of the Yang-Mills theory is

\[
S_{YM} = \frac{1}{4} \int d^2 x \sqrt{g(x)} F_{\mu \nu}^a (F^{\mu \nu})^a.
\]

(8)

We want a path integral formulation which includes also the integration over geometries. Here the CDT formulation is natural: one is summing over geometries which have cylindrical geometry and a time foliation, each geometry being defined by a triangulation and the sum over geometries in the path integral being performed by summing over all triangulations with topology of the cylinder and a time foliation. The coupling of gauge fields to a geometry via dynamical triangulations (where the length of a link is \( a \)) is well known [13]: One uses as plaquettes the triangles. Thus the 2D partition function becomes

\[
Z(\Lambda, g, l', l, \{U_e\}, \{U_i\}) = \sum_T \exp \left[ -4N_T g^2 \Lambda^2 \Delta_G^2 Z^G_T(\beta) \right],
\]

(9)

where the summation is over CDT triangulations \( T \), with an “entrance” boundary consisting of \( l \) links and an “exit” boundary consisting of \( l' \) links, \( \Lambda \) is the lattice cosmological constant, \( N_T \) the number of triangulations in \( T \), and the gauge partition function for a given triangulation \( T \) is defined as

\[
Z^G_T(g, \{U_e\}, \{U_i\}) = \int \prod_{\ell} dU_{\ell} \prod_{p} Z_p[U_p].
\]

(10)

The integration is over all lattice links except the boundary links and \( \prod_p \) is the product over plaquettes (here triangles) in \( T \). For the plaquette action defining \( Z_p[U_p] \) we have again many choices, and for convenience we will use the heat kernel action (2).

We can introduce a transfer matrix \( \hat{T} \), which connects geometry and fields at time label \( l' \) to geometry and fields at time label \( l' + 1 \), and if the (discretized) universe has \( t + 1 \) time labels we can write

\[
Z(\Lambda, g, l', l, \{U_e\}, \{U_i\}) = \langle \{U_e\}, \{U_i\} | T^t | \{U_e\}, \{U_i\} \rangle, \quad T = e^{-a\hat{H}}.
\]

(11)

The one-dimensional geometry at \( l' \) is characterized by the number \( l \) of links (each of length \( a \)), and on these links we have field configurations \( \{U_e\} \). Similarly the geometry at \( l' + 1 \) has \( l' \) links and field configurations \( \{U_e\} \). For fixed \( l \) and \( l' \) the number of plaquettes (triangles) in the spacetime cylinder “slab” between \( l' \) and \( l' + 1 \) is \( l + l' \) and the number of temporal links \( l + l' \). There is a number of possible triangulations of the slab for fixed \( l \) and \( l' \), namely,

\[
N(l', l) = \frac{1}{l + l'} \binom{l + l'}{l}
\]

(12)

For each of these triangulations we can integrate over the \( l + l' \) temporal link variables \( U_{l'}^{(0)} \), as we did for a fixed lattice and we obtain as in that case

\[
\langle U'| \hat{P} e^{-a(l + l')g^2 \Lambda^2 \Delta_G} \hat{\Pi} | U \rangle,
\]

(13)

where \( U' \) and \( U \) are the holonomies corresponding to \( \{U_e\} \) and \( \{U_e\} \), respectively, and \( \hat{P} \) is the projection operator (6) to class functions coming from the last integration over a temporal link \( U_0 \). The factor \( \sqrt{3}/8 \) rather than the factor \( 1/2 \) appears because we are using equilateral triangles rather than squares as in Sec. II. In order to have unified formulas we make a redefinition \( g^2 \sqrt{3}/4 \rightarrow g^2 \) and thus we have the matrix element,

\[
\langle U'| \hat{P} e^{-a(l + l')g^2 \Lambda^2 \Delta_G} \hat{T} | U \rangle.
\]

(14)

If we did not have the matter fields the transfer matrix would be

\[
\langle l'| \hat{T}_{\text{geometry}} | l \rangle = N(l', l) e^{-a(l + l')g^2 \Lambda^2 \Delta_G},
\]

where we have made a redefinition \( \Lambda \sqrt{3}/4 \rightarrow \Lambda \), similar to the one made for \( g^2 \), in order to be in accordance with notations in other articles. The limit where \( a \rightarrow 0 \) and
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L' = al' and L = al are kept fixed has been studied [14] and one finds

\[ \hat{T}_{\text{geometry}} = e^{-a(\hat{R}_{ab} + O(a))}, \quad \hat{H}_{\text{cdt}} = -\frac{d^2}{dL^2}L + \Lambda L. \] (16)

From the definition (11) of \( \hat{H} \) and (14) it follows that

\[ \hat{H} = \hat{H}_{\text{cdt}} + \frac{1}{2} g^2 L \Delta G, \] (17)

acting on the Hilbert space which is the tensor product of the Hilbert space of square integrable class functions on \( G \) and the Hilbert space of the square integrable functions on \( R_+ \) with measure \( d\mu(L) = LdL \).

Since the eigenfunctions of \( \Delta G \) after projection with \( \hat{P} \) are just the characters \( \chi_R(U) \) on \( G \) and they have eigenvalues \( C_{2}(R) \), we can solve the eigenvalue equation for \( \hat{H} \) by writing \( \Psi(L, U) = \psi_R(L)\chi_R(U) \). For \( \hat{H}_{\text{cdt}} \) we have [14,15]

\[ \hat{H}_{\text{cdt}}\psi_n(L, \Lambda) = \epsilon_n\psi_n(L, \Lambda), \quad \epsilon_n = 2n\sqrt{\Lambda}, \quad n > 0, \] (18)

where the eigenfunctions are of the form \( \Lambda p_n(L\sqrt{\Lambda})e^{-\sqrt{\Lambda}L} \), \( p_n(x) \) being a polynomial of degree \( n - 1 \). The corresponding solution for \( \psi_R(L) \) is obtained by the substitution

\[ \Lambda \rightarrow \Lambda_R = \Lambda + \frac{1}{2} g^2 C_2(R), \] (19)

i.e.

\[ \hat{H}\Psi_{n,R} = E(n,R)\Psi_{n,R}, \quad E(n,R) = 2n\sqrt{\Lambda_R}, \quad n > 0, \] (20)

\[ \Psi_{n,R}(L, U) = \Lambda_R p_n(L\sqrt{\Lambda_R})e^{-L\sqrt{\Lambda_R}}\chi_R(U), \] (21)

with the reservation that the correct variable is not really \( \Lambda \) but rather the conjugacy class corresponding to \( U \). In the simplest case of \( SU(2) \) the group manifold can be identified with \( S^3 \) and \( \Delta G \) is the Laplace-Beltrami operator on \( S^3 \). The conjugacy classes are labeled by the geodesic distance \( \theta \) to the north pole and the representations are labeled by \( R = j \) and we have\(^2\)

\[ C_j = j(j + 1), \quad \chi_j(\theta) = \frac{\sin(j + \frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \quad j = 0, 1, 2, 1, \ldots. \] (22)

The above results are also valid in simpler cases. If \( G = U(1) \) where one has

\( U(\theta) = e^{i\theta}, \quad \Delta_G = -\frac{d^2}{d\theta^2}, \) (23)

\[ C_n = n^2, \quad \chi_n(\theta) = e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \ldots, \] (24)

and if \( G = Z_N \), the discrete cyclic group of order \( N \),

\[ U(k) = e^{2\pi k}, \quad (\Delta_G)_{k,k'} = \delta_{k,k'+1} + \delta_{k,k'-1} - 2\delta_{k,k'}, \quad k = 0, \ldots, N - 1, \] (25)

\[ C_n = 2\left(1 - \cos\left(\frac{2\pi}{N} n\right)\right), \quad \chi_n(k) = e^{\frac{2\pi i k}{N}}, \quad n = 0, 1, \ldots, N - 1. \] (26)

IV. THE GROUND STATE OF THE UNIVERSE

In CDT the disk amplitude is defined as

\[ W_{A}(L) = \int_0^\infty dt(L)|e^{-i\hat{H}_{\text{cdt}}L'}|L' \rightarrow 0. \] (27)

It is a version of the Hartle-Hawking wave function. One can calculate \( W_{A}(L) \) [1]:

\[ W_{A}(L) = \frac{e^{-\sqrt{\Lambda}L}}{L}. \] (28)

This function satisfies

\[ \hat{H}_{\text{cdt}}W_{A}(L) = 0, \] (29)

and one can view (29) as the Wheeler-deWitt equation. Formally \( W_{A}(L) \propto \psi_0(L) \) in the notation used in Eq. (18), but it was not included as an eigenfunction in the listing in (18) since it does not belong to the Hilbert space \( L^2(R_+) \) with measure \( LdL \).

If we couple the theory of fluctuating geometries to gauge fields as above, we have to decide what kind of boundary condition to impose in the limit \( L' \rightarrow 0 \) in (27). A possible interpretation of this “singularity” in the discrete setting is that all the vertices of the first time slice at time \( t' = 1 \) have additional temporal links joining a single vertex at time \( t' = 0 \) (see Fig. 2). We can view this as an explicit, discretized, realization of the matter part of the Hartle-Hawking boundary condition.

Denote by \( \{U^{(0)}(\ell)\} \), \( \ell = 1, \ldots, \) \( l \) the gauge fields on these temporal links and by \( \{U_{l}\} \), \( \ell = 1, \ldots, \) \( l \) the gauge fields on the spatial links constituting the first loop at time \( t' = 1 \) and denote by \( U(1) \) the corresponding holonomy at time \( t' = 1 \). The contribution to the matter partition function coming from this first “big bang” part of the universe is then
where we have integrated out the temporal links as the Hartle-Hawking wave function for 2D CDT coupled to matter partition function can now be written (after integrating out the temporal links in the rest of the lattice too) as

$$W(0) = \sum_{U \in \mathcal{G}} e^{i \omega^* \Delta_{U}} |I\rangle,$$

where we have integrated out the temporal links \( \{U_{ij}^{(0)}\} \). The matter partition function can now be written (after integrating out the temporal links in the rest of the lattice too) as the integral over \( t \) holonomies \( U(1), U(2), \ldots, U(t) \),

$$\int \prod_{i=1}^{l} dU(i) \langle U(i) | e^{-i \omega^* \Delta_{U}} |U(i - 1)\rangle,$$

where \( U(0) = I \) and \( l_0 = 0 \). From this expression it is natural to say that the universe starts out in the matter state |I\rangle, or expanded in characters:

$$\langle U|I\rangle = \delta(U - I) = \sum_{R} d_{R} \chi_{R}(U).$$

This wave function is not normalizable if the group has infinitely many representations, but neither is \( W_{\Lambda}(L) \) as we just saw. Combining the two we might define the Hartle-Hawking wave function for 2D CDT coupled to gauge fields as

$$W(L, U) = \int_{0}^{\infty} \langle L, U | e^{-T\hat{H}} | L = 0, U = I \rangle$$

$$= \sum_{R} d_{R} \chi_{R}(U) W_{\Lambda_{R}}(L),$$

where \( \Lambda_{R} \) is defined in Eq. (19). We have explicitly:

$$W(L, k) = \sum_{r} e^{i \pi r \omega} \exp \left( -L \left( \sqrt{\Lambda + g^2 \frac{1 - \cos(2\pi r/n)}} \right) \right),$$

for the \( Z_{N} \) theory.
gauge field at the big bang. With a boundary condition similar to the one proposed by Hartle and Hawking we find that the matter and gauge degrees of freedom become entangled.

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