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RESIDUAL PROPERTIES IN THE THEORY OF POLYNOMIAL MAPS

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Residual Properties in the Theory of Polynomial Maps

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Abstract

This paper provides tools to reduce questions about polynomial maps over a ring R to questions about polynomial maps over the residue fields $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ of the ring R. As an application of this theory, the Abhyankar-Moh Theorem is generalized to arbitrary $\mathbb{Q}$-algebras.

In order to solve questions about polynomial maps, derivations, or other objects in the theory of polynomial maps, over arbitrary commutative rings[1] R, it is often convenient to reduce such questions to the case that R is a field. For example, if one has a polynomial map F over a domain R with $\det J(F) = 1$ and one wants to know if it is invertible, then one only needs to check if F is invertible over the quotient field $Q(R)$ of R.

This paper tries to reduce such questions over a ring R to questions about fields by looking at the residue fields $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R.

Section 1 gives an inventarisation of these so-called “residual properties”. There are two main results here. The first one is that “being locally nilpotent” is a residual property of a derivation for a Noetherian ring R and in dimension two also for arbitrary $\mathbb{Q}$-algebras (Propositions 4 and 6). The second one is that “being a coordinate” is a residual property of a polynomial over an arbitrary $\mathbb{Q}$-algebra in dimension two (Proposition 12). This generalizes a result from Bhatwadekar and Dutta in [BD93].

Section 2 shows the power of these properties by generalizing the Abhyankar-Moh Theorem (see [AM75]) to arbitrary $\mathbb{Q}$-algebras.

1 Residual Properties

Notation 1. Let R be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $\mathfrak{a}$ be an ideal of $R[X]$ and let $\mathfrak{p}$ be a prime ideal of R. Then $\bar{\mathfrak{a}_{\mathfrak{p}}}$ denotes the ideal generated by the image of the ideal $\mathfrak{a}$ in $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]$ under the map induced by the natural homomorphism $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

A similar notation will in the sequel be used for polynomials, polynomial maps, and derivations.

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[1] In this paper, all rings will be commutative and have a unit element.

Proposition 2. *Let R be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $\mathfrak{a}$ be an ideal of $R[X]$. Then the following two statements are equivalent:*

1. $\mathfrak{a} = R[X]$;
2. *for every* $\mathfrak{p} \in \operatorname{Spec}(R)$, $\bar{\mathfrak{a}}_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]$.

Proof. The implication $1 \Rightarrow 2$ is trivial. So assume that for every $\mathfrak{p} \in \operatorname{Spec}(R)$, $\bar{\mathfrak{a}}_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]$.

> Assume that $\mathfrak{a} \neq R[X]$. Then there is some maximal ideal $\mathfrak{m}$ of $R[X]$ such that $\mathfrak{a} \subseteq \mathfrak{m}$. Let $\mathfrak{p} := \mathfrak{m} \cap R$. This is a prime ideal of R.
>
> Using the natural isomorphism between $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $Q(R/\mathfrak{p})$, one can easily see that $1 \in \bar{\mathfrak{a}}_{\mathfrak{p}}$ means that there are an $r \in R \setminus \mathfrak{p}$, polynomials $g_1, \ldots, g_s \in R[X]$, and polynomials $f_1, \ldots, f_s \in \mathfrak{a}$ such that
>
> $$r \equiv g_1 f_1 + \cdots + g_s f_s \pmod{\mathfrak{p}[X]}.$$
>
> Because $\mathfrak{a} \subseteq \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$ this implies, however, that $r \in \mathfrak{m}$. Because also $r \in R$, this contradicts the fact that $r \notin \mathfrak{p} = m \cap R$.

Therefore $\mathfrak{a} = R[X]$. □

The following lemma is Lemma 2.1.15 from [Ess00]. It shows that in order to check if an R-derivation D of $R[X]$ is locally nilpotent, the ring R can assumed to be reduced.

Lemma 3. *Let R be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let $D \in \operatorname{Der}_R(R[X])$. Let η be the nilradical of R and denote by D/η the derivation on $R/\eta[X]$ induced by D. Assume that D/η is locally nilpotent. Then D is locally nilpotent as well.*

Proof. Let R' be the subring of R generated by all coefficients appearing in the polynomials $D(X_1), \ldots, D(X_n)$. Then R' is Noetherian and D restricts to a derivation D' on $R'[X]$. Note that D is locally nilpotent if and only if D' is. Also note that the nilradical η' of R' equals $\eta \cap R'$ and that hence D/η is locally nilpotent if and only if D'/η' is. Therefore it is possible to assume, without loss of generality, that the ring R is Noetherian.

By induction on n it will follow that for every $n \in \mathbb{N}^*$ and every $g \in R[X]$, there exists an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^n[X]$. For $n = 1$, this is just the assumption that D/η is locally nilpotent. Now assume that the claim holds for n and consider $g \in R[X]$. By induction hypothesis $D^N(g) \in \eta^n[X]$ for some $N \in \mathbb{N}$, say $D^N(g) = \sum_{\alpha \in A} c_\alpha X^\alpha$, for certain $c_\alpha \in \eta^n$. Since D/η is locally nilpotent, there are $M_\alpha \in \mathbb{N}$ such that $D^{M_\alpha}(X^\alpha) \in \eta[X]$. Taking $M := N + \max_{\alpha \in A} M_\alpha$, it follows that $D^M(g) \in \eta^{n+1}[X]$.

Because R is Noetherian, its nilradical η is finitely generated and hence there is an $e \in \mathbb{N}$ such that $\eta^e = (0)$. Consequently, for every $g \in R[X]$ there is an $N \in \mathbb{N}$ such that $D^N(g) \in \eta^e[X] = (0)$. Therefore D is locally nilpotent. □

Proposition 4. *Let R be a Noetherian ring, $n \in \mathbb{N}^*$, and consider the polynomial ring $R[X] := R[X_1, \dots, X_n]$. Let $D \in \mathrm{Der}_R(R[X])$. Then the following two statements are equivalent:*

1. *D is locally nilpotent;*

2. *for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the derivation $\bar{D_\mathfrak{p}} \in \mathrm{Der}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X])$ is locally nilpotent.*

Proof. The implication $1 \Rightarrow 2$ is clear. So assume that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the derivation $\bar{D_\mathfrak{p}} \in \mathrm{Der}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X])$ is locally nilpotent.

> Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Consider the ring $R/\mathfrak{p}$ and the derivation $D/\mathfrak{p}$ induced by D on $R/\mathfrak{p}[X]$. Because $Q(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and the derivation $\bar{D_\mathfrak{p}}$ on $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X]$ is locally nilpotent, $D/\mathfrak{p}$ is locally nilpotent as well.

So $D/\mathfrak{p}$ is locally nilpotent for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Since R is Noetherian, its nilradical η is a finite intersection of prime ideals, say $\eta = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$. Because $D/\mathfrak{p}_i$ is locally nilpotent for every $i \in \{1, \dots, s\}$, D/η is locally nilpotent too. Namely, let $g \in R[X]$. Then there is an $N_i \in \mathbb{N}$ such that $D^{N_i}(g) \in \mathfrak{p}_i$, for every $i \in \{1, \dots, s\}$. Taking $N := \max_{i \in \{1,\dots,s\}} N_i$ it follows that $D^N(g) \in \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n = \eta$.

By Lemma 3, D is locally nilpotent. □

In dimension two, the condition that the ring R is Noetherian can be avoided. In order to prove this, the following result is needed ([Ess00], Theorem 1.3.49).

Lemma 5. *Let k be a field of characteristic 0 and take $D \in \mathrm{Der}_k(k[X, Y])$. Assume that $D \neq 0$ and let*

$$d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}.$$

(Here, by convention, the degree of 0 is taken to be $-\infty$.) Then D is locally nilpotent if and only if $D^{d+2}(X) = D^{d+2}(Y) = 0$. □

Proposition 6. *Let R be a $\mathbb{Q}$-algebra and $D \in \mathrm{Der}_R(R[X, Y])$. Then the following two statements are equivalent:*

1. *D is locally nilpotent;*

2. *for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the derivation $\bar{D_\mathfrak{p}} \in \mathrm{Der}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X, Y])$ is locally nilpotent.*

Proof. The implication $1 \Rightarrow 2$ is once again clear, so assume that for every $\mathfrak{p} \in \mathrm{Spec}(R)$, the derivation $\bar{D_\mathfrak{p}} \in \mathrm{Der}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X, Y])$ is locally nilpotent. Take $d := \max\{\deg_X D(X), \deg_X D(Y), \deg_Y D(X), \deg_Y D(Y)\}$.

> Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Just as in the proof of Proposition 4, consider the ring $R/\mathfrak{p}$ and the locally nilpotent derivation $D/\mathfrak{p}$ induced by D on $R/\mathfrak{p}[X, Y]$ and on

$Q(R/\mathfrak{p})[X,Y]$. Note that $Q(R/\mathfrak{p})$ is a field of characteristic 0, since R is a $\mathbb{Q}$-algebra. Hence, by the previous lemma,

$$(D/\mathfrak{p})^{d+2}(X) = (D/\mathfrak{p})^{d+2}(Y) = 0,$$

or, differently said, $D^{d+2}(X) \in \mathfrak{p}[X,Y]$ and $D^{d+2}(Y) \in \mathfrak{p}[X,Y]$.

Hence $D^{d+2}(X) \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathfrak{p} = \eta$, the nilradical of R, and also $D^{d+2}(Y) \in \eta$. This means that D/η is locally nilpotent and hence, by Lemma 3, D is locally nilpotent too. □

In dimension two, the concept of coordinate over a $\mathbb{Q}$-algebra is also a residual concept. In order to prove this, a characterization of coordinates in terms of locally nilpotent derivations is needed. The general form presented here as Proposition 11 comes essentially from [BEM99].

Definition 7. Let R be a ring and $n \in \mathbb{N}^*$. A polynomial $f \in R[X] := R[X_1, \ldots, X_n]$ is called a *coordinate* over R if it is a component of a polynomial automorphism over R, *i.e.*, if there are polynomials $f_1, \ldots, f_{n-1} \in R[X]$ such that $(f_1, \ldots, f_{n-1}, f) \in \mathrm{Aut}_R(R[X])$.

Definition 8. Let R be a ring, $n \in \mathbb{N}^*$, and $R[X] := R[X_1, \ldots, X_n]$. Let D be an R-derivation of $R[X]$. A polynomial $s \in R[X]$ is called a *slice* of D if $D(s) = 1$.

The following lemma is a part of Theorem 3.7 of [BEM99].

Lemma 9. *Let R be a $\mathbb{Q}$-algebra. Then any locally nilpotent R-derivation on $R[X,Y]$ with divergence 0 (i.e., with $\partial_X(D(X)) + \partial_Y(D(Y)) = 0$) and 1 in the ideal generated by $D(X)$ and $D(Y)$ has a slice.* □

Lemma 10. *Let R be a $\mathbb{Q}$-algebra, $f \in R[X,Y]$, and let D be the R-derivation $f_Y\partial_X - f_X\partial_Y$ on $R[X,Y]$. Assume that f is a coordinate in $R[X,Y]$. Then D is locally nilpotent and has a slice.*

Proof. Let $g \in R[X,Y]$ be a polynomial such that (f,g) is an invertible polynomial map over R. Let η denote the nilradical of R. To avoid notational clutter, reduction modulo this nilradical η of R or modulo the ideal $\eta[X,Y]$ of $R[X,Y]$ will be denoted by an overline.

Then $(\bar{f}, \bar{g})$ is an invertible polynomial map over $\bar{R}$ and hence $\bar{R}[X,Y] = \bar{R}[\bar{f}, \bar{g}]$. Now note that

$$\bar{D}(\bar{g}) = \det J(\bar{f}, \bar{g}) \in \bar{R}[X,Y]^* = \bar{R}^*$$

and so $\bar{D}^2(\bar{g}) = 0$. Also $\bar{D}(\bar{f}) = 0$ and therefore $\bar{D}$ is locally nilpotent. By Lemma 3, D is locally nilpotent too.

So in view of Lemma 9, the only thing left to show is that 1 is an element of the ideal generated by $D(X)$ and $D(Y)$. Now $\det J(f,g) \in R[X,Y]^*$. So $g_X f_Y - g_Y f_X$ is invertible. Hence the ideal generated by $D(X) = f_X$ and $D(Y) = -f_X$ contains an invertible element and consequently contains 1. Therefore, as observed, D has a slice. □

Proposition 11. *Let R be a $\mathbb{Q}$-algebra, $f \in R[X,Y]$, and $D := f_Y\partial_X - f_X\partial_Y \in \mathrm{Der}_R(R[X,Y])$. Then the following three statements are equivalent:*

1. *D is locally nilpotent and $(f_X, f_Y) = R[X,Y]$;*
2. *D is locally nilpotent, has a slice, and $R[X,Y]^D = R[f]$.*
3. *f is a coordinate.*

Proof. The equivalence of 1 and 2 is Theorem 3.7 of [BEM99] and the equivalence of 2 and 3 follows from Proposition 2.1 of [Wri81] and Lemma 10. ☐

Now it is possible to prove that "being a coordinate" is a residual property for arbitrary $\mathbb{Q}$-algebras. This extends the result from [BD93], which proves the following statement for a Noetherian domain of characteristic 0 that either contains $\mathbb{Q}$ or for which R/η is seminormal.

Proposition 12. *Let R be a $\mathbb{Q}$-algebra and $f \in R[X,Y]$. Then the following two statements are equivalent:*

1. *f is a coordinate over R in $R[X,Y]$;*
2. *for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $\bar{f}_\mathfrak{p}$ is a coordinate over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ in $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X,Y]$.*

Proof. This is now an easy consequence of the equivalence $1 \Leftrightarrow 3$ from the previous proposition and from Propositions 2 and 6. ☐

The condition that R is a $\mathbb{Q}$-algebra cannot simply be dropped in this proposition. For Bhatwadekar and Dutta have constructed the following example in [BD93]. Take $R := \mathbb{Z}_{2\mathbb{Z}}[2\sqrt{2}]$ and take

$$f := X - 2Y(\sqrt{2}X - Y^2) + \sqrt{2}(\sqrt{2}X - Y^2)^2 - \sqrt{2}(Y - \sqrt{2}(\sqrt{2}X - Y^2))^4.$$

Then $\bar{f}_\mathfrak{p}$ is a coordinate over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, for every prime ideal $\mathfrak{p}$ of R (there are only two), but f itself is not a coordinate over R.

2 The Abhyankar-Moh Theorem

Using the theory in the previous section, the Abhyankar-Moh Theorem ([AM75], Theorem 1.1) can be generalized to arbitrary $\mathbb{Q}$-algebras.

The Abhyankar-Moh Theorem states the following.

Theorem 13. *Let k be a field of characteristic 0 and let $f, g \in k[T]$, $f, g \neq 0$. Assume that $k[f,g] = k[T]$. Then $\deg(f) \mid \deg(g)$ or conversely.*

(Actually, the theorem as formulated in [AM75] is slightly more general. It also holds for $\mathrm{char}(k) \neq 0$ provided that either $\deg(f)$ or $\deg(g)$ is not divisible by $\mathrm{char}(k)$. For the present situation, that information is not needed.) It can also be formulated in the following way ([AM75], Theorem 1.2).

Theorem 14. *Let k be a field of characteristic 0 and let $f \in k[X,Y]$. Assume that $k[X,Y]/(f) \cong k[T]$. Then f is a coordinate in $k[X,Y]$.*

This is the formulation that will actually be generalized.

Theorem 15. *Let R be a $\mathbb{Q}$-algebra and let $f \in R[X,Y]$. Assume that $R[X,Y]/(f) \cong R[T]$. Then f is a coordinate in $R[X,Y]$.*

Proof. Let D be the derivation $f_Y \partial_X - f_X \partial_Y$ on $R[X,Y]$.

> Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X,Y]/(\bar{f}_\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[T]$. Since R is a $\mathbb{Q}$-algebra, $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field of characteristic 0. So Theorem 14 implies that $\bar{f}_\mathfrak{p}$ is a coordinate over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ in $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X,Y]$.

Now Proposition 12 implies that f itself is a coordinate over R. □

Note that [BD93] also gives such a generalization of the Abhyankar-Moh Theorem, but only for Noetherian domains R of characteristic 0 which either contain $\mathbb{Q}$ or which satisfy that R/η is seminormal.

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