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A remark on the Dunkl differential-difference operators

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§1. Introduction

Let $E$ be a Euclidean vector space of dimension $n$ with inner product $(\cdot, \cdot)$. For $\alpha \in E$ with $(\alpha, \alpha) = 2$ we write
\[
(1.1) \quad r_\alpha(\lambda) = \lambda - (\alpha, \lambda)\alpha, \; \lambda \in E
\]
for the orthogonal reflection in the hyperplane perpendicular to $\alpha$.

**Definition 1.1.** A normalized root system $R$ in $E$ is a finite set of non-zero vectors in $E$, normalized by $(\alpha, \alpha) = 2$ for all $\alpha \in R$, such that $r_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.

Let $R \subset E$ be a normalized root system. We write $W = W(R)$ for the group generated by the reflections $r_\alpha$, $\alpha \in R$. Denote by $\mathbb{C}[E]$ the algebra of $\mathbb{C}$-valued polynomial functions on $E$. For $w \in W$, $\xi \in E$, $\alpha \in R$ introduce the operators
\[
(1.2) \quad w, \partial_\xi, \Delta_\alpha : \mathbb{C}[E] \longrightarrow \mathbb{C}[E]
\]
by
\[
(1.3) \quad (wp)(\lambda) = p(w^{-1}\lambda)
\]
\[
(1.4) \quad (\partial_\xi p)(\lambda) = \frac{d}{dt} \{p(\lambda + t\xi)\} |_{t=0}
\]
\[
(1.5) \quad (\Delta_\alpha p)(\lambda) = \frac{p(\lambda) - p(r_\alpha \lambda)}{(\alpha, \lambda)}.
\]

**Remark 1.2.** The operators $\Delta_\alpha$, $\alpha \in R$ were studied by Bernstein, Gel'fand and Gel'fand and are related to the Schubert cells and the cohomology of $G/P$ [BGG]. They are the infinitesimal analogues of the Demazure operators [De 1,2].

Let $R_+ = \{\alpha \in R; (\alpha, \lambda) > 0\}$ for some fixed generic $\lambda \in E$ be a positive subsystem of $R$. 

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Definition 1.3. Suppose for $\alpha \in R$ we have given $k_\alpha \in C$ with $k_w\alpha = k_\alpha \forall w \in W, \forall \alpha \in R$. For $\xi \in E$ the operator

$$D_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi) \Delta_\alpha : C[E] \rightarrow C[E]$$

is called the Dunkl differential-difference operator.

Remark 1.4. It is easy to see that $D_\xi$ is independent of the choice of the positive subsystem $R_+ \subset R$. If we write $q_\alpha = e^{2\pi i k_\alpha}$ then one can think of the operator $D_\xi$ as a $q$-analogue (corresponding to the case $k_\alpha \to 0$) of the directional derivative $\partial_\xi$.

We also write $D_\xi = D_\xi(k)$ to indicate the dependence on $k \in K = \{k = (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R; k_{w\alpha} = k_\alpha \forall w \in W, \forall \alpha \in R\}$.

Theorem 1.5 (Dunkl [Du]): We have $D_\xi D_\eta = D_\eta D_\xi \forall \xi, \eta \in E$.

Let $\mathbb{C}[E^*]$ be the symmetric algebra on $E$. For $\pi \in \mathbb{C}[E^*]$ we write $\partial_\pi$ when we think of $\pi$ as a constant coefficient differential operator on $E$ (rather than a polynomial function on $E^*$). In view of Theorem 1.5 the constant coefficient differential operator $\partial_\pi$ has a well defined $q$-analogue

$$D_\pi : \mathbb{C}[E] \rightarrow \mathbb{C}[E]$$

defined for a monomial $\pi = \xi_1^{d_1} \cdots \xi_n^{d_n}$ by

$$D_\pi = D_\pi(k) = D_{\xi_1}^{d_1} \cdots D_{\xi_n}^{d_n}$$

and extended by linearity.

Theorem 1.6 (Dunkl [Du]): Suppose $\xi_1, \ldots, \xi_n$ is an orthonormal basis for $E$. The $q$-analogue of the Laplacian is given by

$$\sum_{j=1}^n D_{\xi_j}^2 = \sum_{j=1}^n \partial_{\xi_j}^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \{\partial_\alpha - \Delta_\alpha\}.$$

In Section 2 we review the proofs of both theorems as given by Dunkl.

We write $\mathbb{C}[E]^W$ and $\mathbb{C}[E^*]^W$ for the space of $W$-invariants in $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$ respectively. We denote by $\mathbb{A}$ the associative algebra of endomorphisms of $\mathbb{C}[E]$ generated by (multiplication by) $(\xi, \cdot)$ and $D_\eta$ for $\xi, \eta \in E$. Let $\mathbb{A}^W = \{D \in \mathbb{A}; wD = Dw \forall w \in W\}$ be the subalgebra of $W$-invariant operators in $\mathbb{A}$, and denote by

$$\text{Res}(D) : \mathbb{C}[E]^W \rightarrow \mathbb{C}[E]^W, \quad D \in \mathbb{A}^W$$
the restriction of $D$ to $\mathbb{C}[E]^W$. Clearly $\text{Res} : \mathbb{A}^W \rightarrow \text{End}(\mathbb{C}[E]^W)$ is a homomorphism of algebras. Since $wD_\xi w^{-1} = D_{w\xi} \quad \forall w \in W, \forall \xi \in E$ we have $D_\pi \in \mathbb{A}^W \forall \pi \in \mathbb{C}[E^*]^W$.

**Theorem 1.7.** Suppose by the Chevalley theorem that $\mathbb{C}[E]^W = \mathbb{C}[p_1, \ldots, p_n]$ with $p_1, \ldots, p_n$ homogeneous of degrees $d_1 \leq \ldots \leq d_n$. Then the set

$$\{\text{Res}(D_\pi) ; \pi \in \mathbb{C}[E^*]^W\}$$

is a commuting family of differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ containing the operator

$$\text{Res}(\sum_{j=1}^n D_\xi^2) = \sum_{j=1}^n \partial_\xi^2 + 2 \sum_{\alpha \in R_+} k_\alpha \frac{1}{(\alpha, \cdot)} \partial_\alpha.$$  

**Remark 1.8.** The proof of this theorem is a triviality. However it can be reformulated as the complete integrability for the generalized non periodic Calogero-Moser system (both on the quantum mechanical level of differential operators and on the classical mechanical level of symbols). For root systems $R$ of type $A$ the complete integrability of the Calogero-Moser system was first established by Moser by realizing the system as a Lax pair [Mo]. The method of Moser was extended by Olshanetsky and Perelomov to cover the root systems $R$ of classical type [OP]. In the crystallographic case $(\alpha, \beta)^2 \in \mathbb{Z} \forall \alpha, \beta \in R$ the above theorem has been obtained before by Opdam using transcendental methods [HO, He1, Op 1,2, He 2].

Suppose $S \subset R$ is a set of roots in $R$ invariant under $W$. Let $S_+ = S \cap R_+$ and put

$$p_S(\cdot) = \prod_{\alpha \in S_+} (\alpha, \cdot) \in \mathbb{C}[E]$$

$$\pi_S = \prod_{\alpha \in S_+} \alpha \in \mathbb{C}[E^*].$$

Clearly we have

$$wp_S = \chi(w)p_S, w\pi_S = \chi(w)\pi_S \quad \forall w \in W$$

for some one dimensional character $\chi = \chi_S$ of $W$, and conversely every $p \in \mathbb{C}[E]$ with $wp = \chi(w)p \forall w \in W$ is divisible in $\mathbb{C}[E]$ by $p_S$. Although $p_S^{-1}D_{\pi_S}(k)$ need not be an endomorphism of $\mathbb{C}[E]$ it follows that $p_S^{-1}D_{\pi_S}(k)(p) \in \mathbb{C}[E]^W \forall p \in \mathbb{C}[E]^W$, and hence

$$G(1_S, k) := \text{Res}(p_S^{-1}D_{\pi_S}(k)) \in \text{End}(\mathbb{C}[E]^W)$$
is a well defined endomorphism of $\mathbb{C}[E]^W$. We also write
\begin{equation}
G(-1_S, k) := \text{Res}(D_{\pi_S}(k - 1_S) \cdot p_S) \in \text{End}(\mathbb{C}[E]^W)
\end{equation}

where $k - 1_S \in K$ is the multiplicity function by $(k - 1_S)_\alpha = k_\alpha - 1$ for $\alpha \in S$ and $(k - 1_S)_\alpha = k_\alpha$ for $\alpha \in R \setminus S$.

**Theorem 1.9.** The operators (1.16) and (1.17) are differential operators in the Weyl algebra $\mathbb{C}[k, p_1, \ldots, p_n, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}]$ and satisfy the shift relations
\begin{align}
G(1_S, k) \text{Res}(D_{\pi}(k)) &= \text{Res}(D_{\pi}(k + 1_S)) G(1_S, k) \\
G(-1_S, k) \text{Res}(D_{\pi}(k)) &= \text{Res}(D_{\pi}(k - 1_S)) G(-1_S, k)
\end{align}

\forall \pi \in \mathbb{C}[E^*]^W$. Here $(k \pm 1_S)_\alpha = k_\alpha \pm 1$ \forall $\alpha \in S$ and $(k \pm 1_S)_\alpha = k_\alpha$ \forall $\alpha \in R \setminus S$.

The proofs of both Theorem 1.7 and 1.9 will be given in Section 3.

**Remark 1.10.** In the terminology of Opdam the operator (1.16) is a raising operator and the operator (1.17) a lowering operator for the commuting family (1.11). Again in the crystallographic case the above theorem was obtained by Opdam [Op 2]. Recall Macdonald’s (infinitesimal) constant term conjecture, which says that for $\mathcal{R}(s) > 0$
\begin{equation}
\int_E \prod_{\alpha \in R^+} |(\alpha, \lambda)|^{2s} d\gamma(\lambda) = \prod_{j=1}^n \frac{(sd_j)!}{s!},
\end{equation}

where $d\gamma(\lambda) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{4}(\lambda, \lambda)} d\lambda$ is the Gaussian measure on $E$ [Ma]. The same arguments as given in [Op 3, Section 6] show that the evaluation of this integral is equivalent with
\begin{equation}
G(-1, k)(1) = |W| \cdot \prod_{i=1}^n \prod_{j=1}^{m_i} (d_i k - j),
\end{equation}

where $-1 = -1_R$ and $k = k_\alpha \forall \alpha \in R$. In turn this latter formula is related to the normalization of the “multivariable Bessel function associated with $R$” at $\xi = 0$. This normalization problem has been analyzed by Opdam, and the desired formula (1.21) can be obtained [Op 4]. After this one can proceed as in [Op 3, Section 7] to compute the Bernstein-Sato polynomial of the discriminant without the crystallographic restriction in accordance with a conjecture of Yano and Sekiguchi [YS].
§2. The Dunkl differential-difference operators.

Using the bracket \([\cdot, \cdot]\) for the commutator of endomorphisms of \(\mathbb{C}[E]\) we can write for \(\xi, \eta \in E\)

\[
[D_\xi, D_\eta] = I + II + III
\]

with

\[
I = [\partial_\xi, \partial_\eta] = 0
\]

\[
II = \sum_{\alpha \in R_+} k_\alpha \{ (\alpha, \xi)[\Delta_\alpha, \partial_\eta] + (\alpha, \eta)[\partial_\xi, \Delta_\alpha] \}
\]

\[
III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi)(\beta, \eta)[\Delta_\alpha, \Delta_\beta].
\]

**Lemma 2.1.** For \(\xi \in E, \alpha \in R\) we have

\[
[\partial_\xi, \Delta_\alpha] = \frac{(\alpha, \xi)}{(\alpha, \cdot)} \{ r_\alpha \partial_\alpha - \Delta_\alpha \}.
\]

**Proof:** Using the definition \(\Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha)\) we get

\[
[\partial_\xi, \Delta_\alpha] = [\partial_\xi, \frac{1}{(\alpha, \cdot)}](1 - r_\alpha) + \frac{1}{(\alpha, \cdot)}[\partial_\xi, 1 - r_\alpha] = - \frac{(\alpha, \xi)}{(\alpha, \cdot)} \{ r_\alpha \partial_\alpha - \Delta_\alpha \}.
\]

Using (2.5) the second term (2.3) can be rewritten as

\[
II = \sum_{\alpha \in R_+} k_\alpha \{ (\alpha, \xi)[\alpha, \cdot] \} \{ r_\alpha \partial_\alpha - \Delta_\alpha \}(-1 + 1) = 0.
\]

The third term (2.4) can be written as

\[
III = \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta \{ (\alpha, \xi)(\beta, \eta) - (\alpha, \eta)(\beta, \xi) \} \Delta_\alpha \Delta_\beta.
\]

and for the proof of Theorem 1.5 it remains to verify the vanishing of this third term.

**Proposition 2.2.** Suppose \(B(\cdot, \cdot)\) is a bilinear form on \(E\) such that

\[
B(r_\alpha \lambda, r_\alpha \mu) = B(\mu, \lambda) \quad \forall \lambda, \mu \in E, \forall \alpha \in R \cap \text{span } (\lambda, \mu).
\]
If \( w \in W \) is a pure rotation (i.e. \( \text{dim } \text{Im}(w - \text{Id}) = 2 \)) then

\[
\sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)} = 0
\]

and

\[
\sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \Delta_\alpha \Delta_\beta = 0.
\]

**Proof:** Using the definition \( \Delta_\alpha = \frac{1}{(\alpha, \cdot)}(1 - r_\alpha) \) the left hand side of (2.10) can be written as a sum of the following three terms

\[
A = \sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
B = -\sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \left\{ \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha + \frac{1}{(\alpha, \cdot)(\beta, \cdot)} r_\beta \right\}
\]

\[
C = \sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(\alpha, \cdot)(r_\alpha \beta, \cdot)} r_\alpha r_\beta
\]

with the summations over the same index set as in (2.9) and (2.10).

Let \( S = R \cap \text{Im}(w - \text{Id}) \) be the normalized root system of the largest dihedral group \( W(S) \) containing \( w \). If \( w = r_\alpha r_\beta \) then for \( \gamma \in S \) we have \( r_\gamma w r_\gamma = w^{-1} \) and hence \( r_{\gamma \alpha} r_{\gamma \beta} = r_\beta r_\alpha \). We claim that \( r_\gamma A = A \forall \gamma \in S \). Indeed we have

\[
r_\gamma A = \sum_{\alpha, \beta \in R_+ \mid r_\alpha r_\beta = w} k_\alpha k_\beta B(\alpha, \beta) \frac{1}{(r_\gamma \alpha, \cdot)(r_\gamma \beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+, r_\beta r_\alpha = w} k_\alpha k_\beta B(r_\gamma \alpha, r_\gamma \beta) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= \sum_{\alpha, \beta \in r_\gamma R_+, r_\beta r_\alpha = w} k_\alpha k_\beta B(\beta, \alpha) \frac{1}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= A
\]

since the summation in (2.9) is independent of the choice of \( R_+ \). Let \( S_+ = R_+ \cap S \) and put \( p_S = \prod_{\alpha \in S_+} (\alpha, \cdot) \). Then \( p_S \) transforms under the group \( W(S) \) according to the sign character and every polynomial in \( \mathbb{C}[E] \) transforming under \( W(S) \) according to the sign character is divisible in \( \mathbb{C}[E] \) by \( p_S \). Now observe that \( p_S A \in \mathbb{C}[E] \) transforms
under $W(S)$ according to the sign character. Hence $A \in \mathbb{C}[E]$. Since $A$ is homogeneous of degree minus two we have $A = 0$. This proves (2.9).

Since $w = r_{\alpha}r_{\beta} = r_{r_{\alpha}\beta}r_{\alpha}$ and $B(\alpha, \beta) = B(r_{\alpha}\beta, r_{\alpha}\alpha) = -B(r_{\alpha}\beta, \alpha)$ the vanishing of the term (2.12) is clear, and for the term (2.13) we can write $C = -Aw = 0$. Q.E.D.

**Lemma 2.3.** For $\xi, \eta \in E$ fixed the bilinear form

\[(2.14) \quad B(\lambda, \mu) = (\lambda, \xi)(\mu, \eta) - (\lambda, \eta)(\mu, \xi)\]

on $E$ satisfies condition (2.8).

**Proof:** Clearly $B(\mu, \lambda) = -B(\lambda, \mu)$ is an alternating form. For $\lambda \in E, \lambda \neq 0$ we write $\lambda' = \sqrt{2}|\lambda|^{-1}\lambda$ and get

\[B(r_{\lambda'}\lambda, r_{\lambda'}\mu) = B(-\lambda, \mu - (\lambda', \mu)\lambda') = B(-\lambda, \mu) = B(\mu, \lambda).\]

Hence for $\lambda, \mu \in E$ generic we get by continuity

\[B(r_{\nu}\lambda, r_{\nu}\mu) = B(\nu, \lambda) \quad \forall \nu \in \text{span} \langle \lambda, \mu \rangle, (\nu, \nu) = 2. \quad \text{Q.E.D.} \]

The proof of Theorem 1.5 now follows by regrouping the terms in (2.7) as a sum over \(\{\alpha, \beta \in R_+; r_{\alpha}r_{\beta} = w\}\) where $w \in W$ runs over the pure rotations in $W$ and by applying (2.10).

The proof of Theorem 1.6 is just an easy calculation.

\[
\sum_{j=1}^{n} D_{ij}^2 = \sum_{j=1}^{n} \left( \partial \xi_j + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi_j) \Delta_\alpha \right)^2
\]

\[
= \sum_{j=1}^{n} \left\{ \partial^2 \xi_j + \sum_{\alpha \in R_+} k_\alpha (\alpha, \xi_j) (\partial \xi_j \Delta_\alpha + \Delta_\alpha \partial \xi_j) + \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \xi_j) (\beta, \xi_j) \Delta_\alpha \Delta_\beta \right\}
\]

\[
= \sum_{j=1}^{n} \partial^2 \xi_j + \sum_{\alpha \in R_+} k_\alpha (\partial \alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha) + \sum_{\alpha, \beta \in R_+} k_\alpha k_\beta (\alpha, \beta) \Delta_\alpha \Delta_\beta.
\]

The third term vanishes by Proposition 2.2 and because $\Delta^2_\alpha = 0$. Using Lemma 2.1 we get

\[
\partial_\alpha \Delta_\alpha + \Delta_\alpha \partial_\alpha = [\partial_\alpha, \Delta_\alpha] + 2\Delta_\alpha \partial_\alpha = \frac{\langle \alpha, \alpha \rangle}{\langle \alpha, \cdot \rangle} \left\{ r_\alpha \partial_\alpha - \Delta_\alpha \right\} + \frac{2}{\langle \alpha, \cdot \rangle} (1 - r_\alpha) \partial_\alpha = \frac{2}{\langle \alpha, \cdot \rangle} \left\{ \partial_\alpha - \Delta_\alpha \right\}.
\]
§3. The Opdam shift operators.

Recall that $D \in \text{End}(\mathbb{C}[p_1, \ldots, p_m])$ is a differential operator of degree $\leq d$ if and only if

\[(3.1) \quad \text{ad}(p)^{d+1}(D) = 0 \quad \forall p \in \mathbb{C}[p_1, \ldots, p_n].\]

Hence the fact that the operators (1.11), (1.16) and (1.17) are differential operators is clear from

\[(3.2) \quad \text{ad}(p)(D_{\xi}) = \text{ad}(p)(\partial_{\xi}) = -\partial_{\xi}(p)\]

\[(3.3) \quad \text{ad}(p)^2(D_{\xi}) = 0 \quad \forall p \in \mathbb{C}[E]^W, \forall \xi \in E.\]

Hence Theorem 1.7 is an immediate consequence of Theorem 1.5 and Theorem 1.6.

**Theorem 3.1.** For the $q$-analogue of the Laplacian we have

\[(3.4) \quad \text{Res}(p^{-1}_S \circ \left\{ \sum_{j=1}^{n} D_{\xi_j}^2(k) \right\} \circ p_S) = \text{Res}\left(\sum_{j=1}^{n} D_{\xi_j}^2(k + 1_S)\right).\]

**Proof:** First we observe that the left hand side of (3.4) is a well defined endomorphism of $\mathbb{C}[E]^W$. We now use Theorem 1.6 and just calculate term by term. For the first term we get

\[
p^{-1}_S \circ \sum_{j=1}^{n} D_{\xi_j}^2 \circ p_S = \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S^+} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + p^{-1}_S(\sum_{j=1}^{n} \partial_{\xi_j}^2)(p_S)
\]

\[
= \sum_{j=1}^{n} \partial_{\xi_j}^2 + 2 \sum_{\alpha \in S^+} \frac{1}{(\alpha, \cdot)} \partial_{\alpha}.
\]

For the second term we get

\[
p^{-1}_S \circ \left\{ 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} \right\} \circ p_S = 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + p^{-1}_S \cdot \left( 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} \right)(p_S)
\]

\[
= 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\alpha \in R^+, \beta \in S^+} k_{\alpha} \frac{\alpha, (\beta, \cdot)}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\beta \in S^+} k_{\beta} \frac{(\beta, \beta)}{(\beta, \cdot)^2}
\]

\[
+ 2 \sum_{\alpha \in R^+, \beta \in S^+ \alpha \neq \beta} k_{\alpha} \frac{(\alpha, \beta)}{(\alpha, \cdot)(\beta, \cdot)}
\]

\[
= 2 \sum_{\alpha \in R^+} k_{\alpha} \frac{1}{(\alpha, \cdot)} \partial_{\alpha} + 2 \sum_{\beta \in S^+} k_{\beta} \frac{2}{(\beta, \cdot)^2}
\]
by the same argument as in the proof of Proposition 2.2. Finally for the third term we have

\[ p_{S}^{-1} \circ \left\{ 2 \sum_{\alpha \in R_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)} \Delta_{\alpha} \right\} \circ p_{S} = 2 \sum_{\alpha \in R_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \left\{ 1 - p_{S}^{-1} \circ r_{\alpha} \circ p_{S} \right\} \]

\[ = 2 \sum_{\alpha \in R_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \left\{ 1 - \chi_{S}(r_{\alpha}) \right\} \]

\[ = 2 \sum_{\alpha \in S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \left\{ 1 + r_{\alpha} \right\} + 2 \sum_{\alpha \in R_{+} \setminus S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \Delta_{\alpha} \]

\[ = 2 \sum_{\alpha \in S_{+}} k_{\alpha} \frac{2}{(\alpha, \cdot)^{2}} - 2 \sum_{\alpha \in S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \Delta_{\alpha} \]

\[ + 2 \sum_{\alpha \in R_{+} \setminus S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \Delta_{\alpha}. \]

Taking all three terms together yields

\[ p_{S}^{-1} \circ \left\{ \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k) \right\} \circ p_{S} = \sum_{j=1}^{n} \partial_{\xi_{j}}^{2} + 2 \sum_{\alpha \in R_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \partial_{\alpha} + 2 \sum_{\alpha \in S_{+}} \frac{1}{(\alpha, \cdot)^{2}} \partial_{\alpha} \]

\[ + 2 \sum_{\alpha \in S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \Delta_{\alpha} - 2 \sum_{\alpha \in R_{+} \setminus S_{+}} k_{\alpha} \frac{1}{(\alpha, \cdot)^{2}} \Delta_{\alpha}. \quad Q.E.D. \]

**Corollary 3.2.** We have the shift relations

\[ G(1_{S}, k) \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k) \right) = \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k + 1_{S}) \right) G(1_{S}, k) \]

\[ G(-1_{S}, k) \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k) \right) = \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k - 1_{S}) \right) G(-1_{S}, k). \]

**Proof:** Indeed we have

\[ \text{Res} \left( p_{S}^{-1} D_{\pi_{S}}(k) \right) \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k) \right) = \text{Res} \left( \sum_{j=1}^{n} p_{S}^{-1} D_{\pi_{S}}(k) D_{\xi_{j}}^{2}(k) \right) \]

\[ = \text{Res} \left( \sum_{j=1}^{n} p_{S}^{-1} D_{\xi_{j}}^{2}(k) D_{\pi_{S}}(k) \right) \]

\[ = \text{Res} \left( \sum_{j=1}^{n} p_{S}^{-1} D_{\xi_{j}}^{2}(k) p_{S} \right) \text{Res} \left( p_{S}^{-1} D_{\pi_{S}}(k) \right) \]

\[ = \text{Res} \left( \sum_{j=1}^{n} D_{\xi_{j}}^{2}(k + 1_{S}) \right) \text{Res} \left( p_{S}^{-1} D_{\pi_{S}}(k) \right) \]
which proves (3.5). The relation (3.6) is proved similarly. Q.E.D.

**Theorem 3.3.** As endomorphisms of $\mathbb{C}[E]$ the operators

\begin{align}
E &= \frac{1}{2} \sum_{j=1}^{n} (\xi_{j,\cdot})^2 \\
H &= \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial_{\xi_{j,\cdot}} + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha} \right) \\
F &= -\frac{1}{2} \sum_{j=1}^{n} D_{\xi_{j,\cdot}}^2
\end{align}

satisfy the commutation relations of $\mathfrak{sl}(2)$:

\begin{align}
\end{align}

**Proof:** The Euler operator $\sum_{j=1}^{n} (\xi_{j,\cdot}) \partial_{\xi_{j,\cdot}}$ acts as multiplication by $d$ on the space of homogeneous polynomials in $\mathbb{C}[E]$ of degree $d$. Hence the commutation relations $[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$ rephrase that $E$ and $F$ are homogeneous of degree plus and minus two respectively.

Since $[p, \Delta_\alpha] = 0 \quad \forall p \in \mathbb{C}[E]^W, \quad \forall \alpha \in R$ we get

\begin{align}
[E, D_{\xi_{j,\cdot}}] &= [E, \partial_{\xi_{j,\cdot}}] = -\xi_{j,\cdot} \quad \forall \xi \in E,
\end{align}

and therefore

\begin{align}
[E, F] &= -\frac{1}{2} \sum_{j=1}^{n} [E, D_{\xi_{j,\cdot}}] \\
&= \frac{1}{2} \sum_{j=1}^{n} \{ (\xi_{j,\cdot}) D_{\xi_{j,\cdot}} + D_{\xi_{j,\cdot}} (\xi_{j,\cdot}) \} \\
&= \sum_{j=1}^{n} (\xi_{j,\cdot}) D_{\xi_{j,\cdot}} + \frac{1}{2} \sum_{j=1}^{n} [D_{\xi_{j,\cdot}}, (\xi_{j,\cdot})] \\
&= \sum_{j=1}^{n} (\xi_{j,\cdot}) D_{\xi_{j,\cdot}} + \frac{n}{2} + \frac{1}{2} \sum_{j=1}^{n} \sum_{\alpha \in R_+} k_{\alpha} (\alpha, \xi_j) [\Delta_\alpha, (\xi_{j,\cdot})] \\
&= \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial_{\xi_{j,\cdot}} + \sum_{\alpha \in R_+} k_{\alpha} (\alpha, \partial_\cdot) \Delta_\alpha + \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha} r_\alpha \\
&= \sum_{j=1}^{n} (\xi_{j,\cdot}) \partial_{\xi_{j,\cdot}} + \left( \frac{n}{2} + \sum_{\alpha \in R_+} k_{\alpha} \right).
\end{align}
Here we have used that for $\xi \in E$

$$[\Delta_\alpha, (\xi, \cdot)] = -\frac{1}{(\alpha, \cdot)}[r_\alpha, (\xi, \cdot)]$$

$$= -\frac{1}{(\alpha, \cdot)} \{(r_\alpha \xi, \cdot) - (\xi, \cdot)\}r_\alpha$$

$$= (\alpha, \xi)r_\alpha. \quad \text{Q.E.D.}$$

**Proposition 3.4.** Using the inner product $(\cdot, \cdot)$ on $E$ we have an isomorphism between $\mathbb{C}[E]$ and $\mathbb{C}[E^*]$. For $p \in \mathbb{C}[E]$ we write $\pi \in \mathbb{C}[E^*]$ for the corresponding element. For $p \in \mathbb{C}[E]$ homogeneous of degree $d$ we have

$$(3.12) \quad D_\pi = (-1)^d \frac{1}{d!} \ad(F)^d(p).$$

**Proof:** Clearly $\ad(H)D_\pi = -dD_\pi$ and by Theorem 1.5 we have $\ad(F)D_\pi = 0$. Using (3.11) and induction on $d$ (assuming $\pi$ to be a monomial as in (1.9) with $d = d_1 + \cdots + d_n$) it is easy to see that

$$(-1)^d \frac{1}{d!} \ad(E)^d(D_\pi) = p$$

and hence

$$\ad(E)^{d+1}(D_\pi) = 0.$$  

By standard representation theory of $\mathfrak{sl}(2)$ we conclude (3.12). \quad \text{Q.E.D.}

**Corollary 3.5.** For $\pi \in \mathbb{C}[E^*]^W$ we have

$$(3.13) \quad \Res(p_S^{-1} \circ D_\pi (k) \circ p_S) = \Res(D_\pi (k + 1_s)).$$

**Proof:** This is easily derived from Theorem 3.1 and Proposition 3.4. \quad \text{Q.E.D.}

The proof of Theorem 1.9 now goes along the same lines as the proof of Corollary 3.2.

**Remark 3.6.** The above type of arguments to use an $\mathfrak{sl}(2)$ to reduce the computation of higher order operators to those of the second order one go back to Harish-Chandra [Ha].

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References


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