EFFICIENT CSL MODEL CHECKING USING STRATIFICATION

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Abstract. For continuous-time Markov chains, the model-checking problem with respect to continuous-time stochastic logic (CSL) has been introduced and shown to be decidable by Aziz, Sanwal, Singhal and Brayton in 1996 [1, 2]. Their proof can be turned into an approximation algorithm with worse than exponential complexity. In 2000, Baier, Haverkort, Hermanns and Katoen [4, 5] presented an efficient polynomial-time approximation algorithm for the sublogic in which only binary until is allowed. In this paper, we propose such an efficient polynomial-time approximation algorithm for full CSL.

The key to our method is the notion of stratified CTMCs with respect to the CSL property to be checked. On a stratified CTMC, the probability to satisfy a CSL path formula can be approximated by a transient analysis in polynomial time (using uniformization). We present a measure-preserving, linear-time and -space transformation of any CTMC into an equivalent, stratified one. This makes the present work the centerpiece of a broadly applicable full CSL model checker.

Recently, the decision algorithm by Aziz et al. was shown to work only for stratified CTMCs. As an additional contribution, our measure-preserving transformation can be used to ensure the decidability for general CTMCs.

1. INTRODUCTION

Continuous-time Markov chains (CTMC) play an important role in performance evaluation of networked, distributed, and biological systems. The concept of formal verification for CTMCs was introduced by Aziz, Sanwal, Singhal and Brayton in 1996 [1, 2]. Their seminal paper defined continuous-time stochastic logic (CSL) to specify properties over CTMCs. It showed that the model checking problem for CTMCs, which asks whether the CTMC satisfies a given CSL property, is decidable, using algebraic and transcendental number theory. Their proof is constructive, so it can be turned into an approximation procedure.


Key words and phrases: continuous-time Markov chains, continuous stochastic logic, model checking, approximation algorithm, stratification.

* A preliminary version of the paper has appeared in [22].
for the relevant probabilities. However, its complexity may be worse than exponential in the size of the formula.

The characteristic construct of CSL is a probabilistic formula of the form $P_{<p}(\varphi)$, where $p \in [0, 1]$. Here $\varphi$ is a path formula; more concretely, it is a multiple until formula $f_1 U I_1 \wedge f_2 U I_2 \wedge \ldots \wedge U I_{k-1} \wedge f_k$ where $k \geq 2$. The formula $P_{<p}(\varphi)$ expresses a constraint on the probability to reach an $f_k$-state by passing only through (zero or more) $f_1$, $f_2$, ..., $f_{k-1}$-states in the given order (together with a timing constraint indicated by the intervals $I_1, \ldots, I_{k-1}$). The key to solve the model checking problem is to approximate this probability $Pr_s(\varphi)$ closely enough to decide whether it is $< p$. The decision procedure in [2] first decomposes the formula into (up to) $(k - 1)^{k-1}$ many subformulas with suitable timing constraints. For each subformula, it then exploits properties of algebraic and transcendental numbers, but the corresponding algorithm is unfortunately impractical. In 2000, Baier et al. [4] [5] presented an approximate model checking algorithm for the case $k = 2$. This algorithm is based on transient probability analysis for CTMCs. More precisely, it was shown that $Pr_s(\varphi)$ can be approximated, up to an a priori given precision $\varepsilon$, by a sum of transient probabilities in the CTMCs. Their algorithm then led to further development of approximation algorithms for infinite CTMCs [11] [12] and abstraction techniques [15].

Effective model checking of full CSL with multiple until formulas ($k > 2$) is an open problem. This problem is gaining importance e.g. in the field of system biology, where one is interested in oscillatory behavior of CTMCs [6] [19]. More precisely, if one intends to quantify the probability mass oscillating between high, medium and low concentrations (or numbers) of some species, a formula like $P_{>0.2}(\text{high} U I_1 \wedge \text{medium} U I_2 \wedge \text{low} U I_3 \wedge \text{medium} U I_4 \wedge \text{high})$ is needed, but this is not at hand with the current state of the art. In CTL, multiple until formulas like $\forall(\text{high} U \text{medium} U \text{low} U \text{medium} U \text{high})$ do not increase expressivity because they are equivalent to something like $\forall(\text{high} \cup (\text{medium} U \forall(\ldots U \text{high})))$.

In this paper we propose an approximate algorithm for checking CSL with multiple until formulas. We introduce a subclass of stratified CTMCs, on which the approximation of $Pr_s(\varphi)$ can be obtained by efficient transient analysis. Briefly, a CTMC is stratified with respect to $\varphi = f_1 U I_1 \wedge f_2 U I_2 \wedge \ldots \wedge f_k$, if the transitions of the CTMC respect the order given by the $f_i$. This specific order makes it possible to express $Pr_s(\varphi)$ recursively: more precisely, it is the product of a transient vector and $Pr_{\varphi'}(\varphi')$, where $\varphi'$ is a kind of suffix subformula of $\varphi$. Stratified CTMCs are the key element for our analysis: in a stratified CTMC, the problem reduces to a transient analysis, for which efficient implementations using uniformization [10] exist. Thus, we extend the well-known result [5] for the case of binary until to multiple until formulas.

For a general CTMC, we present a measure-preserving transformation to a stratified CTMC. Our reduction is described using a deterministic finite automaton (DFA) over the alphabet $2\{f_1, \ldots, f_k\}$. The DFA accepts the finite word $w = w_1w_2 \ldots w_n$ if and only if the corresponding set of time-abstract paths in the CTMC contributes to $Pr_s(\varphi)$, i.e., it respects the order of the $f_i$. The transformation does not require to construct the full DFA, but only the product of the CTMC and the DFA. We show that the product is a stratified CTMC, and moreover, the measure $Pr_s(\varphi)$ is preserved. This product can be constructed in linear time and space in the size of the CTMC and $k$. Thus our method will be useful as the centerpiece of a full CSL model checker equipped with multiple until formulas.
Recently, the decision algorithm by Aziz et al. was shown to produce erroneous results on some non-stratified CTMCs \cite{13}. Still, their algorithm is correct on stratified CTMCs. As an additional contribution, our measure-preservation theorem ensures the decidability of CSL model checking for general CTMCs.

Overview of the article. Section 2 sets the ground for the paper. In Section 3 we introduce stratified CTMCs formally. The first main result is shown in Section 4: it constructs a DFA for an until formula, and then shows that the product is a stratified CTMC and the relevant measures are preserved. Section 5 discusses the computations in the product CTMC. A model checking algorithm is presented in Section 6. Section 7 discusses related work, and the paper is concluded in Section 8.

2. Preliminaries

This section presents the definition of Markov chains, probability space, transient and steady-state distributions. For details please refer to \cite{20,18,5}.

2.1. Markov Chains.

Definition 2.1. A labeled discrete-time Markov chain (DTMC) is a tuple $D = (S, P, L)$, where $S$ is a finite set of states, $P : S \times S \rightarrow [0,1]$ is a probability matrix satisfying $\sum_{s' \in S} P(s, s') \in \{0,1\}$ for all $s \in S$, and $L : S \rightarrow 2^{AP}$ is a labeling function.

A labeled continuous-time Markov chain (CTMC) is a tuple $C = (S, R, L)$, where $S$ and $L$ are defined as for DTMCs, and $R : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is a rate matrix.

For $A \subseteq S$, define $R(s, A) := \sum_{s' \in A} R(s, s')$, and let $E(s) := R(s, S)$ denote the exit rate of $s$. A state $s$ is called absorbing if $E(s) = 0$. If $R(s, s') > 0$, we say that there is a transition from $s$ to $s'$.

The transition probabilities in a CTMC are exponentially distributed over time. If $s$ is the current state of the CTMC, the probability that some transition will be triggered within time $t$ is $1 - e^{-E(s)t}$. Furthermore, if $R(s, s') > 0$ for more than one state $s'$, the probability to take a particular transition to $s'$ is $\frac{R(s, s')}{E(s)} (1 - e^{-E(s)t})$. The labeling function $L$ assigns to each state $s$ the set of atomic propositions $L(s) \subseteq AP$ which are valid in $s$.

A CTMC $C$ (and also a DTMC) is usually equipped with an initial state $s_{\text{init}} \in S$ or, more generally, an initial distribution $\alpha_{\text{init}} : S \rightarrow [0,1]$ satisfying $\sum_{s \in S} \alpha_{\text{init}}(s) = 1$.

Paths and probabilistic measures. A (sample) path is a right-continuous function $\sigma : \mathbb{R}_{\geq 0} \rightarrow S$ (with the discrete topology on $S$). Then, $\sigma(t)$ denotes the state occupied at time $t$.

For $i \in \mathbb{N}$, let $\sigma_S[i] = s_i$ denote the $(i+1)$-th state visited, and $\sigma_T[i] = t_i$ denote the time spent in $\sigma_S[i]$. For finite paths, $\sigma_T[n]$ is defined to be $\infty$ if $\sigma_S[n]$ is the last (absorbing) state. Let $\text{Path}^C$ denote the set of all (finite and infinite) paths, and $\text{Path}^C(s)$ denote the subset of those paths starting from $s$.

We sometimes use a different notation to describe a path, namely a finite sequence $\sigma = s_0t_0s_1t_1 \ldots s_n$ (meaning that $\sigma_S[i] = s_i$ and $\sigma_T[i] = t_i$ for all $i < n$, and $\sigma_S[n] = s_n$ is an absorbing state), or an infinite sequence $\sigma = s_0t_0s_1t_1 \ldots$ if no absorbing state is hit. The relation between the two notations is: $\sigma(t) = s_i$ where $i$ is the smallest index
with \( t < \sum_{j=0}^{i} t_j \) (as remarked by [23], p. 170), we have to use a strict inequality here for technical reasons, not the non-strict inequality as in [5]).

Let \( s_0, s_1, \ldots, s_k \) be states in \( S \) with \( R(s_i, s_{i+1}) > 0 \) for all \( 0 \leq i < k \). Let \( I_0, I_1, \ldots, I_{k-1} \) be nonempty intervals in \( \mathbb{R}_{\geq 0} \). The cylinder set \( \text{Cyl}(s_0, I_0, \ldots, s_{k-1}, I_{k-1}, s_k) \) is defined by:
\[
\text{Cyl}(s_0, I_0, \ldots, s_{k-1}, I_{k-1}, s_k) := \{ \sigma \in \text{Path}^C | \forall 0 \leq i \leq k. \sigma_S[i] = s_i \wedge \forall 0 \leq i < k. \sigma_T[i] \in I_i \}.
\]

Let \( \mathcal{F}(\text{Path}^C) \) denote the smallest \( \sigma \)-algebra on \( \text{Path}^C \) containing all cylinder sets. For initial distribution \( \alpha : S \to [0,1] \), a probability measure (denoted \( \text{Pr}^C_\alpha \)) on this \( \sigma \)-algebra is introduced as follows: \( \text{Pr}^C_\alpha(C) \) is the unique measure that satisfies: \( \text{Pr}^C_\alpha(\text{Cyl}(s)) = \alpha(s) \), and for \( k > 0 \),
\[
\text{Pr}^C_\alpha(\text{Cyl}(s_0, I_0, \ldots, I_{k-1}, s_k)) = \text{Pr}^C_\alpha(\text{Cyl}(s_0, I_0, \ldots, I_{k-2}, s_{k-1})) \cdot \frac{R(s_{k-1}, s_k)}{E(s_{k-1})} \cdot \eta(I_{k-1})
\]
where \( \eta(I_{k-1}) := \exp(-E(s_k-1) \inf I_{k-1}) - \exp(-E(s_k-1) \sup I_{k-1}) \) is the probability to take a transition during time interval \( I_{k-1} \). (As a consequence, the probability of a cylinder set containing a point interval \([t, t] \) is 0.) If \( \alpha(s) = 1 \) for some state \( s \in S \), we sometimes simply write \( \text{Pr}^C_\sigma \) instead of \( \text{Pr}^C_\alpha \). We omit the superscript \( C \) if it is clear from the context.

**Transient and steady-state probability.** Starting with distribution \( \alpha \), the transient probability vector at time \( t \), denoted by \( \pi(\alpha, t) \), is the probability distribution over states at time \( t \). If \( t = 0 \), we have \( \pi(\alpha, 0)(s') = \alpha(s') \). For \( t > 0 \), the transient probability is given by:
\[
\pi(\alpha, t) = \pi(\alpha, 0)e^{tQ} \quad \text{where} \quad Q := \mathbb{R} - \text{Diag}(E) \quad \text{is the infinitesimal generator matrix.} \quad \text{Diag}(E) \quad \text{denotes the diagonal matrix with} \quad \text{Diag}(E)(s,s) = E(s). \quad \text{The steady-state distribution is defined as the limit} \quad \lim_{t \to \infty} \pi(\alpha, t), \quad \text{which always exists for finite CTMCs.}
\]

### 2.2. Deterministic Finite Automata

**Definition 2.2.** A deterministic finite automaton is a tuple \( \mathcal{B} = (\Sigma, Q, q_{in}, \delta, F) \), where \( \Sigma \) is a finite alphabet, \( Q \) is a finite set of states, \( q_{in} \in Q \) is an initial state, \( \delta : Q \times \Sigma \to Q \) is a partial transition function, and \( F \subseteq Q \) is a set of final states.

We call a finite sequence \( w = w_1w_2 \ldots w_n \) over \( \Sigma \) a word over \( \Sigma \). \( w \) induces at most one path \( \sigma(w) = q_0q_1 \ldots q_n \) in \( \mathcal{B} \) where \( q_0 = q_{in} \) and \( q_i = \delta(q_{i-1}, w_i) \) for \( i = 1, \ldots, n \). This word \( w \), and also the corresponding path \( \sigma(w) \), is accepting if \( \sigma(w) \) exists and \( q_n \in F \).

### 2.3. Continuous Stochastic Logic (CSL)

We consider the branching-time temporal logic Continuous Stochastic Logic (CSL) introduced by Aziz et al. [2], which allows us to specify properties over CTMCs. Its syntax is defined as follows:

\[
\begin{align*}
\Phi & := a \mid \neg \Phi \mid \Phi \land \Phi \mid \mathcal{P}_{\leq p}(\varphi) \\
\varphi & := \Phi_1 U_{I_1} \Phi_2 U_{I_2} \ldots U_{I_{k-1}} \Phi_k
\end{align*}
\]

where \( a \in \text{AP} \) is an atomic proposition, \( I_1, I_2, \ldots \subseteq \mathbb{R}_{\geq 0} \) are nonempty left-closed intervals with rational bounds, \( \leq \in \{<, \leq, \geq, >\} \), \( p \in \mathbb{Q} \cap [0,1] \), and \( k \geq 2 \). We use the abbreviation \( \Diamond a \Phi = (a^{1} \land a^{0}) \) \( U_{1} \Phi \), for an arbitrary atomic proposition \( a \). The syntax of CSL consists of state formulas and path formulas: we use \( \Phi, \Phi_1, \Psi, \Psi_1, \ldots \) for state formulas and \( \varphi, \varphi_1, \psi, \psi_1, \ldots \) for path formulas.
Let $C = (S, R, L)$ be a CTMC with $s \in S$. The semantics of most CSL state formulas is standard: $s \models a$ iff $a \in L(s)$; $s \models \neg \Phi$ iff $s \not\models \Phi$; $s \models \Phi \land \Psi$ iff $s \models \Phi$ and $s \models \Psi$. For probabilistic formulas, we have:

$$s \models P_{\leq p}(\varphi) \text{ iff } \Pr_s\{\sigma \in \text{Path} \mid \sigma \models \varphi\} \leq p$$

where $\Pr_s\{\sigma \in \text{Path} \mid \sigma \models \varphi\}$, or $\Pr_s(\varphi)$ for short, denotes the probability measure of the set of all paths which start with $s$ and satisfy $\varphi$.

The satisfaction relation for CSL path formulas is defined as follows: let $\sigma$ be a path, and let $\varphi = \Phi_1 U_{i_1} \Phi_2 U_{i_2} \cdots \Phi_k$ be a path formula. Then $\sigma \models \varphi$ if and only if there exist real numbers $0 \leq t_1 \leq t_2 \leq \ldots \leq t_{k-1}$ such that $\sigma(t_{k-1}) \models \Phi_k$, and for each integer $0 < i < k$ we have $(t_i \in I_i) \land (\forall t' \in [t_{i-1}, t_i))(\sigma(t') \models \Phi_i)$, where $t_0$ is defined to be $0$ for notational convenience.

For a CSL path formula $\varphi = \Phi_1 U_{[a_1, b_1]} \Phi_2 U_{[a_2, b_2]} \Phi_3$ with $a_2 < a_1$, one can replace the second interval by $[a_1, b_2)$ without changing the set of paths that satisfy the formula. Thus, we shall assume that the left endpoints – and similarly, the right endpoints – of the intervals in multiple until formulas are always nondecreasing.

### 3. Stratified CTMCs

The main challenge of model checking is the computation and the approximation of the probability $\Pr_s(\varphi)$. We now introduce the class of stratified CTMCs. This is the key for the computation of $\Pr_s(\varphi)$. For now, the path formula $\varphi$ contains pairwise different atomic propositions as subformulas. In Section 6.1, we shall see that this definition is easily generalized to formulas containing more complex subformulas.

Let $C = (S, R, L)$ be a CTMC. Let $\varphi = f_1 U_{i_1} f_2 U_{i_2} \cdots f_k$ be a CSL path formula with pairwise different atomic propositions. Moreover, we let $F := \{f_1, f_2, \ldots, f_k\}$, and $\sqsubseteq$ be an order on $F$ such that $f_i \sqsubseteq f_j$ iff $i \leq j$. For a state $s$, if the set $L(s) \cap F$ is not empty, we let $f_{\min}^s := \min_{f \in L(s) \cap F} f$ with respect to the order $\sqsubseteq$. If such $f_j$ does not exist, we define $f_{\min}^s := \bot$.

**Definition 3.1** (Stratified CTMC). We say that $C$ is stratified with respect to $\varphi$ iff for all $s_1, s_2$, it holds that:

- If $f_{\min}^{s_1} = \bot$ or $f_{\min}^{s_1} = f_k$, then $R(s_1, s_2) = 0$.
- Otherwise (i.e., $f_{\min}^{s_1} \neq \bot$ and $f_{\min}^{s_1} \neq f_k$), if $R(s_1, s_2) > 0$ and $f_{\min}^{s_2} \neq \bot$, then $f_{\min}^{s_1} \sqsubseteq f_{\min}^{s_2}$.

A state $s$ with $f_{\min} = \bot$ is a bad state, and a state with $f_{\min} = f_k$ is a good state. (Note that there may be other states satisfying $f_k$ as well.) Both good and bad states are absorbing. The intuition behind Def. 3.1 is that paths reaching bad states will not satisfy $\varphi$, while those reaching good states or other $f_k$-states may satisfy $\varphi$ (provided the timing constraints are also satisfied).

**Example 3.2.** Consider the path formula $\varphi := f_1 U_{[0, 2]} f_2 U_{[2, 4]} f_3 U_{[2, 4]} f_4 U_{[3, 5]} f_5$. The CTMC in Fig. I is not stratified with respect to $\varphi$: we have $R(s_2, s_1) > 0$, however, $f_{\min}^{s_2} = f_4 \not\sqsubseteq f_1 = f_{\min}^{s_1}$. Deleting this edge and the transition out of $s_4$ would result in a stratified CTMC with respect to $\varphi$. □
The notion of stratified CTMCs is the key to an efficient approximation algorithm. The essential idea is that we can reduce the model checking problem to one on a similar, stratified CTMC that preserves the relevant reachability probabilities. Further, our notion of stratified CTMCs solves a semantical problem in [2]: please refer to Section 6.2 for details.

4. Product CTMC

Given a CTMC and a CSL path formula $\varphi$, in this section we construct a stratified CTMC with respect to $\varphi$ preserving the probability to satisfy $\varphi$. We first construct a deterministic finite automaton for $\varphi$ in Subsection 4.1. Then, in Subsection 4.2 we build a product CTMC with the desired property.

4.1. Automaton for a CSL Formula.

For a path formula $\varphi = f_1 U f_2 U f_3 \ldots U f_k$, we first construct a simple deterministic finite automaton (DFA) that describes the required order of $f_1$, $f_2$, ..., $f_k$-states.

**Definition 4.1 (Formula automaton).** Let $\varphi = f_1 U f_2 U f_3 \ldots U f_k$ be a CSL path formula with pairwise different atomic propositions. Then, the *formula automaton* $B_\varphi = (\Sigma, Q, q_{in}, \delta, F)$ is defined by: $\Sigma = 2^{\{f_1, \ldots, f_k\}}$, $Q = \{q_1, q_2, \ldots, q_{k-1}, q_k, \perp\}$ with $q_{in} = q_1$ and $F = \{q_1, \ldots, q_k\}$. For $a \in \Sigma$, the transition relation $\delta$ is defined as follows:

1. $\delta(q, a) = q_j$ if $i < k$; $i \leq j$; $f_i, f_{i+1}, \ldots f_{j-1} \not\in a$; and $f_j \in a$;
2. $\delta(q, a) = \perp$ if $i < k$ and the above clause does not apply;
3. $\perp$ and $q_k$ are absorbing.

As states $\perp$ and $q_k$ have no outgoing transitions, $\delta$ is a partial transition function. Thus formula automata are actually partial DFAs. The words accepted by $B_\varphi$ are finite traces $w \in \Sigma^*$ that can be extended to a trace $ww' \in \Sigma^*$ that satisfies the time-abstract formula of the form $f_1 U f_2 U f_3 \ldots U f_k$. The constructed finite automaton $B_\varphi$ for this special class of formulas is deterministic, the number of states is linear in $k$. The number of transitions is $(k-1)2^k$; however, as we will see later, the product can be constructed in time (and size) linear in the size of the CTMC and in $k$.

**Example 4.2.** In Fig. 2 the formula automaton for $k = 4$ is illustrated. The initial state is $q_1$, final states are marked with a double circle. The transition labels indicate which subsets of $AP$ are acceptable. For example, we have $\delta(q_1, \{f_1\}) = \delta(q_1, \{f_1, f_2\}) = q_1$, as both sets satisfy $f_1$. 

![Figure 1: A non-stratified CTMC.](image.png)
4.2. Product CTMC.

**Definition 4.3** (Product CTMC). Let \( \mathcal{C} = (S, R, L) \) be a CTMC and \( \varphi = f_1 \cup f_2 \cup f_3 \cup f_4 \) a path formula with pairwise different atomic propositions. Let \( B_\varphi \) be as constructed above. The product \( \mathcal{C} \times B_\varphi \) is a CTMC \((S', R', L')\) where:

1. \( S' = S \times Q \),
2. \( R'(s, q_i, (s', q'_{i})) = R(s, s') \) if \( s \models_C f_i \lor f_{i+1} \lor \cdots \lor f_{k-1} \) and \( q' = \delta(q_i, L(s')) \cap \{f_1, \ldots, f_k\} \), and equals 0 otherwise,
3. The labeling function is defined by:
   - \( L'(s, q_i) = L(s) \cap \{f_i, f_{i+1}, \ldots, f_k\} \) for \( 1 \leq i \leq k \),
   - \( L'(s, \bot) = \emptyset \).
4. Given an initial distribution \( \alpha : S \to [0, 1] \) of \( \mathcal{C} \), the initial distribution of the product \( \alpha' : S \times Q \to [0, 1] \) is defined by: \( \alpha'(s, q) \) equals \( \alpha(s) \) if \( q = \delta(q_{in}, L(s)) \cap \{f_1, \ldots, f_k\} \), and equals 0 otherwise.

The product CTMC contains two kinds of absorbing states. In general, states \((s, q)\) with \( s \not\models \bigwedge_{i=1}^k f_i \) are absorbing in the product, as well as states reached through a transition that does not follow the prescribed order of \( f_i \). These two kinds of states can be considered bad states. On the other hand, good states of the form \((s, q_k)\) with \( s \models f_k \) are also absorbing. The behavior after such an absorbing state is irrelevant for the probability to satisfy \( \varphi \).

**Example 4.4.** Consider the CTMC in Fig. 1 and consider the path formula \( \varphi_1 := f_1 \cup f_2 \cup f_3 \cup f_4 \). The path \( \sigma_1 := s_0s_1s_3s_2s_4 \ldots \) does, if \( s_4 \) is reached before time 2, satisfy \( \varphi_1 \); however, the path \( \sigma_2 := s_0s_1s_2s_1s_3s_2s_4 \ldots \) does not. The product of this CTMC with \( B_{\varphi_1} \) is the CTMC depicted on the left of Fig. 3 which is stratified with respect to \( \varphi_1 \). State \((s_4, q_5)\) is a good state – paths reaching this state before time 2 correspond to paths satisfying \( \varphi_1 \) in Fig. 1, while \((s_3, \bot)\) is a bad state.

For the same CTMC in Fig. 1, consider the path formula \( \varphi_2 := f_1 \cup f_2 \cup f_3 \cup f_4 \). The product CTMC \( \mathcal{C} \times B_{\varphi_2} \) is depicted on the right of Fig. 3. This product is stratified with respect to \( \varphi_2 \). The absorbing state \((s_2, q_4)\) is a good state.

\[\square\]
For a CTMC $C = (S, R, L)$ and a state $s \in S$, we use $C|_s = (S', R', L')$ to denote the sub-CTMC reachable from $s$, i.e., $S' \subseteq S$ is the states reachable from $s$, $R'$ and $L'$ are functions restricted to $S' \times S'$ and $S'$, respectively.

**Theorem 4.5** (Measure-preservation theorem). Let $C = (S, R, L)$ be a CTMC and $\varphi = f_1 U_1 f_2 U_2 \ldots f_k$ a path formula. Let $B_\varphi$ denote the formula automaton. For $s \in S$, let $s_B = (s, \delta(q_{in}, L(s)) \cap \{f_1, \ldots, f_k\})$. Then:

1. $C \times B_\varphi|_{s_B}$ is stratified with respect to $\varphi$;
2. $Pr^C(\varphi) = Pr^C_{\times B_\varphi}(\varphi) = Pr^C_{\times B_\varphi}(\varphi)$.

*Proof.* We prove first that $C \times B_\varphi|_{s_B}$ is stratified with respect to $\varphi$. Consider a state $(s, q)$. By definition of the product CTMC, if $(s, q) \not\models_{C \times B_\varphi} \bigvee_{i=1}^{k-1} f_i$, then $s \not\models_C \bigvee_{i=1}^{k-1} f_i$ or $q \in \{q_k, \bot\}$, so state $(s, q)$ is absorbing and therefore trivially satisfies the stratification conditions. Now assume that $(s, q) \models_{C \times B_\varphi} \bigvee_{i=1}^{k-1} f_i$, $q \not\in \{q_k, \bot\}$, and moreover assume $(s', q')$ is a state with $R'((s, q), (s', q')) > 0$ (with $R'$ as in Def. 4.3). By the definition of the transitions of $B_\varphi$, we have $q' = \delta(q, L(s')) \cap \{f_1, \ldots, f_k\}$. Now assume $f_{\min}^{(s', q')} \neq \bot$: it remains to be shown that $f_{\min}^{(s, q)} \subseteq f_{\min}^{(s', q')}$. Let $1 \leq x \leq k$ such that $f_x = f_{\min}^{(s, q)}$, and let $1 \leq y \leq k$ be such that $q = q_y$. The indices $x'$ and $y'$ are defined similarly for $(s', q')$. By definition of transitions of $B_\varphi$ and product CTMC, it is routine to verify that $x = y$ and $x' = y'$. Moreover, in $B_\varphi$, $q' = \delta(q, L(s')) \cap \{f_1, \ldots, f_k\}$ implies that $y' \geq y$, which shows that $x \leq x'$, proving $f_{\min}^{(s, q)} \subseteq f_{\min}^{(s', q')}$. Now we prove the second clause. Obviously, states not reachable from $s_B$ can be safely removed, thus $Pr^C_{\times B_\varphi}(\varphi) = Pr^C_{\times B_\varphi}(\varphi)$. We next prove that $Pr^C_\varphi(\varphi) = Pr^C_{\times B_\varphi}(\varphi)$ by showing that $\sigma \mapsto \sigma_B$ (the canonical mapping from paths in $C$ to paths in $C \times B_\varphi$) preserves the standard probability measures between the probability spaces. To this end, it is enough to show that given a cylinder set $C_B$ over $C \times B_\varphi$, its reverse image $C = \{\sigma|_{\sigma_B} \in C_B\}$ satisfies $Pr^C_\varphi(C) = Pr^C_{\times B_\varphi}(C_B)$.

Stated briefly, we now show that paths in $C$ and in $C \times B_\varphi$ correspond to each other because we only add some (bounded) information about the past to the states.

Let us first describe the canonical mapping $\sigma \mapsto \sigma_B$. Assume given a path $\sigma = s_0 q s_1 t_1 \ldots$ in $C$. The corresponding path in $C \times B_\varphi$ is $\sigma_B = (s_0, q^0)t_1(s_1, q^1)t_1 \ldots$, where $q^0 = \delta(q_{in}, L(s_0) \cap \{f_1, \ldots, f_k\})$ and $q^{i+1} = \delta(q^i, L(s_{i+1}) \cap \{f_1, \ldots, f_k\})$ for all $i \geq 1$, as long as the $(s_i, q^i)$ are not absorbing. However, if $(s_n, q^n)$ is absorbing for some $n$, then $\sigma_B$ is defined to be the finite path $(s_0, q^0)t_0(s_1, q^1)t_1 \ldots (s_n, q^n)$, where $(s_n, q^n)$ is the first absorbing state encountered. Note that $\sigma \models_{\varphi}$ if $\sigma_B \models_{C \times B_\varphi} \varphi$.\]
Let \( C_B = \text{Cyl}((s_0, q^0), I_0, (s_1, q^1), \ldots, (s_n, q^n)) \) and \( C \) be as above. By definition of a cylinder set, \( R_i'((s_i, q^i), (s_{i+1}, q^{i+1})) > 0 \) for all \( i < n \), therefore \( (s_i, q^i) \) is not absorbing (for \( i < n \)) and \( q^{i+1} = \delta(q^i, L(s_{i+1}) \cap \{f_1, \ldots, f_k\}) \). Now assume that some path \( \sigma = s_0' t_0 s'_1 t_1 \ldots \in C \); then it must hold that \( s_0' = s_0 \), \( t_0 \in I_0 \), \( s'_1 = s_1 \), \( t_1 \in I_1 \), \ldots, and \( s'_n = s_n \). Therefore, \( C \subseteq \text{Cyl}(s_0, I_0, s_1, I_1, \ldots, s_n) \). On the other hand, for all paths \( \sigma \in \text{Cyl}(s_0, I_0, s_1, I_1, \ldots, s_n) \), it is easy to prove that \( \sigma \in C_B \). So, \( C \supseteq \text{Cyl}(s_0, I_0, s_1, I_1, \ldots, s_n) \), and together, \( C = \text{Cyl}(s_0, I_0, s_1, I_1, \ldots, s_n) \). It is now an easy calculation to verify that \( \Pr_{C_\sigma}(C) = \Pr_{s_{\sigma}}^{C \times B_\sigma}(C_B) \).

The reverse image of the set of \( C \times B_\varphi \)-paths satisfying \( \varphi \) is exactly the set of \( C \)-paths satisfying \( \varphi \). Since these sets are measurable, both can be decomposed into countable unions of corresponding cylinder sets in \( C \) and \( C \times B_\varphi \), respectively. Thus, the theorem follows. \( \square \)

5. Characterizing the Probability \( \Pr_\alpha(\varphi) \)

For a path formula \( \varphi \), together with a stratified CTMC with respect to \( \varphi \), this section aims at a recursive characterization of the probability \( \Pr_\alpha(\varphi) \) starting from an arbitrary initial distribution \( \alpha \).

We first introduce some notation. For an interval \( I \) and \( 0 \leq x \), we let \( I \cap x \) denote the set \( \{t - x \mid t \in I \wedge t \geq x\} \). For example, \([3, 8) \cap 5 = [0, 3)\). Then, for \( \varphi = f_1 U I_1 f_2 U I_2 \ldots f_k \) and \( x < \sup I_1 \), we let \( \varphi \cap x \) denote the formula \( f_1 U I_1 \cap x f_2 U I_2 \cap x \ldots f_k \). For \( 1 \leq j' \leq k \), define \( f_{j'-1} := \cap_{i=j}^{j' - 1} f_i \); for \( 1 \leq j < k \), define \( f_j := f_j U I_j f_{j+1} U I_{j+1} \ldots f_k \). As a degenerate case of \( \Pr_\varphi(\varphi_j) \), let \( \Pr_\varphi(f_k) := 1 \) if \( s \models f_k \) and \( 0 \) otherwise. For \( \Phi \), we denote by \( C[\Phi] \) the CTMC obtained by \( C \) by making states satisfying \( \Phi \) absorbing – by cutting transitions out of all states satisfying \( \Phi \). Moreover, let \( I_\Phi \) denote the indicator matrix defined by: \( I_\Phi(s, s) = 1 \) if \( s \models \Phi \); and \( I_\Phi(s, s') = 0 \) otherwise.

5.1. Left-Closed Intervals. For the moment, we restrict our attention to until formulas where all timing constraints have the form \( I_i = [a_i, b_i] \). The following theorem characterizes the probability for this case:

**Theorem 5.1.** Let \( \varphi = f_1 U I_1 f_2 U I_2 \ldots f_k \) be a CSL path formula with pairwise different atomic propositions, and assume all \( I_i = [a_i, b_i] \) are left-closed. Let \( C = (S, R, L) \) be a stratified CTMC with respect to \( \varphi \). We write the vector \( (\Pr^C_\alpha(\psi))_{s \in S} \) as \( \Pr^C_{C_\psi}(\psi) \).

1. Assume \( 0 < a_1 \). Then,

\[
\Pr^C_{\alpha}(\varphi) = \pi^{[-f_1]}(\alpha, a_1) \cdot I_{f_1} \cdot \Pr^C_{C_\bullet}(\varphi \cap a_1)
\]

where \( \pi^{[-f_1]}(\alpha, a_1) \) is the transient distribution at time \( a_1 \) in the CTMC \( C[-f_1] \).

2. Assume \( 0 = a_1 = \ldots = a_{j-1} < a_j < b_1 \) for some \( j \in \{2, \ldots, k-1\} \). Then,

\[
\Pr^C_{\alpha}(\varphi) = \pi^{[-f_{j-1}]}(\alpha, a_j) \cdot I_{f_{j-1}} \cdot \Pr^C_{C_\bullet}(\varphi \cap a_j)
\]

3. Assume \( 0 = a_1 = \ldots = a_{j-1} < b_1 < a_j \) for some \( j \in \{2, \ldots, k-1\} \). Let \( j' \leq j \) be the largest integer such that \( b_1 \not\in I_{j'-1} \). Then,

\[
\Pr^C_{\alpha}(\varphi) = \pi^{[-f_{j-1}]}(\alpha, b_1) \cdot I_{f_{j'...j-1}} \cdot \Pr^C_{C_\bullet}(\varphi \cap a_j)
\]
(4) Assume \( 0 = a_1 = \ldots = a_{k-1} \). Let \( j' \leq k \) be the largest integer such that \( b_1 \notin I_{j'-1} \). Then,
\[
\Pr^C_s(\varphi) = \pi^{C[f_1]}(\alpha, b_1) \cdot I_{j'\ldots k} \cdot \Pr^C[\neg f_{j'\ldots k-1}](\varphi'_{j'} \oplus b_1)
\]
\( (5.4) \)

If \( b_1 = \infty \), we replace \( \pi^{C[f_1]}(\alpha, b_1) \) in this equation by the corresponding steady-state distribution.

The key idea of the theorem is a property-driven transient analysis. In the first clause we have \( a_1 > 0 \), thus for any path \( \sigma \) satisfying \( \varphi \) it must hold \( \sigma(t) \models f_1 \) for all \( t \in [0, a_1) \). Thus, we make all states satisfying \( \neg f_1 \) absorbing, and compute the transient distribution \( \pi^{C[\neg f_1]}(\alpha, a_1) \). Furthermore, the multiplication with the matrix \( I_{f_1} \) removes the probabilities in states satisfying \( \neg f_1 \) – thus resulting in a subdistribution. Starting with this subdistribution, the formula will also be reduced by duration \( a_1 \). In the other clauses, we consider the interval \( [0, a_j) \) or \( [0, b_1) \), which is the common prefix of the intervals \( I_1, \ldots, I_{j-1} \). Thus during this time the formula \( f_{1\ldots j} \) must be satisfied. Here the assumption of stratification is crucial: otherwise one might be able jump forward and back between states satisfying \( f_1 \) and \( f_j \), which is illustrated in the following example.

![Figure 4: A CTMC with \( \Pr_{s_0}(f_1 U_{[0,1]} f_2 U_{[0,1]} f_3) = 0 \).](image)

Example 5.2. Consider the CTMC depicted in Fig. 4 and consider the path formula \( \varphi = f_1 U_{[0,1]} f_2 U_{[0,1]} f_3 \). Obviously the probability of the set of paths starting from \( s_0 \) satisfying \( \varphi \) is 0. Since the CTMC is not stratified with respect to \( \varphi \), Thm. 5.1 cannot be applied directly: the product shall be constructed first. In the product CTMC, no states labelled with \( f_3 \) will be reached, thus giving the probability 0, as desired.

Proof of Thm. 5.1. We start with Eqn. 5.1. Let \( a_1 \) and the other notation be as in the theorem. For \( s' \in S \), define the event \( Z_{s'}(s) := \{ \sigma \mid \sigma(a_1) = s' \land \forall t \in [0, a_1) \sigma(t) \models f_1 \} \), consisting of paths which occupy state \( s' \) at time \( a_1 \) and occupy \( f_1 \)-states during the time interval \( [0, a_1) \). Obviously, \( \{ \sigma \mid \sigma \models \varphi \} \subseteq \bigcup_{s' \models f_1} Z_{s'}(s') \). Fix first \( \alpha_s \) as an initial distribution with \( \alpha_s(s) = 1 \) and \( s \models f_1 \). By the law of total probability, we have:
\[
\Pr^C_s(\varphi) = \sum_{s' \models f_1} \Pr^C_s(Z_{s'}(s')) \cdot \Pr^C_s(\varphi \mid Z_{s'}(s'))
\]
\[
= \sum_{s' \models f_1} \pi^{C[\neg f_1]}(s, a_1)(s') \cdot \Pr^C_s(\varphi \mid Z_{s'}(s'))
\]

Strictly speaking, this does not hold always because there may be paths that enter an \( (f_2 \land \neg f_1) \)-state exactly at time \( a_1 \); however, such paths are contained in a (generalization of) cylinder sets like \( \text{Cyl}(s, [a_1, a_1]) \), whose measure is 0.
The latter equality follows from the definition of $Z_3(s')$. By the Markov property of CTMCs:

$$
= \sum_{s'\in S} \pi^{C[-f_1]}(s, a_1)(s') \cdot Pr_s^C(\varphi \land a_1)
$$

$$
= \sum_{s'\in S} \pi^{C[-f_1]}(s, a_1)(s') \cdot 1_{s'\in f_1} \cdot Pr_s^C(\varphi \land a_1)
$$

where $1_{s'\in f_1}$ is 1 if $s' \models f_1$ and 0 otherwise. Note that $Pr_s^C(\varphi) = \sum_{s'\in S} \alpha(s) Pr_s^C(\varphi) = \sum_{s'\in f_1} \alpha(s) Pr_s^C(\varphi)$, thus Eqn. (5.1) follows.

We now jump to the proof of Eqn. (5.3). This proof is more involved, but follows the same lines. Define the event $Z_3(s') := \{\sigma \mid \sigma(b_1) = s' \land \forall t \in [0, b_1), \sigma(t) \models f_{1...j}\}$. Again, $\{\sigma \mid \sigma \models \varphi\} \subseteq \cup_{s'\models f'_{j...j}} Z_3(s')$, and again, fix $\alpha_s$ as an initial distribution with $\alpha_s(s) = 1$ and $s \models f_{1...j}$. We have:

$$
Pr_s^C(\varphi) = \sum_{s'\models f'_{j...j}} Pr_s^C(Z_3(s')) \cdot Pr_s^C(\varphi \mid Z_3(s'))
$$

$$
= \sum_{s'\models f'_{j...j}} \pi^{C[-f_{1...j}]}(s, b_1)(s') \cdot Pr_s^C(\varphi \mid Z_3(s'))
$$

where the latter equality follows from the definition of $Z_3(s')$. Now let $s' \in Z_3(s')$, thus $\sigma(b_1) = s'$, and $\sigma(t) \models f_{1...j}$ for all $0 \leq t < b_1$. Let $\sigma'$ denote the suffix path defined by $\sigma'(x) := \sigma(x + b_1)$.

Now, $\sigma \models \varphi$ implies that at time $b_1$, $\sigma$ has reached a state in a stratum from $q_j', \ldots, q_j$, so $\sigma'$ satisfies $\varphi_j' \land b_1$. On the other hand, every path $\sigma \in Z_3(s')$ whose corresponding $\sigma'$ satisfies $\varphi_j' \land b_1$ also satisfies $\varphi$ (because $C$ is stratified). Again, $Pr_s^C(\varphi \mid Z_3(s')) = Pr_{s'}^C(\varphi_j' \land b_1)$, thus

$$
Pr_s^C(\varphi) = \sum_{s'\models f'_{j...j}} \pi^{C[-f_{1...j}]}(s, b_1)(s') \cdot Pr_s^C(\varphi_j' \land b_1)
$$

$$
= \sum_{s'\models f'_{j...j}} \pi^{C[-f_{1...j}]}(s, b_1)(s') \cdot 1_{s'\models f'_{j...j}} \cdot Pr_{s'}^C(\varphi_j' \land b_1)
$$

However, $C$ needs not be stratified w.r.t. $\varphi_j' \land b_1$, so to simplify the subsequent calculations, we restratify it: $C[-f'_{j...k-1}]$ is stratified w.r.t. $\varphi_j' \land b_1$. Eqn. (5.3) for general initial distribution $\alpha$ follows as in the case of Eqn. (5.1).

The proof for Eqn. (5.2) is similar to the proof for Eqn. (5.3), except that $b_1$ has to be replaced by $a_j$ and $j'$ by 1.

For Eqn. (5.4), we can again make a similar proof. First assume that $j' = k$. In that case, the paths that have reached an $f_k$-state at any time in the interval $I_{k-1} = I_1$ are exactly the paths that satisfy $\varphi$. They have the same probability as the paths in $C[f_k]$ that are in an $f_k$-state exactly at time $b_1$. Therefore,

$$
Pr_s^C(\varphi) = \sum_{s'\models f_k} \pi^{C[f_k]}(s, b_1)(s') = \sum_{s'\in S} \pi^{C[f_k]}(s, b_1)(s') \cdot 1_{s'\models f_k}
$$

(5.5)

With the usual assumption $f_{k-1} = false$, the theorem follows immediately.

If $j' < k$, besides the paths mentioned above, other paths satisfy $\varphi$, namely paths that reach an $f_k$-state during the interval $I_{k-1} \setminus I_1 = [b_1, b_{k-1})$ (and avoid $f_k$-states earlier).
These are the paths that satisfy \((f_1 \land \neg f_k) \ U_1 \ldots \ U_{k-2} \ (f_{k-1} \land \neg f_k) \ U_{k-1} \ \uparrow f_k\). Their probability is, according to Eqn. (5.3),
\[
\sum_{s' \in S} \pi^C[\neg (f_1 \land \neg f_k)](s, b_1)(s') \cdot I_{s' \models f_j' \ldots \neg f_k} \cdot \Pr^C_{s'}[\neg f_j' \ldots \neg f_k](\varphi_{j'} \ominus b_1)
\]
Note that \(C[\neg (f_1 \land \neg f_k)] = C[f_k]\). Adding this term to Eqn. (5.3) produces the desired probability.

We still have to prove Eqn. (5.4) for \(b_1 = \infty\). In that case, all timing constraints are trivial \(([a_i, b_i] = [0, \infty])\) and \(j' = k\). Therefore, \(\Pr^C_s(\varphi)\) is just the probability to reach an \(f_k\)-state eventually, which is exactly \(\lim_{b_1 \to \infty} \pi^C[f_k](s, b_1) I_{f_k} \Pr^C_{s'}[\neg f_j' \ldots \neg f_k](f_k)\).

\[\square\]

### 5.2. Closed Intervals

In Thm. 5.1, we have considered formula \(\varphi\) with left-closed intervals. Now we discuss that a slight generalization of it can be used to handle closed intervals. Thus, below we assume \(I_i = [a_i, b_i]\).

The proof of Thm. 5.1 can be extended easily to hold also for closed intervals. Clause 3 may lead to formulas containing degenerate intervals \([0, 0]\): As \(b_1 \in [a_1, b_1]\), often \(j' = 1\) in this clause. (We have to assume, as an additional simplification of notation, \(I_0 := \emptyset\).) As a consequence, \(\varphi' \ominus b_1 = f_1 U_{[0,0]} f_2 U_{I_2 \ominus b_1} \ldots f_k\).

Further, if the original \(\varphi\) already contained a degenerate interval, say \(a_1 = b_1\), so \(I_1 = \{a_1\}\), applying Clause 1 will also lead to a formula containing \([0, 0]\). These situations can be handled by the following lemma:

**Lemma 5.3.** Let \(\varphi = f_1 U_{I_1} f_2 U_{I_2} \ldots f_k\) be a CSL path formula. Let \(C = (S, R, L)\) be a stratified CTMC with respect to \(\varphi\). Moreover, assume \(I_1 = \ldots = I_{j-1} = [0, 0]\) for \(2 \leq j \leq k\). Then, \(\Pr^C_s(\varphi) = \Pr^C_s(f_j U_{I_j} \ldots f_k)\) for all \(s \in S\).

**Proof.** Assume a path \(\sigma\) satisfies \(\varphi\). The degenerate intervals force \(t_1 = t_2 = \ldots = t_{j-1} = 0\), thus no conditions relating to \(f_1, \ldots, f_{j-1}\) need to be checked. \[\square\]

![Figure 5: A CTMC with \(Pr_{s_0}(f_1 U_{[0,1]} f_2 U_{[1,2]} f_3) \neq Pr_{s_0}(f_1 U_{[0,1]} f_2 U_{[1,2]} f_3)\).](image)

**Example 5.4.** Consider the CTMC in Fig. 5 and the path formula \(\varphi = f_1 U_{[0,1]} f_2 U_{[1,2]} f_3\). Then, \(\Pr_{s_0}(\varphi)\) is the probability to stay in \(s_0\) for at least one time unit \((\psi(0, E(s_0) \cdot 1)\) in the notation of Section (6.3) below), since we can choose \(t_1 = t_2 = 1\) if \(\sigma_{T}[0] \geq 1\). Applying Clause 2 of Thm. 5.1, we get \(j = 2, j' = 1\) and \(\Pr_{s_0}(\varphi) = \pi^C[\neg f_{j} \ldots \neg f_{j-1}](s_0, 1) \cdot I_{f_{j-1}} \cdot \Pr^C_{s'}(\neg f_{j} \ldots \neg f_{j-1})(f_1 U_{[0,0]} f_2 U_{[0,1]} f_3) = \pi^C(s_0, 1) \cdot I \cdot \Pr^C_{s'}(f_2 U_{[0,1]} f_3) = (e^{-2}, 0) \cdot (1, 0)^T = e^{-2}\), the correct value. (In [22], we defined \(j'\) slightly differently, producing \(j' = 2\) and consequently \(\Pr_{s_0}(\varphi) = 0\). Our earlier definition worked only for left-closed intervals.) \[\square\]
Below we apply the theorem to two formulas, and thereby get the well-known result \[5\] for the case of binary until for the case \(k = 2\). As above, \(\mathcal{C}\) is stratified and \(\varphi = f_1 \, U \, I_1 \, f_2 \, U \, I_2 \, \ldots \, U \, f_k\).

(1) **Reachability probability.** Assume that \(I_1 = \ldots = I_{k-1} = [0, b]\). Then, it holds \(Pr_s(\varphi) = Pr_s(\Diamond_{[0, b]} f_k)\), which is the probability to reach an \(f_k\)-state within time \(b\).

(2) **Interval reachability.** Assume that \(I_1 = \ldots = I_{k-1} = [a, b]\) with \(a < b\). Then, it holds \(Pr_s(\varphi) = \sum_{s' \in \{f_1\}} \pi^{[f_1]}(s, a)(s') \cdot Pr_s(\Diamond_{[0, b-a]} f_k)\), which is the interval reachability probability of staying in \(f_1\)-states until time \(a\) and then moving to an \(f_k\)-state before time \(b\) has passed.

### 5.3. Other Intervals

First, the following lemma states properties of the probabilities for binary until with different interval types:

**Lemma 5.5** (Closure of Intervals for Binary Until). Let \(s \in S\). Assume given two nonempty intervals \(I, J\) such that \(\inf I = \inf J\) and \(\sup I = \sup J\). Then, it holds:

1. If \(0 \in I \Leftrightarrow 0 \notin J\), then \(s \models \mathcal{P}_{\leq p}(\Phi \, U \, J)\) iff \(s \models \mathcal{P}_{< p}(\Phi \, U \, J)\) for \(\leq \in \{<, \leq, \geq, >\}\).
2. Otherwise, assume w.l.o.g. \(0 \in I\) and \(0 \notin J\), and assume \(0 < p < 1\). Then, \(s \models \mathcal{P}_{\leq p}(\Phi \, U \, J)\) for \(\geq \in \{\geq, >\}\). Similarly, \(s \models \mathcal{P}_{< p}(\Phi \, U \, J)\) for \(\leq \in \{<, \leq\}\).

The lemma follows immediately from the definition of the measure of cylinder set. To see why we have to treat the case \(\inf I = 0\) separately (not distinguished in \([5]\)), assume that \(\Phi = \mathcal{P}_{\leq 0.1}(f_2 \, U \, [0,1] \, f_1)\) and consider the CTMC depicted in Fig. \([5]\) obviously we have \(s_0 \models \Phi\) as \(s_0 \not\models f_2\). However, \(s_0 \not\models \mathcal{P}_{\leq 0.1}(f_2 \, U \, [0,1] \, f_1)\) as \(s_0\) satisfies \(f_1\) directly. The formula \(\Phi\) is equivalent to \(\mathcal{P}_{\leq 0.1}(f_2 \, U \, [0,1] \, f_1) \lor \neg f_2\). For until formulas with arbitrary multiplicity, we have discussed the case that all of the intervals are left-closed or closed. Other cases can be handled in a way similar to Lemma \([5.5]\).

### 6. Model Checking Algorithm

Let \(\mathcal{C} = (S, R, L)\) be a CTMC, \(s \in S\), and \(\Phi\) be a CSL formula. The model checking problem is to check whether \(s \models \Phi\). In the following two sections, we discuss that the model checking problem is decidable and provide an efficient algorithm for approximate computation of \(Pr_s(\psi)\).

### 6.1. Model Checking CSL is Decidable

The standard algorithm to solve CTL-like model checking problems recursively computes the sets of states satisfying \(\Psi\), denoted by \(\text{Sat}(\Psi)\), for all state subformulas \(\Psi\) of \(\Phi\). For CSL, the cases where \(\Psi\) is an atomic proposition, a negation or a conjunction are given by: \(\text{Sat}(a) = \{s \in S \mid a \in L(s)\}\), \(\text{Sat}(\neg \Psi_1) = S \setminus \text{Sat}(\Psi_1)\) and \(\text{Sat}(\Psi_1 \land \Psi_2) = \text{Sat}(\Psi_1) \cap \text{Sat}(\Psi_2)\).

The case that \(\Psi\) is the probabilistic operator is the challenging part. Let \(\Psi = \mathcal{P}_{\leq p}(\varphi)\) with \(\varphi = \Psi_1 \, U \, I_1 \, \Psi_2 \, U \, I_2 \, \ldots \, U \, f_k\). By the semantics, checking \(\Psi\) is equivalent to checking whether \(Pr_s(\varphi)\) meets the bound \(\leq p\), i.e., whether \(Pr_s(\varphi) \leq p\). Assume that the sets \(\text{Sat}(\Psi_i)\) have been calculated recursively. We replace \(\Psi_1, \ldots, \Psi_k\) by fresh (pairwise different) atomic propositions \(f_1, \ldots, f_k\) and extend the label of state \(s\) by \(f_i\) if \(s \in \text{Sat}(\Psi_i)\). The so
obtained path formula is $\psi := f_1 U_{i_1} f_2 U_{i_2} \ldots f_k$, and obviously we have $\Pr_s(\varphi) = \Pr_s(\psi)$. The steps needed to characterize $\Pr_s(\psi)$ are:

(i) Construct the formula automaton $B_{\psi}$.
(ii) Build the product $C \times B_{\psi}$, which by Thm. 4.5 is a stratified CTMC w.r.t. $\psi$.
(iii) Apply Thm. 5.1 repeatedly to compute $\Pr_s(\psi)$.

Thus, the decidability for the probabilistic formula reduces to checking whether $\Pr_s(\psi) \leq p$ holds true in the product CTMC. After applying Thm. 5.1 a finite number of times, we see that $\Pr_s(\psi)$ reduces to a product of transient probabilities. We can now follow the argumentation in [2]: Although the calculations differ slightly, $\Pr_s(\psi)$ still is a finite sum $\sum_k \eta_k e^{\delta_k}$ (with algebraic $\eta_k$ and $\delta_k$). For such an expression, [2] proved that it can be decided whether it is $\leq p$, for $p \in \mathbb{Q}$. Thus, we still have:

**Theorem 6.1** ([2], Thm. 1). Model checking CSL is decidable.

6.2. **Usefulness of Stratification.** Our notion of stratified CTMCs solves a semantical problem in [2], which we recently pointed out in [13]. Very briefly, Aziz et al. [2] gave an algorithm that did not use the $t_i$ (in the semantics of until formulas) explicitly, which led to incorrect results for non-stratified CTMCs.

Consider the CTMC depicted in Fig. 4 and the formula $\varphi = f_1 U_{[0,1]} f_2 U_{[0,1]} f_3$ in Example 5.2. For this example, the algorithm in [2] calculates the probability that a path satisfies, a.o., the conditions: it stays in $f_1 \cup f_2$-states during time $[0,1)$, thus giving a wrong result. This problem does not occur provided that the CTMC is stratified.

6.3. **Efficient Algorithm for Approximating** $\Pr_s(\psi)$. We first explain how to combine steps (i) and (ii) mentioned above, without having to construct the full automaton $B_{\psi}$. Most parts of the construction of $C \times B_{\varphi}$ depend on $C$ only and do not require much information about $B_{\varphi}$. For example, for the state space, it is enough to generate $k$ copies of every state in $C$, which requires time $O(|S|k)$. When constructing the transitions according to Clause 2 of Def. 4.3 one has to check $q' = \delta(q_i, L(s')) \cap \{f_1, \ldots, f_k\}$, but even this can be done without actually constructing $B_{\varphi}$ by using the definition of $\delta$ (Def. 4.1) directly. Therefore, the overall time complexity to find all transitions of $C \times B_{\varphi}$ is $|R|$ times the number of copies that its source state may have, i.e., $O(|R|k)$, which is also the maximal total number of transitions.

The usual numerical algorithm to compute the matrix exponential $e^{Qt}$ is based on *uniformization* [20]. This algorithm executes most calculations on the uniformized DTMC. For a CTMC, we say that $\lambda$ is a uniformization rate if $\lambda \geq \max_{s \in S}(E(s) - R(s, s))$.

**Definition 6.2.** Let $C = (S, R, L)$ be a CTMC. The uniformized DTMC of $C$ with respect to the uniformization rate $\lambda$ is $uni(C) = (S, P, L)$ where $P(s, s') = R(s, s')/\lambda$ if $s \neq s'$ and $P(s, s) = 1 - P(s, S \setminus \{s\})$.

Let $P$ denote the transition matrix of the uniformized DTMC $uni(C)$, thus it holds that $P = I + Q/\lambda$ where $I$ denotes the identity matrix. For $t > 0$, then:

$$\pi(\alpha, t) = \pi(\alpha, 0)e^{(P-I)t} = \pi(\alpha, 0)e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} P^i = \sum_{i=0}^{\infty} \psi(i, \lambda t)\vec{v}(i) \quad (6.1)$$
In this formula, $\psi(i, \lambda t) = e^{-\lambda t} \cdot \frac{(\lambda t)^i}{i!}$ denotes the $i$-th Poisson probability with parameter $\lambda t$, i.e., the probability to see precisely $i$ transitions within time $t$. The vector $\vec{v}(i)$ is the transient probability of $\text{unit}(\mathcal{C})$ after $i$ transitions, i.e., $\vec{v}(i) = \pi(\alpha, 0) \mathbf{P}^i$. The infinite sum is approximated, by picking $O(\sqrt{\lambda t})$ terms with large $\psi(i, \lambda t)$, using the Fox–Glynn algorithm \cite{FoxGlynn}. To find the $\vec{v}(i)$ for Eqn. (6.1), one requires $O(\lambda t)$ matrix–vector multiplications \cite{Baier}. The following lemma states the complexity of our algorithm:

**Lemma 6.3 (Complexity).** Let $|R|$ denote the number of transitions of $\mathcal{C}$ and $\lambda \in \mathbb{R}_{>0}$ the uniformization rate satisfying $\lambda = \max_{s \in S}(E(s) - R(s, s))$. For each formula $\varphi = f_1 U I_1 f_2 U I_2 \ldots f_k$, the probability $Pr_{s_B}(\varphi)$ can be approximated:

- in time in $O(|R| k \cdot \lambda b)$ if $b = \sup I_{k-1}$ is finite,
- in time in $O(|R| k \cdot \lambda b + (|S|k)^3)$ if $\sup I_{k-1}$ is infinite, where $|S|$ is the number of states in $\mathcal{C}$ and $b = \max\{\inf I_{k-1}\} \cup \{\sup I_i|1 \leq i < k\} \setminus \{\infty\}$.

The space complexity is in $O(|R| k)$.

**Proof.** Recall that the formula automaton $\mathcal{B}_\varphi$ is deterministic, and the size of the product automaton is $O(|R| k)$ which is both linear in the size of the CTMC and the formula. This proves the space complexity.

For the time complexity assume first $b < \infty$ with $b = \sup I_{k-1}$. Applying Thm. 5.1 the probability $Pr_{s_B}(\varphi)$ can be expressed as a sequence of transient probability analyses, which can be efficiently approximated by a sequence of uniformization analyses. The complexity of these analyses is linear in the size of the product automaton, and also linear in $\lambda b$.

For the second case $\sup I_{k-1} = \infty$, by Thm. 5.1 a sequence of transient probability analyses is followed by one steady-state analysis, which can be done with Gaussian elimination for the equation systems $\pi \cdot \mathbf{Q}' = 0$ and $\sum_{s \in S'} \pi(s) = 1$, the complexity of which is $O((|S|k)^3)$. Thus the complexity for this case follows.

Thus, with the notion of stratified CTMC, we achieve polynomial complexity. Our algorithm therefore improves the work of \cite{BBS06}, where only multiple until formulas with suitable timing constraints can be checked polynomially. In the worst case, \cite{BBS06} has to decompose a CSL formula into $O((k - 1)^{k-1})$ formulas with suitable timing, thus resulting in an overall time complexity of $O(|R| \cdot \lambda b (k - 1)^k)$ or $O(|R| k \cdot \lambda b + (|S|k)^3) \cdot (k - 1)^{k-1}$, respectively.

### 7. Related Work

The logic CSL was first proposed in \cite{KPR94}, in which the model checking problem is shown to be decidable. Our paper gives a practical solution: it shows that the relevant probabilities can be approximated efficiently. For the case of binary until path formula, Baier et al. \cite{BBS06} have presented an approximate algorithm for the model checking problem. Their method can be considered a special case of our approach.

Baier et al. \cite{BBS06} defined a logic asCSL that uses so-called programs as path formulas, i.e. regular expressions over state formulas and actions. Programs can express multiple until formulas of the form $\varphi_1 U_{[0, b]} \varphi_2 U_{[0, b]} \cdots U_{[0, b]} \varphi_k$ because asCSL cannot restrict the duration of individual program phases. The model checking algorithm translates the program to an automaton almost equal to the one in Fig. 2. Our work generalizes the method to multiple until formulas with multiple time bounds.
More recently, Donatelli et al. [8] have extended CSL such that path properties can be expressed via a deterministic timed automata (DTA) with a single clock. Chen et al. [7] take this approach further and consider DTA specifications with multiple clocks as well.

In principle, one can translate a multiple until formula to a DTA with a single clock. Its basic structure would look similar to Fig. 2, but Donatelli’s and Chen’s DTAs also include all timing information and would have a size in $O(k^2)$ – an example construction with $k = 4$ is given in Appendix A. To check whether a CTMC satisfies a DTA specification, they build the product of the two, apply the region construction, and then solve a system of integral equations. Chen’s method, applied directly to our specifications, would amount to a complexity in $O(k^4|S|\lambda_c + k^9|S|^3)$, where $c$ is the largest difference between time constraints (roughly comparable to $b$ in Lem. 6.3). Note that our algorithm has only a complexity in $O(|R|k \cdot \lambda b)$ if $b = \sup I_{k-1} < \infty$ or $O(|R|k \cdot \lambda b + |S|^3)$ otherwise.

8. Conclusion

In this paper we have proposed an effective approximation algorithm for CSL with a multiple until operator. We believe that it is the centerpiece of a broadly applicable full CSL model checker.

The technique we have developed in this paper can also be applied to a subclass of PCTL* formulas. Let $\varphi = f_1 U_{I_1} f_2 U_{I_2} \ldots f_k$ be a CSL path formula. As we have seen in the paper, in case of $I_1 = \ldots = I_{k-1} = [0, \infty)$, our multiple until formula $f_1 U_{I_1} f_2 U_{I_2} \ldots f_k$ corresponds to the LTL formula $f_1 U (f_2 U (\ldots (f_{k-1} U f_k) \ldots ))$. In general, $\varphi$ is similar to a step-bounded LTL formula $\varphi = f_1 U_{[i_1,j_1]} f_2 U_{[i_2,j_2]} \ldots f_k$ with $i_1, j_1, \ldots$ integers specifying the step bounds. Such step-bounded until LTL formulas can be first transformed into nested next-state formulas, for example we have: $f_1 U_{[2,3]} f_2 = f_1 \land X(f_1 \land X(f_2 \lor (f_1 \land X(f_2))))$. The approach we have established in this paper can be adapted slightly to handle this kind of formulas in complexity linear in $j_{k-1}$ (assuming $j_{k-1} < \infty$).

We conclude the paper by noting the connection of our DFA-based approach with the classical Büchi-automaton-based LTL model checking algorithm by Vardi and Wolper [21]. The LTL formula $\varphi$ is first transformed into a Büchi automaton – of exponential size in the worst case – accepting exactly the words satisfying $\varphi$. Then, model checking LTL can be reduced to automata-theoretic questions in the product. Instead of Büchi automata accepting infinite runs, we only need DFAs, which is due to the simple form of the multiple until formula: it does not encompass the full expressivity of LTL. This simplification, moreover, allows us to get a DFA whose number of states is only linear in the length of the CSL formula, and the size of the product automaton is then linear in both the size of the CTMC and the length of the CSL formula.

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References


Figure 6: A deterministic CSL\textsuperscript{TA}-timed automaton for $f_1 U_{[a_1,b_1)} f_2 U_{[a_2,b_2)} f_3 U_{[a_3,b_3)} f_4$.

**Appendix A. Translating Fig. 2 to a DTA for CSL\textsuperscript{TA}**

As mentioned in Section 7, Donatelli et al. [8] have extended CSL such that path properties can be expressed via a timed automaton. In Fig. 6 we include a DTA corresponding to the formula $f_1 U_{[a_1,b_1)} f_2 U_{[a_2,b_2)} f_3 U_{[a_3,b_3)} f_4$ with $0 < a_1 < a_2 < a_3 < b_1 < b_2 < b_3$. Dashed lines correspond to transition edges; solid lines to boundary edges. The automaton has a single clock $x$. The state label $q_{i,I}$ indicates that time $t_{i-1}$ has passed and that the current time is in the interval $I$. The guard $x < b_3$ is needed in states without boundary edge to ensure that $q_4$ is not entered too late.

The automaton illustrates that CSL\textsuperscript{TA} may need a DTA with $O(k^2)$ states, where $k$ is the number of phases in the multiple until-formula.